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On some components of moduli space of surfaces of general type

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0. Introduction

Let S be a minimal surface of general type and let $\mathscr{M}(S)$ be the moduli space of surfaces of general type homeomorphic (by an orientation preserving homeomorphism) to S. $\mathscr{M}(S)$ is a quasi projective variety by the well known theorem of Gieseker [Gi].

Let $k_S \in H^2(S, \mathbb{Z})$ be the first Chern class of the canonical bundle of S and let $r(S)$ its divisibility, i.e.

$$r(S) = \max\{r \in \mathbb{N} \mid k_S = rc \text{ for some } c \in H^2(S, \mathbb{Z})\}$$

Obviously if $S' \in \mathscr{M}(S)$ is in the same connected component of S then there exists an orientation preserving diffeomorphism $f: S' \to S$ such that $f^*(k_S) = k_{S'}$ and $r(S) = r(S')$.

Catanese [Ca3] first proved that in general $\mathscr{M}(S)$ is not connected giving homeomorphic surfaces of general type with different divisibility r. Similarly, using the fact that for surfaces with "big monodromy" $r(S)$ is a differential invariant, Friedman, Morgan and Moishezon gave the first examples of surfaces of general type homeomorphic but not diffeomorphic [F-M-M]. Moreover it is possible that $r(S)$ is a differential invariant of minimal surfaces of general type [F-M].

Define $\mathscr{M}_d(S) = \{S' \in \mathscr{M}(S) \mid r(S) = r(S')\}$, it is natural to ask whether $\mathscr{M}_d(S)$ is connected. In this paper we show that the answer is no, more precisely we prove (Theorem 11).

THEOREM A. *For every $k > 0$ there exists a simply connected minimal surface of general type S such that $\mathscr{M}_d(S)$ has at least k connected components.*

From our proof it follows moreover that the k connected components have different dimension. Here we study a particular class of surfaces introduced by Catanese [Ca1], [Ca2], [Ca3] and called "simple bihyperelliptic surfaces".

Denote $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let $\mathcal{O}_X(a, b)$ be the line bundle on X whose sections are bihomogeneous polynomials of bidegree a, b. A minimal surface of general type is said to be simple bihyperelliptic of type $(a, b)(n, m)$ if its canonical model is defined in $\mathcal{O}_X(a, b) \oplus \mathcal{O}_X(n, m)$ by the equation

$$z^2 = f(x, y) \qquad w^2 = g(x, y) \tag{0.1}$$

where f, g are bihomogeneous polynomials of respective bidegree $(2a, 2b)$, $(2n, 2m)$. If $a, b, c, d > 0$ then simple bihyperelliptic surfaces of type (a, b), (c, d) are simply connected ([Ca1] Th. 3.8).

In [Ca1] Th. 3.8, is proved that if S is a simple bihyperelliptic surface of type (a, b), (n, m) with $a, b, n, m \geqslant 4$ then the four numbers a, b, n, m are determined, up permutations induced by changing the role of x and y or f and g in (0.1), by the surface S.

In the same notation of [Ca2], if $a > 2n$, $m > 2b$ let $\hat{N}_{(a,b),(n,m)}$ be the subset of moduli space of simple bihyperelliptic surfaces of type (a, b), (n, m). Here we prove, as conjectured in [Ca2], the following

THEOREM B. *If* $a \geqslant \max(2n + 1,\ b + 2)$, $m \geqslant \max(2b + 1,\ n + 2)$ *then* $\hat{N} = \hat{N}_{(a,b),(n,m)}$ *is a connected component of moduli space.*

Theorem B is an easy consequence of [Ca2] Cor. 4.4 and Lemma 9 that follows from the technical Prop. 7. Theorem A is then an easy consequence of Theorem B and Freedman's results on four-dimensional manifolds.

We remark that an interesting problem is to determine whether simple bihyperelliptic surfaces belonging to different components of $\mathcal{M}_d$ have the same differential structure, a tentative to answer to this problem will be the subject of our next papers.

This work is intended to be a continuation of [Ca1], [Ca2], and [Ca3] where general properties of moduli space of surfaces of general type are investigated. We shall often use the results proved in these papers.

Notation

For a proper algebraic variety Y defined over $\mathbb{C}$ we denote:

$\theta_Y = (\Omega^1_Y)^\vee$ the tangent sheaf.

T^i_Y, $\mathcal{T}^i_Y$ the global and local deformation functors (cf. [Fl]). We recall that if Y is reduced then $T^i = \mathrm{Ext}^i_{O_Y}(\Omega^1_Y, \mathcal{O}_Y)$ and $\mathcal{T}^i = \mathcal{E}xt^i_{O_Y}(\Omega^1_Y, \mathcal{O}_Y)$.

$Def(Y)$ the base space of the semiuniversal deformation of Y.

For $y \in Y$, $M_{y,Y}$ is the maximal ideal of the local ring $\mathcal{O}_{Y,y}$ and $T_{y,Y} = (M_{y,Y}/M^2_{y,Y})^\vee$ is the Zariski tangent space at y.

1. Normal bidouble covers of surfaces and their natural deformations

Let X be a smooth algebraic surface and let $\pi\colon Y \to X$ be a Galois covering with group $G = (\mathbb{Z}/2\mathbb{Z})^2 = \{1, \sigma_1, \sigma_2, \sigma_3\}$. We assume that Y is a normal surface.

Let R_i be the divisorial part of $Fix(\sigma_i) = \{p \in Y \mid \sigma_i(p) = p\}$ and $D_i = \pi(R_i)$. By purity of branch locus the Weil divisor $R = R_i \cup R_2 \cup R_3$ is the set of points where π is branched.

Since Y is normal the direct image sheaf $\pi_* \mathcal{O}_Y$ is locally free and we have a character decomposition

$$\pi_* \mathcal{O}_Y = \mathcal{O}_X \oplus \left(\bigoplus_i \mathcal{O}_X(-L_i) \right)$$

where L_1, L_2, L_3 are line bundle on X and $\mathcal{O}_X \oplus \mathcal{O}_X(-L_i)$ is the σ_i-invariant subsheaf of $\pi_* \mathcal{O}_Y$. We have (cf. [Ca1], § 2)

$$D_k + L_k \equiv L_i + L_j \quad 2L_i \equiv D_j + D_k \quad \{i, j, k\} = \{1, 2, 3\}$$

where $\equiv$ means rational equivalence. If V is the vector bundle $L_1 \oplus L_2 \oplus L_3$ with fibres coordinates w_1, w_2, w_3, then we can realize Y in V as the zero locus of the ideal sheaf $I_Y \subset \mathcal{O}_V$ generated by the six equations

$$\begin{matrix} w_i^2 - x_j x_k = 0 \\ w_k x_k - w_i w_j = 0 \end{matrix} \quad \{i, j, k\} = \{1, 2, 3\} \tag{1.1}$$

where $x_i \in H^0(\mathcal{O}_X(D_i))$ is a section defining D_i.

All these facts are proved in [Ca1], Catanese suppose that Y is a smooth surface but his proof is also valid in our more general situation. It is however easy to see that Y is smooth if and only if the curves D_i are smooth and the divisor $D = D_1 \cup D_2 \cup D_3$ has only ordinary double points as singularities.

G acts on the fibres of V in the following way:

$$\sigma_i : w_i \to w_i \quad w_j \to -w_j \quad w_k \to -w_k$$

and R_i is the subset of Y defined by $x_i = w_j = w_k = 0$.

PROPOSITION 1. *In the notation above are equivalent:*

(a) $D_1 \cap D_2 \cap D_3 = \varnothing$.
(b) R_i *is a Cartier divisor for every* i.
(c) $\dim T_{q,Y} \leqslant 4$ *for every* $q \in Y$
(d) Y *is locally complete intersection in* V.

Proof (a) $\Rightarrow$ (d). If $q \in Y$, $p = \pi(q)$ and $x_k(p) \neq 0$ then Y is locally defined by

$$w_k = \frac{w_i w_j}{x_k}$$
$$w_i^2 = x_j x_k \tag{1.2}$$

$$x_i = \frac{w_j^2}{x_k}$$

(a⇒b). The ideal of R_i is generated by (w_j, w_k, x_i) and if, for example $q \in R_i x_k(\pi(q)) \neq 0$ then from (1.2) it follows that the ideal or R_i is generated in Y by w_j.

(b)⇒(c). If $q \notin R$ then dim $T_{q,Y} = 2$. Suppose $q \in R_i$ and dim $T_{q,Y} = 5$, then w_j, w_k are linearly independent in $T^\vee_{q,Y}$ and the ideal (w_j, w_k, x_i) cannot be principal at q.

(c)⇒(a) If $q \in Y$ and $x_1(q) = x_2(q) = x_3(q) = 0$ then all the equation that define Y are in $M^2_{q,V}$, hence $T_{q,Y} = T_{q,V}$.

(d)⇒(a) Take a point $q \in Y$ such that $x_i(q) = 0$ $(i = 1, 2, 3)$ and let us suppose $I_{Y,q} = (f_1, f_2, f_3)$, this will lead to a contradiction. Since the ideal of Y at q is contained in M^2 (here $M = M_{q,V}$), the vector subspace of M^2/M^3 generated by $I_{Y,q}$ has dimension at most equal to three, but it easy to see that the six equations (1.1) are linearly independent in M^2/M^3. □

Since in the applications we are principally interested to the case where Y has at most rational double points, from now on we always assume that $D_1 \cap D_2 \cap D_3 = \emptyset$.

Let $N_Y = (I_Y/I_Y^2)^\vee$ be the normal sheaf and let $p_i: \mathcal{O}_Y \to \mathcal{O}_{R_i}$ be the projection map.

THEOREM 2. *If $D_1 \cap D_2 \cap D_3 = \emptyset$ then there exists a commutative diagram of $\mathcal{O}_Y$-modules with exact rows and columns.*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & \pi^*V \arrow[r, equal] \arrow[d] & \pi^*V \arrow[d] & & \\
0 \arrow[r] & \theta_Y \arrow[r] \arrow[d, equal] & \theta_V \otimes \mathcal{O}_Y \arrow[r, "\eta"] \arrow[d, "\varphi"] & N_Y \arrow[r, "\mu"] \arrow[d, "\psi"] & \mathcal{T}^1_Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \theta_Y \arrow[r, "\alpha"] & \pi^*\theta_X \arrow[r, "\beta"] \arrow[d] & \bigoplus_i \mathcal{O}_{R_i}(\pi^*D_i) \arrow[r, "\gamma"] \arrow[d] & \mathcal{T}^1_Y \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd} \tag{1.3}$$

The proof of Theorem 2 will be a consequence of the following two lemmas. We first note that $\theta_Y = \mathrm{Der}(\mathcal{O}_Y, \mathcal{O}_Y)$, $\pi^*\theta_X = \mathrm{Der}(\pi^{-1}\mathcal{O}_X, \mathcal{O}_Y)$ and α is defined in the obvious way. Moreover α is an injective map because π is a finite morphism.

If u_1, u_2 are local coordinates on X we set

$$\varphi\left(\frac{\partial}{\partial w_i}\right) = 0,\ \varphi\left(\frac{\partial}{\partial u_i}\right) = \frac{\partial}{\partial u_i}$$

It is clear that $\pi^* V = \ker \varphi$. The upper row is a standard exact sequence [Ar].

LEMMA 3. *There exists a commutative diagram*

$$\begin{array}{ccc} \theta_V \otimes \mathcal{O}_Y & \xrightarrow{\ \eta\ } & \mathcal{H}om_{\mathcal{O}_Y}(I_Y/I_Y^2,\ \mathcal{O}_Y) = N_Y \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi_i} \\ \pi^*\theta_X = Der(\pi^{-1}\mathcal{O}_X,\ \mathcal{O}_Y) & \xrightarrow{\ \beta_i\ } & \mathcal{H}om_{\mathcal{O}_Y}(\pi^*\mathcal{O}_X(-D_i),\ \mathcal{O}_{R_i}) = \mathcal{O}_{R_i}(\pi^* D_i) \end{array} \tag{1.4}$$

Proof. For every $a \in Der(\pi^{-1}\mathcal{O}_X, \mathcal{O}_Y)$ and $f \in \mathcal{O}_X(-D_i)$ we define $\beta_i(a)(f) = p_i(a(f))$ and then we extend by $\mathcal{O}_Y$-linearity. β is a well defined map and $\beta_i \circ \alpha = 0$ since $\pi^* D_i = 2R_i$.

Let $r \in \mathcal{O}_Y$ be a local equation of R_i, if $f \in \mathcal{O}_X(-D_i)$ then $f \in I_Y + (r^2)$ and we can write $f = a + br^2$ with $a \in I_Y$. For $v \in N_Y$ we then define $\psi_i(v)(f) = p_i(v(a))$.

If s is another local equation of R_i and $f = c + ds^2$ then $p_i(v(a - c)) = 0$. In fact we have $s = hr + e$ with $e \in I_Y$ and

$$a - c = ds^2 - br^2 = r(dh^2 r + 2dhe - br) + de^2,$$

since I_Y is a prime ideal necessarily

$$dh^2 r + 2dhe - br \in I_Y$$

and then

$$p_i(v(a - c)) = 0.$$

In order to show that (1.4) commutes it suffices to note that, if for example $x_k \neq 0$, then w_j is a local equation of R_i and $\psi_i(v)(x_i) = p_i(v(x_i - w_j^2/x_k))$. Thus

$$\psi_i\left(\frac{\partial}{\partial w_h}\right)(x_i) = 0 \quad h = 1, 2, 3$$

$$\psi_i\left(\frac{\partial}{\partial u_h}\right)(x_i) = p_i\left(\frac{\partial x_i}{\partial u_h}\right) = \beta_i\left(\frac{\partial}{\partial u_h}\right)(x_i) \quad h = 1, 2 \qquad \square$$

Define $\beta = \bigoplus_i \beta_i$, $\psi = \bigoplus_i \psi_i$.

LEMMA 4. *ψ is a surjective map and* $\ker \psi = \eta(\ker \varphi)$, *in particular* $\ker \psi \subset \ker \mu$ *and we can define γ as in* (1.3).

Proof. By Lemma 3 $\eta(\ker \varphi) \subset \ker \psi$.

If $\psi(v) = 0$ and $x_k \neq 0$ then locally I_Y/I_Y^2 is a free $\mathcal{O}_Y$-module generated by

$$\left(x_i - \frac{w_j^2}{x_k}\right), \left(x_j - \frac{w_i^2}{x_k}\right), \left(w_k - \frac{w_i w_j}{x_k}\right)$$

Moreover

$$v\left(x_i - \frac{w_j^2}{x_k}\right) = w_j h_i, \; v\left(x_i - \frac{w_j^2}{x_k}\right) = w_i h_j$$

If we set

$$v' = v + \frac{h_i x_k}{2}\frac{\partial}{\partial w_j} + \frac{h_j x_k}{2}\frac{\partial}{\partial w_i}$$

then

$$v'\left(x_i - \frac{w_j^2}{x_k}\right) = 0, \; v'\left(x_i - \frac{w_j^2}{x_k}\right) = 0, \; v'\left(w_k - \frac{w_i w_j}{x_k}\right) = h_k$$

then

$$v' - h_k \frac{\partial}{\partial w_k} = 0 \qquad \square$$

Y is locally complete intersection in V, therefore there is an exact sequence

$$0 \to I_Y/I_Y^2 \to \Omega_V^1 \otimes \mathcal{O}_Y \to \Omega_Y^1 \to 0 \tag{1.5}$$

If we apply the functor $\mathscr{H}om$ we get the upper row of (1.3), if we apply Hom we get the exact sequence

$$H^0(0_V \otimes \mathcal{O}_Y) \xrightarrow{H^0(\eta)} H^0(N_Y) \xrightarrow{k} T_Y^1$$

If we apply the left exact functor H^0 to (1.3) we see that $\ker H^0(\psi) \subset Im H^0(\eta) = \ker k$ and there exist a map $\varepsilon : H^0(\bigoplus_i \mathcal{O}_{R_i}(\pi^* D_i)) \to T_Y^1$ such that $\varepsilon \circ H^0(\psi) = k$.

COROLLARY 5. *If $H^1(\pi^*\theta_X) = 0$ then ε is surjective.*

Proof. If $\mathscr{F} = \ker\gamma$ then there exists a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
H^0(\pi^*\theta_X) \arrow[r] \arrow[d, equal] & H^0(\mathscr{F}) \arrow[r] \arrow[d] & H^1(\theta_Y) \arrow[r] \arrow[d] & H^1(\pi^*\theta_X) \\
H^0(\pi^*\theta_X) \arrow[r, "H^0(\beta)"] & H^0\left(\bigoplus_i \mathcal{O}_{R_i}(\pi^*D_i)\right) \arrow[r, "\varepsilon"] \arrow[d] & T_Y^1 \arrow[d] & \\
 & H^0(\mathscr{T}_Y^1) \arrow[r, equal] \arrow[d] & H^0(\mathscr{T}_Y^1) \arrow[d] & \\
H^1(\pi^*\theta_X) \arrow[r] & H^1(\mathscr{F}) \arrow[r] & H^2(\theta_Y) &
\end{tikzcd} \tag{1.6}$$

where the right column is the first part of the cotangent spectral sequence. The conclusion follows by chasing through the diagram. □

We note that $\pi_*\mathcal{O}_{R_i} = \mathcal{O}_{D_i} \oplus \mathcal{O}_{D_i}(-L_i)$ and

$$H^0\left(\bigoplus_i \mathcal{O}_{R_i}(\pi^*D_i)\right) = \bigoplus_i (H^0(\mathcal{O}_{D_i}(D_i)) \oplus H^0(\mathcal{O}_{D_i}(D_i - L_i)))$$

moreover $H^1(\pi^*\theta_X) = H^1(\theta_X) \oplus (\oplus_i H^1(\theta_X(-L_i)))$.

More generally we can include the map ε into an exact sequence of cohomology groups, this can be done as follows. We first prove that $\Omega^1_{Y/X} = \oplus_i \mathcal{O}_{R_i}(-R_i)$, then we consider the exact sequence

$$0 \to \pi^*\Omega^1_X \to \Omega^1_Y \to \bigoplus_i \mathcal{O}_{R_i}(-R_i) \to 0 \tag{1.7}$$

(recall that $\pi^*\Omega^1_X$ is locally free and $(\pi^*\Omega^1_X)^\vee = \pi^*\theta_X$). Applying the functor $\mathrm{Hom}_{\mathcal{O}_Y}(-, \mathcal{O}_Y)$ we get a long exact sequence

$$0 \to H^0(\theta_Y) \to H^0(\pi^*\theta_X) \to \bigoplus_i \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{O}_{R_i}(-R_i), \mathcal{O}_Y) \to T_Y^1 \to H^1(\pi^*\theta_X) \to \cdots \tag{1.8}$$

Since R_i is a Cartier divisor its local equation is a regular element of $\mathcal{O}_Y$, using local commutative algebra ([Ma2] §18, Lemma 2) we have for every $i \geqslant -1$

$$\mathscr{E}xt^{i+1}_{\mathcal{O}_Y}(\mathcal{O}_{R_i}(-R_i), \mathcal{O}_Y) = \mathscr{E}xt^{i}_{\mathcal{O}_{R_i}}(\mathcal{O}_{R_i}(-R_i), \mathcal{O}_{R_i}(R_i))\begin{cases} 0 & \text{if } i \neq 0 \\ \mathcal{O}_{R_i}(\pi^* D_i) & \text{if } i = 0 \end{cases}$$

and (1.8) becomes

$$\begin{aligned} 0 \to H^0(\mathcal{O}_Y) \to H^0(\pi^* \mathcal{O}_X) \to \bigoplus_i H^0(\mathcal{O}_{R_i}(\pi^* D_i)) \xrightarrow{\varepsilon} T^1_Y \xrightarrow{\sigma} H^1(\pi^* \mathcal{O}_X) \\ \to \bigoplus_i H^1(\mathcal{O}_{R_i}(\pi^* D_i)) \to \cdots \end{aligned} \tag{1.9}$$

Let $Def_V(Y)$ be the space of embedded deformations of Y in V. It is well known that the natural map $\hat{k}: Def_V(Y) \to Def(Y)$ is holomorphic and its differential is $k: H^0(N_Y) \to T^1_Y$.

In a neighbourhood of 0 is defined an analytic map

$$\xi: H = \bigoplus_i (H^0(\mathcal{O}_X(D_i)) \oplus H^0(\mathcal{O}_X(D_i - L_i))) \to Def_V(Y)$$

where $\xi(y_i, \gamma_i)$ is the surface in V defined by:

$$\begin{aligned} w_i^2 &= (x_j + y_j + \gamma_j w_j)(x_k + y_k + \gamma_k w_k) \\ w_j w_k &= w_i(x_i + y_i + \gamma_i w_i) \end{aligned} \tag{1.10}$$

DEFINITION. We shall call the deformation of Y defined in (1.10) a natural deformation.

LEMMA 6. *Let* $d\xi: \oplus_i (H^0(\mathcal{O}_X(D_i)) \oplus H^0(\mathcal{O}_X(D_i - L_i))) \to H^0(N_Y)$ *be the differential of* ξ. *Then* $H^0(\psi) \circ d\xi = \rho$ *where*

$$\rho: \bigoplus_i (H^0(\mathcal{O}_X(D_i)) \oplus H^0(\mathcal{O}_X(D_i - L_i))) \to \bigoplus_i (H^0(\mathcal{O}_{D_i}(D_i)) \oplus H^0(\mathcal{O}_{D_i}(D_i - L_i)))$$

is the restriction map.

The proof is a straightforward verification and it is left to the reader.

If $H^1(\mathcal{O}_Y) = 0$ then $H^1(\mathcal{O}_X) = H^1(\mathcal{O}_X(-L_i)) = 0$ and ρ is surjective, the kernel of ε has dimension $h^0(\pi^* \mathcal{O}_X) - h^0(\pi^* \mathcal{O}_Y)$ and since the parameter space H of natural deformations is smooth we have finally

PROPOSITION 7. *If* $H^1(\mathcal{O}_Y) = H^1(\pi^* \mathcal{O}_X) = 0$ *then* $k \circ d\xi = \varepsilon \circ H^0(\psi) \circ d\xi =$

$\varepsilon \circ \rho$ is surjective, the map $\hat{k} \circ \xi$ is smooth and Def(Y) is smooth of dimension

$$\sum_i (h^0(\mathcal{O}_X(D_i)) + h^0(\mathcal{O}_X(D_i - L_i)) - 1) - h^0(\pi^*\theta_X) + h^0(\theta_Y)$$

We remark that if the minimal resolution of Y is of general type then the group of automorphisms of Y is finite [Ma1] and $H^0(\theta_Y) = 0$.

REMARK. If $H^1(\pi^*\theta_X) \neq 0$ (this is true in particular if $H^1(\theta_X) \neq 0$) then in general ε is not surjective; in this case it may be useful to know $\operatorname{Im}\varepsilon = \ker\sigma$. Ziv Ran [Ran] is responsible for an exact sequence where σ appears

$$\cdots \to T_\pi^1 \to T_X^1 \oplus T_Y^1 \xrightarrow{\sigma'} \operatorname{Ext}_\pi^1(\Omega_X^1, \mathcal{O}_Y) \to T_\pi^2 \to \cdots$$

where T_π^1 is the space of first order deformation of the map π and $\operatorname{Ext}_\pi^n(\Omega_X^1, \mathcal{O}_Y)$ is defined as the limit of the spectral sequence $E_2^{p,n-p} = \operatorname{Ext}_{\mathcal{O}_Y}^p(L^{n-p}\pi^*\Omega_X^1, \mathcal{O}_Y)$. It is clear that in our case $\operatorname{Ext}_\pi^n(\Omega_X^1, \mathcal{O}_Y) = H^n(\pi^*\theta_X)$ and $\sigma(x) = \sigma'(0, x)$.

2. Deformations of simple bihyperelliptic surfaces and the space $\hat{N}_{(a,b),(n,m)}$

From now on let S be a simple bihyperelliptic surface of type $(a, b)(n, m)$ with $a, b, n, m \geqslant 3$ and let $\delta : S \to Y$ be the pluricanonical map onto its canonical model Y. Let (0.1) be the equation of Y.

It is well known that Y is a normal surface with at most rational double points as singularities and δ is its minimal resolution.

On Y we have the following exact sequence (cfr. (1.6)):

$$0 \to H^1(\theta_Y) \to T_Y^1 \to H^0(\mathcal{T}_Y^1) \xrightarrow{ob} H^2(\theta_Y)$$

where ob is the obstruction to globalize a first order deformation of the singular points of Y. As a consequence of Proposition 7 we have the following.

COROLLARY 8. *In the notation above Def(Y) is smooth. Def(S) is smooth if and only if $ob = 0$.*

Proof. Let $\pi : Y \to X = \mathbb{P}^1 \times \mathbb{P}^1$ be the projection, then

$$\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{O}_X(-a, -b) \oplus \mathcal{O}_X(-n, -m) \oplus \mathcal{O}_X(-a-n, -b-m)$$

$$\theta_X = \mathcal{O}_X(2, 0) \oplus \mathcal{O}_X(0, 2) \quad \pi_*\pi^*\theta_X = \theta_X \otimes \pi_*\mathcal{O}_Y$$

Since $a, b, n, m \geqslant 3$ we have $h^1(\mathcal{O}_Y) = h^1(\pi^*\theta_X) = 0$ and by Proposition 7 *Def(Y)* is smooth.

Denote by L_Y (resp.: D_Y) the functor of local (resp.: global) deformations of Y, since Y has a finite number of singular points which are R.D.P.'s L_Y is smooth with finite dimensional tangent space $H^0(\mathscr{T}^1_Y)$. Since $Def(Y)$ is smooth, the natural map $\Phi: D_Y \to L_Y$ is smooth if and only if its differential $T^1_Y \to H^0(\mathscr{T}^1_Y)$ is surjective. By a general result ([B-W] Th. 2.14, [Pi]) the smoothness of $Def(S)$ is equivalent to the smoothness of Φ. □

Note that since we have a surjective map $H \to T^1_Y$, the kernel of ob is exactly the subspace of $H^0(\mathscr{T}^1_Y)$ generated by the natural deformations of Y. We shall exhibit later an example where $ob \neq 0$.

LEMMA 9. *The subset $\hat{N} = \hat{N}_{(a,b),(n,m)}$ is open in $\mathscr{M}$ for $a > 2n$, $m > 2b$.*

Proof. We have to prove that simple bihyperelliptic surfaces of type $(a, b)(n, m)$ are stable under small deformations. Let $F: \mathscr{S} \to \Delta$ be a flat family over the complex disk with $S_0 = F^{-1}(0)$ simple bihyperelliptic of type $(a, b)(n, m)$.

Let $F': \mathscr{Y} \to \Delta$ be the corresponding family of canonical models, then Y_0 is a normal bidouble cover of $X = \mathbb{P}^1 \times \mathbb{P}^1$ with, in the notation of Section 1, $L_1 = \mathcal{O}_X(n, m)$, $L_2 = \mathcal{O}_X(a, b)$, $L_3 = \mathcal{O}_X(a + n, b + m)$, $x_1 = f$, $x_2 = g$, $x_3 = 1$.

Then, for $a, b, n, m \geqslant 3$, the surface Y_0 satisfies the hypothesis of Proposition 7 and we can assume, possibly shrinking Δ, that F' is a natural deformation of Y_0.

The natural deformations of Y_0 are defined in $\mathcal{O}_X(a, b) \oplus \mathcal{O}_X(n, m)$ by

$$z^2 = f'(x, y) + w\varphi(x, y)$$
$$w^2 = g'(x, y) + z\psi(x, y)$$

where $f' \in H^0(\mathcal{O}_X(2a, 2b))$, $g' \in H^0(\mathcal{O}_X(2n, 2m))$, $\varphi \in H^0(\mathcal{O}_X(2a - n, 2b - m))$, $\psi \in H^0(\mathcal{O}_X(2n - a, 2m - b))$. If $a > 2n$, $m > 2b$ then $\varphi = \psi = 0$ and the lemma is proved. □

Proof of Theorem B

From [Ca2] Cor. 4.4 we know that for $a \geqslant \max(2n + 1, b + 2)$, $m \geqslant \max(2b + 1, n + 2)$ $\hat{N}$ is a closed irreducible component of $\mathscr{M}$, then we use Lemma 9. □

EXAMPLE 1. Suppose $a > 2n$, $m > 2b$ and let (0.1) be the equations of Y. Denote $D_1 = div(f)$, $D_2 = div(g)$ and suppose moreover that

$$Sing(D_i) \cap D_j = \varnothing, \ \{i, j\} = \{1, 2\}$$

and let $p \in D_1$ be a singular point.

Then $\pi^{-1}(p)$ contains exactly two singular points q_1, q_2 of Y and there exists an involution $\sigma \in G$ such that $\sigma(q_1) = q_2$. σ extends to every natural deformation, in particular every global deformation of Y gives by restriction isomorphic local deformations of (Y, q_1) and (Y, q_2) and Φ cannot be smooth.

More generally one can prove that if $ob = 0$ then D_1 and D_2 are both smooth.

It is interesting to note that in general for $a > 2n$, $m > 2b$ the space $\hat{N}_{(a,b),(n,m)}$ is not closed in $\mathcal{M}$, in fact its closure may contain some minimal resolution of bidouble covers of Segre-Hirzebruch surfaces $\mathbb{F}_{2k}$ ([Ca2] Thm. 4.3). If we consider the irreducible component $\overline{\hat{N}_{(a,b),(n,m)}}$ then in general this is not a connected component of $\mathcal{M}$. (In most cases if $\pi: S \to \mathbb{F}_{2k}$ $(k > 0)$ is a smooth bidouble cover belonging to the closure of $\hat{N}$ the space of natural deformations of S has dimension greater than the dimension of $\hat{N}$.)

From the results of Freedman about the topology of four-manifolds it follows that two simple bihyperelliptic surfaces S_1 and S_2 are homeomorphic (by an orientation preserving homeomorphism) if and only if $K_{S_1}^2 = K_{S_2}^2$, $\chi(\mathcal{O}_{S_1}) = \chi(\mathcal{O}_{S_2})$ and $r(S_1) \equiv r(S_2)$ mod(2) ([Ca1] Prop. 4.4).

For a simple bihyperelliptic surface of type $(a, b)(n, m)$ we have ([Ca1], [Ca3]):

$$K^2 = 8(a + n - 2)(b + m - 2) \quad \chi(\mathcal{O}_S) = \tfrac{1}{8}K^2 + ab + nm$$
$$r(S) = \text{g.c.d.}\ (a + n - 2, b + m - 2) \tag{2.1}$$

and if $a > 2n$, $m > 2b$ then

$$\dim \hat{N} = 4\chi - \tfrac{1}{2}K^2 + 2(a + b + n + m) - 6$$

EXAMPLE 2. Let S_1, S_2 be two simple bihyperelliptic surfaces of respective type (13, 4), (6, 13) and (14, 5), (5, 12). Then these surfaces are homeomorphic, $r(S_1) = r(S_2) = 1$ and they belong to different connected components of $\mathcal{M}$.

In order to prove Theorem A we try to find, for given $k > 0$, k simple bihyperelliptic surfaces which are homeomorphic, with the same divisibility r and belonging to different connected components. We use the following lemma (proved in the appendix of [Ca1]).

LEMMA 10 (Bombieri). *Let $1 > c > 3^{-1/3}$ be a fixed real number, M a positive integer and let $u_i v_i = M$ be k distinct factorizations of M such that $c\sqrt{M} < u_i < v_i < c^{-1}\sqrt{M}$.*

Then there exist positive integers R, S, N and k distinct pairs of integer (z_i, w_i) such that:

$$w_i z_i - 2(u_i + v_i) = N, \quad z_i + 4 < 2Rv_i < 3z_i - 2, \quad w_i + 4 < 2Su_i < 3w_i - 2$$

THEOREM 11. *For every $k > 0$ there exist simply connected surfaces of general type $S_1, \dots, S_k$ orientedly homeomorphic, with $r(S_i) = r(S_j)$ and any two of them are not deformation equivalent to each other.*

Proof. We have to find large positive integers K^2, $\chi(\mathcal{O}_S)$, $r(S)$ such that (2.1) with the inequalities $a \geqslant \max(2n + 1,\ b + 2)$, $m \geqslant \max(2b + 1,\ n + 2)$ has at least k distinct solutions. Fix $1 > c > \max\{2^{-1/2}, 3^{-1/3}\}$ and let u_i, $v_i = M$ be k distinct factorizations with g.c.d. $(u_i, v_i) = 1$ such that $c\sqrt{M} < u_i < v_i < c^{-1}\sqrt{M}$. (We can take for example an integer h such that

$$\binom{2h}{h} > 2k \quad \text{and} \quad M = p_1 p_2 \cdots p_{2h}$$

where $p_1 < p_2 < \cdots < p_{2h}$ are prime numbers such that $p_1^h > c p_{2h}^h$).

Let R, S, N, w_i, z_i be as in Lemma 10 and let S_i be a simple bihyperelliptic surface of type $(a_i, b_i)(n_i, m_i)$ where $a_i = 2RSu_i + Rw_i + 1$, $b_i = 2RSv_i - Sz_i + 1$, $n_i = 2RSu_i - Rw_i + 1$, $m_i = 2RSv_i + Sz_i + 1$.

A computation shows that for every $i = 1, \dots, k$, $K_{S_i}^2 = 128R^2S^2M$, $\chi(\mathcal{O}_{S_i}) = 24R^2S^2M - 2RSN + 2$, $r(S_i) = 4RS$ and $a_i \geqslant \max\{2n_i + 1,\ b + 2\}$, $m_i \geqslant \max\{2b_i + 1, n_i + 2\}$.

These surfaces belong to the same $\mathcal{M}_a$ but they are in distinctly connected components by Theorem B. □

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