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The rationality of the moduli space of Enriques surfaces

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0. Introduction

The purpose of this paper is to prove the rationality of the moduli space $\mathcal{M}$ of Enriques surfaces (defined over $\mathbb{C}$) suggested by Dolgachev [D1]. Recall that $\mathcal{M}$ is described as a Zariski open set of $\mathcal{D}/\Gamma$ where $\mathcal{D}$ is a bounded symmetric domain of type IV and of dimension 10, and Γ is an arithmetic subgroup acting on $\mathcal{D}$ (Horikawa [H]). It is known that $\mathcal{D}/\Gamma$ is a quasi-projective variety (Baily, Borel [B-B]). We shall prove:

THEOREM. *$\mathcal{M}$ is birationally isomorphic to the moduli space $\mathcal{M}_{5,\mathrm{cusp}}$ of plane quintic curves with a cusp.*

It is known that $\mathcal{M}_{5,\mathrm{cusp}}$ is rational ([D1]). Hence we have:

COROLLARY. *$\mathcal{M}$ is rational.*

Let C be a plane quintic curve with a cusp. Let X be a K3 surface with an involution τ obtained as the double cover of $\mathbb{P}^2$ branched at C and the tangent line at the cusp. Then $\mathrm{H}^2(X, \mathbb{Z})^{\langle\tau^*\rangle} \simeq U \oplus D_8$ as lattices. As in the case of Enriques surfaces (Namikawa [Na]), by using the Torelli theorem for K3 surfaces (Piatetskii-Shapiro, Shafarevich [P-S]) and the surjectivity of the period map (Kulikov [K]), we can see that the moduli space of pairs (X, τ) is described as a Zariski open subset of $\mathcal{D}'/\Gamma'$ where $\mathcal{D}'$ is a bounded symmetric domain of type IV and of dimension 10, and Γ' is an arithmetic subgroup (Theorem 3.7). We shall prove that the map from $\mathcal{M}_{5,\mathrm{cusp}}$ to $\mathcal{D}'/\Gamma'$ obtained as above is birational (Theorem 3.21). We remark here that a general K3 surface as above has no fixed point free involution, and hence it is *not* the unramified double cover of any Enriques surfaces. However, forgetting $\mathcal{D}/\Gamma$ and $\mathcal{D}'/\Gamma'$ being moduli spaces, we shall see that there is an equivariant map from $\mathcal{D}$ to $\mathcal{D}'$ with respect to Γ and Γ', and this induces an isomorphism $\mathcal{D}/\Gamma \simeq \mathcal{D}'/\Gamma'$ (Theorem 4.1).

1. Preliminaries

(1.1) A *lattice* L is a free $\mathbb{Z}$-module of finite rank endowed with an integral symmetric bilinear form $\langle\ ,\ \rangle$. If L_1 and L_2 are lattices, then $L_1 \oplus L_2$ denotes the orthogonal direct sum of L_1 and L_2. An isomorphism of lattices preserving the bilinear forms is called an *isometry*. For a lattice L, we denote by $\mathrm{O}(L)$ the group of self-isometries of L. A sublattice S of L is called *primitive* if L/S is torsion free.

A lattice L is *even* if $\langle x, x\rangle$ is even for each $x \in L$. A lattice L is *non-degenerate* if the discriminant $d(L)$ of its bilinear form is non zero, and *unimodular* if $d(L) = \pm 1$. If L is a non-degenerate lattice, the *signature* of L is a pair (t_+, t_-) where $t_\pm$ denotes the multiplicity of the eigenvalues ± 1 for the quadratic form on $L \otimes \mathbb{R}$.

Let L be a non-degenerate even lattice. The bilinear form of L determines a canonical embedding $L \to L^* = \mathrm{Hom}(L, \mathbb{Z})$. The factor group L^*/L, which is denoted by A_L, is an abelian group of order $|d(L)|$. We denote by $l(L)$ the number of minimal generator of A_L. We extend the bilinear form on L to one on L^*, taking value in $\mathbb{Q}$, and define

$$q_L: A_L \to \mathbb{Q}/2\mathbb{Z},\ q_L(x + L) = \langle x, x\rangle + 2\mathbb{Z}\,(x \in L^*).$$

We call q_L the *discriminant quadratic form* of L. We denote by $\mathrm{O}(q_L)$ the group of isomorphisms of A_L preserving the form q_L. Note that there is a canonical homomorphism from $\mathrm{O}(L)$ to $\mathrm{O}(q_L)$.

A non-degenerate even lattice L is called *2-elementary* if $A_L \simeq (\mathbb{Z}/2\mathbb{Z})^{l(L)}$. It is known that the isomorphism class of an even indefinite 2-elementary lattice L is determined by the invariants $(r(L), l(L), \delta(L))$ ([N1], Theorem 3.6.2) where $r(L)$ is the rank of L and

$$\delta(L) = \begin{cases} 0 & \text{if } \langle x, x\rangle \in \mathbb{Z} \text{ for any } x \in L^* \\ 1 & \text{otherwise.} \end{cases}$$

We denote by U the hyperbolic lattice defined by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which is an even unimodular lattice of signature (1, 1), and by A_m, D_n, or E_l an even negative definite lattice associated to the Dynkin diagram of type A_m, D_n or E_l ($m \geqslant 1$, $n \geqslant 4$, $l = 6, 7, 8$). We remark that E_8 is unimodular. Also we denote by $\langle m\rangle$ the lattice of rank 1 defined by the matrix (m). For a lattice L and an integer m, $L(m)$ is the lattice whose bilinear form is the one on L multiplied by m. In section 3, we shall use the fact that both $U(2)$ and D_{4n} are 2-elementary lattices with $l = 2$, $\delta = 0$.

(1.2) A compact complex smooth surface Y is called an *Enriques surface* if the following conditions are satisfied:

(i) the geometric genus $p_g(Y)$ and the irregularity $q(Y)$ vanish;
(ii) if K_Y is the canonical divisor on Y, $2K_Y = 0$.

Note that the unramified double covering of Y defined by the torsion K_Y is a *K3 surface* X, a smooth surface with $q(X) = 0$ and $K_X = 0$. The second cohomology group $\mathrm{H}^2(Y, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{10} \oplus \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ is generated by the canonical class. The free part of $\mathrm{H}^2(Y, \mathbb{Z})$ admits a canonical structure of a lattice induced from the cup product. It is an even unimodular lattice with signature (1, 9) and hence isometric to $U \oplus E_8$ (e.g. [N1], Theorem 1.1.1). In the same way, the lattice $\mathrm{H}^2(X, \mathbb{Z})$ is isometric to $L = U \oplus U \oplus U \oplus E_8 \oplus E_8$. By definition of K3 surface, the Picard group $\mathrm{Pic}(X)$ is a subgroup of $\mathrm{H}^2(X, \mathbb{Z})$ which admits a structure of lattice induced from that of $\mathrm{H}^2(X, \mathbb{Z})$. We call this $\mathrm{Pic}(X)$ the *Picard lattice* of X.

Let Y be an Enriques surface and X its covering K3 surface. Let σ be the covering transformation of X and σ^* the involution of $\mathrm{H}^2(X, \mathbb{Z})$ induced from σ. Then σ^* determines two primitive sublattices

$$M = \{x \in L \mid \sigma^* x = x\}, \quad N = \{x \in L \mid \sigma^* x = -x\}.$$

It is known that $M \simeq U(2) \oplus E_8(2)$ and $N \simeq U \oplus U(2) \oplus E_8(2)$ (e.g. [B-P], §1.2). Let ω_X be the cohomology class of a non-zero holomorphic 2-form on X in $\mathrm{H}^2(X, \mathbb{C})$ which is unique up to constant. This class satisfies the following Riemann condition:

$$\langle \omega_X, \omega_X \rangle = 0, \quad \langle \omega_X, \bar{\omega}_X \rangle > 0$$

where $\langle\ ,\ \rangle$ denotes the bilinear form on $\mathrm{H}^2(X, \mathbb{C})$ induced from the cup product and $\bar{\omega}_X$ the complex conjugation of ω_X. We remark that ω_X is contained in $N \otimes \mathbb{C}$ since there are no global holomorphic 2-form on Y. Put

$$\mathscr{D} = \{[\omega] \in \mathbb{P}(N \otimes \mathbb{C}) \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\} \text{ and } \Gamma = \mathrm{O}(N).$$

Then $\mathscr{D}$ is a union of two copies of bounded symmetric domain of type IV and of dimension 10. By [B-B], the quotient $\mathscr{D}/\Gamma$ is a quasi-projective variety. The correspondence $Y \to [\omega_X] \bmod \Gamma$ defines a well-defined map from the set of isomorphy classes of Enriques surfaces to $(\mathscr{D}/\Gamma)_0 = (\mathscr{D}/\Gamma) \backslash \mathscr{H}$, where $\mathscr{H}$ is a closed irreducible subvariety. The corresponding point $[\omega_X]$ in $(\mathscr{D}/\Gamma)_0$ is called the *period* of Y. It follows from the Torelli theorem for Enriques surfaces that $(\mathscr{D}/\Gamma)_0$ is the coarse moduli space of Enriques surfaces. For more details we refer the reader to [H], and its improvement [Na].

2. The curves related to Enriques surfaces

In this section, we shall devote the observation of Dolgachev [D1, 2] on the relation between Enriques surfaces, plane quintic curves with two nodes and plane quintic curves with a cusp. For the proof of our theorem, we shall only use Proposition 2.2.

(2.1) Let Y be an Enriques surface. A *superelliptic polarization D on Y of degree* 8 is a divisor on Y such that $D = 2(E_1 + E_2)$, E_i an elliptic curve on Y with $E_1 \cdot E_2 = 1$ (see [C-D], Chap. IV, §7). It is known that a general Enriques surface, in the sense of Barth-Peters [B-P], has $2^7 \cdot 17 \cdot 31$ distinct superelliptic polarizations of degree 8 up to automorphisms ([B-P], Theorem 3.9).

Let $\tilde{\mathcal{M}}$ be the moduli space of Enriques surfaces with a superelliptic polarization of degree 8. Then the above implies that the natural map ψ from $\tilde{\mathcal{M}}$ to $\mathcal{M} = (\mathcal{D}/\Gamma)_0$ is of degree $2^7 \cdot 17 \cdot 31$. Note that $2^7 \cdot 17 \cdot 31 = 2^3(2^4 + 1)2^4(2^5 - 1)$ and recall that $2^3(2^4 + 1)$ is equal to the number of even theta characteristics on a smooth curve of genus 4 and $2^4(2^5 - 1)$ is the number of odd theta characteristics on a smooth curve of genus 5. In [D2], Dolgachev gave a map φ from $\tilde{\mathcal{M}}$ to $\mathcal{M}_{5,\mathrm{cusp}}$ of degree $2^7 \cdot 17 \cdot 31$ which factorizes

$$\tilde{\mathcal{M}} \xrightarrow{\varphi_1} \mathcal{X} \xrightarrow{\varphi_2} \mathcal{M}_{5,\mathrm{cusp}}$$

where $\mathcal{X}$ is the moduli space of pointed curves of genus 4 and $\deg \varphi_1 = 2^3(2^4 + 1)$, $\deg \varphi_2 = 2^4(2^5 - 1)$. For the precise definition of φ_1, φ_2, we refer the reader to [D2]. Here we mention that a pointed curve (C, q) of genus 4 is obtained as the normalization of a plane quintic curve with two nodes and q is the point residual to the line passing through the nodes. The last one is naturally appeared as the Hessian of a net of quadrics in $\mathbb{P}^4$. The base locus of this net is the branched curve of the morphism of degree 2 defined by $|D|$ from Y to the intersection of two quadrics in $\mathbb{P}^4$ of rank 3, where (Y, D) is an Enriques surface with a superelliptic polarization D of degree 8. Also a plane quintic curve with a cusp is naturally appeared as the discriminant of the conic bundle associated to a cubic threefold. The last one is constructed from the canonical model of a curve of genus 4. Thus we have two maps of the same degree:

$$\mathcal{M} \xleftarrow{\psi} \tilde{\mathcal{M}} \xrightarrow{\varphi} \mathcal{M}_{5,\mathrm{cusp}}.$$

This and the following suggest that $\mathcal{M}$ may be rational too.

PROPOSITION (2.2) ([D1], §8, (*e*)). *$\mathcal{M}_{5,\mathrm{cusp}}$ is rational.*

Proof. The moduli space of plane quartic curves with a cusp is rational

([D1], Example 5). The same proof implies the assertion. □

3. K3 surfaces with some involution

The purpose of this section is to see that $\mathcal{M}_{5,\mathrm{cusp}}$ is birationally isomorphic to the moduli space of the pairs of K3 surfaces X and its involution τ with $\mathrm{H}^2(X, \mathbb{Z})^{\langle\tau^*\rangle} \simeq U \oplus D_8$.

(3.1) Let C be a plane quintic curve with one cusp q. Let L be the tangent line of C at the cusp. In the following, we assume:

ASSUMPTION (3.2). *L meets C at another distinct two points p_1, p_2.*

Consider the plane sextic curve $C \cup L$. Let X' be the double cover of $\mathbb{P}^2$ branched at $C \cup L$ and X the minimal resolution of X'. Then X' has a rational double point of type E_7 over q and two rational double points of type A_1 over p_1, p_2. The X is a K3 surface which is reconstructed as follows. Taking successive blowing-ups of $\mathbb{P}^2$, we have a rational surface R with the following curves as in Figure 1.

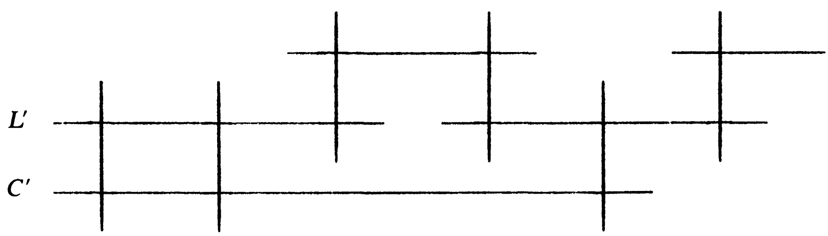


Fig. 1.

where C' and L' are the proper transforms of C and L respectively, and the horizontal lines except C' (resp. the vertical lines) are smooth rational curves with self-intersection number -4 (resp. -1). Then X is obtained as the double cover of R branched at C' and the smooth rational curves with self-intersection number -4 in the Figure 1. Hence X has smooth rational curves as in Figure 2:

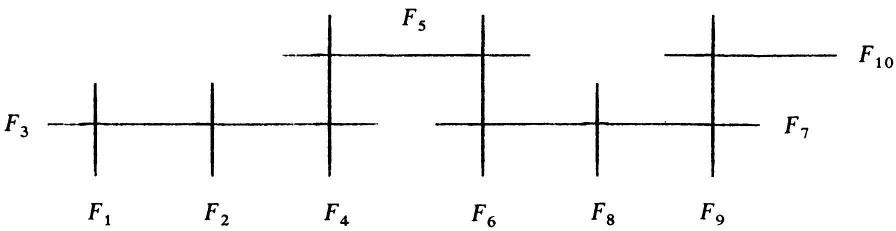


Fig. 2.

Let τ be the covering transformation of $X \to R$. Note that each curve in Figure 2 is preserved by τ.

LEMMA (3.3). $\mathrm{Pic}(X)^{\langle\tau^*\rangle}$ *contains a sublattice isometric to* $U \oplus D_8$.

Proof. The 9 curves except F_{10} in Figure 2 define an elliptic pencil with a singular fibre of type $\tilde{D}_8$ (type I_4^* in the Kodaira's notation) and F_{10} is a section of this pencil. Hence these 10 curves generate a sublattice of $\mathrm{Pic}(X)^{\langle\tau^*\rangle}$ isometric to $U \oplus D_8$. □

Taking an isometry $\mathrm{H}^2(X, \mathbb{Z}) \simeq L = U \oplus U \oplus U \oplus E_8 \oplus E_8$, define

$$S = \{x \in L \mid \tau^* x = x\}, \qquad N' = \{x \in L \mid \tau^* x = -x\}.$$

We remark here that S and N' are primitive in L, i.e. L/S and L/N' are torsion free.

PROPOSITION (3.4). $S \simeq U \oplus D_8$ *and* $N' \simeq U \oplus U(2) \oplus E_8$.

Proof. Obviously $S \supset \mathrm{Pic}(X)^{\langle\tau^*\rangle}$. Note that both S and $U \oplus D_8$ are 2-elementary lattices. It follows from [N2], Theorem 4.2.2 that

$$(22 - r(S) - l(S))/2 = g(C') = 5,$$

$$(r(S) - l(S))/2 = \#\{\text{smooth rational curves fixed by } \tau\} = 4$$

where $r(S)$ (resp. $l(S)$) is the rank of S (resp. the number of minimal generator of A_S). Hence $r(S) = 10$ and $l(S) = 2$. By Lemma 3.3, $\mathrm{Pic}(X)^{\langle\tau^*\rangle}$ contains $U \oplus D_8$ which has the same invariants. Thus we have $S = \mathrm{Pic}(X)^{\langle\tau^*\rangle} \simeq U \oplus D_8$. Then N' is an even indefinite 2-elementary lattice with the invariant $(r(N'), l(N'), \delta(N')) = (12, 2, 0)$ because N' is the orthogonal complement of S in the unimodular lattice L ([N1], Proposition 1.6.1.). Since the isomorphism class of an even indefinite 2-elementary lattice is determined by (r, l, δ) ([N1], Theorem 3.6.2), we have $N' \simeq U \oplus U(2) \oplus E_8$. □

Remark (3.5). Both S and N' are even indefinite 2-elementary lattices and hence the homomorphisms

$$\mathrm{O}(S) \to \mathrm{O}(q_S), \qquad \mathrm{O}(N') \to \mathrm{O}(q_{N'})$$

are surjective ([N1], Theorem 3.6.3). By [N1], Proposition 1.14.1, any $\gamma \in \mathrm{O}(S)$ or $\gamma' \in \mathrm{O}(N')$ can be lifted to an isometry of L. In particular, if γ acts on A_S trivially, then it can be lifted to an isometry acting trivially on N'. Since $S \simeq U \oplus D_8$, $q_S \simeq q_{D_8}$. Hence $A_S \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and $q_S \simeq \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$. A direct calculation shows that $\mathrm{O}(q_S) \simeq \mathbb{Z}/2\mathbb{Z}$ and the generator of $\mathrm{O}(q_S)$ is induced from the isometry ι in $O(S)$ defined by $\iota([F_i]) = [F_i]$, $3 \leqslant i \leqslant 10$, and $\iota([F_1]) = [F_2]$ (Note that the classes $[F_i]$ of F_i ($1 \leqslant i \leqslant 10$) in Figure 2 give a base of S).

DEFINITION (3.6). Now we define:

$$\mathscr{D}' = \{[\omega] \in \mathbb{P}(N' \otimes \mathbb{C}) \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}, \quad \Gamma' = O(N').$$

Then $\mathscr{D}'$ is a union of two copies of bounded symmetric domain of type IV and of dimension 10, and Γ' acts properly discontinuously on $\mathscr{D}'$. By [B-B], $\mathscr{D}'/\Gamma'$ is a quasi-projective variety. Put

$$\mathscr{H}' = \{[\omega] \in \mathscr{D}' \mid \langle \omega, l \rangle = 0 \quad \text{for some } l \in N' \text{ with } \langle l, l \rangle = -2\}.$$

Then Γ' acts on $\mathscr{H}'$. By the same reason as in the case of Enriques surfaces ([H], II, Theorem 2.3), the period $[\omega_X]$ of any K3 surface X as above is contained in $(\mathscr{D}'/\Gamma') \backslash (\mathscr{H}'/\Gamma')$. We remark here that $\mathscr{H}'/\Gamma'$ is a irreducible hypersurface in $\mathscr{D}'/\Gamma'$ (see the following Proposition 3.9). By the similar proof as that of Enriques surfaces ([Na]), using the Torelli theorem for K3 surfaces [P-S] and the surjectivity of the period map [K], we have the following theorem. For our purpose, we do not use this theorem, and hence we omit the proof.

THEOREM (3.7). *$(\mathscr{D}'/\Gamma') \backslash (\mathscr{H}'/\Gamma')$ is the coarse moduli space of the pairs (X, τ) where X is a* K3 *surface and τ is an involution of X with $H^2(X, \mathbb{Z})^{\langle \tau^* \rangle} \simeq U \oplus D_8$.*

DEFINITION (3.8) ([N3], [Na], Theorem 2.15). A vector l in N' with $\langle l, l \rangle = -4$ is called of *even type* if there is a vector m in $S = (N')^{\perp}$ with $\langle m, m \rangle = -4$ and $(m + l)/2 \in L$. By Remark 3.5, the set of (-4) – vectors in N' of even type is invariant under the action of Γ'. Put

$$\mathscr{H}'' = \{[\omega] \in \mathscr{D}' \mid \langle \omega, l \rangle = 0 \text{ for some } (-4)\text{-vector } l \text{ in } N' \text{ of even type}\}.$$

Then by the following proposition, $\mathscr{H}''/\Gamma'$ is an irreducible hypersurface in $\mathscr{D}'/\Gamma'$.

PROPOSITION (3.9). *Let l and l' be two (-2)-vectors in N' (or (-4)-vectors of even type). Then there is an isometry $\gamma \in \Gamma'$ with $\gamma(l) = l'$.*

Proof. This follows from the same proof as in the case of Enriques surfaces ([Na], Theorems 2.13, 2.15 and Proposition 2.16). □

In the following, we shall see that the set

$$(\mathscr{D}'/\Gamma')_0 = (\mathscr{D}'/\Gamma') \backslash (\mathscr{H}'/\Gamma') \cup (\mathscr{H}''/\Gamma')$$

bijectively corresponds to the set of projective isomorphism classes of plane quintic curves with a cusp satisfying the Assumption 3.2.

LEMMA (3.10). *Let X be a* K3 *surface constructed in* (3.1) *and* $[\omega_X]$ *its period. Then* $[\omega_X] \bmod \Gamma' \in (\mathscr{D}'/\Gamma')_0$.

Proof. As mentioned in (3.6), $[\omega_X] \in \mathscr{D}' \backslash \mathscr{H}'$ and hence it suffices to see that $[\omega_X]$ is not contained in $\mathscr{H}''$. Assume $[\omega_X] \in \mathscr{H}''$. Then there exist (-4)-vectors m in S and l in N' with $(m+l)/2 \in L$ and $\langle \omega_X, l \rangle = 0$. Consider the lattice $S \oplus \langle -4 \rangle$ generated by S and l. By adding a vector $(m+l)/2$, we have a sublattice K in $\mathrm{Pic}(X)$ in which $S \oplus \langle -4 \rangle$ is of index 2. Then $d(K) = d(S) \cdot d(\langle -4 \rangle)/[K : S \oplus \langle -4 \rangle]^2 = 4$.

Recall that X has an elliptic pencil π with a section F_{10} and a singular fibre F of type $\tilde{D}_8$ (see the proof of Lemma 3.3). It gives a decomposition

$$S = U \oplus D_8$$

where U is generated by the classes of a fibre and F_{10}, and D_8 is generated by $\{F_i\}_{1 \leqslant i \leqslant 8}$. Since U is unimodular, U is primitive in K, i.e. K/U is torsion free. By using the fact $A_U = \{0\}$ and [N1], Proposition 1.5.1, we can easily see that this U is also the component of a decomposition $K = U \oplus K'$, where K' is a negative definite even lattice of rank 9 with $d(K') = 4$ and $K' \supset D_8$.

If D_8 is not primitive in K', then there is an even lattice M of rank 8 with $K' \supset M \supset D_8$. Since $d(D_8) = 4$, $d(M) = 1$ and hence $M \simeq E_8$. Since E_8 is orthogonal to U, E_8 should be generated by components of F (see [Ko], Lemma 2.2), which is impossible.

If D_8 is primitive in K', then the orthogonal complement of D_8 in K' is isometric to $\langle -2m \rangle$ ($m \in \mathbb{N}$). The primitiveness implies that $K'/(D_8 \oplus \langle -2m \rangle)$ is embedded into $A_{D_8} \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and $A_{\langle -2m \rangle} \simeq \mathbb{Z}/2m\mathbb{Z}$ ([N1], §1.5). Hence $K'/(D_8 \oplus \langle -2m \rangle) \simeq \mathbb{Z}/2\mathbb{Z}$. Therefore we have $m = 2$ by using the equation $d(K') = d(D_8) \cdot d(\langle -2m \rangle)/[K' : D_8 \oplus \langle -2m \rangle]^2$. By [N1], Proposition 1.5.1, K' is obtained from $D_8 \oplus \langle -4 \rangle$ by adding an element $\alpha = x^* + y^*$ where $x^* \in D_8^*$ and $y^* \in \langle -4 \rangle^*$ with $q_{D_8}(x^*) = q_{\langle -4 \rangle}(y^*) = 1$. Note that such x^* and y^* are unique modulo D_8 and $\langle -4 \rangle$ respectively. Hence we can put

$$x^* = -\{[F_1] + [F_2] + 2([F_3] + \cdots + [F_8])\}/2,$$

$$y^* = y/2 \ (y \text{ is a base of } \langle -4 \rangle).$$

Then $\alpha^2 = -2$, $\langle \alpha, [F_i] \rangle = 0$ $(1 \leqslant i \leqslant 7)$ and $\langle \alpha, [F_8] \rangle = 1$. Thus $K' \simeq D_9$, which is also impossible. □

Remark (3.11). If we drop the assumption (3.2), i.e. L tangents to C at a smooth point, then there is a (-4)-vector of even type in $\mathrm{Pic}(X)$. Therefore, in this case, $[\omega_X] \in \mathscr{H}''$.

DEFINITION (3.12). Let X be a K3 surface with $[\omega_X] \in (\mathscr{D}'/\Gamma')_0$. Put

$\Delta = \{\delta \in \mathrm{Pic}(X) \mid \delta^2 = -2, \delta$ is represented by an effective divisor$\}$;
$\Delta(S) = \Delta \cap S$;
$P(X) =$ The connected component of $\{x \in \mathrm{Pic}(X) \otimes \mathbb{R} \mid \langle x, x\rangle > 0\}$ containing an ample class;
$P(S) = P(X) \cap S \otimes \mathbb{R}$;
$C(X) = \{x \in P(X) \mid \langle x, \delta\rangle > 0$ for any $\delta \in \Delta\}$;
$C(S) = \{x \in P(S) \mid \langle x, \delta\rangle > 0$ for any $\delta \in \Delta(S)\}$.

Note that $\overline{C(X)} \cap \mathrm{Pic}(X)$ is nothing but the set of classes of numerically effective divisors. The following is an analogue of [Na], Proposition 4.7.

LEMMA (3.13). $C(S) = C(X) \cap S \otimes \mathbb{R}$.

Proof. Obviously the left hand side contains the right. Let $x \in C(S)$ and $\delta \in \Delta \backslash \Delta(S)$. The primitiveness of S in L implies that $\tau^*(\delta) \neq \delta$. By the Hodge index theorem, $(\delta - \tau^*(\delta))^2 < 0$ and hence $\langle \delta, \tau^*(\delta)\rangle > -2$. Note that $\tau^*(\delta) \neq -\delta$ and $\langle \delta, \tau^*(\delta)\rangle \neq 0, -1$ because $[\omega_X] \in \mathscr{D}' \backslash (\mathscr{H}' \cup \mathscr{H}'')$. Hence $\delta + \tau^*(\delta) \in S$ with $(\delta + \tau^*(\delta))^2 \geqslant -2$. If $(\delta + \tau^*(\delta))^2 = -2$, then $\delta + \tau^*(\delta) \in \Delta(S)$, and hence $\langle x, \delta + \tau^*(\delta)\rangle > 0$. If $(\delta + \tau^*(\delta))^2 \geqslant 0$, then $\delta + \tau^*(\delta) \in \overline{P(X)}$ and hence $\langle x, \delta + \tau^*(\delta)\rangle > 0$. Since $\langle x, \delta\rangle = \langle \tau^*(x), \tau^*(\delta)\rangle = \langle x, \tau^*(\delta)\rangle$, we have $\langle x, \delta\rangle > 0$. □

(3.14). For $\delta \in \Delta(S)$, define $s_\delta \in \mathrm{O}(\mathrm{H}^2(X, \mathbb{Z}))$ by

$$s_\delta(x) = x + \langle x, \delta\rangle \delta.$$

Then the group $W(S)$ generated by $\{s_\delta \mid \delta \in \Delta(S)\}$ acts on $P(S)$ and $\overline{C(S)}$ is a fundamental domain with respect to its action on $\overline{P(S)}$ ([V]).

(3.15) *Surjectivity.* We shall see that each point in $(\mathscr{D}'/\Gamma')_0$ corresponds to a plane quintic curve with a cusp satisfying (3.2). Let X be a K3 surface with its period $[\omega_X] \in (\mathscr{D}'/\Gamma')_0$. First note that X has an involution τ with $\mathrm{H}^2(X, \mathbb{Z})^{\langle \tau^* \rangle} \simeq U \oplus D_8$. In fact, the involution σ of $S \oplus N'$ defined by $\sigma | S = 1_S$ and $\sigma | N' = -1_{N'}$ acts on $\mathrm{A}_S \oplus \mathrm{A}_{N'}$ trivially because S and N' are 2-elementary. Hence by Remark 3.5, we can extend σ to an isometry $\tilde{\sigma}$ of $\mathrm{H}^2(X, \mathbb{Z})$. Obviously $\tilde{\sigma}$ fixes the period of X. Moreover by Lemma 3.13, $\tilde{\sigma}$ preserves effective divisors on X. Therefore it now follows from the Torelli theorem for K3 surfaces [P-S] that $\tilde{\sigma}$ is induced from an involution τ of X with $\tau^* = \tilde{\sigma}$.

LEMMA (3.16). *There exists an elliptic pencil $\pi: X \to \mathbb{P}^1$ with a singular fibre F of type $\tilde{D}_n$ ($n \geqslant 8$) invariant under the action of τ.*

Proof. Consider an orthogonal decomposition $S = U \oplus D_8$ and take $f \in U$ satisfying that $f^2 = 0$ and f is primitive (i.e. $f = me$, $e \in U$, implies $m = \pm 1$). By

Lemma 3.13 and the fact stated in (3.14), we may assume that f is numerically effective, if necessary, replacing f by $\varphi(f)$ where $\varphi \in O(H^2(X, \mathbb{Z}))$ with $\varphi(S) \subset S$. Then f defines an elliptic pencil $\pi: X \to \mathbb{P}^1$ such that f is the cohomology class of a fibre of π ([P-S], §3, Theorem 1). Consider the negative definite sublattice K in {the orthogonal complement of f in $\mathrm{Pic}(X)\}/\mathbb{Z}[f]$ generated by (-2)-elements. Then $K \simeq K_1 \oplus \cdots \oplus K_r$, where K_i is a lattice isometric to A_m, D_n or E_l. By the same proof as that of [Ko], Lemma 2.2, π has singular fibres of type $\tilde{K}_1, \ldots, \tilde{K}_r$. Since $D_8 \subset K$, π has a singular fibre F of type $\tilde{D}_n$ $(n \geqslant 8)$. Since τ^* acts trivially on this D_8, F is invariant under the action of τ. □

LEMMA (3.17). *We keep the same notation as in Lemma* 3.16. *Then* $n = 8$.

Proof. Recall that F consists of the following smooth rational curves:

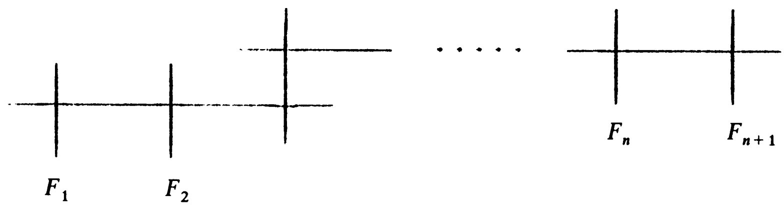


Fig. 3.

If $\tau(F_1) = F_2$ or $\tau(F_1) = F_n$, then $F_1 - F_2$ or $F_1 - F_n$ is (-4)-vector in N' of even type. Since $\langle \omega_X, \mathrm{Pic}(X) \rangle = 0$, this contradicts the assumption $[\omega_X] \in (\mathscr{D}'/\Gamma')_0$. Hence $\tau(F_i) = F_i$ $(1 \leqslant i \leqslant n+1)$. Now if $n \geqslant 9$, then S contains a degenerate lattice of rank $n + 1 \geqslant 10$ generated by components of F, which is impossible. □

Thus the singular fibre F consists of smooth rational curves as in the following:

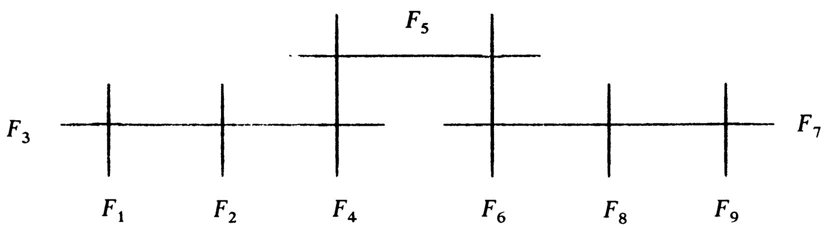


Fig. 4.

It follows from [N2], Theorem 4.2.2 that the set of fixed points of τ is the disjoint union of a smooth curve C of genus 5 and 4 smooth rational curves $E_1, \ldots, E_4$ (see the proof of Proposition 3.4). Since $\tau(F_i) = F_i$ $(1 \leqslant i \leqslant 9)$, F_3 and F_7 are fixed curves of τ. This implies that F_5 is also fixed by τ and τ acts on F_i ($i = 1, 2, 4, 6, 8, 9$) as an involution because the set of fixed points of τ is the disjoint union of smooth curves. Thus we may assume that $E_1 = F_3$, $E_2 = F_5$ and $E_3 = F_7$. Then F_i ($i = 1, 2, 8, 9$) meets either C or E_4. If C meets all F_1, F_2, F_8 and F_9, then $E_4 \cdot F_i = 0$ ($1 \leqslant i \leqslant 9$) and hence $S = U \oplus D_8$ contains a degenerate lattice of rank 10 generated by components of F and E_4. This is a contradiction. Also if C meets only F_1 and F_2, then E_4 meets F_8 and F_9. In this case, $(F_7 + F_8 + F_9 + E_4)^2 = C \cdot (F_7 + F_8 + F_9 + E_4) = 0$. This contradicts the Hodge index theorem. Similarly it does not occur that C meets only F_1 or C meets only F_1 and F_8. Thus we may assume that C meets F_1, F_2, F_8 and E_4 meets F_9. Note that this is the same situation as in Figure 2.

Now taking the quotient $X/\langle\tau\rangle$ and contracting exceptional curves on $X/\langle\tau\rangle$ successively, we have a plane quintic curve with a cusp satisfying the Assumption 3.2.

(3.18) *Injectivity.* Let C and C' be plane quintic curves with a cusp satisfying (3.2). Let X and X' be the corresponding K3 surfaces to C and C' respectively.

PROPOSITION (3.19). *If* $[\omega_X] = [\omega_{X'}]$ *in* $(\mathscr{D}'/\Gamma')_0$, *then* C *is projectively isomorphic to* C'.

Proof. Let $\{F_i\}$ or $\{F_i'\}$ ($1 \leqslant i \leqslant 10$) be smooth rational curves on X or X' as in Figure 2, respectively. It suffices to see that there exists an isomorphism between the pairs (X, τ) and (X', τ') which sends $\{F_i\}$ to $\{F_i'\}$. Let

$$\gamma\colon \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X', \mathbb{Z})$$

be an isometry with $\gamma([\omega_X]) = [\omega_{X'}]$ and $\gamma \circ \tau^* = (\tau')^* \circ \gamma$. By Remark 3.5, if necessary, changing F_1 and F_2, and replacing γ by $\gamma \circ \varphi$ for some $\varphi \in \mathrm{O}(\mathrm{H}^2(X, \mathbb{Z}))$ with $\varphi \circ \tau^* = \tau^* \circ \varphi$ and $\varphi | N' = 1_{N'}$, we may assume that $\gamma([F_i]) = [F_i']$ ($1 \leqslant i \leqslant 10$). Then by the following Lemma 3.20 and Lemma 3.13, $\gamma(C(X)) \subset C(X')$. Therefore it now follows from the Torelli theorem for K3 surfaces [P-S] that γ is induced from an isomorphism as desired. □

LEMMA (3.20). $C(S) = \{x \in P(S) \mid \langle x, [F_i]\rangle > 0,\ 1 \leqslant i \leqslant 10\}$.

Proof. The following proof is an analogue of [Na], Proposition 6.9. We use the same notation as in [V]. Let W be the subgroup generated by reflections $s_{[F_i]}$ ($1 \leqslant i \leqslant 10$). Its Coxeter diagram Σ is defined as follows: the vertices of Σ correspond to $\{F_i\}_{1 \leqslant i \leqslant 10}$ and two vertices F_i and F_j are joined by a simple line iff $F_i \cdot F_j = 1$. Then Σ contains only two parabolic subdiagram $\tilde{D}_8$ and $\tilde{E}_8$ which all have the maximal rank 8. Also Σ contains no Lanner's diagram and no dotted lines. Hence it follows from [V], Theorem 2.6 that W is of finite index in $\mathrm{O}(S)$. Hence the polyhedral cone

$$\{x \in S \otimes \mathbb{R} \mid \langle x, [F_i]\rangle \geqslant 0,\ 1 \leqslant i \leqslant 10\}$$

is contained in $\overline{P(S)}$ (see [V], p. 335. By a direct calculation, we can also see the last assertion without using the above Vinberg's theory). If there exists a smooth rational curve E with $E \neq F_i$ and $[E] \in S$, then $E \cdot F_i \geqslant 0$ for all i and hence $[E] \in \overline{P(S)}$, i.e. $E^2 \geqslant 0$, which is impossible. Similarly there is no smooth rational curve E with $\tau(E) \cdot E = 1$. Now let $\delta \in \Delta(S)$ and $x \in P(S)$. Let D be a irreducible component of δ which is a smooth rational curve. Then the above implies that $D = F_i$ for some i or $\tau(D) \cdot D \geqslant 2$ (Since $[\omega_X] \in (\mathscr{D}'/\Gamma')_0$, $\tau(D) \cdot D \neq 0$). In the latter case, $(D + \tau(D))^2 \geqslant 0$ and hence $\langle x, [D + \tau(D)] \rangle > 0$. Thus $\langle x, \delta \rangle > 0$ for any $x \in P(S)$ with $\langle x, [F_i] \rangle > 0$ $(1 \leqslant i \leqslant 10)$. □

Thus $(\mathscr{D}'/\Gamma')_0$ bijectively corresponds to the set of projective isomorphy classes of plane quintic curves with a cusp satisfying (3.2).

Let $\mathscr{M}_0$ be an open set of $\mathscr{M}_{5,\text{cusp}}$ consisting plane quintic curves with a cusp satisfying (3.2). For any family $\mathscr{C} \to S$ of plane quintic curves with a cusp, we can construct a family $\mathscr{X} \to S$ of K3 surfaces with an involution as above. Associating its period with each member of $\mathscr{X}$, we obtain a holomorphic map from S to $\mathscr{D}'/\Gamma'$. Therefore we have a holomorphic map $\lambda: \mathscr{M}_0 \to (\mathscr{D}'/\Gamma')_0$ which is bijective by the above argument. Let $\bar{\mathscr{M}}_0$ be a compactification of $\mathscr{M}_0$ with normal crossing boundary. Then by Borel's extension theorem [B], λ can be extended to a holomorphic map from $\bar{\mathscr{M}}_0$ to the projective compactification of $\mathscr{D}'/\Gamma'$ due to Bairly-Borel [B-B]. Hence by GAGA, λ is regular. Since λ is smooth and bijective on a Zariski open set, we now conclude:

THEOREM (3.21). *$\mathscr{M}_{5,\text{cusp}}$ is birationally isomorphic to $\mathscr{D}'/\Gamma'$.*

4. Proof of the rationality

The main purpose of this section is to prove the following:

THEOREM (4.1). $\mathscr{D}/\Gamma \simeq \mathscr{D}'/\Gamma'$ *(as quasi-projective varieties).*

Proof. For a lattice $(L, \langle\ ,\ \rangle)$, we denote by $L(1/2)$ the free $\mathbb{Z}$-module L with the symmetric bilinear form $\langle\ ,\ \rangle/2$ valued in $\mathbb{Q}$. Fix an orthogonal decomposition $N = U \oplus U(2) \oplus E_8(2)$. Then $N(1/2) = U(1/2) \oplus U \oplus E_8$ and the sublattice $2U(1/2) = \{2x \mid x \in U(1/2)\}$ of $U(1/2)$ is isometric to $U(2)$. Under this isomorphism, we consider the lattice $N' \simeq U(2) \oplus U \oplus E_8$ as a sublattice $2U(1/2) \oplus U \oplus E_8$ in $U(1/2) \oplus U \oplus E_8$. Let $\mathrm{O}(N(1/2))$ be the group of isomorphisms of $\mathbb{Z}$-module preserving the form $\langle\ ,\ \rangle/2$. Obviously N and $N(1/2)$ define the same bounded symmetric domain $\mathscr{D}$ and $\mathrm{O}(N) = \mathrm{O}(N(1/2))$. Let e_1, e_2 be a base of $U(1/2)$ with the matrix $(\langle e_i, e_j \rangle) = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ and $e_3, \ldots, e_{12}$

a base of $U \oplus E_8$. With respect to this base, any $g \in O(N(1/2))$ has a matrix decomposition

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \begin{matrix} A: U(1/2) \to U(1/2), & B: U \oplus E_8 \to U(1/2) \\ C: U(1/2) \to U \oplus E_8, & D: U \oplus E_8 \to U \oplus E_8. \end{matrix}$$

For $g' \in O(N')$, similarly, $g' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}$ with respect to a base $\{2e_1, 2e_2, e_3, \ldots, e_{12}\}$ of N'.

LEMMA (4.2). *For* $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in O(N(1/2))$, $g' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in O(N')$, *any entries of B and C' are even integers.*

Proof. Put $H = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ and $K = (\langle e_i, e_j \rangle)_{3 \leqslant i,j \leqslant 12}$. Then

$$g \in O(N(1/2)) \Leftrightarrow {}^tAHA + {}^tCKC = H,\ {}^tAHB + {}^tCKD = 0,\ {}^tBHB + {}^tDKD = K.$$

Let $\{b_i\}_{1 \leqslant i \leqslant 10}$ be the first row vector of B, $\{b'_i\}_{1 \leqslant i \leqslant 10}$ the second row vector of B and $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$. Since $U \oplus E_8$ is even, the diagonals of K, ${}^tCKC$ and ${}^tDKD$ are even integers. Then the equation ${}^tAHA + {}^tCKC = H$ implies that $a_1 \cdot a_3 \equiv 0$, $a_2 \cdot a_4 \equiv 0$ and $a_1 \cdot a_4 + a_2 \cdot a_3 \equiv 1 \pmod 2$. Since any entries of ${}^tCKD$ are integers, the equation ${}^tAHB + {}^tCKD = 0$ implies $a_1 \cdot b'_i + a_3 \cdot b_i \equiv 0$, $a_2 \cdot b'_i + a_4 \cdot b_i \equiv 0 \pmod 2$, $1 \leqslant i \leqslant 10$. These imply that b_i and b'_i are even.

Next let $g' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in O(N')$. Let $\{c_i\}_{1 \leqslant i \leqslant 10}$ be the first column vector of C', ${}^t(a_1, a_3)$ the first column vector of A'. Then

$$g'(e_1) = a_1 e_1 + a_3 e_2 + \sum_{i=1}^{10} (c_i/2) e_{i+2}.$$

Note that $N(1/2) = (N')^*$ and e_1, e_2 generate $A_{N'} = (N')^*/N'$. Since g' preserves $A_{N'}$ and $U \oplus E_8$ is unimodular, $\sum (c_i/2) e_{i+2} \in U \oplus E_8$ and hence c_i is even. Similarly any entries of the second column of C are even. □

Let $z = \sum_{i=1}^{12} z_i e_i \in \mathscr{D}$ and $z' = z'_1(2e_1) + z'_2(2e_2) + \sum_{i=3}^{12} z'_i e_i \in \mathscr{D}'$ be homogeneous coordinates. We define a biholomorphic map

$$\varphi: \mathscr{D} \to \mathscr{D}', \quad \varphi(z) = (z_1/2, z_2/2, z_3, \ldots, z_{12}).$$

Also by Lemma 4.2, the homomorphism

$$\psi\colon \mathrm{O}(N(1/2)) \to \mathrm{O}(N'),\ \psi\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \begin{bmatrix} A & B/2 \\ 2C & D \end{bmatrix}$$

is well-defined and an isomorphism of groups. Since $\varphi(g(z)) = \psi(g)(\varphi(z))$ for any $g \in \mathrm{O}(N(1/2))$, φ induces an isomorphism from $\mathscr{D}/\Gamma = \mathscr{D}/\mathrm{O}(N) = \mathscr{D}/\mathrm{O}(N(1/2))$ to $\mathscr{D}'/\Gamma' = \mathscr{D}'/\mathrm{O}(N')$. Since $N(1/2)$ and N' define the same rational structure on $\mathscr{D}$, φ can be extended to an isomorphism between the projective compactifications of $\mathscr{D}/\Gamma$ and $\mathscr{D}'/\Gamma'$ due to Bairly-Borel [B-B]. Thus we have proved Theorem 4.1. □

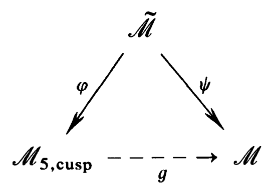
Combining Theorem 4.1 and Theorem 3.21, we have:

THEOREM (4.3). *$\mathscr{D}/\Gamma$ is birationally isomorphic to $\mathscr{M}_{5,\mathrm{cusp}}$.*

By Proposition 2.2, we now conclude:

COROLLARY (4.4). *The moduli space $\mathscr{M}$ of Enriques surfaces is rational.*

Remark (4.5). The author does not know the geometric meaning of the isomorphism in Theorem 4.1, in particular, whether there is a birational isomorphism g from $\mathscr{M}_{5,\mathrm{cusp}}$ to $\mathscr{M}$ forming the following commutative diagram (using the notation of §2), which is conjectured by Dolgachev [D2]:



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