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Stable G_2 bundles and algebraically completely integrable systems

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0. Introduction

The moduli spaces of stable bundles over a smooth curve have been a focus of interest for several years now. One of their remarkable properties was discovered recently by Hitchin. He proved [H1], [H2] that for any simple group G the cotangent bundle of the moduli space of stable principal G-bundles over an algebraic curve is a completely integrable system. Furthermore, for a classical Lie group G he described the exact geometry of the generic level set of this integrable system – it turns out that it can be compactified to a certain Jacobian or Prym variety.

In this paper we study the geometry of the level sets of the corresponding integrable system when the group G is the exceptional Lie group G_2. We prove that in this case the generic level set can be embedded as a Zariski open set in a suitable Prym-Tjurin variety of an algebraic curve endowed with correspondence – a spectral curve.

The paper is organized as follows. In the first section we set up the notations and recall some basic facts about the moduli space of principal Higgs bundles. The second section discusses the Hamiltonian structure on the moduli space $\mathscr{M}$ of Higgs bundles. In Section 3 we associate a spectral curve together with a correspondence on it to the generic level set of $\mathscr{M}$ and identify an interesting abelian subvariety in the Jacobian of this curve – its Prym-Tjurin variety. The last section contains the proof of the main theorem – that the generic level set can be embedded as a Zariski open set in the associated Prym-Tjurin variety. The Appendix explains the seemingly arbitrary choice of correspondence we have made and makes another connection with the theory of integrable systems.

After this work was finished we came across the preprints [D], [BK] where similar results are obtained.

1. Some definitions and basic facts

Let C be a smooth complex algebraic curve of genus g. Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$. Denote by N_G the moduli space of semistable topologically trivial principal G bundles. Recall that N_G is a normal projective algebraic variety of dimension $\dim N_G = (\dim_{\mathbb{C}} G)(g - 1)$. Its smooth locus N_G^{reg} consists of the equivalence classes of the stable bundles whose automorphism group coincides with the center of G. Unlike the vector bundle case the variety N_G can be singular even at a stable point (see e.g. [R]).

The total space $\mathcal{M} = T^*N_G^{\mathrm{reg}}$ of the cotangent bundle of the smooth part N_G^{reg} of the moduli space is in a natural way a holomorphic symplectic manifold. In his paper [H2], Hitchin proved that $\mathcal{M}$ is a completely integrable Hamiltonian system, i.e. that there exists a foliation $\mathcal{H}: \mathcal{M} \to B$ of $\mathcal{M}$ over a vector space B whose fibers are Lagrangian submanifolds in $\mathcal{M}$.

DEFINITION 1.0.1. The Lagrangian fibration $\mathcal{H}: \mathcal{M} \to B$ is said to be algebraically completely integrable if there exists a partial compactification $\tilde{\mathcal{M}}$ of $\mathcal{M}$ over B:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\text{inclusion}} & \tilde{\mathcal{M}} \\ & {}_{\mathcal{H}}\searrow \quad \swarrow_{\tilde{\mathcal{H}}} & \\ & B & \end{array}$$

so that the fibers of $\tilde{\mathcal{H}}$ are isomorphic to abelian varieties, the fibers of $\mathcal{H}$ are Zariski open sets in them and the Hamiltonian vector fields on $\mathcal{M}$ corresponding to functions constant along the fibers of $\mathcal{H}$ extend to global holomorphic vector fields along the fibers of $\tilde{\mathcal{H}}$.

Following Hitchin's ideas from [H1] C. Simpson has constructed [S] a natural compactification $\tilde{\mathcal{H}}: \tilde{\mathcal{M}} \to B$ of $\mathcal{M}$ over B, where $\tilde{\mathcal{M}}$ is the moduli space of semistable principal Higgs G-bundles on M. In the present paper we are describing the fibers of $\tilde{\mathcal{H}}$ in the case $G = G_2$. The geometric picture we obtain generalizes the one obtained by Hitchin [H2] for a classical Lie group G and yields the algebraic complete integrability of $\mathcal{M} = T^*N_{G_2}$.

2. The Hitchin map for the Hamiltonian system $\mathcal{M}$

Let N be the moduli space of stable holomorphic principal G_2-bundles. For each point $E \in N^{\mathrm{reg}}$ the cotangent space T_E^*N is isomorphic to the vector space

$H^0(C, \mathfrak{g}_E \otimes K_C)$, where $\mathfrak{g}_E := E \times_{ad} \mathfrak{g}$. The bundle $\mathfrak{g}_E$ is associated to E by the adjoint representation and hence any Ad-invariant homogeneous polynomial p on $\mathfrak{g}_2$ of degree d gives rise to a map:

$$p: H^0(C, \mathfrak{g}_E \otimes K_C) \to H^0(C, K_C^{\otimes d}).$$

The ring of Ad-invariant polynomials on $\mathfrak{g}_2$ has basis $\{p, q\}$ where p is homogeneous of degree 2 and q is homogeneous of degree 6 (cf. [Sch]). Notice that

$$\dim H^0(C, \mathfrak{g}_E \otimes K_C) = 14(g-1) = \dim(H^0(C, K_C^{\otimes 2}) \oplus H^0(C, K_C^{\otimes 6})).$$

Finally we define the Hitchin map

$$\begin{aligned} \mathscr{H}: T^*N &\to H^0(C, K_C^{\otimes 2}) \oplus H^0(C, K_C^{\otimes 6}) \\ (E, \theta) &\to (p(\theta), q(\theta)) \end{aligned}$$

where $E \in N$ and $\theta \in H^0(C, \mathfrak{g}_E \otimes K_C)$. The fibers of the map $\mathscr{H}$ are Lagrangian subvarieties in $\mathscr{M}$ (see [H1]).

We shall use the standard 7-dimensional representation $\rho: \mathfrak{g}_2 \to \mathrm{End}(V)$ of the Lie algebra $\mathfrak{g}_2$ for a more detailed description of the Hitchin map. For every $E \in N$ denote by $\mathbf{V}_E$ the holomorphic rank 7 vector bundle associated to E by ρ. Since the representation ρ is faithful we have an inclusion of vector bundles

$$0 \to \mathfrak{g}_E \to \mathrm{End}(\mathbf{V}_E),$$

which allows us to consider any $\theta \in H^0(C, \mathfrak{g}_E \otimes K_C)$ as an element of $H^0(C, \mathrm{End}(\mathbf{V}_E) \otimes K_C)$.

Furthermore, the representation ρ is orthogonal, and every matrix from $\mathfrak{g}_2 \overset{\rho}{\hookrightarrow} \mathrm{End}(V)$ has eigenvalues: $0, \lambda_1, \lambda_2, \lambda_3, -\lambda_1, -\lambda_2, -\lambda_3$, which satisfy the condition:

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

A basis $\{p, q\}$ of $\mathbb{C}[\mathfrak{g}_2]^{G_2}$ can be described explicitly in these terms (see [Sch]):

$$p = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \; q = (\lambda_1\lambda_2\lambda_3)^2.$$

3. Spectral curves, correspondences and Prym-Tjurin varieties

3.1

To identify the fiber of the Hitchin map for the Hamiltonian system $\mathcal{M}$ we will need an auxiliary geometric object – the spectral curve. Spectral curves have been a convenient geometric tool in the study of the moduli space of Higgs bundles [H1], [H2], [S] and the moduli space of vector bundles [BNR]. Here we adopt a construction of the spectral covers which was originally developed by V. Kanev [K1] for the study of algebraic tori over function fields and is the most suitable one for dealing with principal bundles.

Let (E, θ) be a principal irreducible Higgs bundle over C. The holomorphic section $\theta \in H^0(C, \mathfrak{g}_E \otimes K_C)$ gives an endomorphism of vector bundles

$$ad_\theta : \mathfrak{g}_E \to \mathfrak{g}_E \otimes K_C.$$

DEFINITION 3.1.1. The Higgs bundle (E, θ) is called regular when the coherent subsheaf $\ker(ad_\theta) \subset \mathfrak{g}_E$ has rank equal to the rank of the Lie algebra $\mathfrak{g}$. Similarly (E, θ) is called regular and semisimple if there exists a Zariski open subset $C_0 \subset C$ for which $\ker(ad_\theta)|_{C_0}$ is a subbundle of Cartan subalgebras in $\mathfrak{g}_E|_{C_0}$.

Let (E, θ) be regular and semisimple G_2 bundle. Denote by $\mathfrak{h}_E$ the saturization of the sheaf $\ker(ad_\theta)$, that is, the unique vector bundle of rank two, contained in this sheaf. The representation ρ defines a multisection $\mathfrak{s}$ of the dual bundle $\mathfrak{h}_E^\vee$ over the open set C_0. The values of $\mathfrak{s}$ at a point $\xi \in C_0$ are the six extremal weights of $\rho: \{\lambda_1(\xi), \dots, \lambda_6(\xi)\} \subset (\mathfrak{h}_E^\vee)_\xi$. Let C' be the Zariski closure of $\mathfrak{s}(C_0)$ in the total space of $\mathfrak{h}_E^\vee$. Following Kanev (see [K1]) we define the spectral curve determined by (E, θ) as the normalization $C'^\vee$ of C'. The curve $C'^\vee$ is a six sheeted cover of C, unramified over the set C_0.

One can construct another geometric model of the spectral cover $C'^\vee$ (see [H1], [BNR]). Start with the natural map:

$$\begin{aligned} i_\theta : \mathfrak{h}_E^\vee &\to K_C \\ \varphi &\to \varphi(\theta). \end{aligned}$$

The image $i_\theta(C')$ is again covering of C which lies in the total space of the vector bundle K_C. For regular and semisimple Higgs bundles the map $i_\theta : C' \to i_\theta(C')$ is clearly a birational isomorphism since in this case the operator $\rho(\theta)(\xi)$ will be regular and semisimple for generic $\xi \in C$, i.e. all of its different extremal eigenvalues will have multiplicity one.

We can look at the covering $i_\theta(C')$ from a different angle. Consider the vector Higgs bundle $(\mathbf{V}_E, \rho(\theta))$ associated to (E, θ) by the representation ρ. Let $X := \mathrm{Tot}(K_C)$ be the total space of K_C, and let $\lambda \in H^0(X, p^*(K_C))$ be the tautological section. We can consider $\rho(\theta)$ as an endomorphism of $\mathbf{V}_E$ with coefficients at the tensor algebra of K_C and we can form the element $\det(\rho(\theta) - \lambda \cdot id_{\mathbf{V}_E}) \in H^0(X, p^*(K_C^{\otimes 7}))$. The zero scheme of the section $\det(\rho(\theta) - \lambda \cdot id_{\mathbf{V}_E})$ is a subscheme in X, finite over C. It has two irreducible components corresponding to the W_{G_2}-orbits in the set of weights of the representation ρ. The first one of them corresponds to the orbit consisting of the zero weight and is just the zero scheme of $\lambda \in H^0(X, p^*(K_C))$. The second one corresponds to the orbit of the six extremal weights and is exactly the curve $i_\theta(C')$.

The curve $i_\theta(C')$ actually depends only on the image of (E, θ) under the Hitchin map. Indeed, let $b = (b_1, b_2) \in B := H^0(C, K_C^{\otimes 2}) \oplus H^0(C, K_C^{\otimes 6})$ and let $(E, \theta) \in \mathscr{H}^{-1}(b)$. By using our explicit base of the ring of invariant polynomials, we can write

$$\det(\rho(\theta) - \lambda \cdot id_{\mathbf{V}_E}) = \lambda \cdot \left(\lambda^6 - p(\theta) \cdot \lambda^4 + \frac{p(\theta)^2}{4} \cdot \lambda^2 - q(\theta)\right)$$

$$= \lambda \cdot \left(\lambda^6 - b_1 \cdot \lambda^4 + \frac{b_1^2}{4} \cdot \lambda^2 - b_2\right). \tag{3.1.1}$$

Therefore $i_\theta(C')$ is simply the zero scheme C_b of the section

$$\lambda^6 - b_1 \cdot \lambda^4 + \frac{b_1^2}{4} \cdot \lambda^2 - b_2 \in H^0(X, p^* K_C^{\otimes 6}). \tag{3.1.2}$$

LEMMA 3.1.1. *For a generic $b \in B$ the curve C_b is smooth and hence is the spectral curve associated with the fiber $\mathscr{H}^{-1}(b)$.*

Proof. Fix $b_1 \in H^0(C, K_C^{\otimes 2})$. Then the formula (3.1.2) determines a linear subsystem $P_{b_1} \subset |p^* K_C|$, isomorphic to $\mathbb{P}(p^* H^0(C, K_C^{\otimes 6}))$. By Bertini's theorem it suffices to prove that the base locus $B_X P_{b_1}$ of P_{b_1} is empty for generic b_1. The section $\lambda^6 - b_1 \cdot \lambda^4 + (b_1^2/4) \cdot \lambda^4 = \lambda^2 \cdot (\lambda^2 - b_1/2)^2$ belongs to P_{b_1}. Therefore $B_X P_{b_1}$ is contained in the union

$$B_{\{\lambda=0\}}(\mathbb{P}(p^* H^0(C, K_C^6))_{|\{\lambda=0\}}) \cup B_{\{\lambda^2 - b_1/2 = 0\}}(\mathbb{P}(p^* H^0(C, K_C^6))_{|\{\lambda^2 - b_1/2 = 0\}}).$$

On the other hand

$$B_{\{\lambda=0\}}(\mathbb{P}(p^* H^0(C, K_C^6))_{|\{\lambda=0\}}) = B_C |K_C^{\otimes 6}|,$$

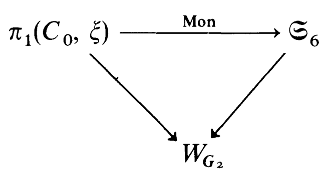
and

$$B_{\{\lambda^2 - b_1/2=0\}}(\mathbb{P}(p^* H^0(C, K_C^6))_{|\{\lambda^2 - b_1/2=0\}} = B_{S_{b_1}}(p^* |K_C|),$$

where $p: S_{b_1} \to C$ is the two sheeted cover of C given by the zero scheme of the section $\lambda^2 - b_1/2$. Using again Bertini's theorem and the fact that the linear system $|K_C^{\otimes 2}|$ does not have base points on C, one easily deduces that S_{b_1} is smooth for generic b_1.

But $p(B_{S_{b_1}} p^* |K_C|) \subset B_C K_C^{\otimes 6} = \varnothing$ and hence $B_X P_{b_1} = \varnothing$ for generic b_1. □

REMARK 3.1.1. The spectral covers described above have a special structure, inherited from the combinatorics of the weight lattice of the root system G_2. For instance, if $\xi \in C_0$ is a base point, one can show that the monodromy map $\text{Mon}: \pi_1(C_0, \xi) \to \mathfrak{S}_6$ factors through the action of the Weyl group W_{G_2}:



(see [K1] for more details).

3.2

Having found the spectral curve C_b attached to the fiber $\mathscr{H}^{-1}(b)$ we proceed with the proof of the complete integrability of $\mathscr{M}$. The key idea, which is due to Hitchin [H1], is to construct an isomorphism between the fiber $\mathscr{H}^{-1}(b)$ and certain abelian subvariety P in the jacobian $J(C_b)$. Such P must arise naturally from the geometry of the cover $\pi: C_b \to C$, which suggests that we look for it among the abelian subvarieties of $J(C_b)$ coming from symmetric correspondences on C_b preserving π. To construct some of the latter we will use the peculiar combinatorial structure of the fibers of $\pi: C_b \to C$. By choosing $(E, \theta) \in \tilde{\mathscr{H}}^{-1}(b)$ we can identify the points in the fiber $\pi^{-1}(\xi)$ over a generic $\xi \in C$ with the extremal weights of the representation $\rho: G_2 \to \mathbf{V}$ with respect to the Cartan subalgebra $\ker ad_{\theta(\xi)} \subset (\mathfrak{g}_E)_\xi$. Therefore the points in $\pi^{-1}(\xi)$ can be labeled (compare with the end of Section 3) as $\lambda_1, \lambda_2, \lambda_3, -\lambda_1, -\lambda_2, -\lambda_3 \in (\ker ad_{\theta(\xi)})^\vee$ where $\lambda_1, \lambda_2, \lambda_3$ are subject to the condition

$$\lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{3.2.1}$$

Since $C_b \subset \text{Tot}(K_C)$ we can add and subtract the points in the fiber of

$\pi: C_b \to C$ to obtain points in the fiber of $K_C \to C$. Moreover, the linearity of the contraction map i_θ and the relation (3.2.1) yield that for any x in the generic $\pi^{-1}(\xi)$ the point $-x$ belongs again to $\pi^{-1}(\xi)$ and that there are exactly two more points $y, z \in \pi^{-1}(\xi)$ with the property $x + y + z = 0$. Denote now by Δ_1, Δ_2, Δ_3 the reduced divisors in $C_b \times C_b$ given by:

$$\begin{aligned}
\Delta_1 &= \{(x, y) \,|\, x = y\}, \\
\Delta_2 &= \{(x, y) \,|\, \pi(x) = \pi(y),\ \exists z \in \pi^{-1}(\pi(x)) \text{ s.t. } x + y + z = 0\}. \\
\Delta_3 &= \{(x, y) \,|\, \pi(x) = \pi(y),\ x = -y\},
\end{aligned}$$

Clearly any divisor of the form $a \cdot \Delta_1 + b \cdot \Delta_2 + c \cdot \Delta_3 \in \mathrm{Div}(C_b \times C_b)$ is a symmetric correspondence (with valence) on the curve C_b which preserves the fibers of π. Consider the divisor $D = 2 \cdot \Delta_2 + 3 \cdot \Delta_3$. Let $\imath: J^0(C_b) \to J^0(C_b)$ be the endomorphism of the Jacobian $J^0(C_b)$ induced by D. In the next section we will show that every connected component of the fiber $\tilde{\mathscr{H}}^{-1}(b)$ is canonically isomorphic to the Prym-Tjurin variety associated with the endomorphism $\imath$.

REMARK 3.2.1. The correspondences Δ_1, Δ_2, Δ_3 are attached to the curve C_b in a functorial way. Moreover, it is easy to see that for the generic pair (C, C_b) the subring of the algebra of correspondences of C_b generated by the Δ_i's coincides with the ring of all symmetric correspondences preserving π. Therefore the correspondence we are looking for, ought to belong to this subring.

A motivation for our particular choice of D is Kanev's theory of algebraic tori over function fields and its relation to the integrable systems. The relevant facts from this theory are collected in the Appendix.

3.3

Let $b \in B$ be a generic point in the sense of Lemma 3.1.1. Denote by J_b the Jacobian $J^0(C_b)$.

CLAIM 3.3.1. *The endomorphism $\imath \in \mathrm{End}(J_b)$ satisfies the cubic equation*

$$(\imath - id_{J_b})(\imath + 5id_{J_b})(\imath - 7id_{J_b}) = 0.$$

Proof. Let $\xi \in C_0$ and let $\pi^{-1}(\xi) = \{x_1, \ldots, x_6\}$. According to the previous section, for every $x \in \pi^{-1}(\xi)$ the points of the fiber $\pi^{-1}(\xi)$ can be labeled as $\pi^{-1}(\xi) = \{x, y, z, x^-, y^-, z^-\}$, so that

$$\begin{aligned}
&\Delta_2(x) = y + z,\ \Delta_2(y) = x + z,\ \Delta_2(z) = x + y, \\
&\Delta_2(x^-) = y^- + z^-,\ \Delta_2(y^-) = x^- + z^-,\ \Delta_2(z^-) = x^- + y^-. \\
&\Delta_3(x) = x^-,\ \Delta_3(y) = y^-,\ \Delta_3(z) = z^-,
\end{aligned}$$

Using these relations and the fact that Δ_3 is an involution it is easy to verify that

$$D(D(x)) + 4D(x) - 5x = 12(x + y + z + x^- + y^- + z^-) = \pi^*(\mathrm{Nm}_\pi(x)).$$

Since the image of the curve C_b under the Abel-Jacobi map generates the Jacobian J_b we get

$$\iota^2 + 4\iota - 5id_{J_b} = 12\pi^* \circ \mathrm{Nm}_\pi. \tag{3.3.1}$$

On the other hand for any $\xi \in C_0$

$$D(\pi^*(\xi)) = D(x + y + z + x^- + y^- + z^-) = 7(x + y + z + x^- + y^- + z^-),$$

i.e.

$$(\iota - 7id_{J_b})(\iota^2 + 4\iota - 5id_{J_b}) = 0. \qquad \square$$

The identity (3.3.1) in the proof of the claim implies that $\mathrm{Im}(\iota^2 + 4\iota - 5id_{J_b})$ is isogeneous to the Jacobian $J^0(C)$. Denote by K the connected component of the identity of the abelian variety $\mathrm{Ker}((\iota - id_{J_b})(\iota + 5id_{J_b}))$. We can consider K as a complement to the "inessential" piece $\pi^*(J^0(C))$ of the Jacobian J_b. Since by definition ι satisfies a quadratic equation on K we can form the Prym-Tjurin variety $P(\iota, K)$ of the pair (K, ι).

DEFINITION 3.3.1. The Prym-Tjurin variety associated with the spectral curve C_b is the variety

$$P_b := P(\iota, K) = \mathrm{Im}(\iota - id_{J_b}).$$

First we prove the following proposition:

PROPOSITION 3.3.1. *For the generic $b \in B$ the Prym-Tjurin variety P_b has dimension* $\dim P_b = 14(g - 1)$.

Proof. The differential $d\iota_0$ of the endomorphism φ of the Jacobian J_b is a nondegenerate linear operator acting on the vector space $H^0(J_b, \mathbb{C})$. It is clear from Claim 3.3.1 and from the definition of the Prym-Tjurin variety that $d\iota_0$ has characteristic polynomial

$$\chi_{d\iota_0}(t) = (t - 7)^g (t + 5)^{\dim P_b}(t - 1)^{g(C_b) - g - \dim P_b}.$$

We can use now Lefschetz fixed point formula for the correspondence D (see [L]) to compute the dimension of P_b. According to this formula

$$D \cdot \Delta_1 = -\operatorname{tr}_{\mathbb{R}}(d\iota_0) + 2 \deg D,$$

where $\operatorname{tr}_{\mathbb{R}}(d\iota_0)$ is the trace of the operator $d\iota_0$ considered as an operator on the underlying real vector space.

By using the adjunction formula on the surface X we can compute the genus $g(C_b) = 36(g-1)+1$ and hence obtain

$$\operatorname{tr}_{\mathbb{R}}(d\iota_0) = 2((36(g-1)+1-g-\dim P_b)\cdot 1 + \dim P_b \cdot (-5) + g \cdot 7).$$

Therefore we can express the dimension of the Prym-Tjurin variety P_b in terms of the number of the fixed points of the correspondence D:

$$\dim P_6 = 7(g-1) + \tfrac{1}{12} D \cdot \Delta_1.$$

To complete the proof of our proposition it remains to prove the following lemma:

LEMMA 3.3.1. *The number of the fixed points of D is* $84(g-1)$.

Proof. It is clear from the construction of our correspondence that its fixed points are exactly the ramification points of the covering $\pi: C_b \to C$. The branch divisor of this covering is the divisor of the discriminant of the polynomial (3.1.2):

$$\operatorname{Discr}(b) = b_2^3(27b_2 - \tfrac{1}{2}b_1^3)^2 \in H^0(C, K_C^{\otimes 30}).$$

Choose local sections $\lambda_1, \lambda_2, \lambda_3$ of K_C such that $\lambda_1, \lambda_2, \lambda_3, -\lambda_1, -\lambda_2, -\lambda_3$ are (locally) the eigenvalues of $\rho(\theta): \mathbf{V}_E \to \mathbf{V}_E \otimes K_C$ for any $(E, \theta) \in \mathscr{H}^{-1}(b)$. Then the discriminant section $\operatorname{Discr}(b)$ can be expressed locally in terms of the λ_i's:

$$\operatorname{Discr}(b) = (\lambda_1\lambda_2\lambda_3)^6(\lambda_1-\lambda_2)^4(\lambda_1-\lambda_3)^4(\lambda_2-\lambda_3)^4 \qquad (3.3.2)$$

It is easy to see that for the generic $b \in B$ its components b_1 and b_2 do not have common zeros and that b_2 and $27b_2 - (1/2)b_1^3$ have only simple zeros. Furthermore, from the definition of D and the representation (3.3.2) of the discriminant, we can determine all possible multiplicities for the fixed points of D. Indeed if $\xi \in C$ is a branch point, then it is a zero of $\operatorname{Discr}(b)$ and hence a zero of one of the factors in the right-hand side of (3.3.2). If, for example, ξ is such that $\lambda_1(\xi) = 0 = -\lambda_1(\xi)$, then since $\lambda_1(\xi) + \lambda_2(\xi) + \lambda_3(\xi) = 0$ we have $\lambda_2(\xi) = -\lambda_3(\xi)$. On the other hand for the generic point η in a small disk around ξ we have

$$D(\lambda_1(\eta)) = 3\cdot(-\lambda_1(\eta)) + 2\cdot\lambda_2(\eta) + 2\cdot\lambda_3(\eta),$$
$$D(\lambda_2(\eta)) = 3\cdot(-\lambda_2(\eta)) + 2\cdot\lambda_1(\eta) + 2\cdot\lambda_3(\eta).$$

By letting $\eta \to \xi$ we get that the point $(\xi, 0) = (\xi, \lambda_1(\xi)) = (\xi, -\lambda_1(\xi)) \in \pi^{-1}(\xi)$ is a fixed point of D of multiplicity 3 and that the points $(\xi, \lambda_2(\xi)) = (\xi, -\lambda_3(\xi)) \in \pi^{-1}(\xi)$ and $(\xi, -\lambda_2(\xi)) = (\xi, \lambda_3(\xi)) \in \pi^{-1}(\xi)$ are not fixed for D. Similarly one can find the multiplicities for the various types of ramification points of C_b. All the possibilities are listed in the table below.

Points over ξ	Multiplicities of D over ξ
$(\xi, 0) = (\xi, \lambda_1(\xi)) = (\xi, -\lambda_1(\xi))$	3
$(\xi, \lambda_2(\xi)) = (\xi, -\lambda_3(\xi))$	0
$(\xi, -\lambda_2(\xi)) = (\xi, \lambda_3(\xi))$	0
$(\xi, 0) = (\xi, \lambda_2(\xi)) = (\xi, -\lambda_2(\xi))$	3
$(\xi, \lambda_1(\xi)) = (\xi, -\lambda_3(\xi))$	0
$(\xi, -\lambda_1(\xi)) = (\xi, \lambda_3(\xi))$	0
$(\xi, 0) = (\xi, \lambda_3(\xi)) = (\xi, -\lambda_3(\xi))$	3
$(\xi, \lambda_1(\xi)) = (\xi, -\lambda_2(\xi))$	0
$(\xi, -\lambda_1(\xi)) = (\xi, \lambda_2(\xi))$	0
$(\xi, \lambda_1(\xi)) = (\xi, \lambda_2(\xi))$	2
$(\xi, -\lambda_1(\xi)) = (\xi, -\lambda_2(\xi))$	2
$(\xi, \lambda_1(\xi)) = (\xi, \lambda_3(\xi))$	2
$(\xi, -\lambda_1(\xi)) = (\xi, -\lambda_3(\xi))$	2
$(\xi, \lambda_2(\xi)) = (\xi, \lambda_3(\xi))$	2
$(\xi, -\lambda_2(\xi)) = (\xi, -\lambda_3(\xi))$	2

From the table and the identity (3.3.2) we see that every $\xi \in C$ which is a zero of b_2 appears as a zero of Discr(b) with multiplicity 3 and that over it lie three fixed points of D with total multiplicity $3 + 0 + 0$. Similarly, over every zero of $27b_2 - (1/2)b_1^3$ there are two fixed points of D with total multiplicity $2 + 2$. Therefore the number of the fixed points of D is

$$D\cdot\Delta_1 = 3\cdot 6(2g - 2) + 4\cdot 6(2g - 2) = 84(g - 1). \qquad \square$$

This finishes the proof of Proposition 3.3.1.

4. The algebraic complete integrability of the Hamiltonian system $\mathcal{M}$

4.1

We will need another interpretation of the algebra $\mathfrak{g}_2$ and the vector space V which we proceed to describe.

Let $\mathbb{O}$ be the Cayley algebra over the complex numbers and let $\mathrm{Aut}(\mathbb{O})$ be its automorphism group. The algebra $\mathbb{O}$ is 8-dimensional and has an involutive $\mathbb{C}$-linear antiautomorphism: $a \to \bar{a}$. Let $\mathbb{O}^\circ$ be the subspace of purely imaginary Cayley numbers, i.e. those $a \in \mathbb{O}$ for which $\bar{a} = -a$. For every element $a \in \mathbb{O}$ denote by $R_a: \mathbb{O} \to \mathbb{O}$ the linear operator of right multiplication by a. Define the bilinear symmetric form $B(\cdot,\cdot)$ on the Cayley algebra by $B(a, b) := \mathrm{tr}(R_{a \cdot \bar{b}})$. This form is nondegenerate on the spaces $\mathbb{O}$ and $\mathbb{O}^\circ$ and is invariant under the group $\mathrm{Aut}(\mathbb{O})$. The group $\mathrm{Aut}(\mathbb{O})$ acts trivially on $\mathbb{C} \cdot 1$ and therefore faithfully on $\mathbb{O}^\circ = \ker \mathrm{tr}$. The identity component $\mathrm{Aut}^\circ(\mathbb{O})$ of $\mathrm{Aut}(\mathbb{O})$ is isomorphic to G_2 and its representation in $SO(\mathbb{O}^\circ, B)$ is exactly the representation ρ. Thus we can identify the vector space V with $\mathbb{O}^\circ$ and the algebra $\rho(\mathfrak{g}_2)$ with the Lie algebra $\mathrm{Der}(\mathbb{O})$ of the derivations of $\mathbb{O}$.

4.2

THEOREM 4.2.1. *$\mathcal{M}$ is an algebraically completely integrable system*

Proof. We will prove that the generic fiber $\mathcal{H}^{-1}(b)$ of the Hitchin map is a Zariski open set in the Prym-Tjurin variety P_b of the fiber.

Let $(E, \theta) \in \mathcal{H}^{-1}(b)$. Let $\mathbf{V}_E^0$ be the unique vector bundle contained in the kernel of the bundle morphism $\rho(\theta): \mathbf{V}_E \to \mathbf{V}_E \otimes K_C$. Consider the vector bundle $\mathbf{V}_E^1$ defined as the quotient

$$0 \to \mathbf{V}_E^0 \to \mathbf{V}_E \to \mathbf{V}_E^1 \to 0.$$

For the generic $\xi \in C$ the multiplicity of zero as an eigenvalue of the linear operator $\rho(\theta)(\xi)$ is equal to the multiplicity of the zero weight in the representation $\rho: \mathfrak{g}_2 \to \mathrm{End}(V)$ and therefore is equal to 1. Consequently $\mathbf{V}_E^1$ is a vector bundle of rank 6 and since $\mathbf{V}_E^0 \subset \ker \rho(\theta)$ we obtain the induced map $\rho(\theta): \mathbf{V}_E^1 \to \mathbf{V}_E^1 \otimes K_C$. As in [H1] we define the eigenvector line bundle corresponding to (E, θ) as the unique line bundle $L_{(E,\theta)}$ contained in the sheaf $\ker(\rho(\theta) - \lambda \cdot id_{\mathbf{V}_E^1}) \subset p^*(\mathbf{V}_E^1)$ (here we use that (E, θ) is regular and semisimple). The degree of $L_{(E,\theta)}$ can be computed via the formula (5.19) of [H1] since the bundle $\mathbf{V}_E$ is in particular a $SO(7, \mathbb{C})$ bundle. The result is $\deg L_{(E,\theta)} = 12(g - 1)$

and thus we obtain a morphism

$$\begin{aligned} \mathscr{H}^{-1}(b) &\to J^{12(g-1)}(C_b) \\ (E, \theta) &\to L_{(E,\theta)}. \end{aligned} \tag{4.2.1}$$

We shall show that any connected component of the range of this morphism is actually contained in a suitable translation of P_b or equivalently that

$$\imath^* L_{(E,\theta)} \otimes L_{(E,\theta)}^{\otimes 5} = \text{const},$$

for every $(E, \theta) \in \mathscr{H}^{-1}(b)$.

Let $L = L_{(E,\theta)}$ for some $(E, \theta) \in \mathscr{H}^{-1}(b)$. Since L is an eigenvector bundle for $\rho(\theta)$ then for generic ξ the fibre $L_{(\xi,\lambda_i(\xi))}$ can be identified with the weight subspace $V_{\lambda_i(\xi)}$. Therefore, using the definition of the correspondence D we can identify the fiber of the bundle $\imath^* L \otimes L^{\otimes 5}$ with some tensor product of weight subspaces of V. For instance, for each $i = 1, 2, 3$ the fibre $(\imath^* L \otimes L^{\otimes 5})_{(\xi,\lambda_i(\xi))}$ can be identified with

$$V_{\lambda_i(\xi)}^{\otimes 5} \otimes V_{\lambda_j(\xi)}^{\otimes 2} \otimes V_{\lambda_k(\xi)}^{\otimes 2} \otimes V_{-\lambda_i(\xi)}^{\otimes 3},$$

where $\{i, j, k\} = \{1, 2, 3\}$.

Consider the morphism of line bundles $\beta: \imath^* L \otimes L^{\otimes 5} \to \pi^* K_C^{\otimes 6}$ defined over each point $(\xi, \lambda_i(\xi))$ of C_b by

$$\begin{aligned} &\beta(v_{\lambda_i}^{\otimes 5} \otimes v_{\lambda_j}^{\otimes 2} \otimes v_{\lambda_k}^{\otimes 2} \otimes v_{-\lambda_i}^{\otimes 3}) \\ &\quad = B(\rho(\theta)(v_{\lambda_i}), v_{-\lambda_i})^{\otimes 3} \otimes B(\rho(\theta)(v_{\lambda_i}), v_{\lambda_j} \cdot v_{\lambda_k}) \otimes B(v_{\lambda_i}, \rho(\theta)(v_{\lambda_j}) \cdot \rho(\theta)(v_{\lambda_k})), \end{aligned}$$

where v_{λ_i} is a weight vector, the form B is the G_2 equivariant form defined in Section 5.1, and the multiplication $v_{\lambda_j} \cdot v_{\lambda_k}$ is the multiplication in the Cayley algebra. We will prove that β is a well defined holomorphic nonzero morphism. Therefore β defines a nonzero holomorphic section of the degree zero line bundle $\imath^* L^* \otimes (L^*)^{\otimes 5} \otimes \pi^* K_C^{\otimes 6}$ which yields $\imath^* L \otimes L^{\otimes 5} \simeq \pi^* K_C^{\otimes 6}$.

The statement that β is well defined and nonzero can be deduced in an obvious way from the following algebraic fact:

CLAIM 4.2.1. Let $\phi \in \mathfrak{g}_2 \subset SO(\mathbb{O}^\circ, B)$ be a regular, semisimple element with eigenvalues $0, \lambda_1, \lambda_2, \lambda_3, -\lambda_1, -\lambda_2, -\lambda_3$ and let $v_0, v_1, v_2, v_3, v_{-1}, v_{-2}, v_{-3}$ be a basis consisting of corresponding eigenvectors in $\mathbb{O}^\circ$. Consider the linear form $\gamma \in (V^*)^{\otimes 3}$, defined by $\gamma(w_1, w_2, w_3) \overset{\text{def}}{=} B(w_1, w_2 \cdot w_3)$. The following properties hold:

1. γ is a skew symmetric, G_2-invariant form on $\mathbb{O}^\circ$.
2. $\gamma(v_1, v_2, v_3) \neq 0$.
3. The forms $B(\phi(.),.) \in (V^*)^{\otimes 2}$ and $\gamma(\phi(.), \phi(.),.) \otimes \gamma(.,.,\phi(.)) \in (V^*)^{\otimes 6}$ determine well defined forms over the factor space $\bar{V} = V/\mathbb{C} \cdot v_0$.

Proof. Fix the eigenvectors v_i for $i = 1, 2, 3$ and normalize $\{v_{-i}\}_{i=1,2,3}$ and v_0, to obtain $B(v_0, v_0) = 1$ and $B(v_i, v_{-i}) = 1$. Writing B in terms of Cayley multiplication $B(x, y) = \frac{1}{2} \cdot (x \cdot \bar{y} + y \cdot \bar{x})$, gives

$$v_0^2 = -1,\ v_i^2 = v_{-i}^2 = 0,\ v_i \cdot v_{-j} = 0 \quad \text{for } i, j = 1, 2, 3 \text{ and } i \neq j.$$

To compute the remaining products of the elements of our basis let us note that since ϕ acts on $\mathbb{O}$ as an algebra derivation we have $\phi(v_2 \cdot v_3) = \phi(v_2) \cdot v_3 + v_2 \cdot \phi(v_3) = (\lambda_2 + \lambda_3) v_2 \cdot v_3 = -\lambda_1 v_2 \cdot v_3$. Thus $v_2 \cdot v_3$ is an eigenvector of ϕ with eigenvalue $-\lambda_1$, i.e. it is proportional to v_{-1}. Therefore for each even permutation $\{i, j, k\}$ of $\{1, 2, 3\}$ we can write

$$v_i \cdot v_j = a_{-k} \cdot v_{-k} \quad \text{and} \quad v_{-i} \cdot v_{-j} = a_k \cdot v_k,$$

where the a's are complex numbers. Furthermore, using the B-orthogonality of v_0 and v_i and the fact that any subalgebra of $\mathbb{O}$ generated by two elements is associative one deduces the relations:

$$v_k \cdot v_{-k} = -1 + \sqrt{-1} v_0,\ v_0 \cdot v_k = \sqrt{-1} v_k,\ v_0 \cdot v_{-k} = -\sqrt{-1} \cdot v_{-k}$$

where $k = 1, 2, 3$. Application of the Moufang identities to the triples $\{v_1, v_2, v_3\}$, $\{v_{-1}, v_{-2}, v_{-3}\}$ and $\{v_{-i}, v_i, v_j\}_{i \neq j}$ results in: $a_1 = a_2 = a_3 = a_+$; $a_{-1} = a_{-2} = a_{-3} = a_-$; $a_+ . a_- = 2$. We are free to choose the base elements $\{v_i\}_{i=1,2,3}$ arbitrarily in the corresponding eigenlines. Therefore we can renormalize our basis so that all of the above multiplication relations remain true and in additon $a_- = 1$, $a_+ = 2$. (For the properties of Cayley algebra used above see [OGV].)

After these general considerations we proceed with the proofs of the three statements of Claim 4.2.1:

1. The invariance of γ is clear from the definition. The skew symmetry of γ follows from the fact that for the purely imaginary Cayley numbers we **have**

$$\begin{aligned}\gamma(w_1, w_3, w_2) &= \operatorname{tr}(w_1 \cdot (w_3 \cdot w_2)) = \operatorname{tr}(w_1 \cdot ((-w_3) \cdot (-w_2))) \\ &= \operatorname{tr}(w_1 \cdot (\overline{w_3} \cdot \overline{w_2})) = -\operatorname{tr}(\overline{w_1} \cdot \overline{(w_2 \cdot w_3)}) \\ &= -\operatorname{tr}(w_1 \cdot (w_2 \cdot w_3)) = -\gamma(w_1, w_2, w_3).\end{aligned}$$

2. Consider the Cayley number $v_2 \cdot v_3$. From the above description of the multiplication in $\mathbb{O}$ we can conclude that $v_2 \cdot v_3$ is a nonzero (since $a_- \neq 0$) eigenvector of ϕ with eigenvalue $-\lambda_1$. This implies that $\gamma(v_1, v_2, v_3) = B(v_1, v_3 \cdot v_2) = a_- \cdot B(v_1, v_{-1}) \neq 0$. It is sufficient to prove that the form $\Gamma = \gamma(\phi(.), \phi(.), .) \otimes \gamma(., ., \phi(.))$ is zero for every triple $(v_{i_1}, v_{i_2}, v_{i_3})$ such that at least one of them is equal to v_0. But this is obvious from the definition of Γ.

This concludes the proof of the claim. □

Thus the morphism (4.2.2) maps any connected component of the fibre $\mathscr{H}^{-1}(b)$ to some component of $\mathrm{Ker}(\iota^* + 5)$, i.e. to a suitable translation of P_b. If $\tilde{\mathscr{H}}: \tilde{\mathscr{M}} \to B$ is the relative compactification of $\mathscr{M}$ discussed at the end of Section 2, then clearly (4.2.2) extends to a morphism

$$\Phi: \tilde{\mathscr{H}}^{-1}(b) \to \mathrm{Ker}^{\to}(i^* + 5)$$
$$(E, \theta) \to L_{(E,\theta)}$$

since in the definition of $L_{(E,\theta)}$ we have not used the semistability of E. On the other hand from the construction of $L_{(E,\theta)}$ it is obvious that $\pi_* L_{(E,\theta)} \simeq \mathbf{V}_E^1$ and that the pushforward of the tautological 1-form λ gives the twisted homomorphism $\rho(\theta): \mathbf{V}_E^1 \to \mathbf{V}_E^1 \otimes K_C$. Next we can recover $(\mathbf{V}_E, \rho(\theta))$ exactly in the same way as in Section 5.17 of [H1]. Taking the frame bundle of $\mathbf{V}_E$ and reducing its structure group to G_2 we obtain (E, θ). This procedure inverts the morphism Φ and hence yields an isomorphism between any connected component of $\tilde{\mathscr{H}}^{-1}(b)$ and P_b. Combined with the fact that the points in P_b which correspond to stable bundles form a Zariski open set this finishes the proof of the complete integrability of $\mathscr{M}$. □

REMARK 4.2.1. When the structure group G is a classical Lie group the generic fiber F of $\mathscr{H}$ is connected. Indeed, if $G = GL(n, \mathbb{C})$ then F is a Zariski open set in the Jacobian of the spectral curve (cf. [H1]) and therefore is irreducible. When G belongs to the series A_n Beauville, Narasimhan and Ramanan proved that F is a Zariski open set in the kernel of the norm map $\mathrm{NM}_\pi: J(C_b) \to J(C)$ and is connected (see [BNR]). If G belongs to one of the series B_n, C_n or D_n, then the spectral curve C_b possesses an involution σ and according to [H1] the fiber F is isomorphic to a Zariski open set in $\ker(1 + \sigma^*)$. In all these cases the involution σ has fixed points and the general theory of Prym varieties yields the connectedness of $\ker(1 + \sigma^*)$ [B], [Sh].

In the case $G = G_2$ the abelian subgroup $\ker(\iota + 5) \subset J(C_b)$ is not connected and so the connectedness of the fiber F is an open problem, which is reflected in the proof of the theorem.

Appendix: Kanev's theory of algebraic tori over function fields

In this appendix we will give a short summary of Kanev's theory which inspired our construction of the Prym-Tjurin variety and will explain how one can motivate our particular choice of the correspondence D in the framework of this theory.

Start with a complex simple Lie group G and let E be a principal G bundle over the curve C. Let $\mathfrak{g}_E := E \times_{ad} \mathfrak{g}$ be the associated bundle of Lie algebras. Suppose we are given the data

(i) A vector subbundle $\mathfrak{h}_E \subset \mathfrak{g}_E$ whose generic fibre is a Cartan subalgebra in $\mathfrak{g}$,
(ii) An irreducible representation $\rho: \mathfrak{g} \to \mathrm{End}(V)$.

As in Section 4.1 we can associate to this data a spectral cover

$$\begin{array}{ccc} \tilde{C} & \dashrightarrow & \mathfrak{h}_E^\vee \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & C & \end{array}$$

whose generic fibre consists of the set of all extremal weights of the representation ρ with respect to the corresponding Cartan subalgebra.

In his work [K1] V. Kanev has shown that in the case $C = \mathbb{P}^1$, $E = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}$ the spectral cover $\bar{C}$ comes equipped with a canonical correspondence $\mathcal{D}$ preserving the fibres of π.

The construction goes as follows. Let $\mathfrak{h} \subset \mathfrak{g}$ be a fixed Cartan subalgebra and let $Q \subset P \subset \mathfrak{h}^\vee$ be its lattices of roots and weights respectively. The Weyl group W acts absolutely irreducibly on Q since $\mathfrak{g}$ is a simple Lie algebra. Therefore the space of the integral valued W-invariant symmetric bilinear forms on Q is isomorphic to $\mathbb{Z}$. Let $(,)$ be a generator (over $\mathbb{Z}$) of this space. Fixing an extremal weight $\lambda \in P$ of the representation ρ allows us to construct a new W-module $N(Q, \lambda)$:

$$N(Q, \lambda) = Q \oplus \mathbb{Z} \cdot l,$$

as a $\mathbb{Z}$-module and the extended W-action is defined by

$$w(l) = w\lambda - \lambda + l, \quad \text{for each } w \in W.$$

Furthermore, the form $(,)$ extends to a symmetric bilinear and W-invariant form on $N(Q, \lambda)$ by $(\beta, l) := (\beta, \lambda)$ for all $\beta \in Q$ and $(l, l) = -1$.

One easily checks that (,) enjoys the properties:

(a) $(\beta, wl) = (\beta, w\lambda)$ for every $\beta \in Q$ and $w \in W$.
(b) The map $\nu: N(Q, \lambda) \to P$ characterized by the property $(\beta, x) = (\beta, \nu(x))$ for any $\beta \in Q$ and $x \in N(Q, \lambda)$ is well defined and W-invariant. It maps the orbit Wl bijectively onto the orbit $W\lambda$.

THEOREM-DEFINITION A1 (cf. [K1])

1. *The monodromy of the spectral covering $\tilde{C} \to C$ factors through the action of the Weyl group W.*
2. *Fix a base point $\xi \in C$ and choose an identification of the fibre $\pi^{-1}(\xi)$ with the set $W\lambda$ of extremal weights of ρ. For any integer $k \in \mathbb{Z}$ define the symmetric divisors $\mathbb{D}_k \subset \tilde{C} \times \tilde{C}$ as the reduced divisors characterized by the properties*
   (i) *If $(x, y) \in \mathbb{D}_k$, then $\pi(x) = \pi(y)$.*
   (ii) *Let $x, y \in \tilde{C}$ be such that $\pi(x) = \pi(y) = \eta$. Chose a path in C/(branch locus of π) which connects ξ and η. Consider the bijective map*

$$\mu: \pi^{-1}(\eta) \tilde{\to} \pi^{-1}(\xi) \tilde{\to} W\lambda \overset{\nu^{-1}}{\tilde{\to}} Wl.$$

   The point *(x, y) belongs to $\mathbb{D}_k$ if and only if $x \neq y$ and $(\mu(x), \mu(y)) = k$. (The last condition is independent on the choice of the path because of the part* 1 *and because* (,) *is W-invariant).*
3. *The divisors $\mathbb{D}_k$ are effective and $\mathcal{D} = \Sigma_{k \in \mathbb{Z}} k \cdot \mathbb{D}_k$ is a symmetric correspondence on $\tilde{C}$ which preserves π.*

Kanev discovered that the correspondence $\mathcal{D}$ is closely related to the completely integrable Hamiltonian systems. He proved the following theorem which stimulated our investigations of the Hitchin system for G_2:

THEOREM A2 (see [K1]). *Suppose E is a trivial principal bundle and $C = \mathbb{P}^1$.*

1. *The endomorphism ι of $J(\tilde{C})$ induced by the correspondence $\mathcal{D}$ satisfies the quadratic equation*

$$(\iota - 1)(\iota - q + 1) = 0,$$

where

$$q = \frac{d + N - 1}{r},$$

with $N = \deg \mathscr{D}$, $r = \operatorname{rank} \mathfrak{g}$ *and* d *is the degree of the spectral covering* $\tilde{C} \to C$.

2. *If the bundle* $\mathfrak{h}_E$ arises from a Lax pair with spectral parameter on $\mathbb{P}^1$, then the flow is linearized on the Prym-Tjurin variety $P(\tilde{C}, \iota)$.

We can apply Theorem A1 to the case $G = G_2$, ρ-the standard seven dimensional representations of G_2 and $(\mathfrak{h}_E, \mathfrak{g}_E)$ a pair arising from a regular and semisimple Higgs bundle. Fixing a Weyl chamber in the fibre $(\mathfrak{h}_E)_\xi$ over the base point ξ we have the corresponding simple roots $\{\alpha_1, \alpha_2\}$. The highest weight for the representation ρ then coincides with the first fundamental weight $\lambda = 2\alpha_1 + \alpha_2$. The form (,) is uniquely determined by the normalization $(\lambda, \lambda) = -8$. The multiplication table for its extension on $N(Q, \lambda)$ is shown in Table 1 . Next, from the definition of the extended action of the Weyl group on $N(Q, \lambda)$ we obtain that the elements of Wl are:

$$l_1 = l,\ l_2 = -\alpha_1 - \alpha_2 + l,\ l_3 = -3\alpha_1 - 2\alpha_2 + l,\ l_4 = l - \alpha_1,$$
$$l_5 = -3\alpha_1 - \alpha_2 + l,\ l_6 = -4\alpha_1 - 2\alpha_2 + l.$$

Table 1. The multiplication table for the form (,)

	l	α_1	α_2
l	-1	-1	0
α_1	-1	-2	3
α_2	0	3	-6

Following the recipe given in the above theorem we can see from Table 2 below that $\mathscr{D} = 2 \cdot \mathbb{D}_2 + 3 \cdot \mathbb{D}_3$. Furthermore by writing explicitly the isomorphism

Table 2. The values of the form (,) on the orbit Wl

	l_1	l_2	l_3	l_4	l_5	l_6
l_1	-1	0	2	0	2	3
l_2	0	-1	0	2	3	2
l_3	2	0	-1	3	2	0
l_4	0	2	3	-1	0	2
l_5	2	3	2	0	-1	0
l_6	3	2	0	2	0	-1

$$v: \{l_1, \ldots, l_6\} \to \{\lambda_1, \lambda_2, \lambda_3, -\lambda_1, -\lambda_2, -\lambda_3\}$$

we see that $\mathbb{D}_2 = \Delta_2$ and $\mathbb{D}_3 = \Delta_3$ which motivates our choice of the correspondence D.

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