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Inversion techniques and combinatorial identities— Strange evaluations of basic hypergeometric series

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0. Introduction

In 1973, a general pair of inverse series relations was discovered by Gould and Hsu [12] which has shown to have high potential for wide applications in interpolation theory, series expansion of implicit functions and combinatorial computation. However, during the next ten years, Gould-Hsu inversions did not arouse the attention from the combinatorial world although its specific examples were rediscovered and rephrased in terms of Lagrange inversions, and used to establish hypergeometric relations in Gessel and Stanton's paper [10]. Fifteen years later, it was realized by Hsu and the author [8] that the Hagen-Rothe and Abel identities are trivially implied by the Vandermonde convolution and the binomial theorem, respectively. This pair of reciprocal relations can be used to treat most hypergeometric identities in a unified way. This was partially motivated by Gessel-Stanton's work [10, 11] and has been fulfilled in the author's recent work [6, 7].

At the same time, the q-analogue of Gould-Hsu inversions was established by Carlitz [5]. It may be stated as follows: Let (a_i) and (b_i) be two complex sequences such that the polynomials defined by

$$\lambda(x; n) = \prod_{k=1}^{n} (a_k + q^x b_k) \tag{0.1a}$$

differ from zero for $-x$, $n \in \mathbb{N}_0$ (the set of non-negative integers) with the convention $\lambda(x; 0) = 1$. Then there hold inverse relations:

$$f(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \lambda(-k; n)g(k), \tag{0.1b}$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{a_{k+1} + q^{-k}b_{k+1}}{\lambda(-n; k+1)} f(k). \tag{0.1c}$$

By interchanging a and b this reciprocal pair can be reformulated in an equivalent version.

$$f(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \lambda(k; n) g(k) \tag{0.1d}$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{a_{k+1} + q^k b_{k+1}}{\lambda(n; k+1)} f(k) \tag{0.1e}$$

provided that the polynomials defined by (0.1a) do not vanish for non-negative integers x and n.

Unfortunately, the q-series inversions due to Carlitz have been neglected completely. In fact, most theorems and relations demonstrated in [3] and [11] are immediate consequences of that reciprocal pair. For example, the famous *Bailey pair* (cf. [3, 14]), which has an important application to the certification of Rogers-Ramanujan identities

$$\begin{cases} \beta(n) = \sum_{k=0}^{n} \alpha(k)/(q; q)_{n-k}(qx; q)_{n+k} \\ \alpha(n) = \dfrac{1 - xq^{2n}}{1 - x} \sum_{k=0}^{n} (-1)^{k+n} q^{\binom{n-k}{2}} \beta(k)(x; q)_{n+k}/(q; q)_{n-k} \end{cases}$$

could be re-expressed as a very special version of (0.1d)–(0.1e). Similar to the observation (on the Bailey pair) by Andrews [3], one implication of Carlitz' theorem is that for every relation in one form of (0.1b–e) there is a companion of the dual version. To prove each is to prove both. Two free parameter-sequences involved in (0.1) make it quite flexible in practice. To save space in writing, the inverse pairs (0.1b)–(0.1c) and (0.1d)–(0.1e) will be referred to as the C-pair and the C'-pair, respectively.

In the present paper, we will show that (0.1) plays the same part for the basic hypergeometric series as the Gould-Hsu inversions for the ordinary hyper-geometric identities. By telescoping the q-Saalschutz summations into some members of C-pairs properly, we will demonstrate, in a straightforward way, that most of the formulas displayed in [3, 11] correspond to their dual members. Furthermore, the same procedure can often be performed for the reformulations of the dual relations. This frequently produces strange evalu-ations, whose process is like the "chain-reaction" that begins with the q-Saalschutz theorem.

Let $|q| < 1$. As usual $_r\Phi_s$ basic hypergeometric series with base q is defined

by (cf. e.g., [9])

$$
{}_r\Phi_s\left[\begin{array}{c} a_1,\, a_2,\ldots,a_r \\ b_1,\ldots,b_s \end{array}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1;\, q)_k(a_2;\, q)_k \cdots (a_r;\, q)_k}{(b_1;\, q)_k(b_2;\, q)_k \cdots (b_s;\, q)_k} \frac{z^k}{(q;\, q)_k} \{(-1)^k q^{\binom{k}{2}}\}^{1-r+s}
$$

(0.2a)

whenever the series converges (e.g., if $|z| < 1$) where the q-shifted factorials are defined by

$$
(x;\, q)_\infty = \prod_{k=0}^{\infty} (1 - xq^k),\ (x;\, q)_n = (x;\, q)_\infty/(xq^n;\, q)_\infty.
$$

(0.2b)

The q-Saalschutz theorem (cf. [4, 9, 14]), which can be taken as the starting point, may be stated as

$$
{}_3\Phi_2\left[\begin{array}{c} q^{-n},\, a,\, b \\ c,\, q^{1-n}ab/c \end{array}; q\right] = \left[\begin{array}{c} c/a,\, c/b \\ c,\, c/ab \end{array}; q\right]_n
$$

(0.3)

where the factorial-fraction is shortened as

$$
\left[\begin{array}{c} a_1,\, a_2,\ldots,a_r \\ b_1,\, b_2,\ldots,b_s \end{array}; q\right]_n = \frac{(a_1;\, q)_n(a_2;\, q)_n \cdots (a_r;\, q)_n}{(b_1;\, q)_n(b_2;\, q)_n \cdots (b_s;\, q)_n}.
$$

To carry out the exchange between the C-pairs and q-series, we need some factorial transformations (cf. Appendix I in [9]), for example

$$
(x;\, q)_{kn} = [x, xq, \ldots, xq^{k-1};\, q^k]_n
$$

(0.4a)

$$
(x^k;\, q^k)_n = [x, x\omega_k, \ldots, x\omega_k^{k-1};\, q]_n, \qquad \text{where } \omega_k = \exp(2\pi i/k)
$$

(0.4b)

$$
(xq^k;\, q)_n = (x;\, q)_n(xq^n;\, q)_k/(x;\, q)_k
$$

(0.4c)

$$
(xq^{-k};\, q)_n = q^{-nk}(x;\, q)_n(qx^{-1};\, q)_k/(q^{1-n}x^{-1};\, q)_k.
$$

(0.4d)

They will be used without indication.

1. Basic analogues of Gauss-Vandermonde theorems and their dual relations

Taking n to infinity in (0.3), we have the q-analogue of Gauss's theorem

$$
{}_2\Phi_1\left[\begin{array}{c} a,\, b \\ c \end{array}; c/ab\right] = \left[\begin{array}{c} c/a,\, c/b \\ c,\, c/ab \end{array}; q\right]_\infty.
$$

(1.1a)

Its terminating version reduces to the q-analogue of Chu-Vandermonde formula

$$
{}_2\Phi_1\left[\begin{matrix} q^{-n}, & a \\ & c \end{matrix}; q^n c/a\right] = \left[\begin{matrix} c/a \\ c \end{matrix}; q\right]_n \tag{1.1b}
$$

which can be restated, by reversing the summation index:

$$
{}_2\Phi_1\left[\begin{matrix} q^{-n}, & a \\ & c \end{matrix}; q\right] = a^n \left[\begin{matrix} c/a \\ c \end{matrix}; q\right]_n. \tag{1.1c}
$$

The last two identities constitute a C-pair or C'-pair if they are telescoped.

In the following we will demonstrate some examples which produce several non-trivial evaluations dual to the specifications of (1.1b–c) when the latter are properly embedded in the C-pairs.

First, consider one special formula from (1.1b)

$$
{}_2\Phi_1\left[\begin{matrix} q^{-n}, & -q^{-n} \\ & -c \end{matrix}; q^{2n} c\right] = \left[\begin{matrix} q^n c \\ -c \end{matrix}; q\right]_n. \tag{1.2a}
$$

Its reformulation in accordance with the base change from q to $q^{1/2}$ yields the C-pair:

$$
\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \left[\begin{matrix} -q^{1/2} \\ -c \end{matrix}; q^{1/2}\right]_k q^{\binom{k}{2}} c^k = \left[\begin{matrix} c, & cq^{1/2} \\ & c^2 \end{matrix}; q\right]_n,
$$

$$
\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \left[\begin{matrix} c, & cq^{1/2} \\ & c^2 \end{matrix}; q\right]_k = \left[\begin{matrix} -q^{1/2} \\ -c \end{matrix}; q^{1/2}\right]_n q^{\binom{n}{2}} c^n.
$$

The latter can be expressed as an identity due to Gessel-Stanton [11, Eq. (5.15)]

$$
{}_3\Phi_2\left[\begin{matrix} q^{-n}, & c, & cq^{1/2} \\ & 0, & c^2 \end{matrix}; q\right] = c^n \left[\begin{matrix} -q^{1/2} \\ -c \end{matrix}; q^{1/2}\right]_n. \tag{1.2b}
$$

Next, the same process for summation

$$
{}_2\Phi_1\left[\begin{matrix} q^{-n}, & -q^{-n} \\ c^{-1}q^{-2n} \end{matrix}; q\right] = \left[\begin{matrix} -qc \\ q^{1+n}c \end{matrix}; q\right]_n = \left[\begin{matrix} q^2 c^2 \\ qc, & q^2 c^2 \end{matrix}; q^2\right]_n \tag{1.3a}
$$

generates the following *C*-pair:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2})} \frac{1 - c^{-1}q^{-1/2}}{(c^{-1}q^{-1/2-n}; q^{1/2})_{k+1}} (-q^{1/2}; q^{1/2})_k q^{-k/2}$$

$$= \begin{bmatrix} qc^2 \\ qc, q^{3/2}c \end{bmatrix}; q \end{bmatrix}_n q^{\binom{n+1}{2}},$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (c^{-1}q^{-1/2-k}; q^{1/2})_n \begin{bmatrix} qc^2 \\ qc, q^{3/2}c \end{bmatrix}; q \end{bmatrix}_k q^{\binom{k+1}{2}}$$

$$= \frac{1 - cq^{1/2}}{1 - cq^{(n+1)/2}} (-q^{1/2}; q^{1/2})_n.$$

The last relation can be restated as

$$_3\Phi_2 \begin{bmatrix} 0, qc^2, q^{-n} \\ cq^{(2-n)/2}, cq^{(3-n)/2} \end{bmatrix}; q \end{bmatrix} = \begin{bmatrix} -q^{1/2}, q^{1/2}c \\ q^{1/2}/c, qc \end{bmatrix}; q^{1/2} \end{bmatrix}_n. \tag{1.3b}$$

Again, this identity can be rewritten in the form

$$_3\Phi_2 \begin{bmatrix} 0, q^n c^2, q^{-n} \\ -q^{1/2}c, -qc \end{bmatrix}; q \end{bmatrix} = \begin{bmatrix} -q^{1/2} \\ -c \end{bmatrix}; q^{1/2} \end{bmatrix}_n \begin{bmatrix} -c \\ -qc \end{bmatrix}; q \end{bmatrix}_n q^{\binom{n+1}{2}/2} c^n \tag{1.4a}$$

which creates another *C'*-pair:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2})} (c^2 q^k; q)_n \begin{bmatrix} c^2 \\ -q^{1/2}c, -qc \end{bmatrix}; q \end{bmatrix}_k$$

$$= c^n q^{n/2 + 3\binom{n}{2}/2} \begin{bmatrix} -q^{1/2} \\ -c \end{bmatrix}; q^{1/2} \end{bmatrix}_n \begin{bmatrix} -c, c^2 \\ -qc \end{bmatrix}; q \end{bmatrix}_n,$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1 - c^2 q^{2k}}{(c^2 q^n; q)_{k+1}} \begin{bmatrix} -q^{1/2} \\ -c \end{bmatrix}; q^{1/2} \end{bmatrix}_k \begin{bmatrix} -c, c^2 \\ -qc \end{bmatrix}; q \end{bmatrix}_k q^{k/2 + 3\binom{k}{2}/2} c^k$$

$$= \begin{bmatrix} c^2 \\ -q^{1/2}c, -qc \end{bmatrix}; q \end{bmatrix}_n.$$

The last relation can be reformulated, under the base change from q to q^2, as a strange evaluation

$$_5\Phi_5 \begin{bmatrix} c, qc^{1/2}, -qc^{1/2}, -q^{-n}, q^{-n} \\ 0, c^{1/2}, -c^{1/2}, q^{1+n}c, -q^{1+n}c \end{bmatrix}; -q^{1+2n}c \end{bmatrix} = \begin{bmatrix} q^2 c^2 \\ -qc, -q^2 c^2 \end{bmatrix}; q^2 \end{bmatrix}_n.$$

$$\tag{1.4b}$$

Finally, an equivalent version of the q-Vandermonde convolution

$$
{}_2\Phi_1\left[\begin{array}{c} q^{-n},\ aq^n \\ qa/b \end{array}; q\right] = (-qa/b)^n q^{\binom{n}{2}}\left[\begin{array}{c} b \\ qa/b \end{array}; q\right]_n \tag{1.5a}
$$

can also be telescoped and gives rise to a C'-pair as follows:

$$
\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}(aq^k; q)_n \left[\begin{array}{c} a \\ qa/b \end{array}; q\right]_k = (-a/b)^n q^{n^2}\left[\begin{array}{c} a,\ b \\ qa/b \end{array}; q\right]_n,
$$

$$
\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1-aq^{2k}}{(aq^n; q)_{k+1}}\left[\begin{array}{c} a,\ b \\ qa/b \end{array}; q\right]_k (a/b)^k q^{k^2} = \left[\begin{array}{c} a \\ qa/b \end{array}; q\right]_n.
$$

This time the dual relation corresponds to the following basic closed formula

$$
{}_5\Phi_5\left[\begin{array}{c} a,\ qa^{1/2},\ -qa^{1/2},\ b,\ q^{-n} \\ 0,\ a^{1/2},\ -a^{1/2},\ qa/b,\ q^{1+n}a \end{array}; q^{1+n}a/b\right] = \left[\begin{array}{c} qa \\ qa/b \end{array}; q\right]_n \tag{1.5b}
$$

which in fact contains (1.4b) as its special case when $b = -q^{-n}$.

By embedding another expression of q-Vandermonde theorem

$$
{}_2\Phi_1\left[\begin{array}{c} q^{-n},\ a \\ q^{1-n}b \end{array}; q\right] = \left[\begin{array}{c} a/b \\ b^{-1} \end{array}; q\right]_n
$$

in the C-pair, one can find that this is a self-reciprocal relation, i.e., its dual relation has the same formulation under the parameter-replacement.

From the examples just demonstrated, one can regard the 'C-pair' as a "black box". The embedding (or telescoping) operation on C-pairs is just like the "input-output process" in the black box. And the derivation from (1.3) to (1.4) looks like the "chain-reaction" which has high potential for a proliferation of q-series identities. Because the whole technique consists of only trivial transformations between q-series and the C-pairs, numerous known and unknown evaluations of basic hypergeometric series could, and should be created in an almost mechanical manner. Hence the greedy mathematician might expect the "chain-reaction" to go forward without breaking, producing countless combinatorial identities.

2. Strange evaluations associated with q-Pfaff-Saalschutzian formulae

For convenience in specifying parameters, the q-Saalschutz theorem (0.3) can

also be stated in the symmetric form

$$
{}_3\Phi_2\left[\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix}; q\right] = \left[\begin{matrix} q/e,\ d/a,\ d/b,\ d/c \\ d,\ qa/e,\ qb/e,\ qc/e \end{matrix}; q\right]_\infty \tag{2.1}
$$

where $qabc = de$ and one of a, b and c is q^{-n}, n being a non-negative integer.

Denote by $\left(\begin{matrix} r,s,t \\ u,v \end{matrix}\right)$ the expoential type of basic hypergeometric series ${}_3\Phi_2\left[\begin{matrix} aq^{rn},\ bq^{sn},\ cq^{tn} \\ dq^{un},\ eq^{vn} \end{matrix}\right]$ with zeros being omitted. As the central part of the present paper, the complete list of C-pairs associated with the Saalschutz series will be demonstrated in this section. The embedding machinery shows that a number of strange basic hypergeometric formulae are just the dual relations of the Saalschutz theorem.

The most striking example is the C-pair of the q-Saalschutz theorem and Jackson's q-analogue of the Dougall-Dixon formula.

$$
(-1,\ +1)\quad {}_3\Phi_2\left[\begin{matrix} aq^n,\ qa/bc,\ q^{-n} \\ qa/b,\ qa/c \end{matrix}; q\right] = (qa/bc)^n \left[\begin{matrix} b,\ c \\ qa/b,\ qa/c \end{matrix}; q\right]_n. \tag{2.2a}
$$

This is a reformulation of the Saalschutz theorem which can be telescoped and corresponds to the C'-pair:

$$
\sum_{k=0}^n (-1)^k \left[\begin{matrix} n \\ k \end{matrix}\right] q^{\binom{n-k}{2}}(aq^k; q)_n \left[\begin{matrix} a,\ qa/bc \\ qa/b,\ qa/c \end{matrix}; q\right]_k
$$
$$
= (qa/bc)^n q^{\binom{n}{2}} \left[\begin{matrix} a,\ b,\ c \\ qa/b,\ qa/c \end{matrix}; q\right]_n, \tag{2.2b}
$$

$$
\sum_{k=0}^n (-1)^k \left[\begin{matrix} n \\ k \end{matrix}\right] \frac{1 - aq^{2k}}{(aq^n; q)_{k+1}} (qa/bc)^k q^{\binom{k}{2}} \left[\begin{matrix} a,\ b,\ c \\ qa/b,\ qa/c \end{matrix}; q\right]_k = \left[\begin{matrix} a,\ qa/bc \\ qa/b,\ qa/c \end{matrix}; q\right]_n. \tag{2.2c}
$$

The last relation may be stated as the q-Dougall-Dixon theorem

$$
{}_6\Phi_5\left[\begin{matrix} a,\ qa^{1/2},\ -qa^{1/2},\ b,\ c,\ q^{-n} \\ a^{1/2},\ -a^{1/2},\ qa/b,\ qa/c,\ q^{n+1}a \end{matrix}; q^{1+n}a/bc\right] = \left[\begin{matrix} qa,\ qa/bc \\ qa/b,\ qa/c \end{matrix}; q\right]_n. \tag{2.2d}
$$

For the original Saalschutz formula, the associated C-pair

$$
\left(\begin{matrix} -1 \\ -1 \end{matrix}\right)\quad {}_3\Phi_2\left[\begin{matrix} q^{-n},\ a,\ b \\ c,\ q^{1-n}ab/c \end{matrix}; q\right] = \left[\begin{matrix} c/a,\ c/b \\ c,\ c/ab \end{matrix}; q\right]_n \tag{2.3a}
$$

$$
_3\Phi_2 \begin{bmatrix} q^{-n}, & c/a, & c/b \\ c, & q^{1-n}c/ab \end{bmatrix} = \begin{bmatrix} a, & b \\ c, & ab/c \end{bmatrix}_n
$$
(2.3b)

is self-reciprocal. This is the only trivial example deduced from the Saalschutz series-telescoping.

The identity

$$
\left(\frac{-1}{2 \times (-1/2)} \right) \ _3\Phi_2 \begin{bmatrix} q^{-n}, & ae, & q^{1/2}e/a \\ q^{(1-n)/2}e, & q^{(2-n)/2}e \end{bmatrix} = \begin{bmatrix} a, & q^{1/2}/a \\ q^{1/2}e, & e^{-1} \end{bmatrix}_n
$$
(2.4a)

can be telescoped into the *C*-pair:

$$
\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (e^{-1}q^{-k}; q^{1/2})_n \frac{[ae, q^{1/2}e/a; q]_k}{(eq^{1/2}; q^{1/2})_{2k}} q^{\binom{k+1}{2}}
$$
$$
= \begin{bmatrix} a, & q^{1/2}/a \\ eq^{1/2} \end{bmatrix}_n,
$$
(2.4b)

$$
\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1 - e^{-1}q^{-k/2}}{(e^{-1}q^{-n}; q^{1/2})_{k+1}} \begin{bmatrix} a, & q^{1/2}/a \\ eq^{1/2} \end{bmatrix}_k
$$
$$
= q^{\binom{n+1}{2}} \begin{bmatrix} ae, & q^{1/2}e/a \\ eq^{1/2}, & eq \end{bmatrix}_n .
$$
(2.4c)

Under the base change from q to q^2, (2.4c) can be rewritten as the q-analogue of the Whipple formula due to Jain (cf. [13, 15]).

$$
_4\Phi_3 \begin{bmatrix} q^{-n}, & -q^{-n}, & a, & q/a \\ -q, & e, & q^{1-2n}e^{-1} \end{bmatrix} = \begin{bmatrix} ae, & qe/a \\ e, & eq \end{bmatrix}_n .
$$
(2.4d)

Another Saalschutz summation of the same exponential-type

$$
_3\Phi_2 \begin{bmatrix} q^{-n}, & x, & -y \\ (xyq^{1-n})^{1/2}, & -(xyq^{1-n})^{1/2} \end{bmatrix} = (-y)^n \begin{bmatrix} xy^{-1}q^{1-n} \\ xyq^{1-n} \end{bmatrix}_n
$$
(2.5a)

can be embedded in (0.1c) and generates the *C*-pair:

$$
\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{[x, -y; q]_k}{(xyq^{1/n}; q^2)_k} = (-y)^n q^{\binom{n}{2}} \begin{bmatrix} xy^{-1}q^{1-n} \\ xyq^{1-n} \end{bmatrix}_n
$$
(2.5b)

$$
\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (xyq^{-1-k}; q^2)_n \frac{1 - xyq^{n-1}}{1 - xyq^{-k-1}} \frac{(xy^{-1}q^{1-k}; q^2)_k}{(xyq^{1-k}; q^2)_k} q^{\binom{k}{2}} y^k = [x, -y; q]_n
$$
(2.5c)

where a new strange summation formula (2.5c) is established which does not fit into the general theory of basic hypergeometric series.

We next consider

$$(-1, 2 \times 1/2) \quad {}_3\Phi_2 \left[\begin{array}{c} q^{-n}, \, aq^{n/2}, \, aq^{(1+n)/2} \\ abq^{1/2}, \, aq/b \end{array} ; q \right] = a^n q^{\binom{n+1}{2}/2} \frac{[b, \, q^{1/2}/b; \, q^{1/2}]_n}{[abq^{1/2}, \, qa/b; \, q]_n}.$$

$$(2.6a)$$

This special Saalschutzian summation can be telescoped into C'-pair:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (aq^k; \, q^{1/2})_n \frac{(a; \, q^{1/2})_{2k}}{[abq^{1/2}, \, qa/b; \, q]_k}$$

$$= a^n q^{\binom{3n}{2}/6} \frac{[a, \, b, \, q^{1/2}/b; \, q^{1/2}]_n}{[abq^{1/2}, \, qa/b; \, q]_n}$$

$$(2.6b)$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{3k}{2}/6} \frac{1 - aq^{3k/2}}{(aq^n; \, q^{1/2})_{k+1}} \frac{[a, \, b, \, q^{1/2}/b; \, q^{1/2}]_k}{[abq^{1/2}, \, qa/b; \, q]_k} a^k$$

$$= \left[\begin{array}{c} a, \, aq^{1/2} \\ abq^{1/2}, \, qa/b \end{array} ; q \right]_n$$

$$(2.6c)$$

where (2.6c) can be expressed as the q-analogue of a strange hypergeometric evaluation due to Gosper (cf. Gessel & Stanton: [10, Eq. (1.2)] and [11, Eq. (6.8)]).

$$\sum_{k=0}^{n} \frac{(q^{-n}; \, q)_k}{(aq^{1/2+n}; \, q^{1/2})_k} \frac{1 - aq^{3k/2}}{1 - a} \frac{[a, \, b, \, q^{1/2}/b; \, q^{1/2}]_k}{[q, \, abq^{1/2}, \, qa/b; \, q]_k} q^{nk + \binom{k+1}{2}/2} a^k$$

$$= \left[\begin{array}{c} aq^{1/2}, \, aq \\ abq^{1/2}, \, qa/b \end{array} ; q \right]_n.$$

$$(2.6d)$$

The identity

$$(2 \times (-1/2), \, +1) \quad {}_3\Phi_2 \left[\begin{array}{c} aq^n, \, q^{(1-n)/2}, \, q^{-n/2} \\ bq^{1/2}, \, qa/b \end{array} ; q \right] = q^{-\binom{n}{2}/2} \frac{[b, \, q^{1/2}a/b; \, q]_n}{[b, \, q^{1/2}a/b; \, q^{1/2}]_n},$$

$$(2.7a)$$

gives rise to the following C'-pair, after replacing q by q^2

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ 2k \end{bmatrix} q^{\binom{n-2k}{2}} (aq^{2k}; \, q^2)_n \left[\begin{array}{c} a, \, q \\ qb, \, q^2a/b \end{array} ; q^2 \right]_k = \frac{[a, \, b, \, qa/b; \, q^2]_n}{[b, \, qa/b; \, q]_n}$$

$$(2.7b)$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1-aq^{3k}}{(aq^n; q^2)_{k+1}} \frac{[a, b, qa/b; q^2]_k}{[b, qa/b; q]_k}$$

$$= \begin{cases} 0, & (n\text{-odd}) \\ \begin{bmatrix} q, a \\ qb, q^2a/b \end{bmatrix}_m, & (n = 2m) \end{cases} \tag{2.7c}$$

where (2.7c) may be reformulated as the q-analogue of another strange hypergeometric evaluation due to Gosper (cf. Gessel & Stanton: [10, Eq. (1.3)] and [11, Eq. (6.13)]).

$$\sum_{k=0}^{n} \frac{(q^{-n/2}; q^{1/2})_k}{(aq^{1+n/2}; q)_k} \frac{1-aq^{3k/2}}{1-a} \frac{[a, b, q^{1/2}a/b; q]_k}{[q^{1/2}, b, q^{1/2}a/b; q^{1/2}]_k} q^{-(n-\frac{k}{2}+1)/2}$$

$$= \begin{cases} 0, & (n - \text{odd}) \\ q^{-(\frac{n+1}{2})/2} \begin{bmatrix} q^{1/2}, qa \\ q^{1/2}b, qa/b \end{bmatrix}_m, & (n = 2m). \end{cases} \tag{2.7d}$$

Another version reads as

$$\begin{pmatrix} 2\times(-1/2) \\ -1 \end{pmatrix} \,_3\Phi_2 \begin{bmatrix} q^{-n/2}, q^{(1-n)/2}, q^{1/2}c^2/b \\ q^{3/2-n}/b, q^{1/2}c^2 \end{bmatrix}; q \end{bmatrix}$$

$$= \begin{bmatrix} bq^{-1/2} \\ c^2 \end{bmatrix}; q^{1/2} \Big]_n \begin{bmatrix} c^2 \\ bq^{-1/2}; q \end{bmatrix}_n. \tag{2.8a}$$

After performing the base change on (2.8a), from q to q^2, we have the C-pair:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ 2k \end{bmatrix} (bq^{-1-2k}; q^2)_n \begin{bmatrix} q, qc^2/b \\ qc^2, q^3/b \end{bmatrix}; q^2 \Big]_k q^{(\frac{2k+1}{2})} = \begin{bmatrix} q^{-1}b \\ c^2 \end{bmatrix}; q \Big]_n (c^2; q^2)_n \tag{2.8b}$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{(\frac{n-k}{2})} \frac{1-bq^{k-1}}{(bq^{-1-n}; q^2)_{k+1}} \begin{bmatrix} bq^{-1} \\ c^2 \end{bmatrix}; q \Big]_k (c^2; q^2)_k$$

$$= \begin{cases} 0, & (n - \text{odd}) \\ q^{(\frac{n+1}{2})} \begin{bmatrix} q, qc^2/b \\ qc^2, q^3/b \end{bmatrix}; q^2 \Big]_m, & (n = 2m) \end{cases} \tag{2.8c}$$

where (2.8c) may be restated as the q-Watson formula due to Andrews (cf. [2]

or [3, Eq. (4.6)]).

$$
{}_4\Phi_3\left[\begin{array}{c} q^{-n},\, b,\, c,\, -c \\ c^2,\, (q^{1-n}b)^{1/2},\, -(q^{1-n}b)^{1/2} \end{array}; q\right] = \begin{cases} 0, & (n - \text{odd}) \\ \left[\begin{array}{c} q,\, qc^2/b \\ qc^2,\, q/b \end{array}; q^2\right]_m & (n = 2m). \end{cases} \tag{2.8d}
$$

Again, consider the Saalschutzian formula

$$
{}_3\Phi_2\left[\begin{array}{c} q^{-n},\, -q^{-n},\, b/c \\ -b,\, q^{1-2n}/c \end{array}; q\right] = \left[\begin{array}{c} b,\, bq,\, c^2 \\ c,\, cq,\, b^2 \end{array}; q^2\right]_n. \tag{2.9a}
$$

Its reformulation under the base change from q to $q^{1/2}$ leads to the following C-pair:

$$
\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{[-q^{1/2},\, b/c;\, q^{1/2}]_k}{[-b,\, c^{-1}q^{1/2-n};\, q^{1/2}]_k} q^{-k/2} = q^{\binom{n}{2}} \left[\begin{array}{c} b,\, bq^{1/2},\, c^2 \\ c,\, cq^{1/2},\, b^2 \end{array}; q\right]_n, \tag{2.9b}
$$

$$
\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (c^{-1}q^{-k};\, q^{1/2})_n \left[\begin{array}{c} b,\, bq^{1/2},\, c^2 \\ cq,\, cq^{1/2},\, b^2 \end{array}; q\right]_k q^{\binom{k+1}{2}}
$$

$$
= \left[\begin{array}{c} -q^{1/2},\, c,\, b/c \\ -b,\, cq^{1/2} \end{array}; q^{1/2}\right]_n. \tag{2.9c}
$$

The last relation may be rewritten as (cf. [3, Eq. (4.3)] or [11, Eq. (4.22)]),

$$
{}_4\Phi_3\left[\begin{array}{c} q^{-n},\, b,\, bq^{1/2},\, c^2 \\ cq^{(1-n)/2},\, cq^{(2-n)/2},\, b^2 \end{array}; q\right] = \left[\begin{array}{c} -q^{1/2},\, c,\, b/c \\ -b,\, cq^{1/2},\, c^{-1} \end{array}; q^{1/2}\right]_n. \tag{2.9d}
$$

The identity

$$
\left(\begin{array}{c} 2 \times (-1/2) \\ 2 \times (-1/2) \end{array}\right) {}_3\Phi_2\left[\begin{array}{c} a,\, q^{-n/2},\, q^{(1-n)/2} \\ a^{1/2}q^{(1-n)/2},\, a^{1/2}q^{(2-n)/2} \end{array}; q\right] = (-a^{1/2})^{-n} \left[\begin{array}{c} a^{1/2} \\ a^{-1/2} \end{array}; q^{1/2}\right]_n \tag{2.10a}
$$

may be telescoped into the following C-pair after replacing q by q^2:

$$
\sum_{k=0}^{n} \begin{bmatrix} n \\ 2k \end{bmatrix} (a^{-1}q^{-2k};\, q)_n \left[\begin{array}{c} q,\, a^2 \\ qa,\, q^2a \end{array}; q^2\right]_k q^{\binom{2k+1}{2}} = (-a)^{-n}(a;\, q)_n \tag{2.10b}
$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{(a; q)_k}{(a^{-1}q^{1-n}; q)_k} a^{-k} = \begin{cases} 0, & (n - \text{odd}) \\ q^{\binom{m}{2}} \begin{bmatrix} q, a^2 \\ a, qa \end{bmatrix}; q^2 \end{bmatrix}_m & (n = 2m). \end{cases} \tag{2.10c}$$

where (2.10c) is equivalent to the q-analogue of the terminating Kummer theorem.

$$_2\Phi_1 \begin{bmatrix} q^{-n}, a \\ a^{-1}q^{1-n} \end{bmatrix}; -q/a \end{bmatrix} = \begin{cases} 0, & (n - \text{odd}) \\ \begin{bmatrix} q, a^2 \\ a, qa \end{bmatrix}; q^2 \end{bmatrix}_m & (n = 2m). \end{cases} \tag{2.10d}$$

In addition, the Saalschutz summation

$$_3\Phi_2 \begin{bmatrix} a, q^{-n}, -q^{-n} \\ a^{1/2}q^{1/2-n}, -a^{1/2}q^{1/2-n} \end{bmatrix}; q \end{bmatrix} = a^{-n} \begin{bmatrix} qa \\ -qa \end{bmatrix}; q^2 \end{bmatrix}_n \tag{2.11a}$$

can be used to establish the C-pair, after replacing q by $q^{1/2}$:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{[a, -q^{1/2}; q^{1/2}]_k}{(aq^{1/2-n}; q)_k} q^{-k/2} = a^{-n}q^{\binom{n}{2}} \begin{bmatrix} q^{1/2}a \\ q^{1/2}/a \end{bmatrix}; q \end{bmatrix}_n, \tag{2.11b}$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (aq^{-1/2-k}; q)_n \begin{bmatrix} q^{1/2}a \\ q^{3/2}/a \end{bmatrix}; q \end{bmatrix}_k q^{\binom{k+1}{2}} a^{-k} = q^{-n/2}[a, -q^{1/2}; q^{1/2}]_n. \tag{2.11c}$$

The last one may be reformulated as a new strange evaluation

$$_2\Phi_1 \begin{bmatrix} aq^{1/2}, q^{-n} \\ a^{-1}q^{3/2-n} \end{bmatrix}; q/a \end{bmatrix} = q^{-n/2} \frac{[a, -q^{1/2}; q^{1/2}]_n}{(aq^{-1/2}; q)_n} \tag{2.11d}$$

which differs from the q-analogues of the formulae due to Vandermonde and Kummer.

We consider

$$\binom{3 \times (-1/3)}{-1} \, _3\Phi_2 \begin{bmatrix} q^{-n/3}, q^{(1-n)/3}, q^{(2-n)/3} \\ q/x, q^{1-n}x \end{bmatrix}; q \end{bmatrix} = \frac{(x^{-1}; q^{1/3})_{2n}}{(q^{1/3}/x; q^{1/3})_n (x^{-1}; q)_n}. \tag{2.12a}$$

The following C-pair can be derived from it, under the base change from q to q^3:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ 3k \end{bmatrix} (x^{-3}q^{-3k}; q^3)_n \begin{bmatrix} q, q^2 \\ q^3x^3, q^3/x^3 ; q^3 \end{bmatrix}_k q^{\binom{3k+1}{2}} = \frac{(x^{-3}; q)_{2n}}{(q/x^3; q)_n},$$

(2.12b)

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{(qx^{-3}; q)_{2k}}{(x^{-3}q^{3-n}; q^3)_k (qx^{-3}; q)_k}$$

$$= \begin{cases} q^{\binom{2}{2}} \begin{bmatrix} q, q^2 \\ x^3, q^3x^{-3} ; q^3 \end{bmatrix}_m, & (n = 3m) \\ 0, & (\text{otherwise}). \end{cases}$$

(2.12c)

The last one can be modified as a strange evaluation

$$\sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q^{3-n}/x^3; q^3)_k} \frac{[q/x^3, q^2/x^3; q^2]_k}{[q, q/x^3; q]_k} q^k = \begin{cases} \begin{bmatrix} q, q^2 \\ x^3, q^3/x^3 ; q^3 \end{bmatrix} & (n = 3m) \\ 0, & (\text{otherwise}) \end{cases}$$

(2.12d)

which has not appeared before.

For the cubic root ω of the unit, the Saalschutz theorem may be specialized as

$$_3\Phi_2 \begin{bmatrix} q^{-n}, \omega q^{-n}, \omega^2 q^{-n} \\ A, q^{1-3n}/A \end{bmatrix} ; q \end{bmatrix} = \frac{(A^3; q^3)_{2n}}{(A; q)_{3n}(A^3; q^3)_n}.$$

(2.13a)

Rewriting it after a base change from q to $q^{1/3}$ gives the following C-pair:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{(q; q)_k}{(q^{1/3}, A, A^{-1}q^{1/3-n}; q^{1/3})_k} q^{-2k/3} = q^{\binom{2}{2}} \frac{(A^3; q)_{2n}}{(A^3, q)_n(A; q^{1/3})_{3n}},$$

(2.13b)

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (A^{-1}q^{-k}; q^{1/3})_n \frac{(A^3; q)_{2k}}{(A^3; q)_k(Aq^{1/3}; q^{1/3})_{3k}} q^{\binom{k+1}{2}}$$

$$= \frac{1-A}{1-Aq^{2n/3}} \frac{(q; q)_n}{(q^{1/3}, A; q^{1/3})_n}.$$

(2.13c)

The last one can be restated as a strange evaluation due to Gessel and Stanton (cf. [11, Eq. (6.28)])

$$\sum_{k=0}^{n} \frac{(q^{-n}, q)_k}{(A^3; q)_k} \frac{(A^3; q)_{2k}}{(Aq^{(1-n)/3}; q^{1/3})_{3k}} \frac{q^k}{(q; q)_k} = \frac{1-A}{1-Aq^{2n/3}} \frac{(q; q)_n}{(q^{1/3}, A, A^{-1}; q^{1/3})_n}$$

(2.13d)

The identity

$$\binom{3 \times (-1/3)}{2 \times (-1/2)} \ _3\Phi_2 \left[\begin{matrix} q^{-n/3}, q^{(1-n)/3}, q^{(2-n)/3} \\ q^{3/4-n/2}, q^{5/4-n/2} \end{matrix} ; q \right] = \frac{(q^{1/12-n/6}; q^{1/3})_{n-1}}{(q^{3/4-n/2}; q)_{n-1}}$$

(2.14a)

yields the following C-pair after the base change from q to q^3 has been performed:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ 3k \end{bmatrix} (q^{3/4-3k}; q^{3/2})_n \begin{bmatrix} q, q^2 \\ q^{9/4}, q^{15/4} ; q^3 \end{bmatrix}_k q^{\binom{3k+1}{2}}$$

$$= \frac{(q^{-3/4}; q^{3/2})_n (q^{1/4-n/2}; q)_{n-1}}{(q^{9/4-3n/2}; q^3)_{n-1}},$$

(2.14b)

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{(q^{3/4}; q^{3/2})_k (q^{1/4-k/2}; q)_k}{(q^{3/4-n}; q^{3/2})_k (q^{9/4-3k/2}; q^3)_k}$$

$$= \begin{cases} q^{\binom{m}{2}} \begin{bmatrix} q, q^2 \\ q^{3/4}, q^{9/4} ; q^3 \end{bmatrix}_m, & (n = 3m) \\ 0, & \text{(otherwise)}. \end{cases}$$

(2.14c)

Again, the last one is a new strange summation which does not fit in the usual q-series relations.

It should be pointed out that most C-pairs displayed above contain some particularly interesting relations among the terminating basic hypergeometric series, because of the free parameters involved there. For example, the C-pair (2.2) implies the equivalence between the q-analogues of Saalschutzian theorem and Dixon's formula:

$$_3\Phi_2 \left[\begin{matrix} aq^n, qa^{1/2}/b, q^{-n} \\ qa^{1/2}, qa/b \end{matrix} ; q \right] = (qa^{1/2}/b)^n \left[\begin{matrix} a^{1/2}, b \\ qa^{1/2}, qa/b \end{matrix} ; q \right]_n,$$

(2.15a)

$$
{}_4\Phi_3\left[\begin{matrix} a, \ -qa^{1/2}, \ b, \ q^{-n} \\ -a^{1/2}, \ qa/b, \ q^{n+1}a \end{matrix}; q^{1+n}a^{1/2}/b\right] = \left[\begin{matrix} qa, \ qa^{1/2}/b \\ qa^{1/2}, \ qa/b \end{matrix}; q\right]_n. \tag{2.15b}
$$

3. Reversal Embedding on q-Saalschutz Theorem

Sometimes, the embedding technique explored in last two sections does not work directly for some terminating series in the form $\Sigma_{k=0}^n w(n, k)$. But the reversal of that series, $\Sigma_{k=0}^n w(n, n - k)$, can be telescoped into the C-pairs. In this case, some new interesting identities are established. We demonstrate a few examples.

The Saalschutz theorem (2.1) may be modified as:

$$
{}_3\Phi_2\left[\begin{matrix} q^{-n/2}, \ q^{-(1+n)/2}, \ q^{-n+1/2}/ab \\ q^{-n+1/2}/a, \ q^{-n+1/2}/b \end{matrix}; q\right] = \frac{[a, \ b; \ q^{1/2}]_n}{[q^{1/2}a, \ q^{1/2}b; \ q]_n} \, q^{\binom{n+1}{2}/2}. \tag{3.1}
$$

Note that

$$
[q^{-(1+n)/2}; q^{1/2}]_{2n-2k} = (-1)^n q^{(2k^2+3k)/2 - (n^2+3n)/4}
$$
$$
* \, (q; q^{1/2})_n (q^{-k-1/2}; q^{1/2})_n / (q; q^{1/2})_{2k}. \tag{3.2}
$$

The reversal of (3.1) can be reformulated as

$$
\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (q^{-k-1/2}; q^{1/2})_n \frac{[q, \ q^{1/2}a, \ q^{1/2}b; \ q]_k}{(q; q^{1/2})_{2k}(q^{1/2}ab; \ q)_k} \, q^{\binom{k+1}{2}}
$$
$$
= \left[\begin{matrix} a, \ b \\ q \end{matrix}; q^{1/2}\right]_n \left[\begin{matrix} q \\ q^{1/2}ab \end{matrix}; q\right]_n \tag{3.3a}
$$

which can be telescoped into (0.1b) and gives rise to the dual relation

$$
\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1 - q^{-k/2}}{(q^{-n-1/2}; \ q^{1/2})_{k+1}} \left[\begin{matrix} a, \ b \\ q \end{matrix}; q^{1/2}\right]_k \left[\begin{matrix} q \\ q^{1/2}ab \end{matrix}; q\right]_k
$$
$$
= q^{\binom{n+1}{2}} \frac{[q, \ q^{1/2}a, \ q^{1/2}b; \ q]_n}{(q; \ q^{1/2})_{2n}(q^{1/2}ab; \ q)_n}. \tag{3.3b}
$$

Replacing the base q by q^2, we can restate this relation as the q-analog of Watson's formula (cf. [13] and [15])

$$
{}_4\Phi_3\left[\begin{matrix} a, \ b, \ q^{-n}, \ -q^{-n} \\ q^{-2n}, \ (qab)^{1/2}, \ -(qab)^{1/2} \end{matrix}; q\right] = \left[\begin{matrix} qa, \ qb \\ q, \ qab \end{matrix}; q^2\right]_n. \tag{3.4}
$$

Similarly, consider the Saalschutzian summation for $t = 0$ and 1:

$$_3\Phi_2 \left[\begin{array}{c} q^{-(1+2n+t)/3},\ q^{-(2n+t)/3},\ q^{-(-1+2n+t)/3} \\ q^{-n}x,\ q^{1-n-t}/x \end{array} ; q \right]$$

$$= \frac{[q^{(2t-1)/3}x,\ q^{(2-t)/3}/x;\ q^{1/3}]_n}{[q^t x,\ q/x;\ q]_n}\, q^{n(t+n)/3}. \tag{3.5}$$

Noticing that

$$[q^{-y-2n/3};\ q^{1/3}]_{3n-3k} = (-1)^k\, q^{3ky-2ny+(k^2+k/3)-(2n^2+n)/3}$$

$$* (q^{y+1/3};\ q^{1/3})_{2n}(q^{-y-k};\ q^{1/3})_n/(q^{1/3+y};\ q^{1/3})_{3k} \tag{3.6}$$

we can rewrite the reversal of (3.5) in the form

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (q^{-1/3-t/3-k};\ q^{1/3})_n\, \frac{[q,\ qx^{-1},\ q^t x;\ q]_k}{(q^{(2+t)/3};\ q^{1/3})_{3k}}\, q^{\binom{k+1}{2}}$$

$$= \frac{[q^{(2t-1)/3}x,\ q^{(2-t)/3}x^{-1};\ q^{1/3}]_n}{(q^{(2+t)/3};\ q^{1/3})_{2n}}\, (q;\ q)_n \tag{3.7a}$$

which can be telescoped into (0.1b) and brings about the dual relation

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-q^{-(1+t+2k)/3}}{(q^{-n-(1+t)/3};\ q^{1/3})_{k+1}}\, \frac{[q^{(2t-1)/3}x,\ q^{(2-t)/3}x^{-1};\ q^{1/3}]_k}{(q^{(2+t)/3};\ q^{1/3})_{2k}}\, (q;\ q)_k$$

$$= q^{\binom{n+1}{2}} \frac{[q,\ qx^{-1},\ q^t x;\ q]_n}{(q^{(2+t)/3};\ q^{1/3})_{3n}}. \tag{3.7b}$$

Replacing the base q by q^3, we can reformulate this summation in terms of q-hypergeometric series

$$_6\Phi_5 \left[\begin{array}{c} q,\ q^{2t-1}x,\ q^{2-t}x^{-1},\ q^{-n},\ \omega q^{-n},\ \omega^2 q^{-n} \\ q^{(1+t)/2},\ -q^{(1+t)/2},\ q^{(2+t)/2},\ -q^{(2+t)/2},\ q^{-t-3n} \end{array} ; q \right]$$

$$= \left[\begin{array}{c} q^3,\ q^3 x^{-1},\ q^{3t}x \\ q^{1+t},\ q^{2+t},\ q^{3+t} \end{array} ; q^3 \right]_n \tag{3.8}$$

which is the unified formulation of the q-analogues of the strange hypergeometric evaluations due to Gosper (cf. [10, Eqs. (1.4–1.5)] and [11, Eqs. (6.20,

6.24)]):

$$
{}_5\Phi_4\left[\begin{array}{c} q^{-1}x,\ q^2x^{-1},\ q^{-n},\ \omega q^{-n},\ \omega^2 q^{-n} \\ -q,\ q^{1/2},\ -q^{1/2},\ q^{-3n} \end{array}; q\right] = \left[\begin{array}{c} x,\ q^3x^{-1} \\ q,\ q^2 \end{array}; q^3\right]_n, \tag{3.9a}
$$

$$
{}_5\Phi_4\left[\begin{array}{c} qx,\ qx^{-1},\ q^{-n},\ \omega q^{-n},\ \omega^2 q^{-n} \\ -q,\ q^{3/2},\ -q^{3/2},\ q^{-1-3n} \end{array}; q\right] = \left[\begin{array}{c} q^3x^{-1},\ q^3x \\ q^2,\ q^4 \end{array}; q^3\right]_n. \tag{3.9b}
$$

4. *C*-pairs from Jackson's *q*-Dougall-Dixon theorem

Jackson's *q*-analogue of the Dougall-Dixon theorem (2.2d) may be rewritten in a symmetric manner

$$
{}_6\Phi_5\left[\begin{array}{c} a,\ qa^{1/2},\ -qa^{1/2},\ b,\ c,\ d \\ a^{1/2},\ -a^{1/2}, qa/b,\ qa/c,\ qa/d \end{array}; qa/bcd\right] = \left[\begin{array}{c} qa,\ qa/bc,\ qa/bd,\ qa/cd \\ qa/bcd,\ qa/b,\ qa/c,\ qa/d \end{array}; q\right]_\infty \tag{4.1}
$$

which is valid even for the non-terminating case (this fact will be confirmed in the next section). Similar to section 2, the *C*-pairs associated with this theorem are demonstrated as follows.

First, one form of the *q*-Dougall-Dixon summation

$$
{}_6\Phi_5\left[\begin{array}{c} a,\ qa^{1/2},\ -qa^{1/2},\ q^{1/2}a/c^2,\ q^{(1-n)/2},\ q^{-n/2} \\ a^{1/2},\ -a^{1/2},\ q^{1/2}c^2,\ q^{(1+n)/2}a,\ q^{1+n/2}a \end{array}; q^nc^2\right]
$$
$$
= \left[\begin{array}{c} q^{1/2}a \\ c^2 \end{array}; q^{1/2}\right]_n\left[\begin{array}{c} c^2 \\ q^{1/2}a \end{array}; q\right]_n \tag{4.2a}
$$

may be reformulated after a base change from q to q^2 and generates the C'-pair:

$$
\sum_{k=0}^{n}\begin{bmatrix} n \\ 2k \end{bmatrix}\frac{1-aq^{4k}}{(aq^n;\ q)_{2k+1}}\left[\begin{array}{c} q,\ a,\ qa/c^2 \\ qc^2 \end{array}; q^2\right]_k c^{2k}q^{\binom{2k}{2})} = \left[\begin{array}{c} a \\ c^2 \end{array}; q\right]_n\left[\begin{array}{c} c^2 \\ qa \end{array}; q^2\right]_n, \tag{4.2b}
$$

$$
\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}q^{\binom{n-k}{2}}(aq^k;\ q)_n\left[\begin{array}{c} a \\ c^2 \end{array}; q\right]_k\left[\begin{array}{c} c^2 \\ qa \end{array}; q^2\right]_k
$$
$$
= \begin{cases} 0, & (n-\text{odd}) \\ q^{\binom{2}{2}}c^n\left[\begin{array}{c} q,\ a,\ qa/c^2 \\ qc^2 \end{array}; q^2\right]_m, & (n=2m). \end{cases} \tag{4.2c}
$$

The last relation can be restated as the q-analogue of Watson's formula (cf. [2] or [3, Eq. (4.6)])

$$
{}_4\Phi_3\left[\begin{array}{c} q^{-n},\ aq^n,\ c,\ -c \\ c^2,\ (qa)^{1/2},\ -(qa)^{1/2} \end{array}; q\right] = \begin{cases} 0, & (n-\text{odd}) \\ c^n\left[\begin{array}{c} q,\ qa/c^2 \\ qa,\ qc^2 \end{array}; q^2\right]_m, & (n=2m) \end{cases}
\tag{4.2d}
$$

which is equivalent to (2.8d) under parameter replacement.

Again, from (4.1) we have

$$
{}_6\Phi_5\left[\begin{array}{c} -c,\ q(-c)^{1/2},\ -q(-c)^{1/2},\ c/a,\ q^{-n},\ -q^{-n} \\ (-c)^{1/2},\ -(-c)^{1/2},\ -qa,\ -cq^{n+1},\ cq^{n+1} \end{array}; aq^{2n+1}\right]
$$

$$
= \left[\begin{array}{c} qa,\ q^2a,\ q^2c^2 \\ qc,\ q^2c,\ q^2a^2 \end{array}; q^2\right]_n.
\tag{4.3a}
$$

Its modification in accordance with the replacement of q by $q^{1/2}$, leads to the following C'-pair:

$$
\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}\frac{1-c^2q^{2k}}{(c^2q^n;\ q)_{k+1}}\frac{1-c}{1-cq^k}\left[\begin{array}{c} -q^{1/2},\ c/a,\ -c \\ -q^{1/2}a \end{array}; q^{1/2}\right]_k (q^{1/2}a)^k q^{\binom{k}{2}}
$$

$$
= \left[\begin{array}{c} aq^{1/2},\ aq,\ c^2 \\ cq^{1/2},\ cq,\ a^2q \end{array}; q\right]_n,
\tag{4.3b}
$$

$$
\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}q^{\binom{n-k}{2}}(c^2q^k;\ q)_n\left[\begin{array}{c} q^{1/2}a,\ qa,\ c^2 \\ q^{1/2}c,\ qc,\ qa^2 \end{array}; q\right]_k
$$

$$
= \frac{1-c}{1-cq^n}\left[\begin{array}{c} -q^{1/2},\ c/a,\ -c \\ -q^{1/2}a \end{array}; q^{1/2}\right]_n (q^{1/2}a)^n q^{\binom{n}{2}}.
\tag{4.3c}
$$

The latter may be reformulated as (cf. [3, Eq. (4.3)] or [11, Eq. (4.22)])

$$
{}_4\Phi_3\left[\begin{array}{c} q^{-n},\ aq^{1/2},\ aq,\ c^2q^n \\ cq^{1/2},\ cq,\ a^2q \end{array}; q\right] = (aq^{1/2})^n\frac{1-c}{1-cq^n}\left[\begin{array}{c} -q^{1/2},\ c/a \\ -q^{1/2}a,\ c \end{array}; q^{1/2}\right]_n
\tag{4.3d}
$$

which is in turn equivalent to (2.9d).

Next, similar to (4.2a), the q-Dougall-Dixon theorem (4.1) may also be specialized as

$$
{}_6\Phi_5\left[\begin{array}{c} a,\ qa^{1/2},\ -qa^{1/2},\ q^{-n/3},\ q^{(1-n)/3},\ q^{(2-n)/3} \\ a^{1/2},\ -a^{1/2},\ q^{1+n/3}a,\ q^{(2+n)/3}a,\ q^{(1+n)/3}a \end{array}; q^na\right] = \frac{(a;\ q)_n(q^{1/3}a;\ q^{1/3})_n}{(a;\ q^{1/3})_{2n}}.
\tag{4.4a}
$$

After the base change from q to q^3 it can be used to produce the C'-pair

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ 3k \end{bmatrix} \frac{1 - aq^{6k}}{(aq^n; q)_{3k+1}} [q, q^2, a; q^3]_k q^{\binom{3k}{2}} a^k = \frac{(a; q^3)_n(a; q)_n}{(a; q)_{2n}}, \qquad (4.4b)$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (aq^k; q)_n \frac{(a; q^3)_k(a; q)_k}{(a; q)_{2k}}$$

$$= \begin{cases} q^{\binom{m}{2}} a^m(q, q^2, a; q^3)_m, & (n = 3m) \\ 0, & \text{(otherwise)}. \end{cases} \qquad (4.4c)$$

The last relation is exactly the Andrews' strange evaluation (cf. [3, Eq. (4.7)] and [11, Eq. (4.32)])

$$\sum_{k=0}^{n} \frac{(a; q^3)_k}{(q; q)_k} \frac{(q^{-n}, aq^n; q)_k}{(a, aq; q^2)_k} q^k = \begin{cases} a^m \begin{bmatrix} q, q^2 \\ aq, aq^2 \end{bmatrix} ; q^3 \end{bmatrix}_m, & (n = 3m) \\ 0, & \text{(otherwise)}. \end{cases} \qquad (4.4d)$$

Finally, it is easy to derive the following evaluation, from (4.1)

$${}_6\Phi_5 \begin{bmatrix} A, qA^{1/2}, -qA^{1/2}, q^{-n}, \omega q^{-n}, \omega^2 q^{-n} \\ A^{1/2}, -A^{1/2}, Aq^{1+n}, \omega^2 Aq^{1+n}, \omega Aq^{1+n} \end{bmatrix} ; Aq^{3n+1} \end{bmatrix} = \frac{(qA; q)_{3n}(q^3 A^3; q^3)_n}{(q^3 A^3; q^3)_{2n}}. \qquad (4.5a)$$

Replacing q and A in the above by $q^{1/3}$ and $A^{1/3}$, respectively, results in the following C'-pair:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1 - A^{1/3}q^{2k/3}}{(Aq^n; q)_{k+1}} \frac{(A^{1/3}; q^{1/3})_k}{(q^{1/3}; q^{1/3})_k} (q; q)_k (Aq)^{k/3} q^{\binom{k}{2}}$$

$$= \frac{(A; q)_n(A^{1/3}; q^{1/3})_{3n+1}}{(A; q)_{2n+1}}, \qquad (4.5b)$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}(Aq^k; q)_n \frac{(A^{1/3}; q^{1/3})_{3k+1}}{(A; q)_{2k+1}} (A; q)_k$$

$$= \frac{1 - A^{1/3}q^{2n/3}}{1 - Aq^{2n}} \frac{(q; q)_n(A^{1/3}; q^{1/3})_n}{(q^{1/3}; q^{1/3})_n} (Aq)^{n/3} q^{\binom{n}{2}}. \qquad (4.5c)$$

The last relation reads as another strange q-series identity of Andrews (cf. [3,

Eq. (4.5)] and [11, Eq. (4.26)])

$$
{}_5\Phi_4\left[\begin{array}{c} q^{-n},\ Aq^n,\ A^{1/3}q^{1/3},\ A^{1/3}q^{2/3},\ A^{1/3}q \\ A^{1/2}q,\ -A^{1/2}q,\ A^{1/2}q^{1/2},\ -A^{1/2}q^{1/2} \end{array};q\right]
$$

$$
= \frac{1-A}{1-A^{1/3}}\ \frac{1-A^{1/3}q^{2n/3}}{1-Aq^{2n}}\ \frac{(q;\,q)_n(A^{1/3};\,q^{1/3})_n}{(q^{1/3};\,q^{1/3})_n(A;\,q)_n}\ (Aq)^{n/3}. \tag{4.5d}
$$

5. Compositions Based on the C-pairs

For a given terminating summation

$$
U(n) = \sum_{k=0}^{n} c(n,\,k)V(k) \tag{5.1a}
$$

consider the formal composition (series-rearrangement)

$$
\sum_{k=0}^{n} a(n,\,k)U(v) = \sum_{i=0}^{n} V(i)\sum_{k=i}^{n} a(n,\,k)c(k,\,i) \tag{5.1b}
$$

which follows from the exchange between summation indices. If the original series (5.1a) can be embedded in one member of the *C*-pair, then another member would provide the necessary reference for manipulating the series-transformation. Here we demonstrate a triple of examples of this kind of composition.

First, consider the formal series composition

$$
\sum_{k=0}^{n}\left[\begin{array}{c} a,\ qa^{1/2},\ -qa^{1/2},\ d,\ e,\ q^{-n} \\ q,\ a^{1/2},\ -a^{1/2},\ qa/d,\ qa/e,\ q^{n+1}a \end{array};q\right]_k (q^{1+n}a/de)^k\ {}_3\Phi_2\left[\begin{array}{c} aq^k,\ qa/bc,\ q^{-k} \\ qa/b,\ qa/c \end{array};q\right]
$$

$$
= \sum_{i=0}^{n} \frac{(qa)_{2i}}{(q)_i}\left[\begin{array}{c} qa/bc,\ d,\ e,\ q^{-n} \\ qa/b,\ qa/c,\ qa/d,\ qa/e,\ q^{n+1}a \end{array};q\right]_i (-1)^i q^{-\binom{i}{2}}(q^{1+n}a/de)^i
$$

$$
*\ {}_6\Phi_5\left[\begin{array}{c} aq^{2i},\ q^{1+i}a^{1/2},\ -q^{1+i}a^{1/2},\ q^i d,\ q^i e,\ q^{-n+i} \\ q^i a^{1/2},\ -q^i a^{1/2},\ q^{1+i}a/d,\ q^{1+i}a/e,\ q^{1+n+i}a \end{array};q^{1+n-i}a/de\right].
$$

By means of *C*-pair (2.2), this relation reads as *Watson's q-analogue of Whipple's transform* (e.g., cf. [4, 9, 14]):

$$
{}_8\Phi_7\left[\begin{array}{c} a,\ qa^{1/2},\ -qa^{1/2},\ b,\ c,\ d,\ e,\ q^{-n} \\ a^{1/2},\ -a^{1/2},\ qa/b,\ qa/c,\ qa/d,\ qa/e,\ q^{n+1}a \end{array};q^{2+n}a^2/bcde\right]
$$

$$
= \left[\begin{array}{c} qa,\ qa/de \\ qa/d,\ qa/e \end{array};q\right]_n\ {}_4\Phi_3\left[\begin{array}{c} q^{-n},\ d,\ e,\ qa/bc \\ qa/b,\ qa/c,\ de/q^{1+n}a \end{array};q\right]. \tag{5.2}
$$

When the $_4\Phi_3$ in the above reduces to a $_3\Phi_2$ series, it can be evaluated by the q-Saalschutz theorem. In this case we get a balanced summation formula: *Jackson's q-analogue of Dougall's formula* (see [4, 9, 14] also)

$$
_8\Phi_7 \left[\begin{array}{c} a, qa^{1/2}, -qa^{1/2}, b, c, d, e, q^{-n} \\ a^{1/2}, -a^{1/2}, qa/b, qa/c, qa/d, qa/e, q^{n+1}a \end{array} ; q \right] = \left[\begin{array}{c} qa, qa/bc, qa/bd, qa/cd \\ qa/bcd, qa/b, qa/c, qa/d \end{array} ; q \right]_n
$$

(5.3)

provided that $q^{1+n}a^2 = bcde$. When n tends to infinity, it admits the non-terminating version of (4.1).

Next, based on C-pair (2.6), we can carry out the operation

$$
\sum_{k=0}^{n} \frac{1 - aq^{3k/2}}{1 - a} q^{-\binom{k}{2}/2} a^{-k} \frac{(a; q^{1/2})_k}{(q; q)_k} \frac{[q^{-n}, a/d, q^{1/2+n}ad; q]_k}{[q^{1/2}d, q^{-n}/d, q^{1/2+n}a; q^{1/2}]_k}
$$

$$
\times {}_3\Phi_2 \left[\begin{array}{c} q^{-k}, q^{k/2}a, q^{(1+k)/2}a \\ q^{1/2}ab, qa/b \end{array} ; q \right]
$$

(5.4a)

$$
= \sum_{i=0}^{n} (-1)^i q^{\binom{i+1}{2}} \left[\begin{array}{c} a, q^{1/2}a \\ q, q^{1/2}ab, qa/b \end{array} ; q \right]_i \sum_{k=i}^{n} \frac{1 - aq^{3k/2}}{1 - a} q^{-ki-\binom{k}{2}/2} a^{-k}
$$

$$
\times \frac{(aq^i; q^{1/2})_k}{(q; q)_{k-i}} \frac{[q^{-n}, a/d, q^{1/2+n}ad; q]_k}{[q^{1/2}d, q^{-n}/d, q^{1/2+n}a; q^{1/2}]_k}.
$$

(5.4b)

The inner sum on the last line can be rewritten, under the variable substitution $k = n - r$, as

$$
q^{-ni-\binom{n}{2}} a^{-n} \frac{1 - aq^{3n/2}}{1 - a} \frac{(aq^i; q^{1/2})_n}{(q; q)_{n-i}} \frac{[q^{-n}, a/d, q^{1/2+n}ad; q]_n}{[q^{1/2}d, q^{-n}/d, q^{1/2+n}a; q^{1/2}]_n}
$$

$$
* \sum_{r=0}^{n-i} \frac{1 - a^{-1}q^{3(r-n)/2}}{1 - a^{-1}q^{-3n/2}} \frac{(q^{-n+i}; q)_r}{(a^{-1}q^{-i-(n-1)/2}; q^{1/2})_r}
$$

$$
* \frac{[q^{-n/2}d^{-1}, q^{(1+n)/2}d, q^{-3n/2}a^{-1}; q^{1/2}]_r}{[q, q^{1-n}d/a, q^{1/2-2n}a^{-1}d^{-1}; q]_r} q^{-ri+\binom{r}{2}/2} a^{-r}
$$

(5.4c)

where in view of (2.6d) the last summation is equal to

$$
q^{\binom{n}{2}/2} \left[\begin{array}{c} q^{1/2-3n/2}a^{-1}, q^{1-3n/2}a^{-1} \\ q^{1-n}d/a, q^{1/2-2n}a^{-1}d^{-1} \end{array} ; q \right]_{n-i}.
$$

(5.4d)

By means of the substitution of (5.4c –d) into (5.4b), the latter can be evaluated as

$$
(d/a)^n \begin{bmatrix} q^{1/2}a \\ q^{1/2}d \end{bmatrix}_{2n}; q^{1/2} \Big]_{2n} {}_3\Phi_2 \begin{bmatrix} q^{-n}, \ a/d, \ q^{n+1/2}ad \\ qa/b, \ q^{1/2}ab \end{bmatrix}; q \Big]. \tag{5.4e}
$$

After equating (5.4a) and (5.4e), and simplifying with the help of Saalschutz's theorem, we can establish the strange q-hypergeometric formula due to Gessel and Stanton (cf. [10, Eq. (1.7)] and [11, Eq. (1.4)])

$$
\sum_{k=0}^{n} \begin{bmatrix} q^{-n}, \ a/d, \ q^{1/2+n}ad \\ q, \ q^{1/2}ab, \ qa/b \end{bmatrix}; q \Big]_k \frac{1-aq^{3k/2}}{1-a} \begin{bmatrix} a, \ b, \ q^{1/2}/b \\ q^{1/2}, \ q^{-n}/d, \ q^{1/2+n}a \end{bmatrix}; q^{1/2} \Big]_k q^{k/2}
$$

$$
= \begin{bmatrix} q^{1/2}a \\ q^{1/2}d \end{bmatrix}; q^{1/2} \Big]_{2n} \begin{bmatrix} q^{1/2}bd, \ qd/b \\ q^{1/2}ab, \ qa/b \end{bmatrix}; q \Big]_n. \tag{5.4}
$$

Another strange evaluation due to Gessel and Stanton (cf. [10, Eq. (1.8)] and [11, Eq. (6.14)])

$$
\sum_{k=0}^{n} \begin{bmatrix} a, \ b, \ q^{1/2}a/b \\ q^{1/2}ad, \ q^{1+n/2}a, \ q^{1-n/2}/d \end{bmatrix}; q \Big]_k \frac{1-aq^{3k/2}}{1-a} \begin{bmatrix} q^{1/2}/d, \ q^{n/2}ad, \ q^{-n/2} \\ q^{1/2}, \ b, \ q^{1/2}a/b \end{bmatrix}; q^{1/2} \Big]_k q^{k/2}
$$

$$
= \begin{cases} 0, & (n-\text{odd}) \\ \begin{bmatrix} q^{1/2}, \ qa, \ bd, \ q^{1/2}ad/b \\ q^{1/2}b, \ qa/b, \ d, \ q^{1/2}ad \end{bmatrix}; q \Big]_m, & (n = 2m) \end{cases} \tag{5.5}
$$

can be revisited through the C-pair (2.7) in a similar way.

In fact, the left hand side of (5.5) can be restated, according to (2.7a), in the form of (5.1)

$$
\sum_{k=0}^{n} \frac{1-aq^{3k/2}}{1-a} q^{\binom{k+1}{2}/2} \frac{(a; q)_k}{(q^{1/2}; q^{1/2})_k} \frac{[q^{-n/2}, \ q^{1/2}/d, \ q^{n/2}ad; \ q^{1/2}]_k}{[q^{1+n/2}d, \ q^{1/2}ad, \ q^{1-n/2}/d; \ q]_k}
$$

$$
\times {}_3\Phi_2 \begin{bmatrix} aq^k, \ q^{(1-k)/2}, \ q^{-k/2} \\ q^{1/2}b, \ qa/b \end{bmatrix}; q \Big] \tag{5.5a}
$$

$$
= \sum_{i=0}^{n} q^{\binom{2i+1}{2}/2} \begin{bmatrix} a \\ q, \ q^{1/2}b, \ qa/b \end{bmatrix}; q \Big]_i \sum_{k=2i}^{n} \frac{1-aq^{3k/2}}{1-a} q^{-ki-\binom{k+1}{2}/2}
$$

$$
\times \frac{(aq^i; q)_k}{(q^{1/2}; q^{1/2})_{k-2i}} \frac{[q^{-n/2}, \ q^{1/2}/d, \ q^{n/2}ad; \ q^{1/2}]_k}{[q^{1+n/2}a, \ q^{1/2}ad, \ q^{1-n/2}/d; \ q]_k}. \tag{5.5b}
$$

The inner sum on the last line may be rewritten, under the variable substitution $k = n - r$, as

$$q^{-ni-\binom{n+1}{2}} \frac{1 - aq^{3n/2}}{1 - a} \frac{(aq^i; q)_n}{(q^{1/2}; q^{1/2})_{n-2i}} \frac{[q^{-n/2}, q^{1/2}/d, q^{n/2}ad; q^{1/2}]_n}{[q^{1+n/2}a, q^{1/2}ad, q^{1-n/2}/d; q]_n}$$

$$* \sum_{r=0}^{n-2i} \frac{1 - a^{-1}q^{3(r-n)/2}}{1 - a^{-1}q^{-3n/2}} \frac{(q^{i-n/2}; q^{1/2})_r}{(a^{-1}q^{1-n-i}; q)_r}$$

$$* \frac{[q^{-3n/2}a^{-1}, q^{-n/2}d, q^{1/2-n}a^{-1}d^{-1}; q]_r}{[q^{1/2}, q^{-n/2}d, q^{1/2-n}a^{-1}d^{-1}; q^{1/2}]_r} q^{-ri-\binom{n-r+1}{2}/2} \qquad (5.5c)$$

where in view of (2.7d) the last summation can be evaluated as

$$\begin{cases} 0, & (n - \text{odd}) \\ q^{-\binom{n+1}{2}/2} \left[\begin{matrix} q^{1/2}, q^{1-3m}a^{-1} \\ q^{1/2-m}d, q^{1-2m}a^{-1}d^{-1} \end{matrix} ; q \right]_{m-i} & (n = 2m). \end{cases} \qquad (5.5d)$$

By means of the substitution of (5.5c–d) into (5.5b), the latter equals

$$q^{-m/2}d^m \left[\begin{matrix} q^{1/2}, qa \\ d, q^{1/2}ad \end{matrix} ; q \right]_m {}_3\Phi_2 \left[\begin{matrix} q^{-m}, q^{1/2}d^{-1}, q^mad \\ q^{1/2}b, qa/b \end{matrix} ; q \right] \qquad (5.5e)$$

for $n = 2m$, and vanishes for n being odd. In accordance with Saalschutz's theorem, the last relation reduces to the right member of (5.5).

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