COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 92, nº 1 (1994), p. 1-22
<http://www.numdam.org/item?id=CM_1994__92_1_1_0>

Compositio Mathematica **92**: 1–22, 1994.

Algebraic cycles and Hodge theory on generalized Reye congruences

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Received 14 January 1992; accepted in final form 10 April 1993

0. Introduction

Let (x, y) denote a system of bihomogeneous coordinates on $P^{n+1} \times P^{n+1}$ and σ the involution of $P^{n+1} \times P^{n+1}$ defined by $\sigma(x, y) = (y, x)$. In this paper we will check Grothendieck's generalized Hodge conjecture for the general element of a family $\pi: \mathscr{X} \to U$ of n-dimensional complete intersections $\{X_t\}_{t \in U}$ in $P^{n+1} \times P^{n+1}$ of σ-invariant divisors of bidegree (1, 1) so that a general X_t admits a fixed point free involution σ.

Our construction is motivated by the following classical 2-dimensional example (the Reye congruence [2] ex. VIII.19 p. 106).

Let P be a linear system of quadrics in P^3 of projective dimension 3 satisfying

(1) $\bigcap_{Q \in P} \mathrm{Q} = \varnothing$
(2) if l is a line in P^3 which is the vertex of a quadric $Q \in P$, then no other quadric of P contains l.

Let $Y \subset Gr(P^1, P^3)$ denote the variety of lines which are contained in the intersection of all quadrics from a one dimensional linear subspace of P. Then Y is an Enriques surface which is isomorphic to the quotient of the complete intersection in $P^3 \times P^3$ of four σ-invariant divisors of bidegree (1, 1).

We give a similar higher dimensional construction in section 1, together with a description of the topology and the cohomology of such varieties.

In section 2 we study the monodromy representation of the fundamental group $\pi_1(U, t)$ on $H^n(X_t, \mathbf{Q})$ for the family $\pi: \mathscr{X} \to U$. If V^+ and V^- denote the spaces of vanishing cycles respectively invariant and antiinvariant under σ, the main result is that V^+ and V^- are irreducible $\pi_1(U, t)$-subspaces of $H^n(X_t, \mathbf{Q})$.

In section 3 we recall Grothendieck's generalized Hodge conjecture (GGHC)

Work partially supported by M.U.R.S.T. Progetto Nazionale Geometria Algebrica and Science Program (contract SCI-0398-C(A)).

and introduce a suitable family

$$\begin{array}{ccc} T & \longrightarrow & X_t \\ \downarrow & & \\ F & & \end{array} \qquad (*)$$

of algebraic 1-cycles on the general X_t in $\mathscr{X} \to U$.

By using the infinitesimal cylinder map we show in section 4 that the morphism of Hodge structures deduced from $(*)$

$$\Psi: H^n(X_t) \to H^{n-2}(F)$$

is non-trivial and this, together with the irreducibility of $V^{\pm}$ under the global monodromy representation, allows to conclude that the GGHC holds for t general in U.

I would like to thank F. Bardelli for his helpful suggestions and encouragement.

1. Generalized Reye congruences

We start by giving the following generalization of the classical Reye congruence.

Let P be a linear system of quadrics in P^{n+1} of projective dimension $n+1$. We impose the following conditions on P, which are satisfied if P is generic enough:

(i) $\bigcap_{Q \in P} Q = \varnothing$,
(ii) if l is a line in P^{n+1} which is the vertex of a quadric $Q \in P$, then there exists no $(n-2)$-dimensional linear system of quadrics in P containing l.

Let Y be the variety of lines $l \subset P^{n+1}$ contained in the quadrics of some $(n-1)$-dimensional linear system in P i.e. $Y = \{l \subset P^{n+1}: \exists L \subset P$ $\operatorname{proj\,dim} L = n-1\; L \supset l\}$.

If we let (x, y) be a system of bihomogeneous coordinates on $P^{n+1} \times P^{n+1}$ and σ be the involution defined by $(x, y) \to (y, x)$ we have the following:

PROPOSITION 1.1. *Y is isomorphic to $X/\langle\sigma\rangle$ where $X \subset P^{n+1} \times P^{n+1}$ is a smooth connected n-dimensional complete intersection of $n+2$ divisors of bidegree $(1,1)$ invariant under σ.*

Proof. Let X be the subvariety of $P^{n+1} \times P^{n+1}$ of pairs (x, y) such that x and y are polar with respect to all the quadrics of P. If we let $Q_0, \ldots, Q_{n+1}$ be a basis of P, we can describe X as the intersection of the divisors $\{xQ_iy^T = 0\}$ $i = 0, \ldots, n+1$. These divisors are invariant under σ. By the Jacobian criterion, X is smooth and n-dimensional at a point (x, y) if and only if the line $\langle x, y\rangle \subset P^{n+1}$ is not contained in the vertex of a quadric of P. This eventuality is excluded by condition (ii). By the Lefschetz hyperplane sections theorem (L.h.s.t.) we know that $h^0(X) = h^0(P^{n+1} \times P^{n+1}) = 1$, hence $X \subset P^{n+1} \times P^{n+1}$ is a smooth connected n-dimensional complete intersection. The fixed point set of the involution σ of $P^{n+1} \times P^{n+1}$ is the diagonal. The induced action of σ on X is fixed point free because $xQ_ix^T = 0\, \forall i$ contradicts (i). We can construct a map $p: X \to Y$ such that $p((x, y))$ is the line $\langle x, y\rangle$. In fact if $(x, y) \in X$, the quadrics of P through x and y contain the line $\langle x, y\rangle$. The subspace

$$L = \{Q \in P: xQx^T = 0 = yQy^T\}$$

has codimension two, whence $\langle x, y\rangle \in Y$. Conversely, let l be a line of Y. The system P induces on l a pencil of 0-dimensional quadrics and there is exactly one pair of points (x, y) polar with respect to all the quadrics of this pencil, whence p induces an isomorphism between $X/\langle\sigma\rangle$ and Y.

In what follows we will study the varieties described before. From now on let X and Y be as in Prop. 1.1, $p: X \to Y$ the natural projection map, $p^*: H^n(Y, \mathbf{Q}) \to H^n(X, \mathbf{Q})$ and $p_*: H_n(X, \mathbf{Q}) \to H_n(Y, \mathbf{Q})$. If M is any module or vector space on which σ acts, we denote by M^+ and M^- the subspaces of invariant and antiinvariant elements of M with respect to σ. Since $p: X \to Y$ is an unramified double cover and Y is smooth, we have the following:

PROPOSITION 1.2. (1) $\chi_{\text{top}}(X) = 2\chi_{\text{top}}(Y)$ *and* $H^i(Y, \mathbf{Q}) = H^i(X, \mathbf{Q})^+$;

(2) *the canonical bundle* K_X *is trivial and* K_Y *is trivial in case n is odd;*

(3) $H^n(X, \mathbf{Q})^+$ *and* $H^n(X, \mathbf{Q})^-$ *are perpendicular with respect to the cup product;*

(4) $H^{n,0}(X) \subseteq H^n(X, \mathbf{C})^+$ *or* $H^n(X, \mathbf{C})^-$ *in case n is respectively odd or even;*

(5) *the cup product over* $H^n(X, \mathbf{Q})$ *restricts to a non-degenerate alternating bilinear form on* $H^n(X, \mathbf{Q})^+$ *and on* $H^n(X, \mathbf{Q})^-$.

Proof. (1) is obvious, being p unramified;

(2) by definition of X in $P^{n+1} \times P^{n+1}$ and the adjunction formula the triviality of K_X follows $\forall n$. If n is odd, the component of degree n of the Todd class of T_X is zero. The relations $c_i(T_X) = c_i(p^*T_Y) = p^*c_i(T_Y)$ and the injectivity of p^* imply $(Td(T_Y))_n = 0$. Hence, by the Hirzebruch-Riemann-Roch formula, $\chi(O_Y) = 0$. By the L.h.s.t. we have $h^{i,0}(X) = 0$ and consequently $h^{i,0}(Y) = 0$ when $0 < i < n$. The relation $\chi(O_Y) = 0$ implies $h^{n,0}(Y) = 1$. Let $\alpha \in H^{n,0}(Y)$ be a generator, then $p^*(\alpha)$ is a generator of $H^{n,0}(X)$, hence $\operatorname{div} p^*(\alpha) = 0$ and α cannot vanish at any point of Y, hence $K_Y = 0$;

(3) we can get the statement by computing the cup product between

elements $a = (a + \sigma^*(a))/2$ and $b = (b - \sigma^*(b))/2$ of $H^n(X, \mathbf{Q})^+$ and $H^n(X, \mathbf{Q})^-$ respectively;

(4) since p^* maps $H^n(Y, \mathbf{C})$ isomorphically onto $H^n(X, \mathbf{C})^+$ and $H^{n,0}(Y)$ onto $H^{n,0}(X)^+$, we have $\mathbf{C} \simeq H^{n,0}(Y) \simeq H^{n,0}(X) \subset H^n(X, \mathbf{C})^+$ if n is odd. If n is even $0 = H^{n,0}(Y) = H^{n,0}(X)^+$ hence, by the non triviality of $H^{n,0}(X)$, the statement follows;

(5) follows from (3) and the non-degeneracy of the cup product.

From the above facts we get the following:

CONCLUSION 1.3. In the case n is odd (even) $H^n(X, \mathbf{Q})^-(H^n(X, \mathbf{Q})^+)$ is a $\mathbf{Q}$-Hodge substructure of $H^n(X, \mathbf{Q})$, perpendicular to $H^{n,0}(X)$ with respect to the cup product.

As regards the topology of the varieties X and Y, we can state the following:

PROPOSITION 1.4.

$$b^i(X) = \begin{cases} k+1 & i = 2k,\ 2n - 2k \quad k = 0, \dots, [(n-1)/2] \\ 0 & i = 2k+1 \quad k = 0, \dots, n-1 \quad i \neq n \end{cases}$$

$$b^i(Y) = \begin{cases} (k+1)/2 & i = 2k,\ 2n - k \quad k \text{ odd}, \quad 0 < k \leqslant [(n-1)/2] \\ (k+2)/2 & i = 2k,\ 2n - k \quad k \text{ even}, \quad 0 \leqslant k \leqslant [(n-1)/2] \\ 0 & i = 2k+1 \quad k = 0, \dots, n-1 \quad i \neq n \end{cases}$$

where $[(n-1)/2]$ means the greatest integer less or equal than $(n-1)/2$.

Proof. Since X is a smooth n-dimensional complete intersection of very ample hypersurfaces in $P^{n+1} \times P^{n+1}$, we know, by the L.h.s.t., that $H^i(X, \mathbf{Q}) \sim \mathrm{H}^i(\mathrm{P}^{n+1} \times \mathrm{P}^{n+1}, \mathbf{Q})\ 0 \leqslant i < n$ and, by the hard Lefschetz theorem, that $b^i(X) = b^{2n-1}(X)$. If $i \neq n$ and ω_1 and ω_2 denote the Poincare' duals of the hyperplane sections respectively of the first and the second P^{n+1} restricted to X, by the Kunneth formula, we have that $H^i(X)$ ($\neq 0$ only if i is even) is spanned by $\langle \omega_1^{i/2}, \omega_1^{i/2-1} \wedge \omega_2, \dots, \omega_2^{i/2} \rangle$, hence the first part of the statement.

Changing basis, if we let $\omega^+ = (\omega_1 + \omega_2)/2$ and $\omega^- = (\omega_1 - \omega_2)/2$, we have: if $i = 2k$, k even, $H^i(X) = H^i(X)^+ \oplus H^i(X)^- = \langle (\omega^+)^{i/2}, (\omega^+)^{i/2-2} \wedge (\omega^-)^2, \dots, (\omega^-)^{i/2} \rangle \oplus \langle (\omega^+)^{i/2-1} \wedge \omega^-, \dots, \omega^+ \wedge (\omega^-)^{i/2-1} \rangle$ hence $b^i(Y) = \dim H^i(X)^+ = (k+2)/2$; if $i = 2k$, k odd, $H^i(X) = H^i(X)^+ \oplus H^i(X)^- = \langle (\omega^+)^{i/2}, \dots, (\omega^+) \wedge (\omega^-)^{i/2-1} \rangle \oplus \langle (\omega^+)^{i/2-1} \wedge \omega^-, \dots, (\omega^-)^{i/2} \rangle$ hence $b^i(Y) = (k+1)/2$; if i is odd, $b^i(Y) = b^i(X) = 0$. □

REMARK 1.5. $b^n(X)$ can be computed, once we know $c_n(X)$, from the relation $\chi_{\mathrm{top}}(X) = c_n(X)$ since all the $b^i(X)$'s for $i \neq n$ are known. If we denote by $T_{P^{n+1} \times P^{n+1}|X}$ the tangent bundle to $P^{n+1} \times P^{n+1}$ restricted to X, by T_X the tangent bundle to X and by $N_{X|P^{n+1} \times P^{n+1}}$ the normal bundle to X in

$P^{n+1} \times P^{n+1}$, the value of $c_n(X)$ comes from the following relation on the Chern polynomials

$$C(T_{P^{n+1}\times P^{n+1}|X}) = C(T_X)C(N_{X|P^{n+1}\times P^{n+1}})$$

By means of the values $b^i(Y)$ for $i \neq n$ previously computed and the relation $\chi_{\mathrm{top}}(X) = 2\chi_{\mathrm{top}}(Y)$ we get

$$b^n(Y) = \begin{cases} \dfrac{b^n(X)}{2} - \dfrac{n}{4} & n = 2k \;\; k \text{ even} \\ \dfrac{b^n(X)}{2} - \dfrac{n+2}{4} & n = 2k \;\; k \text{ odd} \\ \dfrac{b^n(X)}{2} + \dfrac{n+3}{4} & n = 2k+1 \;\; k \text{ even} \\ \dfrac{b^n(X)}{2} + \dfrac{n+1}{4} & n = 2k+1 \;\; k \text{ odd} \end{cases}$$

Now we want to construct a family of complete intersections admitting a fixed point free involution σ like in Prop. 1.1; let (x, y) be bihomogeneous coordinates in $P^{n+1} \times P^{n+1}$ and $\Delta = \{(x, y) \in P^{n+1} \times P^{n+1} : x = y\}$ the subspace of fixed points of σ. Let $R = H^0(P^{n+1} \times P^{n+1}, \mathcal{O}(1, 1))$ be the set of (1, 1)-forms on $P^{n+1} \times P^{n+1}$. We consider the following decomposition: $R = S \oplus A$ where $S = \{S_0, \dots, S_{N = \frac{(n+1)(n+4)}{2}}\}$ is the subspace of σ-invariant (1, 1)-forms and $A = \{A_{N+1}, \dots, A_{(n+2)^2-1}\}$ is the subspace of (1, 1)-forms of $P^{n+1} \times P^{n+1}$ antiinvariant under σ. We define the following maps: $v_2 : P^{n+1} \times P^{n+1} \to P(S^*)$ by

$$v_2(x, y) = (\dots, S_i(x, y), \dots)_{i=0,\dots,N}$$

and the Segre embedding $\eta : P^{n+1} \times P^{n+1} \to P(R^*)$ by

$$\eta(x, y) = (S_0(x, y), \dots, S_N(x, y), A_{N+1}(x, y)(, \dots, A_{(n+2)^2-1}(x, y))$$

We get the following commutative diagram

$$\begin{array}{ccc} P^{n+1} \times P^{n+1} & \xrightarrow{\eta} & \Sigma \\ & \searrow^{v_2} & \downarrow \pi \\ & & Z \end{array}$$

where Σ is a smooth variety isomorphic to $P^{n+1} \times P^{n+1}$, v_2 and the projection π are finite morphisms of degree 2 onto Z and Z is smooth off $\operatorname{Sing} Z = v_2(\Delta) = \pi(\Sigma \cap \operatorname{Ann}\langle A_i \rangle_{i=N+1,\dots,(n+2)^2-1})$.

The image by v_2 of a smooth complete intersection of $n+2$ symmetric divisors of bidegree (1, 1) $X \subset P^{n+1} \times P^{n+1}$ is given by $Y = Z \cap L$, where L is the $n(n+3)/2$-projective dimensional linear subspace of $P(S^*)$ defined by $L = \operatorname{Ann}\langle Q_0, \ldots, Q_{n+1}\rangle$. We get therefore

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \Upsilon = \Sigma \cap \pi^{-1} L \\ & {\scriptstyle v_2} \searrow & \downarrow {\scriptstyle \pi} \\ & & Y \end{array}$$

The morphisms $v_2: X \to Y$ and $\pi: \Upsilon \to Y$ have degree 2 and fibres of cardinality one exactly over the points of $L \cap \operatorname{Sing} Z$. We will choose L generic in such a way that $L \cap \operatorname{Sing} Z = \varnothing$ and L and Z are transversal at each point of $Z \cap L$. In particular Y will be smooth and the double coverings $v_2: X \to Y$ and $\pi: \Upsilon \to Y$ will be unramified. We get therefore a parametrization of the varieties X's by the points of a Zariski open subset U of $\operatorname{Gr}((n+1)(n+2)/2, S^*) \simeq \operatorname{Gr}(n+2, S) = \operatorname{Gr}(P^{n+1}, P(S))$ with the generic one smooth. We denote $\pi: \mathscr{X} \to U$ the family of smooth complete intersections of $n+2$ symmetric divisors of bidegree (1,1) $\subset P^{n+1} \times P^{n+1}$ admitting the involution σ.

2. The monodromy action

The aim of this section is to study the monodromy action of $\pi_1(U, t)$ on $H_n(X_t, \mathbf{Q})$ for the family $\pi: \mathscr{X} \to U$ previously constructed. Following the notations introduced in section 1, we consider an $(n+1)(n+2)/2$-dimensional linear space $L' \subset P(S^*)$ satisfying the following conditions:

2.0.1. L' is transversal to Z at all points of $L' \cap (Z \backslash \operatorname{Sing} Z)$.

2.0.2. $L' \cap \operatorname{Sing} Z$ is a finite set of $M = 2^{n+1}$ distinct points $P'_1, \ldots, P'_M$ along which L' and $\operatorname{Sing} Z$ intersect transversely.

Define $W_S = L' \cap Z$ and its counterimages $W = v_2^{-1}(W_S)$ and $W_R = \pi^{-1}(W_S)$. For this choice of L' we get:

PROPOSITION 2.1. (1) *W is a smooth complete intersection of $n+1$ divisors belonging to $P(S)$.*

(2) *W_R is a smooth complete intersection of $n+1$ hyperplane sections of Σ, symmetric with respect to the hyperplanes given by $\{\operatorname{Ann} A_i\} i = N+1, \ldots, (n+2)^2 - 1$, and intersecting $\operatorname{Ann}\langle A_{N+1}, \ldots, A_{(n+2)^2-1}\rangle$ in exactly $2^{(n+1)}$ points (the images of the points of W fixed by σ).*

Proof. (1) Since the smoothness of $W \backslash \Delta$ follows from the smoothness of $W_S \backslash \operatorname{Sing} Z$, we are interested in studying the points $(\bar{x}, \bar{x}) \in W \cap \Delta$. Here the tangent space to $W \subset P^{n+1} \times P^{n+1}$ has dimension $n+1$; the transversality of $W = \bigcap_{i=0}^{n} V(Q_i)$ and Δ follows from condition 2.0.2 and the isomorphism $v_2 : \Delta \to \mathrm{Sing}\, Z$, hence the statement.

(2) follows from (1) and the definition of η. □

We define the dual variety D_S of W_S in L'^V by

$$D_S = \check{W}_S \cup \left(\bigcup_{i=1}^{2^{n+1}} H_i \right)$$

where $\check{W}_S$ is the closure of the set of hyperplanes in L' which are tangent at some point of $W_S \backslash \mathrm{Sing}\, W_S$ and H_i is the set of hyperplanes of L' passing through P'_i; in the same way we define the dual variety D_R of W_R in $(\pi^{-1}L')^V$ by $D_R = \check{W}_R = \{$hyperplanes in $\pi^{-1}L'$ which are tangent at some point of $W_S\}$.

We want to study the homology group $H_n(X_t, \mathbf{Q})$ for a general variety X_t of the family previously constructed and to do this we choose a pencil $\mathscr{P}_l$ of hyperplanes of W_S by choosing a line l in L'^V such that l and D_S are transversal at each point of $l \cap D_S$. If we consider the counterimages by v_2 we get a pencil of hypersurfaces of W with these properties:

(1) there are exactly $M = 2^{n+1}$ hypersurfaces X_{R_i} $i = 1, \dots, M$ with an ordinary double point at $P_i = v_2^{-1}(P'_i)$ which is a fixed point of σ and no other singular point;
(2) if r is the number of points of $l \cap \check{W}_S$, we have r hypersurfaces X_{T_i} with 2 ordinary double points P_i^1 and P_i^2 interchanged by σ and no other singular point;
(3) all the other hypersurfaces $X_t$of the pencil are smooth.

We now fix a base point $t \in l^* = l \backslash \{R_1, \dots, R_M, T_1, \dots, T_r\}$ and let δ_i be the vanishing cycle attached to the singularity $P_i \in X_{R_i}$ $i = 1, \dots, M$ and δ_i^1, δ_i^2 the vanishing cycles attached to the singularities P_i^1 and P_i^2 of X_{T_i} $i = 1, \dots, r$.

We choose orientations on the δ_i^j's in such a way that

$$\sigma_*(\delta_i^1) = \delta_i^2 \quad i = 1, \dots, r.$$

A local computation shows that

$$\sigma_*(\delta_i) = (-1)^{n+1}\delta_i \quad i = 1, \dots, M.$$

By the hard Lefschetz theorem, $H_n(X_t, \mathbf{Q}) = V \oplus I$ where V is spanned by the vanishing cycles introduced above and I ($\neq 0$ only if n is even) is the space spanned by the invariant cycles $[\omega^+]^{(n/2)-i} \cdot [\omega^-]^i$ $i = 0, \dots, n/2$, Poincare' duals of the restrictions to X_s of the cohomology classes ω^+ and ω^- where the multiplication stands for the intersection pairing in homology ([7] 4.1.8 p. 30). If we denote $\delta_i^{\pm} = (\delta_i^1 \pm \delta_i^2)/2$ we have the following decompositions:

$H_n(X_t, \mathbf{Q})^+ = V^+ \oplus I^+$ where

$$V^+ = \begin{cases} \langle \delta_i, \delta_k^+ \rangle & i = 1, \ldots, M \quad k = 1, \ldots, r \quad n \text{ odd} \\ \langle \delta_k^+ \rangle & k = 1, \ldots, r \quad n \text{ even} \end{cases}$$

and

$$I^+ = \begin{cases} [\omega^+]^{(n/2)-2i} \cdot [\omega^-]^{2i} & i = 0, \ldots, \frac{n-2}{4}, \quad n = 2k \ k \text{ odd} \\ [\omega^+]^{(n/2)-2i} \cdot [\omega^-]^{2i} & i = 0, \ldots, \frac{n}{4}, \quad n = 2k \quad k \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$H_n(X_t, \mathbf{Q})^- = V^- \oplus I^-$ where

$$V^- = \begin{cases} \langle \delta_i, \delta_k^- \rangle & i = 1, \ldots, M \quad k = 1, \ldots, r \quad n \text{ even} \\ \langle \delta_k^- \rangle & k = 1, \ldots, r \quad n \text{ odd} \end{cases}$$

and

$$I^- = \begin{cases} [\omega^+]^{(n/2)-(2i+1)} \cdot [\omega^-]^{2i+1} & i = 0, \ldots, \frac{n-2}{4}, \quad n = 2k \quad k \text{ odd} \\ [\omega^+]^{(n/2)-(2i+1)} \cdot [\omega^-]^{2i+1} & i = 0, \ldots, \frac{n}{4} - 1, \quad n = 2k \quad k \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Now we can state the

2.2. MONODROMY THEOREM. *V^+ and V^- are simple submodules for the global monodromy representation on $H_n(X_t, \mathbf{Q})$.*

In order to prove the theorem we recall:

2.3.The Picard-Lefschetz formulas ([7] 6.3.3, p. 40)

$$\rho_i(\gamma) = \gamma + \varepsilon(\gamma, \delta_i)\delta_i$$

and

$$\tau_i(\gamma) = \gamma + 2\varepsilon(\gamma, \delta_i^+)\delta_i^+ + 2\varepsilon(\gamma, \delta_i^-)\delta_i^-$$

where

$$\varepsilon = (-1)^{(n+1)(n+2)/2},\ \gamma \in H_n(X_t, \mathbf{Q}),\ \rho_i\colon H_n(X_t, \mathbf{Q}) \to H_n(X_t, \mathbf{Q})$$

is the Picard-Lefschetz monodromy transformation associated to an elementary loop r_i in l^* based at s and encircling R_i but no other R_j for $i \neq j$ or T_k. Now $\tau_i\colon H_n(X_t, \mathbf{Q}) \to H_n(X_t, \mathbf{Q})$ is the monodromy transformation associated to an elementary loop t_i in l^* based at s and encircling T_i but no other T_j for $i \neq j$ or R_k;

2.4. The following propositions:

If we denote with the same symbols t_i and r_i the homotopy classes of t_i and r_i in $\pi_1(l^*, t)$ and in $\pi_1(\check{L}' \backslash D_S, t)$, the following relations hold in $\pi_1(\check{L}' \backslash D_S, t)$:

(1) $\forall i, j = 1, \dots, M$, $r_i r_j = r_j r_i$ ([1] Prop. 3.6, p. 179);
(2) *t_i, t_j are mutually conjugate $\forall i, j = 1, \dots, r$ and there exists $u \in \pi_1(\check{L}' \backslash D_S, t)$ such that $u(\delta_i^{\pm}) = \pm\delta_j^{\pm}$* ([7] 7.3.5, p. 46).

We also need

LEMMA 2.5. (1) $\forall i, j = 1, \dots, M$

$$(\delta_i, \delta_j) = \begin{cases} 0 & n \text{ odd} \\ 0 & n \text{ even} \quad \delta_i, \delta_j \text{ linearly independent} \\ \pm(-1)^{n/2}2 & n \text{ even} \quad \delta_i, \delta_j \text{ linearly dependent.} \end{cases}$$

(2) *$\forall \delta_h$, $h = 1, \dots, M$ there exists $\delta_i^{\pm}$ such that $(\delta_h, \delta_i^{\pm}) \neq 0$ (δ_i^+* if n is odd, *δ_i^- if n is even).*

Proof. (1) By 2.4(1) we know that $\forall i,j\ \rho_i\rho_j = \rho_j\rho_i$, hence for each $\gamma \in H_n(X_t, \mathbf{Q})$ we have:

$$\rho_i\rho_j(\gamma) = \gamma + \varepsilon(\gamma, \delta_j)\delta_j + \varepsilon(\gamma, \delta_i)\delta_i + \varepsilon^3(\gamma, \delta_j)(\delta_j, \delta_i)\delta_i$$
$$\rho_j\rho_i(\gamma) = \gamma + \varepsilon(\gamma, \delta_i)\delta_i + \varepsilon(\gamma, \delta_j)\delta_j + \varepsilon^2(\gamma, \delta_i)(\delta_i, \delta_j)\delta_j,$$

which gives

$$(\gamma, \delta_j)(\delta_j, \delta_i)\delta_i = (\gamma, \delta_i)(\delta_i, \delta_j)\delta_j. \qquad (*)$$

By definition of vanishing cycle, the self-intersection number

$$(\delta_i, \delta_i) = \begin{cases} 0 & n \text{ odd} \\ (-1)^{n/2}2 & n \text{ even.} \end{cases}$$

In case n is odd, if δ_i, δ_j are linearly dependent, the assertion is obvious; if they are independent, then there exists $\tilde{\gamma} \in H_n(X_t, \mathbf{Q})$ such that $(\tilde{\gamma}, \delta_i) = 0$ and $(\tilde{\gamma}, \delta_j) \neq 0$, hence, by $(*)$, the assertion follows.

If n is even and δ_i, δ_j are non zero and linearly dependent in $H_n(X_t, \mathbf{Q})$, we note that $\delta_i = \pm\delta_j$, in fact, there exist two rational numbers $a \neq 0$ and $b \neq 0$ such that $a\delta_j + b\delta_i = 0$, we have that

$$(\delta_i, \delta_j) = -\frac{b}{a}(\delta_i, \delta_i) = -\frac{b}{a}(-1)^{n/2}2 = -\frac{a}{b}(\delta_j, \delta_j) = -\frac{a}{b}(-1)^{n/2}2,$$

which gives $a = \pm b$ i.e. $\delta_i = \pm\delta_j$ and $(\delta_i, \delta_j) = \pm(-1)^{n/2}2$.

If δ_i, δ_j are independent, $(*)$ implies that for each $\gamma \in H_n(X_t, \mathbf{Q})$

$$(\gamma, \delta_j)(\delta_j, \delta_i) = (\gamma, \delta_i)(\delta_i, \delta_j) = 0,$$

but if we choose, for example $\gamma = \delta_i$, we get $(\delta_i, \delta_j)^2 = \pm 2(\delta_i, \delta_j) = 0$ and the assertion follows.

(2) Let n be odd. In (1) it is proved that, given a vanishing cycle δ_h, $(\delta_h, \delta_j) = 0 \forall j$. If $(\delta_h, \delta_i^+) = 0 \forall i = 1, \ldots, r$ the intersection pairing in $H_n(X_t, \mathbf{Q})^+$ would be degenerate, but this is a contradiction.

If n is even, we note first the following facts.

Let l be the line in L'^V introduced at the beginning of this section, corresponding to the $\mathscr{P}_l$ of hypersurfaces $\{X_s\}_{s \in l}$ of W admitting the involution σ.

(i) If α is a hyperplane of the pencil $\mathscr{P}_l$ passing through the image by v_2 of a fixed point P_α of W, by the symmetry of W_R and the meaning of π^{-1}, we have that $\pi^{-1}(\alpha)$ is tangent to W_R at $\eta(P_\alpha)$ i.e. $\cup H_i$ belongs to $\check{W}_R \cap \{A_i = 0\}_{i = N+1, \ldots, (n+2)^2 - 1}$.

(ii) If $\beta \in \mathscr{P}_l$ is tangent to W_S at a point $P_\beta \notin \operatorname{Sing} W_S$, its counterimage $\pi^{-1}(\beta)$ is tangent to W_R at the two counterimages P_β^1 and P_β^2 of P_β. This implies that its corresponding point $B \in \check{W}_R$ is double for $\check{W}_R$, otherwise there would exist only one tangency point between $\pi^{-1}(\beta)$ and W_R.

To prove the assertion, we construct a general Lefschetz pencil of hyperplane sections of W_R whose fibres no longer admit the involution σ. In particular, we may consider L'^V as a subspace of $(\pi^{-1}L')^V$ and consequently the line l as a line in $L'^V \subset (\pi^{-1}L')^V$; we choose a line l' in $(\pi^{-1}L')^V$ 'close enough to l' with the following properties:

—l' is contained in $(\pi^{-1}L')^V$ but not in L'^V;

—$l \cap l' = t$ where $t \in l \setminus \{R_1, \ldots, R_M, T_1, \ldots, T_r\}$;

—l' and D_R are transversal at each point of $l' \cap D_R$.

Let R_i', T_{i1}' and T_{i2}' be the points of $l' \cap D_R$ 'close' to R_i and $T_i \in l \cap D_R$ and δ_i', δ_{i1}', δ_{i2}' the corresponding vanishing cycles. By choosing a suitable path in $(\pi^{-1}L')^V$, we can construct a $(n+1)$-chain Γ whose boundary is given by $\delta_i' - \delta_i$. This implies that δ_i' and δ_i are homologous in X_t. The same argument shows that δ_{i1}' and δ_{i2}' are homologically equivalent to δ_i^1 and δ_i^2 respectively and, as a consequence, $(\delta_i, \delta_j^1) = (\delta'_i, \delta'_{j1})$; $(\delta_i, \delta_j^2) = (\delta'_i, \delta'_{j2})$; $(\delta_i^1, \delta_j^2) = (\delta'_{i1}, \delta'_{j2})$.

To show that for each given δ_h there exists a δ_i^- such that $(\delta_h, \delta_i^-) \neq 0$, we show that $(\delta_h, \delta_i^1) \neq 0$, since

$$\begin{aligned}(\delta_h, \delta_i^-) &= (\delta_h, \delta_i^1) - (\delta_h, \delta_i^2) = (\delta_h, \delta_i^1) - (\sigma_* \delta_h, \sigma_* \delta_i^2) \\ &= (\delta_h, \delta_i^1) - (-\delta_h, \delta_i^1) = 2(\delta_h, \delta_i^1).\end{aligned}$$

Suppose there exist no such δ_i^1's and, correspondingly, no δ_{i1}''s such that $(\delta'_h, \delta'_{i1}) \neq 0$. By our choice of l', corresponding to a Lefschetz pencil $\mathscr{P}_{l'}$, we know by the classical Lefschetz theory that, if we denote by $l'^* = l' \backslash \{R_i', T_{j1}', T_{j2}'\}_{i=1,\ldots,M; j=1,\ldots,r}$, there exists an element $u \in \pi_1(l'^*, t)$ such that $u(\delta_h') = \delta_{i1}'$, but, by the Picard Lefschetz formulas, this implies that there exists at least a $\delta_k' \neq \delta_h'$ $k \neq h$ such that $(\delta'_h, \delta'_k) \neq 0$ and this δ_k' must be one of the δ_{i1}' by 2.5 (1). This gives a contradiction.

Proof of the monodromy theorem. Suppose n is odd. We know that V^- is spanned by $\langle \delta_i^- \rangle$. If $F \subset V^-$ is a nontrivial π_1-invariant subspace, by the non-degeneracy of the intersection pairing on V^-, there exists $x \in F$ and some δ_i^- such that $(x, \delta_i^-) \neq 0$, but then, by the Picard Lefschetz formulas and the π_1-invariance of F, it follows that $\delta_i^- \in F$ and 2.4 (2) implies that $F = H_n(X_t, \mathbf{Q})^-$.

If $F \in V^+$ is a nontrivial π_1-invariant subspace and $x \in F$, always by the nondegeneracy of the intersection pairing, there exists a vanishing cycle δ such that $(x, \delta) \neq 0$. By the same arguments as before, $\delta \in F$. If $\delta = \delta_h$ then by 2.5 (2) there exists a δ_i^+ such that $(\delta_h, \delta_i^+) \neq 0$ and $\delta_i^+ \in F$. By 2.4 (2) and the π_1-invariance of F all the δ_i^+'s belongs to F. To finish the proof we note that $\forall \delta_k$ there exists a cycle δ_s^+ such that $(\delta_k, \delta_s^+) \neq 0$ and by applying the transformation ρ_k to δ_s^+, we conclude that $\delta_k \in F$ $k = 1, \ldots, M$ and we are done.

The same proof holds if n is even by changing plus into minus.

3. A family of algebraic one cycles

In this section we want to construct a family of algebraic one cycles on the general variety X_t of the family $\mathscr{X} \to U$ introduced in section 1. Let $X = X_t$ denote the variety we have fixed. If $Q_0, \ldots, Q_{n+1}$ denotes a fixed basis for the linear system P satisfying the imposed generality conditions and W the smooth $(n+1)$-dimensional complete intersection given by $W=\{(x,y)\subset P^{n+1}\times P^{n+1}: xQ_iy^T=0,\ i=1,\dots,n+1\}$, we can think of X as the hypersurface of W given by $\{xQ_0y^T=0\}$ or, equivalently, as the hypersurface given by $\{xQ'_0y^T=0\}$ where Q'_0 represents a rank n quadric of P which, together with $Q_1,\dots,Q_{n+1}$, spans the linear system.

Let $F_n(X)$ be the variety of such quadrics i.e. the variety of quadrics in P^{n+1} of rank n containing X.

PROPOSITION 3.1. *$F_n(X)$ is a $(n-2)$-dimensional variety whose singular locus, given by $\{Q\in F_n(X)\colon \operatorname{rank} Q<n\}$ has dimension $(n-5)$.*

Proof. It is well known that the dimension of the affine variety of quadrics in P^{n+1} of rank n is $[(n+2)(n+3)/2]-3$ and that its singular locus is given by the quadrics of rank strictly smaller than n. For a general choice of the $(n+1)$-dimensional linear system P, the assertion follows. □

Let $Q'_0\in F_n(X)$; after a projective automorphism we can always arrange $xQ'_0y^T=\Sigma_{i=0}^{n-1}x_iy_i$ so that it is immediate to see that $xQ'_0y^T=0$ contains a $P^1\times P^{n+1}$ given by $x_0=\cdots=x_{n-1}=0$ and the corresponding $P^{n+1}\times P^1$ under the involution σ.

Let us denote $C_1=X\cap(P^1\times P^{n+1})$ and $C_2=X\cap(P^{n+1}\times P^1)$.

PROPOSITION 3.2. *C_1 and C_2 are smooth rational curves on X, complete intersections in W of the hyperplanes $x_0=\cdots=x_{n-1}=0$ and $y_0=\cdots=y_{n-1}=0$ respectively.*

Proof. We will prove the assertion for $C=C_1$; the same proof holds for $C_2=\sigma(C_1)$ interchanging x with y. Let $\Gamma=P^1\times P^{n+1}$ and W as above. By the exact sequence

$$0\to H^0(P^{n+1}\times P^{n+1},\mathcal{I}_\Gamma(1,1))\to H^0(P^{n+1}\times P^{n+1},\mathcal{O}(1,1))\to H^0(\Gamma,\mathcal{O}_\Gamma(1,1))\to 0$$

knowing that $h^0(P^{n+1}\times P^{n+1},\mathcal{O}(1,1))=(n+2)^2$ and $h^0(\Gamma,\mathcal{O}_\Gamma(1,1))=2(n+2)$, we have $h^0(P^{n+1}\times P^{n+1},\mathcal{I}_\Gamma(1,1))=n(n+2)$ hence, in the space of all divisors of bidegree (1, 1) in $P^{n+1}\times P^{n+1}$, the space of symmetric divisors of $P^{n+1}\times P^{n+1}$ containing Γ has projective dimension $\geqslant n(n+2)+[(n+2)(n+3)/2]-(n+2)^2-1=[(n+2)(n-1)/2]-1\geqslant n+1=\dim P\ \forall n\geqslant 3$ hence we can choose a quadric Q'_0 in $P(S)$ containing Γ which is not linearly dependent on the quadrics defining W and such that $C_1=\Gamma\cap W$ is a smooth complete intersection. As regards the rationality, let ω_1 and ω_2 denote the first Chern classes of the hyperplane bundles of the first and the second P^{n+1} respectively. It is immediate to see that

(1) C is algebraically equivalent to $\omega_1^n(\omega_1+\omega_2)^{n+1}$ in $P^{n+1}\times P^{n+1}$;

(2) $\Omega^{2n+2}_{P^{n+1}\times P^{n+1}} \simeq -(n+2)\omega_1 - (n+2)\omega_2$;
(3) $\det N_{W|P^{n+1}\times P^{n+1}} \simeq (n+1)\omega_1 + (n+1)\omega_2$ where $N_{W|P^{n+1}\times P^{n+1}} = \bigoplus_1^{n+1} \mathcal{O}(1,1)$;
(4) $\det N_{C|W} = n\omega_1$ where $N_{C|W} = \bigoplus_1^n \mathcal{O}(1,0)$.

By the adjunction formula we have $\Omega^{n+1}_W = -\omega_1 - \omega_2$ and the class of Ω_C in $\mathrm{Pic}(C)$ is represented by the divisor $[(n-1)\omega_1 - \omega_2]_C$. Hence, by restriction to C, we see that $\deg \Omega_C = \deg([(n-1)\omega_1 - \omega_2][\omega_1^n(\omega_1+\omega_2)^{n+1}]) = -2$ and the assertion follows. □

By taking a desingularization $\tilde{F}$ of $F_n(X)$ as a parameter space for such quadrics Q'_0, we get a family of curves

$$\begin{array}{ccc} T & \longrightarrow & X_t \\ {\scriptstyle f}\big\downarrow & & \\ \tilde{F} & & \end{array}$$

By the presence of the involution σ which, given a point $Q'_0 \in F_n(X_t)$, interchanges the $P^1 \times P^{n+1} \subset Q'_0$ into $P^{n+1} \times P^1$, we can consider a Stein factorization of the map f

$$\begin{array}{ccc} T & \xrightarrow{\ p\ } & F = F_1 \cup F_2 \\ {\scriptstyle f}\big\downarrow & \swarrow {\scriptstyle h} & \\ \tilde{F} & & \end{array}$$

where $h: F \to \tilde{F}$ is an unramified double cover, F is smooth and has two irreducible components F_1 and F_2. In fact, there is no closed path $\gamma: [0,1] \to \tilde{F}$ which, lifted to a path $\tilde{\gamma}$ on F, admits $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ lying on distinct sheets of the covering $h: F \to \tilde{F}$.

We will think, from now on, of the family $T \to F$ as the union of two families of curves on X_t, $\{(C_1)_r\}_{r\in F_1}$ and $\{(C_2)_s\}_{s\in F_2}$, interchanged by the action of σ.

If i denotes the map from F to the component of the Hilbert scheme parametrizing such curves on X_t, it is not difficult to see that i is generically injective.

PROPOSITION 3.3. *If we let C be a rational curve of one of the above families (for example $C = (C_1)_s$) then*

(1) $\det N_{C|X} = \mathcal{O}(-2)$;
(2) $N_{C|W} = \bigoplus_1^n \mathcal{O}(1)$;
(3) $N_{X|W|C} = \mathcal{O}(n+2)$;
(4) $N_{C|X} = \bigoplus_1^{n-2} \mathcal{O} \oplus \mathcal{O}(-2)$;

Proof. (1) This follows from the adjunction formula and the triviality of Ω^n_X;

(2) C is a complete intersection in W of n divisors of bidegree (1, 0) hence $N_{C|W} = \bigoplus_1^n \mathcal{O}(1, 0)$. If ω_1 and ω_2 are the Chern classes introduced in the proof of Prop. 3.2, the restriction to C of ω_1 is equivalent to $\omega_1^{n+1}(\omega_1 + \omega_2)^{n+1} = \omega_1^{n+1}\omega_2^{n+1}$. Therefore it has degree 1 and $\mathcal{O}_C(1, 0) = \mathcal{O}_C(1)$;

(3) in the same way, being $N_{X|W|C} = \mathcal{O}_C(1, 1)$, the restriction to C of $\omega_1 + \omega_2$ has degree $n + 2$;

(4) as seen in Prop. 3.1, we know that the parameter space F for our family of curves has dimension $n - 2$ hence, in the generic point, the tangent space to the component of the Hilbert scheme parametrizing such curves must have dimension greater or equal than $n - 2$, thus $h^0(N_{C|X}) \geqslant n - 2$. If $N_{C|X} = \bigoplus_{i=1}^{n-1} \mathcal{O}(a_i)$ denotes the decomposition of the $(n - 1)$-bundle $N_{C|X} \to C$, by the Riemann-Roch formula we get

$$\chi(N_{C|X}) = \sum_{i=1}^{n-1} h^0(\mathcal{O}(a_i)) - \sum_{i=1}^{n-1} h^0(\mathcal{O}(-2 - a_i)) = n - 3$$

hence $\Sigma_{i=1}^{n-1} h^0(\mathcal{O}(-2-a_i)) = \Sigma_{i=1}^{n-1} h^0(\mathcal{O}(a_i)) + 3 - n \geqslant n - 2 + 3 - n = 1$. Th implies there exists at least an index j, $1 \leqslant j \leqslant n - 1$, such that $a_j \leqslant -2$. L us consider on C the normal bundle sequence

$$0 \to \bigoplus_{i=1}^{n-1} \mathcal{O}(a_i) \to \bigoplus_{i=1}^{n} \mathcal{O}(1) \to \mathcal{O}(n + 2) \to 0;$$

by tensoring with $\mathcal{O}(-1)$ we get the corresponding cohomology exact sequence

$$0 \to \bigoplus_{i=1}^{n-1} H^0(\mathcal{O}(a_i - 1)) \to \bigoplus_{i=1}^{n} H^0(\mathcal{O}) \xrightarrow{f} H^0(\mathcal{O}(n + 1))$$
$$\to H^1(\mathcal{O}(a_i - 1)) \to 0.$$

We want to prove the injectivity of f. If this is the case, $\bigoplus_{i=1}^{n-1} H^0(\mathcal{O}(a_i - 1) = 0$, hence $\forall i$ we get $a_i < 1$. By this and the previous relations $\Sigma_{i=1}^{n-1} a_i = -2$ and $\exists j$ such that $a_j \leqslant -2$ the assertion follows.

If $\Sigma_{i=0}^{n-1} x_i y_i = 0$ and $x_0 = \cdots = x_{n-1} = 0$ are respectively the equations of X and C in W, we see that the map

$$f: \bigoplus_{i=1}^{n} H^0(\mathcal{O}) \to H^0(\mathcal{O}(n + 1))$$

is computed by:

$$(\alpha_1, \ldots, \alpha_n) \to \sum_{i=1}^{n} \alpha_i y_i$$

where y_i denote the restrictions to C of the n sections $y_1, \ldots, y_n \in H^0(P^{n+1} \times P^{n+1}, \mathcal{O}(0, 1))$. If the sections y_i were linearly independent, f would be injective. Let us tensor by $\mathcal{O}(0, 1)$ the exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_{P^{n+1} \times P^{n+1}} \to \mathcal{O}_C \to 0;$$

in cohomology we get

$$0 \to H^0(\mathcal{I}_C \otimes \mathcal{O}(0, 1)) \to H^0(\mathcal{O}_{P^{n+1} \times P^{n+1}}(0, 1)) \xrightarrow{r} H^0(\mathcal{O}_C(0, 1)) \to 0$$

where $H^0(P^{n+1} \times P^{n+1}, \mathcal{O}(0, 1))$ is spanned by the $n+2$ independent sections $y_0, \ldots, y_{n+1}$ and, using the same argument as in (1) and (2), $\mathcal{O}_C(0, 1) \simeq \mathcal{O}_C(n+1)$ whence $H^0(\mathcal{O}_C(0, 1)) \cong H^0(\mathcal{O}_C(n+1))$. Being the curve C defined as the complete intersection in $P^{n+1} \times P^{n+1}$ of $n+1$ symmetric forms of type (1, 1) and n forms of type (1, 0), $H^0(\mathcal{I}_C \otimes \mathcal{O}(0, 1)) = 0$ hence the restriction of the n independent sections $y_1, \ldots, y_n \in H^0(P^{n+1} \times P^{n+1}, \mathcal{O}(0, 1))$ to C remain independent and we are done. □

Given our family

$$\begin{array}{ccc} T & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \\ F & & \end{array}$$

let us consider the induced morphism of Hodge structure of type $(-1, -1)$

$$p_* q^*: H^n(X, \mathbf{Q}) \to \mathrm{H}^{n-2}(\mathrm{F}, \mathbf{Q})$$

(the so called 'integration over the fibre'). In the next section we prove the nontriviality of

$$\Phi = p_* q^*_{|(n-1,1)}: H^{n-1,1}(X) \to H^{n-2,0}(F).$$

which implies the nontriviality of $p_* q^*$.

4. The infinitesimal cylinder map

In order to study the relations between these families of curves and the cohomology of X, let us consider the cylinder map

$$\begin{aligned} q_* p^*: H_{n-2}(F) &\to H_n(X), \\ [\gamma] &\to [\textstyle\bigcup_{t \in \gamma} C_t] \end{aligned}$$

the restriction to $H^{n-1,1}(X)$ of the corresponding map in cohomology

$$\Phi = p_* q^*_{|(n-1,1)}\colon H^{n-1,1}(X) \to H^{n-2,0}(F).$$

and the composition map

$$\tau^* = r\Phi\colon H^1(\Omega_X^{n-1}) \to H^0(\Omega_F^{n-2}) \to \Omega_{F,0}^{n-2}.$$

where r denotes the restriction to $0 \in F$.

To give a formula for τ^*, if $C = C_0$, let

(1) $\psi^*\colon H^1(X, \Omega_X^{n-1}) \to H^1(\Omega_{X|C}^{n-1}) \to H^1(C, \bigwedge^{n-2} N^*_{C|X} \otimes \Omega_C^1) \cong H^0(\bigwedge^{n-2} N_{C|X})^*$ be the composition of the restriction map together with the map induced by the exact sequence

$$0 \to \bigwedge^{n-1} N^*_{C|X} \to \Omega_{X|C}^{n-1} \to \bigwedge^{n-2} N^*_{C|X} \otimes \Omega_C^1 \to 0;$$

(2) $\eta^*\colon H^0(\bigwedge^{n-2} N_{C|X})^* \to \bigwedge^{n-2} H^0(N_{C|X})^*$ be the dual of the natural map $\eta\colon \bigwedge^{n-2} H^0(N_{C|X}) \to H^0(\bigwedge^{n-2} N_{C|X})$;

(3) $\rho^*\colon \bigwedge^{n-2} H^0(N_{C|X})^* \to \Omega_{F,0}^{n-2}$ be the dual of the map induced by the Kodaira Spencer map ([6] Def. 4, p. 150).

By the same arguments as in ([4] Thm. 2.25, p. 827) we have the following:

PROPOSITION 4.1. $\tau^* = \rho^* \eta^* \psi^*$.

Proof. Let $\Delta \in F$ be a polycylinder with coordinates $t_1, \ldots, t_{n-2}$, $t = 0$ its origin and let us choose local coordinates $z, w_1, \ldots, w_{n-1}$ on X such that $C = C_0$ is given by $w_1 = \cdots = w_{n-1} = 0$. Locally, C_t will be given by $w_i = f_i(z, t)$, where $f_i(z, t)$ is holomorphic and, by the condition $f_i(z, 0) = 0$, we can write

$$f_i(z, t) = \sum_j \left.\frac{\partial f_i(z, t)}{\partial t_j}\right|_{t=0} t_j + [2]$$

where [2] are terms of order $\geqslant 2$ in t.

If $\xi \in H^{n-1,1}(X)$, locally, we can write

$$\xi = \sum_{i=1}^{n-1} (-1)^{i+1} \xi_i(z, w) dz \wedge d\bar{z} \wedge dw_1 \wedge \cdots \wedge \widehat{dw_i} \wedge \cdots \wedge dw_{n-1} + [n-1]$$

where $[n-1]$ are terms which either do not involve dz or do not involve $d\bar{z}$.

By definition of $\Phi = p_* q^*_{|(n-1,1)}\colon H^{n-1,1}(X) \to H^{n-2,0}(F)$, we get $\Phi(\xi) =$

$(\int_{C_t} \det A(z, t) dz \wedge d\bar{z}) dt_1 \wedge \cdots \wedge dt_{n-2}$, where

$$A = \begin{bmatrix} \xi_1 & \cdots & \cdots & \xi_{n-1} \\ \dfrac{\partial f_1}{\partial t_1} & \cdots & \cdots & \dfrac{\partial f_{n-1}}{\partial t_1} \\ \vdots & \cdots & \cdots & \vdots \\ \dfrac{\partial f_1}{\partial t_{n-2}} & \cdots & \cdots & \dfrac{\partial f_{n-1}}{\partial t_{n-2}} \end{bmatrix}.$$

The composition of Φ with the restriction map gives therefore $\tau^*(\xi) = (\int_C \det A(z, 0) dz \wedge d\bar{z}) dt_1 \wedge \cdots \wedge dt_{n-2}$.

On the other hand $\psi^*(\xi) \in H^0(\bigwedge^{n-2} N_{C|X})^*$ is the element which, by Kodaira-Serre duality, corresponds to

$$\sum_{i=1}^{n-1} (-1)^{i+1} \xi_i(z) dz \wedge d\bar{z} \otimes dw_1 \wedge \cdots \wedge \widehat{dw_1} \wedge \cdots$$
$$\wedge\, dw_{n-1} \in H^1(C, \Omega^1_C \otimes \textstyle\bigwedge^{n-2} N^*_{C|X})$$

i.e.

$$\psi^*(\xi) = \left(\alpha \to \int_C \alpha \otimes \sum_{i=1}^{n-1} (-1)^{i+1} \xi_i(z) dz \wedge d\bar{z} \otimes dw_1 \wedge \cdots \wedge \widehat{dw_i} \wedge \cdots \wedge dw_{n-1}\right)$$

$\forall \alpha \in H^0(\bigwedge^{n-2} N_{C|X})$.

Furthermore $\rho: \bigwedge^{n-2} T_{F,0} \to \wedge^{n-2} H^0(N_{C|X})$ acts as follows:

$$\rho\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_{n-2}}\right) = \sum_{i=1}^{n-1} \zeta_i(z) \frac{\partial}{\partial w_1} \wedge \cdots \wedge \widehat{\frac{\partial}{\partial w_i}} \wedge \cdots \wedge \frac{\partial}{\partial w_{n-1}}$$

where $(-1)^{i+1}\zeta_i$ is the cofactor of the element ξ_i in the matrix A. Therefore we have

$$\rho^*\eta^*\psi^*(\xi): \textstyle\bigwedge^{n-2} T_{F,0} \xrightarrow{\rho} \bigwedge^{n-2} H^0(N_{C|X}) \xrightarrow{\eta} H^0(\bigwedge^{n-2} N_{C|X}) \to \mathbf{C}$$
$$\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_{n-2}} \to \int_C \det A(z, 0) dz \wedge d\bar{z}$$

which, thought as an element of $\Omega^{n-2}_{F,0}$ is exactly $\Phi^*(\xi)$. □

What we want to do now, is to describe η^* and ψ^* in our situation. We will follow the notations previously introduced supposing, for example, $0 \in F_1$

and consequently $C = (C_1)_0$.

To describe η^* let us consider the dual map $\eta: \bigwedge^{n-2} H^0(N_{C|X}) \to H^0(\bigwedge^{n-2} N_{C|X})$.

By Prop. 3.3(4) the map η may be written

$$\bigwedge^{n-2} H^0(\textstyle\bigoplus_{i=1}^{n-2} \mathcal{O}e_i \oplus \mathcal{O}(-2)) \to H^0(\bigwedge^{n-2}(\bigoplus_{i=1}^{n-2} \mathcal{O}e_i \oplus \mathcal{O}(-2)))$$
$$\simeq H^0(\mathcal{O}(e_1 \wedge \cdots \wedge e_{n-2})).$$

Since $e_1 \wedge \cdots \wedge e_{n-2}$ gets mapped to $e_1 \wedge \cdots \wedge e_{n-2}$, η is an isomorphism of one dimensional vector spaces.

To study ψ^* let us consider

$$0 \to N^*_{C|W} \to T^*_W \otimes \mathcal{O}_C \to T^*_C \to 0 \qquad \text{(a)}$$

and the induced sequence

$$0 \to \bigwedge^n N^*_{C|W} \to \Omega^n_W \otimes \mathcal{O}_C \to \bigwedge^{n-1} N^*_{C|W} \otimes \Omega^1_C \to 0 \qquad \text{(ȧ)}$$

which tensored with $N_{X|W}$ gives

$$0 \to \bigwedge^n N^*_{C|W} \otimes N_{X|W} \to \Omega^n_W \otimes N_{X|W} \otimes \mathcal{O}_C \to \bigwedge^{n-1} N^*_{C|W} \otimes \Omega^1_C \otimes N_{X|W} \to 0. \qquad \text{(ä)}$$

Furthermore let us consider the sequence

$$0 \to N^*_{X|W} \to T^*_W \otimes \mathcal{O}_X \to T^*_X \to 0 \qquad \text{(b)}$$

which taking exterior n-powers and tensoring with $N_{X|W}$ induces

$$0 \to \Omega^{n-1}_X \to \Omega^n_W \otimes N_{X|W} \otimes \mathcal{O}_X \to \Omega^n_X \otimes N_{N|W} \to 0. \qquad \text{(b̈)}$$

Let us put (ä) and (b̈) $\otimes\, \mathcal{O}_C$ into the following diagram:

$$\begin{array}{ccccccccc}
 & & (*) & & (\ddot{a}) & & & & \\
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \bigwedge^{n-1} N^*_{C|X} & \xrightarrow{\alpha} & \bigwedge^n N^*_{C|W} \otimes N_{X|W} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^{n-1}_X \otimes \mathcal{O}_C & \longrightarrow & \Omega^n_W \otimes N_{X|W} \otimes \mathcal{O}_C & \longrightarrow & \Omega^n_X \otimes N_{X|W} \otimes \mathcal{O}_C & \longrightarrow & 0\,(\ddot{b}) \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega^1_C \otimes \bigwedge^{n-2} N^*_{C|X} & \xrightarrow{\beta} & \bigwedge^{n-1} N^*_{C|W} \otimes \Omega^1_C \otimes N_{X|W} & \longrightarrow & \Omega^n_X \otimes N_{X|W} \otimes \mathcal{O}_C & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The isomorphism α, which is equivalent to $\det N_{C|W} \cong \det N_{C|X} \otimes \det N_{X|W} \otimes \mathcal{O}_C$, comes from the exact sequence

$$0 \to N_{C|X} \to N_{C|W} \to N_{X|W} \otimes \mathcal{O}_C \to 0.$$

A simple diagram chase now shows that the map β is injective.

By means of adjunction, the isomorphism α and the duality on bundles, we obtain isomorphisms

(i) $\Omega^1_C \otimes \bigwedge^{n-2} N^*_{C|X} \cong \Omega^n_X \otimes \bigwedge^{n-1} N_{C|X} \bigwedge^{n-2} N^*_{C|X} \cong \Omega^n_X \otimes N_{C|X}$;

(ii) $\bigwedge^{n-1} N^*_{C|W} \otimes \Omega^1_C \otimes N_{X|W} \cong N_{C|W} \otimes \bigwedge^n N^*_{C|W} \otimes \Omega^1_C \otimes N_{X|W} \cong N_{C|W} \otimes \bigwedge^{n-1} N^*_{C|X} \otimes N^*_{X|W} \otimes \Omega^1_C \otimes N_{X|W} \cong N_{C|W} \otimes \Omega^n_X$

which allow us to replace the last line of the preceding diagram with

$$0 \to \Omega^n_X \otimes N_{C|X} \to \Omega^n_X \otimes N_{C|W} \to \Omega^n_X \otimes N_{X|W} \otimes \mathcal{O}_C \to 0.$$

To describe ψ^* let us consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{n-1}_X & \longrightarrow & \Omega^n_W \otimes N_{X|W} & \longrightarrow & \Omega^n_X \otimes N_{X|W} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^n_X \otimes N_{C|X} & \longrightarrow & \Omega^n_X \otimes N_{C|W} & \longrightarrow & \Omega^n_X \otimes N_{X|W} \otimes \mathcal{O}_C & \longrightarrow & 0 \end{array}$$

and the induced diagram

$$\begin{array}{ccccccccc} \to & H^0(\Omega^n_W \otimes N_{X|W}) & \to & H^0(\Omega^n_X \otimes N_{X|W}) & \to & H^1(\Omega^{n-1}_X) & \to & H^1(\Omega^n_W \otimes N_{X|W}) & \to \\ & \downarrow & & \downarrow & & \downarrow{\scriptstyle \psi^*} & & \downarrow & \\ \to & H^0(\Omega^n_X \otimes N_{C|W}) & \to & H^0(\Omega^n_X \otimes N_{X|W} \otimes \mathcal{O}_C) & \to & H^1(\Omega^n_X \otimes N_{C|X}) & \to & H^1(\Omega^n_X \otimes N_{C|W}) & \to \\ & & & & & \wr\| & & & \\ & & & & & H^0(\bigwedge^{n-2} N_{C|X})^* & & & \end{array}$$

PROPOSITION 4.2. *In our situation ψ^* is a non trivial surjective map.*

Proof. The above diagram becomes

$$\begin{array}{ccccccccc} \to & H^0(\Omega^n_W \otimes \mathcal{O}_X(1,\ 1)) & \to & H^0(\mathcal{O}_X(1,\ 1)) & \to & H^1(\Omega^{n-1}_X) & \to & H^1(\Omega^n_W \otimes \mathcal{O}_X(1,\ 1)) & \to \\ & \downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \psi^*} & & & \\ \to & H^0(\bigoplus_1^n \mathcal{O}(1)) & \to & H^0(\mathcal{O}_C(n+2)) & \stackrel{a}{\to} & H^1(N_{C|X}) & \to & 0 & \end{array}$$

where a and b are surjective maps. The surjectivity of a is obvious; let us consider the following commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_{P^{n+1}\times P^{n+1}}(1,\ 1)) & & \\ \downarrow r & \searrow c & \\ H^0(\mathcal{O}_X(1,\ 1)) & \xrightarrow{b} & H^0(\mathcal{O}_C(1,\ 1)) \cong H^0(\mathcal{O}_C(n+2)) \end{array}$$

We know $h^0(\mathcal{O}_{P^{n+1}\times P^{n+1}}(1,\ 1)) = (n+2)^2$ while an easy computation shows that $\dim \operatorname{Ker} c = n(n+2) + n + 1 = n^2 + 3n + 1$. Then $\dim \operatorname{Im} c = (n+2)^2 - (n^2 + 3n + 1) = n + 3 = h^0(\mathcal{O}_C(1,\ 1))$ i.e. c and hence b are surjective. By the non triviality of $H^1(N_{C|X})$ and the surjectivity of a and b the assertion follows. □

By Proposition 4.1 and 4.2 we get the following:

CONCLUSION. $\Phi = p_* q^*_{|(n-1,1)}: H^{n-1,1}(X) \to H^{n-2,0}(F)$ is not trivial.

5. The GGHC and the conclusion

Let us recall the GGHC for a smooth connected complex projective variety X ([5], p. 300). We denote $F^{\bullet}H^i(X, \mathbf{C})$ the Hodge filtration defined by

$$F^p H^i(X, \mathbf{C}) = \bigoplus_{\substack{p' \geqslant p \\ p'+q=1}} H^{p',q}(X)$$

and $F'^{\bullet}H^i(X, \mathbf{Q})$ the arithmetic filtration defined by

$$F'^p H^i(X, \mathbf{Q}) = \left\{ \eta \in \mathrm{H}^i(X,\ \mathbf{Q}) : \begin{array}{l} \exists \text{a Zariski closed set } Z \subseteq X \\ \text{with } \operatorname{codim} Z \geqslant p \text{ and } \eta|_{X\setminus Z} = 0 \end{array} \right\}.$$

We note that $F^{\bullet}H^i$ is defined over $\mathbf{C}$ and generally it is not induced by a corresponding filtration defined over $\mathbf{Q}$ while $F'^{\bullet}H^i$ is defined over $\mathbf{Q}$ and it can be shown, by using standard exact sequences and [3] thm. 8.2.7–8.2.8 p. 40, that $F'^p H^i$ is the space spanned by the images of the Gysin morphisms $H^{i-2q}(Y, \mathbf{Q}) \to H^i(X, \mathbf{Q})$ for any desingularization Y of closed subschemes $Z \subseteq X$ of pure codimension $q \geqslant p$. As a consequence, we have that $F'^p H^i(X, \mathbf{Q})$ spans a $\mathbf{Q}$-Hodge substructure of $H^i(X, \mathbf{C})$ contained in $F^p H^i(X, \mathbf{C}) \cap H^i(X, \mathbf{Q})$. In particular, if we denote $F^p M^i$ the maximal $\mathbf{Q}$-Hodge substructure of $F^p H^i(X, \mathbf{C})$, we have that $F'^p H^i(X, \mathbf{Q}) \subseteq F^p M^i$.

The GGHC states that this is an equality, i.e. $F^p M^i \subseteq F'^p H^i(X, \mathbf{Q})$.

For $i = 2p$, this is nothing else that the usual Hodge conjecture for rational cohomology classes of type (p, p). In fact in this case $F^p M^{2p}$ coincides with

$H^{p,p}(X,\mathbf{C}) \cap H^{2p}(X,\mathbf{Q})$ and $F'^{p}H^{2p}(X,\mathbf{Q})$ is the space of $2p$-cohomology classes supported by subvarieties of X of codimension $\geqslant p$, therefore exactly p (or, equivalently, which are Poincaré duals of such subvarieties).

If $i \neq n$, we saw in Prop. 1.5 that $H^i(X)$ is spanned by the Poincaré duals of intersections of hyperplane sections of $P^{n+1} \times P^{n+1}$ restricted to X, hence, in this case, the GGHC is easily checked.

If $i = n$, let us consider the family

$$\begin{array}{ccc} T & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \\ F & & \end{array}$$

of algebraic one cycles on X introduced in section 3 and the induced sequence

$$H_{n-2}(F, \mathbf{Q}) \to H_n(T, \mathbf{Q}) \to H_n(q(T), \mathbf{Q}) \to H_n(X, \mathbf{Q}).$$

It is not difficult to see that $\dim q(T) = n - 1$.

PROPOSITION 5.1. (1) *The maximal* $\mathbf{Q}$*-Hodge substructure* F^1M^n *contained in* $F^1H^n(X,\mathbf{C}) \cap H^n(X,\mathbf{Q})$ *is* $H^n(X,\mathbf{Q})^-$ *if n is odd and* $V^+ \oplus I^+ \oplus I^-$ *if n is even.*

(2) *If n is even, the maximal* $\mathbf{Q}$*-Hodge substructure* $F^{n/2}M^n = H^{n/2,n/2}(X,\mathbf{C}) \cap H^n(X,\mathbf{Q})$ *is* $I^+ \oplus I^-$.

The proof is a straightforward consequence of the π_1-invariance of the maximal $\mathbf{Q}$-Hodge substructures F^iM^n and Thm. 2.2.

PROPOSITION 5.2. *The image of the Gysin morphism*

$$\lambda: H^{n-2}(\widetilde{q(T)}, \mathbf{Q}) \xrightarrow{\sim} H_n(\widetilde{q(T)}, \mathbf{Q}) \to H_n(q(T), \mathbf{Q}) \to H_n(X, \mathbf{Q}) \xrightarrow{\sim} H^n(X, \mathbf{Q}),$$

where $\widetilde{q(T)}$ *denotes a desingularization of* $q(T)$, *coincides with* $H^n(X,\mathbf{Q})^-$ *if n is odd. If n is even,* $V^+ \subseteq \operatorname{Im}\lambda \subseteq V^+ \oplus I^+ \oplus I^-$.

Proof. We know the following facts:

— λ is nontrivial by section 4;
— $\operatorname{Im}\lambda$ is contained in $F'^1H^n(X,\mathbf{Q})$ since $\operatorname{codim} q(T) = 1$;
— $\operatorname{Im}\lambda$ generates a $\mathbf{Q}$-Hodge substructure of $H^n(X)$ invariant under monodromy and contained in $F^1H^n(X,\mathbf{C}) \cap H^n(X,\mathbf{Q})$.

If n is odd, by 5.1 (1), $\operatorname{Im}\lambda \subseteq H^n(X,\mathbf{Q})^- = F^1M^n$. By the above facts and the irreducibility of $H^n(X,\mathbf{Q})^-$ under monodromy they must coincide.

If n is even, by 5.1 (1), $\operatorname{Im}\lambda \subseteq V^+ \oplus I^+ \oplus I^-$. We know by the previous results that $\operatorname{Im}\lambda \cap V^+ \neq 0$ hence, by the same irreducibility argument as before, $V^+ \subseteq \operatorname{Im}\lambda$.

CONCLUSION 5.3. (1) In case n is even, the classical Hodge conjecture holds;

(2) the GGHC holds for F^1M^n.

Proof. (1) By the meaning of $I^+ \oplus I^-$, the classical Hodge conjecture

$$H^{n/2,n/2}(X, \mathbf{C}) \cap H^n(X, \mathbf{Q}) = F^{n/2}M^n = I^+ \oplus I^- \subseteq F'^{n/2}H^n(X, \mathbf{Q})$$

is exactly the assertion 5.1 (2).

(2) If n is odd, by 5.1 (1) and 5.2, we get $F^1M^n = H^n(X, \mathbf{Q})^- = \operatorname{Im}\lambda \subseteq F'^1H^n(X, \mathbf{Q})$. If n is even, by 5.2 we know that $V^+ \subseteq \operatorname{Im}\lambda \subseteq F'^1H^n(X, \mathbf{Q})$. On the other hand, $I^+ \oplus I^- = F'^{n/2}H^n(X, \mathbf{Q})) \subseteq F'^1H^n(X, \mathbf{Q})$ hence,

$$F^1M^n = V^+ \oplus I^+ \oplus I^- \subseteq F'^1H^n(X, \mathbf{Q}).$$

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