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Birational morphisms of regular schemes

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1. Introduction

Zariski proved around 1944 that every birational morphism between smooth surfaces over a field k is a composition of blowing-ups at closed points. Later, around 1966 Shafarevich proved the same theorem for regular schemes of dimension 2 without base field. This generalization is important for arithmetic geometry. Danilov generalized the Zariski's theorem to relative dimension 1 by studying the relative canonical divisor. He also left the regular scheme case as an open question, which was answered respectively in [4] and [8]. On the other hand, some results in higher dimension appeared around 1981 (see [2], [6], [7], [10]). But all the authors required the algebraic varieties have an algebraically closed base field. This paper is devoted to the generalization of Schaps and Teicher's results to regular schemes without base field. Although the main technical tool is the theory of ramification index of regular local rings, which we shall review in the first section, our central ideas are based on [7] and [10]. In Section 2, we shall establish some general lemmas of birational morphisms for regular schemes, which is the same as [7] if the schemes have an algebraically closed base field. But for regular schemes without base field, new difficulties arise. For example, we can not use determinant and "test curve" since a subscheme may have no rational point. Section 3 contains the proofs of the following theorems

THEOREM 3.1. *Let $f: X \to Y$ be a proper birational morphism of regular schemes such that* $\dim(f^{-1}(y)) \leqslant 3$ *for any $y \in Y$ and $S(f)$ regular. If $E(f)$ has only two nonsingular components, then f is a composition of two blowing-ups with regular centers.*

THEOREM 3.2. *Let $f: X \to Y$ be a proper birational morphism between regular schemes of dimension three such that $E(f)$ has only three normally crossing nonsingular components. If $S(f)$ is a regular subscheme of codimension 2, then*

(a) *f consists of three blowing-ups, or*
(b) *f is formed by blowing up $S(f)$ and then blowing up two intersecting curves in different orders at different intersection points.*

All the schemes and rings we consider in this paper are integral and Noetherian, and all the morphisms are finite type. By point, we mean a point of any codimension, which may not be a closed point. It is interesting to note that even for the complex algebraic varieties some of our results are still new.

2. Preliminary

Let A, B be two local rings with the same quotient field $Q(A)$ and $Q(B)$. (A, B) means A dominates B, and A is a quotient ring of a finitely generated B-algebra. In this paper, we always assume A and B are regular, v_A denote the normalized discrete valuation of $Q(A)$ determined by A. If P is a prime ideal of height 1 of B such that $P = (f)B$, we define $e(ABP) = v_A(f)$, which is independent of the choice of prime element f and $e(ABP) \geqslant 1$. If $e(ABP) = 1$, then B/P is regular.

Let $P \subset B$ be a prime ideal of height r such that B/P regular and $P = (x_1, x_2, \ldots, x_r)B$, where $(x_1, x_2, \ldots, x_r)$ is a part of minimal basis of the maximal ideal N of B. Let $B' = B[x_2/x_1, \ldots, x_r/x_1]$ and Q a prime ideal of B' such that $N \subset Q$, then B'_Q is the blowing-up of B with center P. If $Q = NB'$, then B'_Q is uniquely determined by B and P, which will be denoted by $B_{[P]}$. If $P = N$, we shall write $Z(B)$ for $B_{[N]}$. The following properties of blowing-up are often used later (see [1] for the proof).

PROPOSITION 1.1. (1) $(x_1)B'_Q$ *is prime ideal and* $(x_1^k)B'_Q \cap B = P^k$ *for any positive integer.*

(2) $\dim B = \dim B'_Q + \mathrm{tr.deg}(k(B'_Q)/k(B))$, $\dim B = \dim B'_Q \Leftrightarrow k(B'_Q)/k(B)$ *is an algebraic extension.*

(3) *There are* $y_2, \ldots, y_m \in B'_Q$ *such that* $(x_1, y_2, \ldots, y_m, x_{r+1}, \ldots, x_n)$ *is the minimal basis of* QB'_Q. *If* $x_i/x_1 \in QB'_Q$, *then we can take* $y_i = x_i/x_1$.

For any (A, B), we define the ramification index of A over B to be $r(AB) = v_A(d(A/B))$ where $d(A/B)$ is the Jacobian ideal of A over B which is a non-zero (principal) ideal of A. Let $e(AB) = \max\{v_A(\Pi x_i) \mid (x, \ldots, x_r)$ it is a minimal basis of $N\}$ and $r(ABC) = v_A(d(B/C))$ if A dominates B and B dominates C. We summarize the properties of $r(AB)$ in the following proposition (see [5] for the proof).

PROPOSITION 1.2. *Let* (A, B) *and* (B, C) *be pairs of regular local rings. Then*

(1) $r(AB) \geqslant e(AB) - \dim(A)$, $r(Z(B)B) = \dim B - 1$.

(2) $r(ABC) = \Sigma_{P \in C(B/C)} r(B_P C_{P \cap C}) e(ABP)$, *where* $C(B/C) = \{P \in \operatorname{spec} B \mid ht\, P = 1, ht\, P \cap C > 1\}$.

(3) $r(AC) = r(AB) + r(ABC)$.

(4) $r(ABC) = 0$ *if and only if* $B = C$.
$r(AB) = 0$ *if and only if* $A = B$.

(5) *If* $r(ABC) = 1$, *then* (i) $C(B/C) = \{P\}$; (ii) $e(ABP) = 1$; (iii) $r(B_P C_{P \cap C}) = 1$; (iv) B/P *is regular*; (v) $ht(P \cap C) = 2$.

(6) $r(AB) = r(Z(A)AB)$, $r(ABC) \geqslant r(BC)$.

REMARK. The above statements about the ramification index of local rings are almost verbatim from [4] except that (6) and (1). By the definition, (6) is clear since A dominates B and $v_A = v_{Z(A)}$. (1) was presented in [4] under the assumption that A is a discrete valuation ring. But the general case is clearly the consequence of [4] since $r(AB) = r(Z(A)AB) = r(Z(A)B) - r(Z(A)A) = r(Z(A)B) - \dim A + 1$.

Let $f: X - \cdots \rightarrow Y$ be a birational map of regular schemes and $S(f) = \{y \in Y | f^{-1}$ is not well-defined at $y\}$, $E(f) = f^{-1}(S(f))$. If $U \subset X$ is the open set where f is well-defined, and $D \subset X$ is a subscheme, we define $f[D] = \overline{f(U \cap D)}$. For the convenience, we give the following algebraic form of Zariski Main Theorem.

PROPOSITION 1.3. *Let* $A \supset B$ *be two Noetherian local domains such that* A *dominates* B *and* $Q(A) = Q(B)$. *If* B *is a unique factorization domain and* $A \neq B$, *then* $C(A/B) \neq \emptyset$.

Proof. Since $A \neq B$, there exists $a \in A$ such that $a \notin B$. We can write $B = \bigcap_{htP=1} B_P$ since B is normal. Hence $a \notin B_P$ for some prime ideal P of height 1. On the other hand, $P = (b)$ since B is a UFD. Let P_A be the minimal prime ideal of A which contains b, then $ht\, P_A = 1$. We claim $P_A \in C(A/B)$. In fact, A_{P_A} dominates $B_{P_A \cap B}$. If $ht(P_A \cap B) = 1$, then $A_{P_A} = B_{P_A \cap B}$ since A_{P_A} and $B_{P_A \cap B}$ are the discrete valuation rings of $Q(A) = Q(B)$. Thus $A_{P_A} = B_{P_A \cap B} = B_P$, which implies $a \in A \subset A_{P_A} = B_P$. We get a contradiction. So $ht(P_A \cap B) > 1$, i.e. $C(A/B) \neq \emptyset$.

The following more geometric proposition is suggested by the referee, which is often used in this paper.

PROPOSITION 1.4. *Let* $f: X \rightarrow Y$ *be a birational morphism of regular schemes with* $E(f) = \bigcup_i^n D_i$ *and* $W = S(f)$. *If* x *is a point on the intersection of* $D_1, \ldots, D_s$ *and* $y = f(x) \in W$, *then we have*

$$r(\mathcal{O}_x \mathcal{O}_y) \geqslant r_1 + r_2 + \cdots + r_s,$$

where $r_i = r(A_i B_i)$ *and* A_i, B_i *is the local ring of generic point of* D_i *and* $f(D_i)$, *respectively.*

In particular, $r(\mathcal{O}_x \mathcal{O}_y) = r_1 + r_2 + \cdots + r_s$ if and only if each D_i is regular at x.

Proof. Let $A = \mathcal{O}_x$, $B = \mathcal{O}_y$, then each D_i correspond to a prime ideal $P_i \in C(A/B)$ such that $P_i \cap B$ is the prime ideal of B which corresponds to $f(D_i)$, so $C(A/B) = \{P_1, \ldots, P_s\}$. By the Proposition 1.2(2), we have (note $v_A = v_{Z(A)}$)

$$r(AB) = r(Z(A)AB) = \sum_{i=1}^{s} r(A_{P_i}B_{P_i \cap B})e(Z(A)AP_i).$$

It is clear that $A_i = A_{P_i}$, $B_i = B_{P_i \cap B}$ and $P_i = (x_i)$, where x_i is the local equation of D_i at x. So $r(AB) = \Sigma_{i=1}^{s} r_i v_A(x_i) \geqslant \Sigma_{i=1}^{s} r_i$, since each D_i contains x. It is easy to see that $r(AB) = \Sigma_{i=1}^{s} r_i$ if and only if $v_A(x_i) = 1$, i.e. each D_i is regular at x.

REMARK. In this paper, we use the following facts without mention. (1) If $W = S(f)$ is regular at y and $E(f)$ with normal crossings, then we can suppose that $m_x = (x_1, \ldots, x_{n'})$ and $m_y = (y_1, \ldots, y_{c'})$ such that $x_1, \ldots, x_s$ are the local equations of $D_1, \ldots, D_s$ and $(y_1, \ldots, y_c)\mathcal{O}_y = I_W$ is the ideal corresponded by W at y, where $c = \operatorname{codim} W$, $n' = \operatorname{codim} x$ and $c' = \operatorname{codim} y$ ($n' \leqslant c' \leqslant \dim X$ since x, y may not be closed points). (2) Let σ_W be the blowing-up with center W and $f_1 = \sigma_W^{-1} f$, we write $y_i = x_1 \cdots x_s q_i$ with $q_i \in \mathcal{O}_x$ and $1 \leqslant i \leqslant c$, then f_1 is well-defined if $v_x(q_i) = 0$ for some $1 \leqslant i \leqslant c$.

2. Ramification argument

Let $f: X \to Y$ be a proper birational morphism of regular schemes. The normal discrete valuation determined by the local ring of a point x is denoted by v_x. I_W denotes the ideal of a closed subscheme W at one point. If $E(f) = \bigcup_1^n D_i$, and A_i, B_i is the local ring of D_i, $f(D_i)$. Then the ramification divisor of f is defined by $R(f) = \Sigma_{i=1}^{n} r_i D_i$, where $r_i = r(A_i B_i)$.

LEMMA 2.1. *Let $W \subset S(f)$ be regular subscheme and Y_1 be the blowing-up with center W. If $f_1: X \dashrightarrow Y_1$ is the induced birational map, and f_1^{-1} does not collapse the exceptional divisor of Y_1, then*

$$\operatorname{codim} f(E) - \operatorname{codim} E \geqslant 1$$

where E is a irreducible component of the locus where f_1 is not well-defined.

Proof. Let $G \subset X \times_Y Y_1$ be the graph of f_1, with projections p_1 and p_2 on X and Y_1, $f_1 = p_2 p_1^{-1}$. By Zariski's Main Theorem, there must be a divisor $\bar{E}$ in G such that $p_1(\bar{E}) = E$. We have $\operatorname{codim} p_2(\bar{E}) \geqslant 2$ since f_1^{-1} does not collapse the exceptional divisor. By $\bar{E} \subset E \times_Y p_2(\bar{E})$, we obtain $\operatorname{codim} f(E) - \operatorname{codim} E \geqslant 1$.

LEMMA 2.2. *Let x be a point lying on a unique component D_1 of $E(f)$ and $f(D_1) \subset W$. Suppose that the multiplicity of D_1 in $f^{-1}(W)$ is at least b and* codim $W = c'$. *Let* $f_1: X \dashrightarrow Y_1$ *be the map to the blowing-up of* W, *then* f_1 *is well-defined at* x *if*

(i) $r_1 = bc' - 1$, *or*
(ii) $r_1 = bc'$, *and* f_1^{-1} *does not collapse the exceptional divisor.*

Proof. Since $C(\mathcal{O}_x/\mathcal{O}_{f(x)}) = \{I_{D_1}\}$, *by Proposition* 1.4, *we have* $r_1 = r(\mathcal{O}_x\mathcal{O}_{f(x)})$. Let $m_x = (t_1, \ldots, t_n)$, $m_{f(x)} = (y_1, \ldots, y_n)$ such that $(t_1)\mathcal{O}_x = I_{D_1}$ and $(y_1, \ldots, y_{c'})\mathcal{O}_{f(x)} = I_W$, then $y_i = q_i t_1^b$, $q_i \in \mathcal{O}_x$, $i = 1, \ldots, c'$, we obtain

$$r_1 = r(\mathcal{O}_x\mathcal{O}_{f(x)}) \geqslant e(\mathcal{O}_x\mathcal{O}_{f(x)}) - n$$
$$\geqslant bc' + \sum_{i=1}^{c'} v_x(q_i) + \sum_{i=c'+1}^{n} v_x(y_i) - n,$$

(i) If $r_1 = bc' - 1$ then there exist q_i such that $v_x(q_i) = 0$. Thus f_1 is well-defined at x.

(ii) If $r_1 = bc'$, then $v_x(q_i) \leqslant 1$ and $v_x(y_{c'+j}) = 1$ where $j = 1, \ldots, n - c'$. If there exist q_i such that $v_x(q_i) = 0$, we complete the proof, otherwise, we have $v_x(q_i) = 1$ and $v_x(y_{c'+j}) = 1$ for all $1 \leqslant i \leqslant c'$. Let E be a component of the locus where f_1 is not well-defined such that E contains x. If P is the prime ideal of E at x, then $P \supseteq (q_1, \ldots, q_{c'})$. Let $P' = P \cap \mathcal{O}_{f(x)}$. By Lemma 2.1, $ht\,P' - ht\,P \geqslant 1$ and $P' \supseteq (q_1, \ldots, q_{c'}) \cap \mathcal{O}_{f(x)} \supseteq I_W$. So we can write $P' = (y_1, \ldots, y_{c'}, z_{c'+1}, \ldots, z_{c'+k})$, $ht\,P' = c' + k$. By 1.2(1),

$$r((\mathcal{O}_x)_P(\mathcal{O}_{f(x)})_{P'}) \geqslant e((\mathcal{O}_x)_P(\mathcal{O}_{f(x)})_{P'}) - htP.$$

By the choice of x, we know that D_1 is the unique component through E. Thus we have $v_P(t_1) > 0$ and $r_1 = r((\mathcal{O}_x)_P(\mathcal{O}_{f(x)})_{P'})$, i.e. $r_1 \geqslant c' + bc' + k - ht\,P$, which implies $ht\,P \geqslant ht\,P'$. We get a contradiction, hence f_1 is well-defined at x.

REMARK. The above two lemmas are in fact the generalization of Schaps' Lemma 1.4 and Lemma 2.2 in [7].

LEMMA 2.3. *Let* x *be a point lying on only two components* D_1 *and* D_2 *of* $E(f)$. *Suppose that* $f(D_i) = W_i$ *is the point of codimension* c_i *such that* $c_2 \geqslant c_1$. *Suppose that* D_i *has multiplicity* b_i *in* $f^{-1}(W_i)$ *and* W_i *is regular. Let* f_i *be the map to the blowing-up of* W_i.

(i) *If* $r_1 + r_2 \leqslant b_1c_1 + b_2c_2 - 1$, *then either* f_1 *or* f_2 *is well-defined at* x.
(ii) *If* $r_1 + r_2 = b_1c_1 + b_2c_2 - 1$, *then* f_1 *is well-defined at the generic point of* $D_1 \cap D_2$.

Proof. (i) Let $m_x = (t_1, t_2, \ldots, t_n)$, $m_{f(x)} = (y_1, \ldots, y_{c_1}, \ldots, y_{c_2}, \ldots, y_n)$ such that $(t_1)\mathcal{O}_x = I_{D_1}$, $(t_2)\mathcal{O}_x = I_{D_2}$, $(y_1, \ldots, y_{c_1})\mathcal{O}_{f(x)} = I_{W_1}$, $(y_1, \ldots, y_{c_2})\ \mathcal{O}_{f(x)} = I_{W_2}$. Then

$y_i = t_1^{b_1} t_2^{b_2} q_i$ $(i = 1, 2, \ldots, c_1)$, $y_i = t_2^{b_2} q_i$ $(i = c_1 + 1, \ldots, c_2)$ for $q_i \in \mathcal{O}_x$. By $r_1 + r_2 = r(\mathcal{O}_x \mathcal{O}_{f(x)}) \geqslant e(\mathcal{O}_x \mathcal{O}_{f(x)}) - n$, we have

$$b_1 c_1 + b_2 c_2 - 1 \geqslant r_1 + r_2 \geqslant b_1 c_1 + b_2 c_2 - n + \sum_{i=1}^{c_2} v_x(q_i) + \sum_{i=c_2+1}^{n} v_x(y_i)$$

i.e.

$$c_2 - 1 \geqslant \sum_{i=1}^{c_2} v_x(q_i).$$

Thus there exist q_i such that $v_x(q_i) = 0$, i.e. either $I_{W_1}\mathcal{O}_x$ or $I_{W_2}\mathcal{O}_x$ is a principle ideal. Hence either f_1 or f_2 is well-defined at x.

(ii) Let ξ be the generic point of $D_1 \cap D_2$ and $m_\xi = (t_1, t_2)$, $m_{f(\xi)} = (y_1, \ldots, y_{c_1}, \ldots, y_{c_2}, \ldots, y_{m_1})$, where $m_1 = \operatorname{codim} f(\xi)$. So we can write

$$y_i = q_i t_1^{b_1} t_2^{b_2}, \quad (i = 1, 2, \ldots, c_1)$$

$$y_j = q_j t_2^{b_2}, \quad (j = c_1 + 1, \ldots, c_2)$$

Since

$$r_1 + r_2 = r(\mathcal{O}_\xi \mathcal{O}_{f(\xi)}) \geqslant e(\mathcal{O}_\xi \mathcal{O}_{f(\xi)}) - 2$$

and

$$e(\mathcal{O}_\xi \mathcal{O}_{f(\xi)}) \geqslant b_1 c_1 + b_2 c_2 + \sum_{i=1}^{c_2} v_\xi(q_i) + m_1 - c_2,$$

we have

$$c_2 + 1 \geqslant m_1 + \sum_{i=1}^{c_2} v_\xi(q_i).$$

If f_1 is not well defined at ξ, then $v_\xi(q_i) \geqslant 1$ $(i = 1, 2, \ldots, c_1)$ which implies $c_2 + 1 \geqslant m_1 + c_1 + \Sigma_{i=c_1+1}^{c_2} v_\xi(q_i)$. So we have $c_1 \leqslant 1$ since $m_1 \geqslant c_2$. But c_1 must be bigger than 1 by the Zariski's Main Theorem. Thus f_1 has to be well-defined at ξ.

LEMMA 2.4. *Let $f: X \to Y$ be a birational morphism of regular schemes and $W \subseteq S(f)$. Suppose that $f^{-1}(W)$ is a divisor and $\sigma_W: Y_1 \to Y$ is the blowing-up with center W. Let $D = \sigma_W^{-1}(W)$, $f_1 = \sigma_W^{-1} f$ and $y_1 \in D$ such that $y = \sigma_W(y_1)$ is* *a regular point of W. If v_{y_1} has a center x on X, then*

(1) *x lies on a unique component of $E(f)$.*
(2) *f_1 is well defined at x, and*
(3) *If $\sigma_W(y_1)$ is the generic point of W, then $f_1[V] = D$, where V is the component of $E(f)$ on which x lies.*

Proof. Let codim $x = k$, codim $y_1 = n_1$, codim $y = n$, codim $W = m$, and $m_y = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$ such that $I_W = (x_1, \ldots, x_m)\mathcal{O}_y$, then we have $n \geqslant n_1 \geqslant 1$ and $f(x) = y$. Let $t \in m_x$ such that $(t)\mathcal{O}_x = I_{f^{-1}(W)}$, $x_i = q_i t$ and $I_W \mathcal{O}_x = (t)(q_1, \ldots, q_m)$, where $i = 1, 2, \ldots, m$. Then we have

$$r(Z(\mathcal{O}_{y_1})\mathcal{O}_y) = r(Z(\mathcal{O}_{y_1})\mathcal{O}_x) + r(Z(\mathcal{O}_{y_1})\mathcal{O}_x\mathcal{O}_y)$$
$$r(Z(\mathcal{O}_{y_1})\mathcal{O}_x) \geqslant r(Z(\mathcal{O}_x)\mathcal{O}_x) = k - 1$$

and

$$r(Z(\mathcal{O}_{y_1})\mathcal{O}_x\mathcal{O}_y) \geqslant r(\mathcal{O}_x\mathcal{O}_y) \geqslant e(\mathcal{O}_x\mathcal{O}_y) - k,$$

which imply

$$r(Z(\mathcal{O}_{y_1})\mathcal{O}_y) \geqslant v_x\left(\prod_{i=1}^{n} x_i\right) - 1 \geqslant \sum_{i=1}^{m} v_x(q_i) + m(v_x(t) - 1) + n - 1. \tag{I}$$

On the other hand,

$$r(Z(\mathcal{O}_{y_1})\mathcal{O}_{y_1}\mathcal{O}_y) = \sum_{P \in C(\mathcal{O}_{y_1}/\mathcal{O}_y)} r((\mathcal{O}_{y_1})_P(\mathcal{O}_y)_{P\cap\mathcal{O}_y})e(Z(\mathcal{O}_{y_1})\mathcal{O}_{y_1}P).$$

Let P be the prime ideal of D at y_1, then $C(\mathcal{O}_{y_1}/\mathcal{O}_y) = \{P\}$. Since y_1 is the regular point of D, we have $e(Z(\mathcal{O}_{y_1})\mathcal{O}_{y_1}P) = 1$ and $r(Z(\mathcal{O}_{y_1})\mathcal{O}_{y_1}\mathcal{O}_y) = r((\mathcal{O}_{y_1})_P(\mathcal{O}_y)_{P\cap\mathcal{O}_y}) = m - 1$, which implies

$$r(Z(\mathcal{O}_{y_1})\mathcal{O}_y) = r(Z(\mathcal{O}_{y_1})\mathcal{O}_{y_1}) + m - 1 = n_1 + m - 2. \tag{II}$$

By (I) and (II), we obtain

$$m - 1 \geqslant \sum_{i=1}^{m} v_x(q_i) + m(v_x(t) - 1).$$

Since $v_x(t) - 1 \geqslant 0$ and $\Sigma_{i=1}^{m} v_x(q_i) \geqslant 0$, we have

$$v_x(t) = 1, \ \sum_{i=1}^{m} v_x(q_i) \leqslant m - 1.$$

Thus there exists q_i such that $v_x(q_i) = 0$, and we complete the proof of (1) since $v_x(t) = 1$ and (2) since $v_x(q_i) = 0$.

(3) If $\sigma_W(y_1)$ is the generic point of W, then we take y_1 to be the generic point of D and x is the center of v_{y_1} on X. By (2), f_1 is well defined at x. Thus we have

$$r(Z(\mathcal{O}_{y_1})\mathcal{O}_x) + r(Z(\mathcal{O}_{y_1})\mathcal{O}_x\mathcal{O}_{y_1} = r(Z(\mathcal{O}_{y_1})\mathcal{O}_{y_1}).$$

Since $\operatorname{codim} y_1 = 1$, $\mathcal{O}_{y_1}$ is a discrete valuation ring of rank 1. Hence $Z(\mathcal{O}_{y_1}) = \mathcal{O}_{y_1}$ and $r(\mathcal{O}_{y_1}\mathcal{O}_x) = 0$, which implies $\mathcal{O}_x = \mathcal{O}_{y_1}$, and x is a point of codimension 1. We complete the proof of (3).

REMARK. The above lemma is in fact an analogue of Schaps' test curve lemma. Test curve lemma is based on the fact that two closed points have the same residue field, which is not true in our case. But the conclusion of Schaps' test curve lemma is still true in our case. The primary difference in our proof of this lemma is the way we look for x, which is the center of v_{y_1} (of course, it may not be a closed point).

LEMMA 2.5. *Let $f: X \to Y$, W, f_1, Y_1, and D be as in Lemma 2.4. Let $E(f) = \bigcup_{i\in I} V_i$, if, in addition, f is proper and W is regular, then*

(1) *there is a V_j, $j \in I$, such that $f_1[V_j] = D$.*
(2) $S(f_1) = \bigcup_{i\neq j} f_1[V_i]$.
(3) $f_1: X - \bigcup_{i\neq j} V_i \cong Y_1 - \bigcup_{i\neq j} f_1[V_i]$.

Proof. (1) Let y_1 be the generic point of D, then v_{y_1} has a center on X since f is proper. So there is a V_j such that $f_1[V_j] = D$ by the Lemma 2.4(3).

(2) For any $y_1 \in S(f_1)$, v_{y_1} has a center x on X such that f_1 is well defined at x. By Zariski's Main Theorem, V_j can not pass through x. Thus $y_1 = f_1(x) \in \bigcup_{i\neq j} f_1[V_i]$. On the other hand, $f_1^{-1}(D - S(f_1)) \subseteq V_j - \bigcup_{i\neq j} V_i$. So

$$S(f_1) = \bigcup_{i\neq j} f_1[V_i].$$

(3) It is enough to prove that f_1 is well defined at any $x \in V_j - \bigcup_{i\neq j} V_i$. Let $m_{f(x)} = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$ such that $(x_1, \ldots, x_m)\mathcal{O}_{f(x)} = I_W$. Let $t_j \in m_x$ such that $(t_j)\mathcal{O}_x = I_{V_j}$ and $v_x(t_j) = 1$. Suppose that $x_i = q_i t_j$, where $i = 1, 2, \ldots, m$. Then

$$m - 1 = r_j = r(\mathcal{O}_x\mathcal{O}_{f(x)}) \geqslant e(\mathcal{O}_x\mathcal{O}_{f(x)}) - \dim \mathcal{O}_x$$

implies that there is a q_i such that $v_x(q_i) = 0$. Thus f_1 is well defined at x.

3. Birational morphisms with small exceptional divisor

In this section, we discuss the birational morphisms whose exceptional divisor $E(f)$ has only two or three irreducible components. Let $f: X \to Y$ be a proper birational morphism of regular schemes. We suppose that $E(f)$ has only two components V_1 and V_2 at first, and $S(f)$ is regular. Let $\sigma: \bar{Y} \to Y$ be the blowing-up with center $S(f)$ and $f_1 = \sigma^{-1}f$. Let $D_1 = \sigma^{-1}(S(f))$, $\Delta = S(f_1)$, $\delta = \text{codim } S(f)$ and $l = \text{codim } \Delta$. Then, by the Lemma 2.5(3), we can suppose that $\Delta = f_1[V_2]$ and $f_1: V_1 - V_2 \to D_1 - \Delta$ is an isomorphism. Now we give some lemmas, which are useful in the proof of the theorems.

LEMMA 3.1. *Let E be a component of the locus on where f_1 is not well defined. Suppose that $t =$ codim E and $s =$ codim $f(E)$, then*

$$t + l - s \leqslant 1.$$

In particular, if $\dim f^{-1}(y) \leqslant k$ *for any $y \in Y$ and $l = k + 1$, then f_1 is well defined everywhere.*

Proof. Let G be the graph of f_1, with projections p_1 and p_2 on X and $\bar{Y}$. By Zariski's Main Theorem, there is a divisor $\bar{E}$ of G such that $p_1(\bar{E}) = E$ and $p_2(\bar{E}) \subseteq \Delta$, which imply $t + l - s \leqslant 1$. If $\dim f^{-1}(y) \leqslant k$ and $l = k + 1$, then $t - s = -k$ and $p_2(\bar{E}) = \Delta$. Thus $f(E) = \sigma(\Delta) = f(V_2)$. But $\dim f^{-1}(y) \leqslant k$ implies $k \geqslant \text{codim } f(V_2) - \text{codim } V_2 = s - 1$, hence we have $t = s - k \leqslant 1$, which is impossible.

LEMMA 3.2. $R(f) = r_1 V_1 + r_2 V_2 = (\delta - 1)V_1 + (\delta + l - 2)V_2$.

Proof. Let $\sigma_\Delta^{-1}: \bar{Y}_1 \to \bar{Y}$ be the blowing-up of the regular points of Δ, and let $\sigma_\Delta^*(D_1) = \bar{\Delta} + \bar{D}_1$, with $\bar{D}_1 = \sigma_\Delta^{-1}[D_1]$. Consider

$$f_2 = \sigma_\Delta^{-1}\sigma^{-1}f: X \dashrightarrow \bar{Y}_1.$$

If B be the local ring of Δ, then $Z(B)$ is the local ring of $\bar{\Delta}$ such that $v_{Z(B)} = v_B$ has a center x on X. By Lemma 2.4 and Zariski's Main Theorem, f_1 is well defined at x, and $x \in V_2 - V_1$ such that $f_1(x)$ is the generic point of Δ. Let U be a neighborhood of x on which f_1 is well defined. Since $v_{Z(B)}$ has a center $x \in U$, we are in the situation of Lemma 2.4 for $f_1|_U$, so f_2 is well defined at x, and $f_2(x)$ is the generic point of $\bar{\Delta}$. Thus f_2 is an isomorphism of codimension 1, and

$$R(f) = f_2^*(R(\sigma\sigma_\Delta) = (\delta - 1)V_1 + (\delta + l - 2)V_2.$$

The following lemma is a generalization of Moishezon's theorem (see [8] for the proof).

LEMMA 3.3. *Let $f: X \to Y$ be a proper birational morphism of regular schemes such that $E(f)$ is an irreducible regular scheme. Then $S(f)$ is regular, and f is the blowing-up of $S(f)$.*

Proof of Theorem 3.1

By the Lemma 3.3, it is enough to prove f_1 is well defined. It is clear $2 \leqslant \delta \leqslant 4$ and $2 \leqslant l \leqslant 3$ by the Lemma 3.1.

Case I. $l = 2$. If $x_0 \in V_1 \cap V_2$, let $m_{x_0} = (x_1, \ldots, x_n)$ and $m_{f(x_0)} = (y_1, \ldots, y_\delta, y_{\delta+1}, \ldots, y_n)$ such that $(y_1, \ldots, y_\delta)\mathcal{O}_{f(x_0)} = I_{S(f)}$. Let $t_0 \in m_{x_0}$ such that $(t_0)\mathcal{O}_{x_0} = I_{E(f)}$, then

$$I_{S(f)}\mathcal{O}_{x_0} = (y_1, \ldots, y_\delta)\mathcal{O}_{x_0} = (t_0)(q_1, \ldots, q_\delta)\mathcal{O}_{x_0}$$

where $q_1, \ldots, q_\delta \in \mathcal{O}_{x_0}$. Since $x_0 \in V_1 \cap V_2$, we have $v_{x_0}(t_0) \geqslant 2$. Thus, by the Proposition 1.2(1),(2), we have

$$r_1 + r_2 = r(\mathcal{O}_{x_0}\mathcal{O}_{f(x_0)}) \geqslant \delta + \sum_{i=1}^{\delta} v_{x_0}(q_i),$$

which implies (since $l = 2$ by Lemma 3.2) that $\delta - 1 \geqslant \Sigma_{i=1}^{\delta} v_{x_0}(q_i)$ i.e., $I_{S(f)}\mathcal{O}_{x_0}$ is a principal ideal. So f_1 is well defined at x_0.

If $x_0 \in V_2 - V_1$, then, by using the Lemma 2.2(ii) in the case in which $b = 1$ and $c' = r_1 = \delta$ since $l = 2$, f_1 is well defined at x_0.

Case II. $l = 3$. If f_1 is not well defined everywhere, let E be an irreducible component of the locus where f_1 is not well defined, and $t = \operatorname{codim} E$, $s = \operatorname{codim} f(E)$. Let $x_0 \in E$ such that $f(x_0)$ is a regular point of $f(E)$. Suppose that $A = \mathcal{O}_{x_0}$, $B = \mathcal{O}_{f(x_0)}$, and P is the prime ideal of E at x_0, then we have $t = \dim A_P$ and $s = \dim B_{P \cap B}$. Let $m_B = (y_1, \ldots, y_\delta, \ldots, y_s, \ldots, y_n)$ such that $(y_1, \ldots, y_\delta)B = I_{S(f)}$ and $(y_1, \ldots, y_s)B = P \cap B$. Let $h \in m_A$ such that $(h)A = I_{E(f)}$, then

$$I_{S(f)}A = (h)(q_1, \ldots, q_\delta)A, \; q_1, \ldots, q_\delta \in A.$$

By the choice of E, we know $v_P(q_i) > 0$ and $v_P(h) > 0$. From the Proposition 1.2(2), since $r(A_P B_{P \cap B}) = r(Z(A_P)A_P B_{P \cap B})$, it is known that

$$r(A_P B_{P \cap B}) = \begin{cases} r_1 + r_2 & \text{if } v_P(h) > 1 \\ r_2 & v_P(h) = 1. \end{cases}$$

On the other hand, $e(A_P B_{P\cap B}) \geqslant v_P(h)\delta + s$ and $r(A_P B_{P\cap B}) \geqslant e(A_P B_{P\cap B}) - t$. So we have

$$s - t \leqslant \begin{cases} (2 - v_P(h))\delta & v_P(h) > 1 \\ 1 & v_P(h) = 1. \end{cases}$$

But the Lemma 3.1 implies $s - t \geqslant 2$, we obtain a contradiction. Thus f_1 is well defined everywhere, and we complete the proof of the theorem by Case I and Case II.

COROLLARY 3.1. *Let $f: X \to Y$ be a proper birational morphism of regular schemes whose dimension $\leqslant 4$. Suppose that $E(f)$ has only two regular components, and $S(f)$ is regular. Then f is a composition of two blowing-ups with regular centers.*

This is the main theorems of [6] and [10] when X and Y have an algebraically closed base field.

Proof of Theorem 3.2

Let $E(f) = V_1 \cup V_2 \cup V_3$ and $D_1 = \sigma^{-1}(S(f))$. By the Lemma 2.5, we can suppose that $S(f_1) = f_1[V_2] \cup f_1[V_3] = \Delta$ and $f_1.\ V_1 - V_2 - V_3 \cong D_1 - \Delta$.

(1) If Δ is a closed point, the Lemma 3.1 implies that f_1 is well defined everywhere. Using the Theorem 3.1, we complete the proof.

(2) If Δ is two intersecting curves, i.e. $\operatorname{codim} f_1[V_i] = 2$, $i = 2, 3$ and $f_1[V_2] \neq f_1[V_3]$. Let A_i be the local ring of $f_1[V_i]$, and let x_i be the center of v_{A_i} on X. By the Lemma 2.4 and Zariski's Main Theorem, x_i lies on a unique component V_i of $E(f)$ such that f_1 is well defined at x_i, where $i = 2, 3$. So $A_i = \mathcal{O}_{f_1(x_i)}$, $\mathcal{O}_{x_i} \neq A_i$, which imply $r(Z(A_i)\mathcal{O}_{x_i}) = 0$ by $1 = r(Z(A_i)A_i) = r(Z(A_i)\mathcal{O}_{x_i}) + r(Z(A_i)\mathcal{O}_{x_i}A_i)$ and $r(Z(A_i)\mathcal{O}_{x_i}A_i) \geqslant r(\mathcal{O}_{x_i}A_i) > 0$. Thus (since $Z(A_i) = \mathcal{O}_{x_i}$)

$$R(f) = V_1 + 2V_2 + 2V_3.$$

If f_1 is not well defined everywhere, then, since f_1^{-1} does not collapse the exceptional divisor D_1, there is a point $x \in X$ of codimension 2 on which f_1 is not well defined, and $f(x)$ is a closed point. Let $m_{f(x)} = (x_1, x_2, x_3)$ such that $(x_1, x_2)\mathcal{O}_{f(x)}$ is the ideal of $S(f)$ at $f(x)$. Then

$$x_i = q_i t,\ q_i \in m_x, \quad i = 1, 2,$$

where $(t)\mathcal{O}_x$ is the ideal of $E(f)$ at x. By the Proposition 1.4, it is known that

$$r(\mathcal{O}_x\mathcal{O}_{f(x)}) = \begin{cases} r_i & \text{if } v_x(t) = 1 \\ r_i + r_j & \text{if } v_x(t) = 2,\ i \neq j \\ r_1 + r_2 + r_3 & \text{if } v_x(t) = 3. \end{cases}$$

On the other hand, $r(\mathcal{O}_x\mathcal{O}_{f(x)}) \geqslant 2v_x(t) + 1$, so we have a contradiction. Thus f_1 is well defined everywhere, which completes the proof by [4] since the fiber dimension of f_1 is bounded by 1.

(3) If Δ is an irreducible curve, then $\sigma(\Delta)$ is a closed point by the Lemma 3.1, i.e., $f(V_2) = f(V_3) = \sigma(\Delta) = y$, and Δ is the fiber of σ at y. Let A be the local ring of Δ, and let x be the center of v_A on X, then f_1 is well defined at x such that x lies on a unique component $V_i \neq V_1$ by the Lemma 2.4 and Zariski's Main Theorem. Let $V_i = V_2$ and $\mathcal{O}_{f_1(x)} = A$, then $r(Z(A)\mathcal{O}_x) = 0$, i.e., x is a point of codimension 1, since $1 = r(Z(A)A) = r(Z(A)\mathcal{O}_x) + r(Z(A)\mathcal{O}_x A)$ and $\mathcal{O}_x \neq \mathcal{O}_{f_1(x)}$. On the other hand, $v_y = v_A$, so x is also a center of v_y on X such that $\overline{\{x\}} = V_2$. Let σ_y be the blowing-up with center y and $D_2 = \sigma_y^{-1}(y)$, then V_2 is birationally equivalent to D_2 under $g = \sigma_y^{-1}f$. It is clear that g^{-1} is an isomorphism on $D_2 - D_2 \cap g[V_1] - g[V_3]$. Thus, if $g[V_3]$ is a closed point, then g is well defined everywhere by the Lemma 3.1, we complete the proof. So, in the following proof, we always suppose that $g[V_3]$ is a curve. It is not difficult to prove

$$R(f) = V_1 + 2V_2 + 3V_3.$$

Now we prove that g is well-defined on $X - V_1 \cap V_3$ at first. For any $x \in X - V_1 \cap V_3$, if $x \notin V_1$, then g is well defined at x by the Lemma 2.2 and Lemma 2.3(i). So we only need to consider $x \in V_1 \cap V_2$. Let $m_{f(x)} = (x_1, x_2, x_3)$ such that $(x_1, x_2)\mathcal{O}_{f(x)}$ is the ideal of $S(f) = f(V_1)$ at $f(x)$, and let $x_1 = q_1t_1t_2$, $x_2 = q_2t_1t_2$, and $x_3 = q_3t_3$ such that $(t_1)\mathcal{O}_x$ and $(t_2)\mathcal{O}_x$ are the ideals of V_1 and V_2 at x. By the Proposition 1.2(1),(2), we have

$$v_x(q_1) + v_x(q_2) + v_x(q_3) \leqslant 1.$$

If $v_x(q_3) = 0$, then $(x_1, x_2, x_3)\mathcal{O}_x$ is generated by x_3, hence g is well defined at x. If $v_x(q_3) > 0$, then $v_x(q_1) = v_x(q_2) = 0$, which implies that $(x_1, x_2)\mathcal{O}_x$ is a principal ideal. So there is a prime ideal $Q \subset \mathcal{O}_{f(x)}[x_2/x_1]$ such that $\mathcal{O}_x$ dominates $\mathcal{O}_{f(x)}[x_2/x_1]_Q$. Let $\bar{\mathcal{O}}_{f(x)} = \mathcal{O}_{f(x)}[x_2/x_1]_Q$, then

$$3 = r_1 + r_2 = r(\mathcal{O}_x\mathcal{O}_{f(x)}) = r(\mathcal{O}_x\bar{\mathcal{O}}_{f(x)}) + r(\mathcal{O}_x\bar{\mathcal{O}}_{f(x)}\mathcal{O}_{f(x)}).$$

But

$$r(\mathcal{O}_x\bar{\mathcal{O}}_{f(x)}\mathcal{O}_{f(x)}) = v_x(d(\bar{\mathcal{O}}_{f(x)}/\mathcal{O}_{f(x)})) = v_x(x_1) = 2,$$

so $r(\mathcal{O}_x\bar{\mathcal{O}}_{f(x)}) = 1$. On the other hand, by the Proposition 1.1(3), there is a $y_2 \in \bar{\mathcal{O}}_{f(x)}$ such that (x_1, y_2, x_3) is a minimal basis of $\bar{m}_{f(x)} \subset \bar{\mathcal{O}}_{f(x)}$. Thus we have

$$r(\mathcal{O}_x\bar{\mathcal{O}}_{f(x)}) \geqslant v_x(x_1) + v_x(y_2) + v_x(x_3) - 3 = v_x(q_3) + 1,$$

which is impossible. So g is well defined on $X - V_1 \cap V_3$.

Now we consider the graph G of $f_1|_{V_1}$ with projections p_1 and p_2. By the Zariski's Connectedness Theorem, $p_2^{-1}\sigma^{-1}(y)$ is connected, whose image in V_1 is $V_1 \cap (V_2 \cap V_3)$, which is connected.

If $V_1 \cap V_2 \neq \varnothing$, then $V_1 \cap V_2 \cap V_3 \neq \varnothing$, i.e., there is a point $x \in V_1 \cap V_2 \cap V_3$. We hope to prove that g is well defined at x (which implies g is well defined everywhere by the Lemma 3.1). Let $m_x = (t_1, t_2, t_3)$ and $m_y = m_{f(x)} = (x_1, x_2, x_3)$ such that $(t_i)\mathcal{O}_x$ is the ideal of V_i at x_i, and let

$$x_1 = q_1t_1t_2t_3, \quad x_2 = q_2t_1t_2t_3, \quad x_3 = q_3t_2t_3.$$

If either $v_x(q_1)$ or $v_x(q_2)$ is not zero, then $6 = r(\mathcal{O}_z\mathcal{O}_{f(x)}) \geqslant e(\mathcal{O}_z\mathcal{O}_{f(x)}) - 3$ implies $v_x(q_3) = 0$. It means g is well defined at x. So we assume $v_x(q_1) = v_x(q_2) = 0$, i.e., $(x_1, x_2)\mathcal{O}_x$ is a principal ideal. Thus there is a prime ideal $Q \subset \mathcal{O}_{f(x)}[x_2/x_1]$ such that $\mathcal{O}_x$ dominates $\bar{\mathcal{O}}_{f(x)}$ $(= \mathcal{O}_{f(x)}[x_2/x_1]_Q)$, and $\bar{\mathcal{O}}_{f(x)}$ dominates $\mathcal{O}_{f(x)}$. By the Proposition 1.1, there is a $y_2 \in \bar{\mathcal{O}}_{f(x)}$ such that (x_1, y_2, x_3) is a minimal basis of $\bar{m}_{f(x)}$, and $d(\bar{\mathcal{O}}_{f(x)}/\mathcal{O}_{f(x)}) = (x_1)\bar{\mathcal{O}}_{f(x)}$. Since

$$\begin{aligned} r(\mathcal{O}_x\mathcal{O}_{f(x)}) &= r(\mathcal{O}_x\bar{\mathcal{O}}_{f(x)}) + r(\mathcal{O}_x\bar{\mathcal{O}}_{f(x)}\mathcal{O}_{f(x)}) \\ &= r(\mathcal{O}_x\bar{\mathcal{O}}_{f(x)}) + v_x(d(\bar{\mathcal{O}}_{f(x)}/\mathcal{O}_{f(x)})) \\ &= r(\mathcal{O}_x\bar{\mathcal{O}}_{f(x)}) + v_x(x_1), \end{aligned}$$

we have $r(\mathcal{O}_x\bar{\mathcal{O}}_{f(x)}) = 3$. So $1 + v_x(x_3) \leqslant 3$ by the Proposition 1.2(1), i.e., $v_x(q_3) = 0$, and g is well defined at x. Now since g is a proper birational morphism, and $S(g)$ is two intersecting curves $g(V_1)$ and $g(V_3)$, it is easy to know $R(g) = V_1 + V_3$, and $g(V_1) = \sigma_y^{-1}[S(f)]$ is regular because σ_y is a birational morphism from it onto a regular curve. Let $\sigma_g(V_1)$ be the blowing-up with center $g(V_1)$, then it is known that g is factorized through $\sigma_g(V_1)$ by the same reason as before. Hence f is a composition of three blowing-ups.

We complete the proof of our theorem by considering the case $V_1 \cap V_2 = \varnothing$.

In this case, g is well defined on V_2 and $V_2 - (V_2 \cap V_3) \cong D_2 - g[V_3]$. So $g[V_3] = C_3$ is regular since $V_2 \cap V_3$ is nonsingular and g^{-1} maps C_3 onto $V_2 \cap V_3$. Let σ_{C_3} be the blowing-up with center C_3 and $h = \sigma_{C_3}^{-1}\sigma_y^{-1}f$. If $\sigma_{C_3}^{-1}[D_2] = \bar{D}_2$ and $\sigma_{C_3}^{-1}(C_3) = \bar{D}_3$, then $h[V_2] = \bar{D}_2$, $h[V_3] = \bar{D}_3$. For any $x \in (V_2 \cup V_3) - V_1$, if $g(x) \notin C_3$, it is clear that h is well defined at x. If $g(x) \in C_3$, let $m_{g(x)} = (x_1, x_2, x_3)$ such that $(x_1, x_2)\mathcal{O}_{g(x)}$ is the ideal of C_3 at $g(x)$, then $x_1 = q_1 t_3$ and $x_2 = q_2 t_3$, where $(t_3)\mathcal{O}_x$ is the ideal of V_3 at x. Thus

$$r(\mathcal{O}_x \mathcal{O}_{f_2(x)}) \geqslant v_x(q_1) + v_x(q_2).$$

On the other hand, $R(g) = V_1 + V_3$, so $r(\mathcal{O}_x \mathcal{O}_{g(x)}) = 1$ and either $v_x(q_1)$ or $v_x(q_2)$ is zero. Thus h is well defined at x. By Zariski's Main Theorem, h is an isomorphism on $X - V_1$. Let $q \in S(h)$ be a point of codimension 2, and let x be the center of v_q on X. It is clear $x \in V_1$. We claim $x \notin V_3$ (which implies that h is well defined at x and $q = h(x) \in h[V_1]$). In fact, if $x \in V_1 \cap V_3$, then $f(x) = y = \sigma_y \sigma_{C_3}(q)$, and $r(Z(\mathcal{O}_q)\mathcal{O}_y) \geqslant 5$ since

$$r(Z(\mathcal{O}_q)\mathcal{O}_y) = r(Z(\mathcal{O}_q)\mathcal{O}_x) + r(Z(\mathcal{O}_q)\mathcal{O}_x\mathcal{O}_y) \geqslant 1 + r(\mathcal{O}_x\mathcal{O}_y) = 1 + r_1 + r_3 = 5.$$

On the other hand, since h is an isomorphism on V_2, and $h(V_2) = \bar{D}_2$, we know $q \notin \bar{D}_2$, which implies

$$r(Z(\mathcal{O}_q)\mathcal{O}_y) = r(Z(\mathcal{O}_q)\mathcal{O}_q) + r(Z(\mathcal{O}_q)\mathcal{O}_q\mathcal{O}_y) = 1 + r(\mathcal{O}_q\mathcal{O}_y) = 4.$$

We obtain a contradiction. Let G be the graph of h, with projections p_1 and p_2. If E is an irreducible component of the locus on which h is not well defined, then there is a divisor $\bar{E}$ of G such that $p_1(\bar{E}) = E$ and $p_2(\bar{E}) \subset S(h)$. By Lemma 2.1, we have $f(E) = y$, hence $p_2(\bar{E})$ has to be a closed point, which implies codim $\bar{E} \geqslant$ codim $E \geqslant 2$, it is impossible. Thus h is well defined everywhere, and we complete the proof of our theorem.

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