COMPOSITIO MATHEMATICA

VALERY ALEXEEV

General elephants of ℚ-Fano 3-folds

General elephants of $\mathbb{Q}$-Fano 3-folds

VALERY ALEXEEV

Department of Mathematics, University of Utah, Salt Lake City, Utah, 84112.

Let X be a normal 3-fold and $\pi\colon X \to S$ a proper morphism with $\pi_* \mathcal{O}_X = \mathcal{O}_S$ such that the anticanonical divisor $-K_X$ is relatively π-ample. A general elephant (or simply a g.e.) in this article means a general element of a linear system $L \subset |-K_X + \pi^*(d)|$, where $d \in \mathrm{Div}(S)$ is "an ample enough" Cartier divisor on S. We start by listing several well-known results on general elephants in various situations.

0.1. THEOREM (Shokurov, [Sh1]; Reid, [R1]). *Let X be a Fano 3-fold with Gorenstein canonical singularities. Then a g.e. $D \in |-K_X|$ is a $K3$ surface with Du Val singularities only.*

0.2. THEOREM (Reid, [R2]). *Let $P \in X$ be a germ of a terminal singularity. Then a g.e. $D \in |-K_X|$ has Du Val singularity.*

0.3. THEOREM (Kollár-Mori, [KoMo], Theorem 1.7). *Assume that X has only terminal singularities and that $\pi\colon X \to S$ is a contraction of an irreducible curve. Then a g.e. $D \in |-K_X + \pi^*(d)|$ has only Du Val singularities.*

0.4. In this paper we consider the case of $\mathbb{Q}$-Fano 3-folds, i.e. 3-folds with $\mathbb{Q}$-factorial terminal singularities, ample anticanonical divisor and $\rho(X) = 1$. Although Theorems 0.1–0.4 suggest that a g.e. in this case should also have only Du Val singularities, we do not prove this. Instead, we show that if a $\mathbb{Q}$-Fano 3-fold X has a big anticanonical system, in the sense that the image of the rational map $\varphi_{|-K|}$ has dimension 3, then X is birational to some $\mathbb{Q}$-Fano on which a g.e. has only Du Val singularities. We also prove that in this case X is birational to a Gorenstein Fano variety with canonical singularities and a free anticanonical system (Theorems 4.3 and 4.8).

0.5. Chapters 1–3 contain the main technique used later for proofs.

In Section 1 we introduce a new version of the Minimal Model Program appropriate for linear systems and prove all the necessary statements.

In Section 2 we prove some basic results on the behavior of the anticanonical system on $\mathbb{Q}$-Fano and Del Pezzo fibration.

Section 3 contains a generalization of the well-known "double projection" of Fano to the cases of $\mathbb{Q}$-Fano and Del Pezzo fibration.

In Section 5 we give another application of the main technique to partial resolutions of 3-dimensional terminal singularities.

0.6. NOTATION AND TERMINOLOGY. All varieties are defined over an algebraically closed field of characteristic zero. For a normal variety X we don't distinguish between a Weil divisor D and the corresponding reflexive sheaf $O_X(D)$. K and $-c_1$ denote the canonical Weil divisor on X.

When we say that a variety X has, say, canonical singularities, it means that singularities are not worse than that, so X can be actually nonsingular. Also we use standard notations and definitions, widely accepted in papers on the Minimal Model Program. For example, the shorthand *nef* for a divisor D as usual means that $DC \geqslant 0$ for any curve $C \subset X$.

In the main part of the paper all varieties are supposed to be 3-dimensional, although many definitions and theorems in Section 1 are valid for any dimension.

1. Categories $\mathbb{Q}$LSt and $\mathbb{Q}$LSc

The Minimal Model Program (MMP) has been developed over the last 15 last years by efforts of many mathematicians, see [KMM], [Ko2], [M2], [Sh3], [W2] for an introduction and references. There are two well known variants of MMP: the terminal/canonical and the log terminal/log canonical.

In both versions one first proves the basic theorems: the Cone theorem, the Contraction theorem, the Flip theorems I and II. If one has these theorems for the specific category (for example in the terminal/canonical version for 3-folds) one can then use a simple algorithm to get the main result: the existence of the minimal model of a variety in the chosen category.

Below we briefly recall MMP in the mentioned versions, as well as generalizations due to V. V. Shokurov. After this we introduce a new variant of MMP: the categories $\mathbb{Q}$LSt and $\mathbb{Q}$LSc and prove all the basic theorems for these categories in dimension three.

We use [KMM], [Sh2] and [Sh3] as the sources for references for technical results.

Reminder of the minimal model program

1.1. DEFINITION. A $\mathbb{Q}$-divisor on a normal variety X is a formal sum $D = \Sigma\, d_i D_i$, where D_i are distinct Weil divisors, i.e. algebraic subvarieties of codimension 1, and $d_i \in \mathbb{Q}$.

1.2. DEFINITION. A ℚ-divisor $D = \Sigma d_i D_i$ is said to be a ℚ-Cartier divisor of some $N > 0$ a multiple $ND = \Sigma N d_i D_i$ is a Cartier divisor, i.e. $N d_i$ are integers and ND is locally defined by one equation.

1.3. DEFINITION. A log divisor is a ℚ-Cartier divisor of the form $K + B$, $B = \Sigma b_i B_i$, where K is the canonical Weil divisor and B_i are distinct Weil divisors.

1.4. DEFINITION. For any birational morphism $f: Y \to X$ we can define a divisor $K_Y + B_Y$, where

$$B_Y = \sum b_i f^{-1}(B_i) + \sum \beta_j E_j,$$

$f^{-1}(B_i)$ are strict transforms of B_i, E_j are exceptional divisors of the morphism f.

Here the coefficients b_i and β_j can be chosen in different ways. Two standard choices are:

terminal/canonical   all $b_i = 0$, i.e. $B = \varnothing$ and all $\beta_j = 0$.

log terminal/log canonical $0 \leqslant b_i \leqslant 1$ and $\beta_j = 1$ for any i, j.

1.5. DEFINITION. Let $f: Y \to X$ be a *good* resolution of singularities, i.e. Y is nonsigular and the strict transforms of B_i and E_j are nonsingular and cross normally. We have the usual formula for the pull back of the log divisor

$$K_Y + B_Y = f^*(K + B) + \sum \gamma_j E_j, \quad \gamma_j \in \mathbb{Q}.$$

The coefficients γ_j are called *discrepancies* (resp. *log discrepancies*).

The divisor $K + B$ is terminal (resp. canonical) and the pair X has terminal (resp. canonical) singularities if $\gamma_j > 0$ (resp. $\gamma_j \geqslant 0$) for any j with the choice of coefficients b_i and β_j as in the terminal/canonical version.

The divisor $K + B$ is log terminal (resp. log canonical) and the pair (X, B) has log terminal (resp. log canonical) singularities if $\gamma_j > 0$ (resp. $\gamma_j \geqslant 0$) for any j with the choice of coefficients b_i and β_j as in the log terminal/log canonical version.

Obviously, $K + B$ terminal or canonical implies log terminal.

1.6. REMARK. In the terminal/canonical or in the log terminal/log canonical version under the additional assumption that all $b_i < 1$, these definitions do not depend on the choice of the good resolution f.

1.7. REMARK. In the case when some b_i equal 1 the definition of log

canonical $K + B$ also does not depend on a choice of f, but one has two different definitions of log terminal:

$K + B$ is said to be log terminal if there is *at least one* good resolution with $\gamma_j > 0$;

$K + B$ is said to be Kawamata log terminal if all $\gamma_j > 0$ *for any* good resolution;

Also in this case some of the basic theorems do not hold without some extra conditions. The options include assuming that X is $\mathbb{Q}$-factorial or that B is LSEPD (locally supports an effective principal divisor), see [Sh3] for further details.

The *Cone Theorem* states that if a divisor $K + B$ is terminal/canonical or log terminal/log canonical + extra conditions then the part of Kleiman-Mori cone of curves

$$\overline{NE(X)} \cap \{C \,|\, C \cdot (K + B) < 0\}$$

is locally finitely generated. The 1-dimensional faces of this part of the cone are called extremal rays. Hence, if $K + B$ is not nef, one has at least one extremal ray.

The *Contraction Theorem* states that under the above same assumptions every extremal ray R gives a contraction morphism $\mathrm{cont}_R \colon X \to Z$ and one of the following holds:

(i) (Mori fiber space, see Definition 2.1) $\dim Z < \dim X$ and the restriction of $-(K_X + B)$ to the general fiber is ample;

(ii) (divisorial type) $\mathrm{cont}_R \colon X \to Z$ is a contraction of a single prime divisor;

(iii) (flipping type) the exceptional set of cont_R has codimension $\geqslant 2$.

Flip conjecture I. In the situation (iii) there is a birational map, called a flip,

$$X \overset{t}{\dashrightarrow} X^+$$

with f and f^+ mapping to Z.
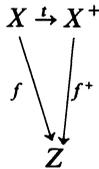
such that the exceptional set of f^+ has codimension $\geqslant 2$, $\rho(X^+/Z) = \rho(X/Z) = 1$ and $K_X + B^+$ is f^+-ample, where B^+ is the image of B.

Flip conjecture II. A sequence of flips terminates.

Given that the definitions and theorems appropriate to the chosen category are established, the Minimal Model Program (MMP) works as follows:

Minimal model program. Start with X and $K + B$ which is terminal/canonical or log terminal/log canonical + extra assumptions. Or, if $K + B$ is neither

of those, first find a good resolution and start with $K_Y + B_Y$.

If $K + B$ is not nef, find an extremal ray and the corresponding contraction morphism. If cont_R is of type (iii), make the flip. Proceed until you get either a variety with $K + B$ nef (a minimal model) or a contraction of type (i), i.e. a (log) Mori fiber space.

In either case of a divisorial contraction or a flip one stays in the chosen category. This follows from the Negativity of Contractions ([Sh3], 1.1) and from the Increasing of Discrepancies under a Flip ([Sh2], [KMM] 5-1-11).

$\mathbb{Q}$*-factoriality.* The variety X is $\mathbb{Q}$-factorial if for any Weil divisor D, some multiple ND is a Cartier divisor, i.e. on X all $\mathbb{Q}$-divisors are $\mathbb{Q}$-Cartier divisors. If we start with a $\mathbb{Q}$-factorial variety then all the constructions preserve this property.

Relative MMP. The whole construction works in the same way if all the varieties and morphisms are defined over a fixed scheme S/k, $k = 0$.

For example, if $K + B$ on X has arbitrary singularities, then applying MMP over X one gets a minimal model $f: Y \to X$, i.e. $K_Y + B_Y$ is f-nef and has (log) terminal singularities. In a non $\mathbb{Q}$-factorial case the situation is a little more subtle. In fact, sometimes one needs flips even in the case of a divisorial type contraction.

(Log) *canonical model.* If $K + B$ is nef, big and is terminal/canonical or log terminal/log canonical $+ B$ is LSEPD, then by the Base Point Free theorem ([Sh2], [KMM] 3-1-1) some multiple of $K + B$ gives a morphism to a variety Z with $K_Z + B_Z$ (relatively) ample and (log) canonical.

For example, applying the relative MMP over X one gets $g: Z \to X$ such that $K_Z + B_Z$ is g-ample and has (log) canonical singularities.

What is proved about MMP. In the terminal/canonical version the Flip conjecture I is proved only in dimension 3 and the Flip conjecture II in dimension $\leqslant 4$. In the log terminal/log canonical version the Flip conjecture I is proved only in dimension 3 and the Flip conjecture II is still an open question. (Since this paper was written, Y. Kawamata proved termination in the log terminal case.)

Generalizations (compare the remark on the definition of B_Y in [Sh3], after 1.1). We can choose coefficients b_i and β_j in different ways. Three conditions should be satisfied:

(a) After a divisorial contraction, one should stay in the same category. If we define b_i and β_j in some uniform way, then by the Negativity of Contractions ([Sh3], 1.1) the sufficient condition for this is

$$b_i \leqslant \beta_i$$

for the component B_i which is contracted.

(b) After a flip, one should stay in the same category. If we define b_i and β_j in some uniform way, this again follows from the Increasing of Discrepancies under a Flip ([Sh2], [KMM], 5-1-11).

(c) In the "ideal" case when X is smooth and all B_i are smooth and cross normally we should certainly have that $K + B$ is (log) terminal, otherwise we do not have enough good resolutions of singularities. This gives a condition $b_i \leqslant 1$. Moreover, it would be even better to have all $b_i < 1$, otherwise we need several definitions of log terminality and some extra conditions, see Remark 1.7.

For example, one can choose

$$0 \leqslant b_i \leqslant c \quad \text{and} \quad \beta_j = c \quad \text{for any } i, j,$$

where $c < 1$ is some constant.

One might also experiment with some "nonuniform" choices of b_i and β_j.

1.8. Now we are ready to introduce two categories, to which we apply the Minimal Model Program.

The inequalities $b_i \leqslant \beta_i$ in (a) above might suggest that it is impossible to work at the same time with varieties with terminal/canonical singularities and nonempty B. Nevertheless, this is exactly what we do.

Instead of working with $K + B$, where $B = \Sigma B_i$ is $\mathbb{Q}$-divisor, we work with $K + L$, $L = \Sigma l_i L_i$, where L_i are *movable* linear systems. For a birational morphism $f: Y \to X$ we define $K_Y + L_Y$, where $L_Y = \Sigma l_i f^{-1} L_i$ (where $0 \leqslant l_i \leqslant +\infty$), i.e. we set all β_j in Definition 1.4. to zero.

1.9. We say that $K_X + L$ is terminal (resp. canonical) if for any *fixed good* resolution $f: Y \to X$ of singularities (this means that X is nonsingular and all L_i are free), in the formula

$$K_Y + B_Y = f^*(K_X + B) + \sum \gamma_j E_j,$$

one has $\gamma_j > 0$ (resp. $\gamma_j \geqslant 0$) for *general* members $B_i \in L_i$ of the linear systems. Note that this condition does not depend on the choice of the resolution f.

The meaning of this definition is that on any fixed resolution divisors "do not see" the "log part" of $K_X + B$, because the L_i are movable. We also does not need to check the condition $\beta_j \geqslant b_j$ because in our case none of the components of B_i is contractible.

1.10. Finally, we introduce the categories

(QLSt) $\mathbb{Q}$-factorial varieties such that $K + L = K + \Sigma l_i L_i$ (where $0 \leqslant l_i \leqslant +\infty$) is terminal.

(QLSc) $\mathbb{Q}$-factorial varieties such that $K + L = K + \Sigma l_i L_i$ (where $0 \leqslant l_i \leqslant +\infty$) is canonical, and K is terminal.

1.11. The connection with the terminal/canonical category

Obviously, if $K + L$ is terminal then so is K. Hence all varieties in $\mathbb{Q}$LSt and $\mathbb{Q}$LSc have $\mathbb{Q}$-factorial terminal singularities.

If R is an extremal ray with $(K + L) \cdot R < 0$ and R is not of flipping type, then obviously $L \cdot R \geqslant 0$, so $K \cdot R < 0$ and cont_R is also a contraction in the terminal/canonical category. Hence the only new transformations in $\mathbb{Q}$LSt and $\mathbb{Q}$LSc are new flips. For the terminal/canonical category they could be flips, flops or antiflips.

Relations with the log terminal/log canonical category

1.12. LEMMA. *Suppose that a variety* X *and* $K + L = K + \Sigma l_i L_i$ *are as in* $\mathbb{Q}$LSt *or* $\mathbb{Q}$LSc, $B_i \in L_i$ *are general elements and that all* $l_i \leqslant 1$. *Then* $K + \Sigma l_i B_i$ *is log terminal.*

Proof. Let $f: Y \to X$ be a good resolution of singularities of X and L. It is obviously also a good resolution of singularities for $K + \Sigma l_i B_i$. Then in the categories $\mathbb{Q}$LSt, $\mathbb{Q}$LSc

$$K_Y + B_Y = f^*(K + B) + \sum \gamma_j E_j, \quad \text{and} \quad \gamma_j \geqslant 0.$$

In the log terminal/log canonical version the corresponding discrepancies will be equal to $1 + \gamma_j > 0$. $\square$

Basic theorems in $\mathbb{Q}$LSt *and* $\mathbb{Q}$LSc *for the dimension three*

1.13. LEMMA. *Let* X *and* $K + L$ *be in* $\mathbb{Q}$LSt *or* $\mathbb{Q}$LSc. *We divide* L *into two parts:*

$$K + L = K + \sum_{l_i \leqslant 1} l_i L_i + \sum_{l_k > 1} l_k L_k.$$

Then for any L_k *with* $l_k > 1$, *the base locus of* L_k

$$\dim \text{Bs } L_k < 1.$$

In $\mathbb{Q}$LSt *the same is true also for all* L_k *with* $l_k = 1$. *In particular these* L_k *are nef.*

Proof. Let $C \subset \text{Bs } L_k$ be a base curve of one of L_k. Then one of discrepancies of the divisors lying over C (namely the exceptional divisor of the first blowup of C) satisfies

$$\gamma = 1 - l_k \cdot \text{mult}_C L_k - \cdots < 0,$$

and therefore $K + L$ is not canonical. In the case $l_k = 1$, the same argument shows that $K + L$ is not terminal.

1.14. CONE THEOREM. *By the previous lemma,*

$$\overline{NE(X)} \cap \{C \cdot (K + L) < 0\} \subset \overline{NE(X)} \cap \left\{ C \cdot \left(K + \sum_{l_i \leqslant 1} l_i L_i \right) < 0 \right\}$$

Now, for any $\varepsilon > 0$,

$$\overline{NE(X)} \cap \left\{ C \cdot \left(K + \sum_{l_i \leqslant 1} (l_i - \varepsilon) L_i \right) < 0 \right\}$$

is locally finitely generated by Lemma 1.12 and [KMM], 4-2-1. This implies the statement. □

1.15. CONTRACTION THEOREM. *Let be an extremal ray for $K + L$ and D be a Cartier divisor such that D is nef and $D \cdot R = 0$. Then $|mD|$ is free for $m \gg 0$, because $mD - (K + L)$ is nef and big, and so is $mD - (K + \Sigma_{l_i \leqslant 1}(l_i - \varepsilon)L_i)$ by*

Lemma 1.13. Now the statement follows from Lemma 1.12 and the standard Base Point Free theorem, [KMM], 3-1-1.

1.16. *Flip conjecture I.* Suppose, that R is an extremal ray for $K + L$ of flipping type. Then R is also an extremal ray for $K + \Sigma_{l_i \leqslant 1}(l_i - \varepsilon)L_i$ by Lemma 1.13, so the flip exists by Lemma 1.12 and [Sh3]. □

1.17. *Flip conjecture II.* The proof is actually the original Shokurov's proof for the category terminal/canonical, [Sh2], with minor changes, as in [Ko1].

First, it is easy to see that in Definition 1.9 the number of discrepancies for $K + L$ with $0 \leqslant \gamma_j < 1$ is finite and does not depend on the good resolution $f: Y \to X$. The proof is the same as in the terminal/canonical category because discrepancies "do not see" free linear systems.

We now define the difficulty $d_{SH}(K + L)$ to be the finite nonnegative integer

$$d_{SH}(K + L) = \sum_{n_i \geqslant 0} \# \left\{ \gamma_j | \gamma_j < 1 - \sum_{n_i, l_i > 0} n_i l_i \right\}.$$

After a flip, corresponding discrepancies by the Increasing of Discrepancies under a Flip (Sh], [KMM], 5-11-11) satisfy $\gamma_j^+ \geqslant \gamma_j$; and for divisors that lie over the exceptional curves of $f^+: Y^+ \to X$, we have the strict inequality $\gamma_j^+ > \gamma_j$. But the variety Y^+ has terminal singularities, and hence is nonsingular

in codimension two, so that one of these discrepancies is

$$\gamma_j^+ = 1 - \sum \text{mult}_C(B_i^+) \cdot l_i.$$

Therefore $d_{SH}(K^+ + L^+) < d_{SH}(K + L)$, and the sequence of flips in QLSt or QLSc terminates. □

1.18. *Canonical model over* X. If $f: Y \to X$ is a relative terminal model, then $K_Y + L_Y$ is f-nef and

$$(K_Y + L_Y) - \left(K + \sum_{l_i < 1} l_i L_i \right) = \sum_{l_i \geqslant 1} l_i L_i$$

is f-nef and big by Lemma 1.13; so by the Base Point Free Theorem ([Sh2], [KMM], 3-1-1), $K_Y + L_Y$ is f-free and gives a birational morphism to the canonical model over X.

Now we can use the Minimal Model Program for QLSt and QLSc.

1.19. REMARK. In 1.10 we can give the definition, not requiring Q-factoriality. In this case we get the same difficulties with the coefficients $l_i = 1$ as in Remark 1.7.

1.20. Let X be a variety with terminal singularities, and suppose that L is a linear system such that locally, in the neighborhood of any fixed point of L, we have $L \sim -K_X$ and a g.e. $S \in L$ is irreducible.

Let $f: Y \to X$ be a good resolution of singularities of the variety X and of the linear system L. Write

$$\begin{aligned} K_Y &= f^*K_X + \sum a_j E_j \\ f^*L &= L_Y + \sum r_j E_j \quad \text{with } L_Y \text{ free.} \end{aligned} \tag{1.20.1}$$

Where a_j and $r_j \in \mathbb{Q}$.

1.21. LEMMA. *The following are equivalent:*

   (i) *a g.e.* $S \in L$ *is a normal surface with Du Val singularities;*
   (ii) $r_j \leqslant a_j$ *for all* j;
   (iii) $r_j < a_j + 1$ *for all* j;
   (iii) *for a g.e.* S, *the divisor* $K + S$ *is canonical;*
   (iv) *for a g.e.* S, *the divisor* $K + S$ *is log terminal.*

Proof. By (1.20.1) we have the following numerical equality:

$$-K_Y = L_Y + \sum (r_j - a_j)E_j$$

Let ω_X be any rational differential form of top degree over the function field $k(X)$. Then the Weil divisor K_X is the divisor of zeros and poles of ω_X. The divisor K_Y is the divisor of zeros and poles of ω_X if considered as a differential form over $k(Y) = k(X)$. This shows that there is (a linear!) equality

$$-K_Y = L_Y + \sum n_j E_j,$$

where n_j are integers. By Lemma 1.1 of [Sh3] (Negativity of Contractions) we have $n_j = r_j - a_j$.

Therefore (ii) and (iii) are equivalent, because $(r_j - a_j)$ are integers.

From 1.20.1,

$$K_Y + S_Y = f^*(K + S) + \sum (a_j - r_j)E_j, \qquad (1.20.1)$$

so (ii) is equivalent to (iv) and (iii) to (v) by definition.

(ii), (iii) $\Rightarrow$ (i). By [Sh3], Lemma 3.6, S is normal. Now $f: S_Y \to S$ is a resolution of singularities, and by the adjunction formula

$$K_{S_Y} = f^* K_S + \sum (a_j - r_j)E_j|_{S_Y},$$

so if $a_j - r_j \geqslant 0$, then (i) holds by the definition of Du Val singularities.

(i) $\Rightarrow$ (iii). For exceptional divisors E_j with $\dim f(E_j) = 1$ the inequality $r_j \leqslant a_j + 1$ can be easily proved by induction. The case $\dim f(E_j) = 0$ is proved in [Kaw2], Lemma 8.8. $\qquad\square$

1.22. LEMMA. *Suppose that $K + L$ is terminal, with L any movable linear system. Then L has at most isolated nonsingular base points P_i such that $\mathrm{mult}_{P_i} L = 1$.*

Proof. Indeed, let P be a base point of L_Y and suppose that P is a singular point of X of index m, $g: Y \to X$ be a resolution of singularities and, as usual,

$$K_Y = g^* K_X + \sum a_j E_j$$
$$g^* L = L_Y + \sum r_j E_j$$

Then by [Kaw3] there is a coefficient $a_j = 1/m$. Since all $r_j \geqslant 1/m$ this implies that there is a discrepancy $\gamma_j = a_j - r_j \leqslant 0$ and $K + L$ is not terminal.

Also, $K + L$ also has no base curves or surfaces, again by the condition $a_j - r_j = 1 - r_j > 0$.

Therefore the set-theoretic base locus of L is a finite set $\{P_i\}$ of nonsingular points of X and $r_j = \mathrm{mult}_{P_j} L = 1$ by $a_j - r_j = 2 - r_j > 0$. $\qquad\square$

1.23. LEMMA. *Suppose that X has only terminal singularities and K + L is canonical. Then in the neighborhood of any base point P of L we have L ∼ − K_X.*

Proof. We only have to consider singular points of X. Let P be one of these points. As in the previous proof there is a divisor E_j with minimal coefficient $a_j = 1/m$, where m is the index of P. In order L to be canonical one should have $r_j = 1/m$, and since the local class group $Cl_P X \simeq \mathbb{Z}/m$ generated by K_X, one gets $L = -K_X$.   □

2. Basic results on the anticanonical system

In this chapter we recall some known facts about terminal singularities and Q-Fanos and prove several basic results on the linear system $|-K_X + \pi^*(d)|$ for Q-Fano 3-folds and Del Pezzo fibrations.

2.1. DEFINITION. A (strict) Mori fiber space is a variety X with a morphism $\pi: X \to S$ such that

  (i) K is terminal;
  (ii) X is Q-factorial;
  (iii) $\pi = \text{cont}_R$ for some extremal ray R for K (in particular $\rho(X/S) = 1$) and dim $S <$ dim X;
  (iv) $-K$ is π-ample.

In the case dim $S = 0, 1$ or 2 respectively we call $\pi: X \to S$ a (strict) Q-Fano, a (strict) Del Pezzo fibration or a (strict) conic bundle respectively.

The most important numerical information about the coherent cohomology of a variety is often contained in the Riemann-Roch formula. In our case one has the following:

2.2. RIEMANN-ROCH FORMULA (Barlow-Reid-Fletcher, [R2,§9]). Let X be a 3-dimensional variety with Q-factorial canonical singularities and D be a Weil divisor on X. Then

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12} D(D - K_X)(2D - K_X)$$

$$+ \frac{1}{12} Dc_2(X) + \sum_Q c_Q(D),$$

where

$$c_Q(D) = -i \cdot \frac{r^2 - 1}{12r} + \sum_{j=1}^{i-1} \frac{\overline{bj}(r - \overline{bj})}{2r},$$

if the singular point Q is a cyclic quotient singularity and the divisor D has type $i(1/r(a, -a, 1))$ at this point, b satisfies $ab \equiv 1 \bmod r$ and $\bar{}$ denotes the residue modulo r. Every noncyclic corresponds to a basket of cyclic points.

2.3. PROPOSITION (Kawamata, [Kaw1]). *For a* $\mathbb{Q}$*-Fano one has* $c_1 c_2 = -K c_2 > 0$.

2.4. COROLLARY (Kawamata, [Kaw1]). *For a* $\mathbb{Q}$*-Fano one has*

$$\frac{1}{24} c_1 c_2 + \frac{1}{24} \sum \left(r - \frac{1}{r} \right) = 1$$

and, in particular, all $r \leqslant 24$.

2.5. COROLLARY. *For a* $\mathbb{Q}$*-Fano one has*

$$\chi(O_X(xc_1)) = \frac{x(x + 1)(2x + 1)}{12} c_1^3 + \frac{x}{12} c_1 c_2 + 1 + \sum_Q c_Q(D)$$

2.6. THEOREM (Kawamata, [Kaw1]). *There is an absolute constant* N *such that* $c_1^3 < N$ *for all* $\mathbb{Q}$*-Fanos. This fact together with Corollary 2.4 implies that the family of all* $\mathbb{Q}$*-Fanos is bounded.*

2.7. VANISHING THEOREM (Kawamata-Fiehweg, [KMM], 1-2-5). *Let* X *be a normal variety and* $\pi: X \to S$ *be a proper morphism. Suppose that a log canonical divisor* $K + B = K_X + \Sigma b_i B_i$ *with* $0 \leqslant b_i < 1$ *is weakly log terminal and that* D *is a* $\mathbb{Q}$*-Cartier Weil divisor. If* $D - (K + B)$ *is* π*-nef and* π*-big, then*

$$R^i \pi_* O_X(D) = 0 \quad \text{for} \quad i > 0.$$

2.8. COROLLARY (simple vanishing). *Let* X *be a* $\mathbb{Q}$*-factorial variety with terminal singularities and* D *be a Weil divisor. If* $D - K$ *is ample, then*

$$H^i(O_X(D)) = 0 \quad \text{for} \quad i > 0.$$

We are going to draw some conclusion from these results. But Riemann-Roch formula 2.2 seems to be quite inconvenient to work with directly. In order to simplify it we make the following definition:

2.9. DEFINITION. For a given function $f(t)$ and four arguments of the form $a, a - e, b, b - e$, *the double difference is defined by the formula*

$$\Delta\Delta_f(a, a - e, b, b - e) = f(a) - f(a - e) - f(b) + f(b - e)$$
$$= (f(a) - f(b)) - (f(a - e) - f(b - e))$$

2.10. EXAMPLE. It is easy to see that

$$\Delta\Delta_1 = \Delta\Delta_t = 0$$

$$\Delta\Delta_{t^2} = a^2 - (a - e)^2 - b^2 + (b - e)^2 = 2(a - b)e$$

$$\Delta\Delta_{t^3} = a^3 - (a - e)^3 - b^3 + (b - e)^3 = 3(a - b)e(a + b - e).$$

2.11. EXAMPLE. For the regular part of the Riemann-Roch formula

$$\chi_{\text{reg}}(t) = \frac{1}{6}t^3 + \frac{1}{4}c_1t^2 + \frac{1}{12}(c_1^2 + c_2)t + \chi(0)$$

one has

$$\Delta\Delta_{\chi,\text{reg}}(a, a - e, b, b - e) = \frac{1}{2}(a - b)e(a + b - e + c_1).$$

A kind of surprise is the following

2.12. LEMMA. *For the part of Riemann-Roch formula* 2.2, *arising from singularities one has*

$$\Delta\Delta_{\chi,\text{sing}}(-K, -K - E, E, 0) \geqslant 0$$

for any E. Here K denotes, as usual, the canonical class.

Proof. We have to check that for every r with $0 \leqslant a < r$ and $0 \leqslant i < r$

$$(c(r, a, r - 1) - c(r, a, r - 1 - i)) - (c(r, a, i) - c(r, a, 0)) \geqslant 0$$

Let us denote $\bar{x}(r - \bar{x})/2r$ by $F(x)$. Then this inequality is equivalent to the following

$$\sum_{j=0}^{r-2} F(bj) - \sum_{j=0}^{r-1-i-1} F(bj) - \sum_{j=0}^{i-1} F(bj)$$

$$= \sum_{j=r-i-1}^{r-2} F(bj) - \sum_{j=0}^{i-1} F(bj) = \sum_{j=2}^{i+1} F(bj) - \sum_{j=0}^{i-1} F(bj)$$

$$= (F(b(i + 1)) - F(i)) + (F(bi)) - F(0)) \geqslant 0$$

It is easy to verify the second inequality, using the fact that the function $F(z + bi) - F(z)$ obviously has a maximum at the point $z = 0$. $\qquad\square$

2.13. LEMMA. *Let X be a $\mathbb{Q}$-Fano variety. Then*

(i) $h^0(O_X(-K_X)) > \frac{1}{2}c_1^3 - 1$;

(ii) *if moreover, the basket of cyclic singulariaties does not contain a point with* $r = 2$ *then* $h^0(O_X(-K_X)) > \frac{1}{2}c_1^3$.

2.14. COROLLARY. (i) *If the degree of a $\mathbb{Q}$-Fano variety $d = c_1^3 \geqslant 2$ then the anticanonical system $|-K|$ is not empty;*

(ii) *If moreover, the basket of cyclic singularities does not contain a point with* $r = 2$ *then this system is not empty without any conditions on the degree.*

Proof of Lemma 2.13. From the vanishing theorem and the Riemann-Roch formula we have

$$h^0(O_X(-K_X)) = \frac{1}{2}c_1^3 + \frac{1}{12}c_1c_2 + 1 + \sum_Q c_Q(D)$$

and for all singular points $i(1/r(a, -a, 1))$, $i = r - 1$. From Proposition 2.3 $c_1c_2 > 0$, and we need only to estimate $\Sigma_Q c_Q(D)$.

$$\sum_Q c_Q(D) = -(r-1)\cdot\frac{r^2-1}{12r} + \sum_{j=1}^{r-2}\frac{\overline{bj}(r-\overline{bj})}{2r}$$

$$= -(r-1)\cdot\frac{r^2-1}{12r} + \sum_{j=1}^{r-1}\frac{j(r-j)}{2r} - \frac{b(r-b)}{2r}$$

Here

$$\sum_{j=1}^{r-1}\frac{j(r-j)}{2r} = \frac{r^2-1}{12}.$$

The number b is coprime to r, therefore if all $r \neq 2$ then $(1/2r)b(r-b) \leqslant (1/8r)r^2 - 1$ and

$$\sum_Q c_Q(D) \geqslant -\frac{1}{24}\sum\left(r - \frac{1}{r}\right) \geqslant -1$$

so case (ii) is done. For $r = 2$

$$c_Q(D) = -1 \times \frac{2^2-1}{12\cdot 2} = -2 \times \frac{2^2-1}{24\cdot 2};$$

so, at least we always have

$$\sum_Q c_Q(D) \geqslant -2\frac{1}{24}\sum\left(r - \frac{1}{r}\right) \geqslant -2.$$

2.15. REMARK. Something like Lemma 2.13 is necessary because of the following example due to A. R. Fletcher.

2.16. EXAMPLE (Fletcher, [Fl1], [Fl2]). A general weighted complete intersection $X_{12,14}$ in $\mathbf{P}(2,3,4,5,6,7)$ is a ℚ-Fano with $h^0(-K_X) = 0$. This variety has the following isolated singularities: 1 of type $\frac{1}{5}(4,1,2)$, 2 of type $\frac{1}{3}(2,1,1)$, and 7 of type $\frac{1}{2}(1,1,1)$. The degree $d = (-K_X)^3 = \frac{1}{30}$.

2.17. LEMMA. *Let* $\pi\colon X \to S$ *be a Del Pezzo fibration. Then* $|-K_X + \pi^*(d)|$ *is nonempty for a sufficiently ample* $d \in \mathrm{Div}(S)$.

Proof. Indeed, if X_0 is a general fiber (a smooth Del Pezzo surface), then for sufficiently ample d one has

$$H^0(X, -K_X + \pi^*(d)) \twoheadrightarrow H^0(X_0, -K_X + \pi^*(d)) = H^0(X_0, -K_{X_0})$$

and the group on the right is not trivial.

2.18. THEOREM. *Let* X *be a strict* ℚ-*Fano. Then a g.e. of* $|-K|$ *is irreducible and reduced. If we further assume that* $h^0(-K_X) \geqslant 3$ *then the linear system* $|-K_X|$ *does not have base components and is not composed of a pencil.*

Proof. Assume the opposite, that either

(a) $|-K|$ has a base component $E \neq -K$, or
(b) $|-K|$ is composed of a pencil, $|-K| = |nE|$ and $n \geqslant 2$.

In either case there is an effective divisor E such that

$$(h^0(O(-K)) - h^0(O(-K-E))) - (h^0(O(E)) - h^0(O(0))) = 0$$

Since $\mathrm{rk}\,\mathrm{Pic}(X) = 1$, all higher cohomologies of these sheaves vanish by Corollary 2.8 and all h^0 coincide with χ, so the following double difference vanishes:

$$\Delta\Delta_\chi(-K, -K-E, E, 0) = \Delta_{\chi,\mathrm{reg}} + \Delta_{\chi,\mathrm{sing}} = 0$$

For the regular part, by Example 2.11, we have

$$\Delta\Delta_{\chi,\mathrm{reg}} = (-K-E)E(-K) > 0$$

and $\Delta\Delta_{\chi,\text{reg}} \geqslant 0$ by Lemma 2.12. This contradiction proves the theorem.   □

2.19. THEOREM. *Let* $\pi\colon X \to S$ *be a Del Pezzo fibration. Then for sufficiently ample* $d \in \text{Div}(S)$ *the linear system* $|-K + \pi^*(d)|$ *has no base components and is not composed of a pencil (in particular, a g.e. is irreducible and reduced).*

Proof. Assume the opposite, that $|-K + \pi^*(d)|$ has a base component E for any sufficiently ample d. In particular,

$$h^0(-K + \pi^*(3d)) = h^0(-K - E + \pi^*(3d)),$$

and

$$h^0(E + \pi^*(d)) = h^0(\pi^*(d)).$$

But for sufficiently ample $d \in \text{Div}(S)$ the higher cohomology of all these divisors vanish by 2.8 (it is important that here $\rho(X/S) = 1$). Therefore the following double difference vanishes:

$$\Delta\Delta_{\chi}(-K + \pi^*(3d), \ -K - E + \pi^*(3d), \ E + \pi^*(d), \ \pi^*(d)) = 0$$

But by the example 2.11.

$$\Delta\Delta_{\chi,\text{reg}} = E(-K - E + \pi^*(2d))(-K + \pi^*(2d)) > 0,$$

and $\Delta\Delta_{\chi,\text{sing}} \geqslant 0$ again by Lemma 2.12. We again achieved a contradiction.

Finally, $|-K + \pi^*(d)|$ is, evidently, not composed of a pencil.   □

2.20. REMARK. Theorems 2.18, 2.19 also hold for a slightly wider class of varieties: with canonical K and canonical $K + \varepsilon| - K|$.

3. Double projection

In the theory of smooth Fano 3-folds there is a well-known method, called "the double projection method", which was introduced in classical papers of Fano.

If $X \subset \mathbb{P}^N$ is a smooth projective 3-fold such that $|-K_X|$ is linearly equivalent to a hyperplane section then the method is first to prove the existence of a line l on $X \subset \mathbb{P}^N$ and then to consider a rational map – given by a linear system of hyperplanes tangent to X along l, i.e. the rational map $\phi = \phi_{|f^*(-K_X) - 2E|}\colon Y -- \to Z$, where Y is the blowup of l in X and E is the exceptional divisor of this blowup. One can then show that the map ϕ is a composite of a flop $t\colon Y -- \to Y^+$ and an extremal contraction $g\colon Y^+ \to Z$ and

either Z is a Fano variety of index >1, or $\dim Z < \dim Y^+$ and g is a Del Pezzo fibration or a comic bundle. This method was used by Fano and Iskovskikh to classify smooth Fano 3-folds with $\rho(X) = 1$, see [I1, I2].

A variation of the same construction is to consider the "double projection" from a point (therefore one does not need to prove the existence of a line $l \subset X \subset \mathbb{P}^N$ which is very nontrivial). This approach was used by several authors, for example, in [M3].

In this chapter we introduce a generalization of the double projection to the case of a ℚ-Fano variety, and more generally, to the case of Mori fibrations. In this new situation there is no hope to have a good projective embedding because $-K_X$ is not necessarily a Cartier divisor. Therefore we change the point of view and start not with a line or a point, but with an arbitrary movable linear system $L \sim -K_X$ such that a g.e. of L has worse than Du Val singularities.

In the classical Fano construction, for example, L is the linear system of all hyperplane sections on X passing (at least) twice through the line l. All these surfaces are not normal, so they certainly have worse than Du Val singularities.

3.1. Construction

Let $\pi: X \to S$ be a Mori fiber space, $L \subset |-K_X + \pi^*(d)|$ and suppose that the system L is movable and a general element $B \in L$ has singularities worse, that Du Val singularities. By Lemma 1.21 this means that $K + L$ is not canonical.
   Let

$$c = \min\{x \,|\, \text{such that } K + cL \text{ is canonical}\}$$

Obviously, $0 < c < 1$ and the divisor $K + cL$ is not terminal, so there is a terminal model $f: Y \to X$, which is crepant, i.e.

$$K_Y + cL_Y = f^*(K + cL)$$

Since $c < 1$, $-(K + cL)$ is ample, therefore

$$\kappa_{\mathrm{num}}(Y/S, K_Y + cL_Y) = -\infty.$$

Let us apply the Minimal Model Program to $K_Y + cL_Y$ in the category ℚLSt (relative over S). We will get another Mori fiber space $f_1: X_1 \to S_1$, where S_1 is a variety over $S: S_1 \to S$.

1st Case. S_1 is not isomorphic to S.

2nd Case. $S_1 \simeq S$.

Consider the second case. Varieties X_1 and X are certainly birational. I claim that this birational map cannot be extended to an isomorphism. Indeed, $K_X + cL_{X_1}$ is terminal by construction, but $K_X + cL_X$ is not even canonical.

I claim that the anticanonical system $|-K_{X_1} + \pi^*(d_1)|$ splits into the strict transform of L, i.e. L_{X_1} and some additional nonzero effective divisor.

Indeed, for $f: Y \to X$, we have

$$L_Y + \sum (r_j - a_j)E_j \subset |-K_Y + \pi^*(d_1)| \tag{3.1.1}$$

and, since f is crepant for $K + cL$, we have $a_j = cr_j$ for all j and hence so $r_j - a_j > 0$ since $c < 1$.

This means that all $\rho(Y) - \rho(X)$ exeptional divisors in this splitting appear with positive coefficients. Here it is crucial that 1.21.1 holds not only for divisors modulo numerical equivalence but actually for Weil divisors modulo rational equivalence. Now I claim that the birational map $Y - \to X_1$ does not contract all of these exceptional divisors E_j.

Take a resolution of indeterminacy of $Y - \to X_1$. We have a diagram
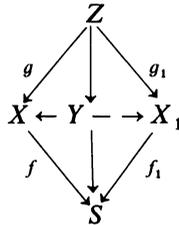
Suppose that, on the contrary, $Y - \to X_1$ contracts divisors that are exceptional for $Y \to X$. Since $\rho(X/Y) = \rho(X_1/S) = 1$, we have $\rho(Z/X) = \rho(Z/X_1)$ and this means that the exceptional sets of $g: Z \to X$ and $g_1: Z \to X_1$ are the same union of divisors; let us denote them $G_1, \ldots, G_s$. Let H, H_1 be ample divisors on X and X_1 respectively. Then g^*H is g_1-nef. By Negativity of Contractions ([Sh3], 1.1) this implies that in $\text{Div}_{\mathbb{Q}}(Z)$ modulo $\pi_Z^*(\text{Div}_{\mathbb{Q}}(S))$

$$g^*H = \alpha g_1^*H_1 - \sum \gamma_k G_k \quad \text{and} \quad \gamma_k \geq 0.$$

On the other hand,

$$g_1^*H_1 = \alpha' g^*H - \sum \gamma_k' G_k \quad \text{and} \quad \gamma_k' \geq 0.$$

This is possible only if all $\gamma_k = \gamma_k' = 0$ and $g^*H = \alpha g_1^*H_1 + \pi_Z^*(d_S)$. But in this case the varieties X and X_1 should coincide, since the morphisms g and g_1 are

given by the relative linear systems $|Ng^*H|$ and $|Ng_1^*H_1|$ over S, i.e.

$$X = \text{Proj} \bigoplus_{n=0}^{\infty} (fg)_* O_Z(nNg^*H)$$

and

$$X_1 = \text{Proj} \bigoplus_{n=0}^{\infty} (f_1 g_1)_* O_Z(nNg_1^*H_1)$$

for large divisible N. This contradiction proves the statement.

We stress that in this proof it is important that $\rho(X/S) = \rho(X_1/S) = 1$. $\square$

In Chapter 4 we shall use this splitting to prove the termination of the sequence of birational transformations.

3.2. VERSION. For the morphism $f: Y \to X$ we can contract all but one exceptional divisors (making flips on the way). So we get $\bar{f}: \bar{Y} \to X$, $\rho(\bar{Y}/X)$, $|-K_{\bar{Y}}| = L_{\bar{Y}} + (r-a)E$, where E is an exceptional divisor of $\bar{f}$. $K_{X_1} + cL_{\bar{Y}}$ is canonical and $-(K_{\bar{Y}} + cL_{\bar{Y}}) = -(K_X + cL)$ is a pulling back of an ample divisor from X. So $\overline{NE(\bar{Y}/S)}$ is a 2-dimensional cone, which has two extremal rays R_1 and R_2 such that

$$R_1 \cdot (K_{\bar{Y}} + cL_{\bar{Y}}) \geqslant 0 \quad \text{(actually, } = 0\text{)}$$

$$R_2 \cdot (K_{\bar{Y}} + cL_{\bar{Y}}) < 0$$

$$R_1 \cdot E < 0$$

We should also have $R_2 \cdot E > 0$, otherwise $-E$ would be nef, which is impossible.

We contract R_2, and if it is of flipping type, we make a flip. After the flip we again have two extremal rays satisfying the same conditions as above. Continuing the Minimal Model Program, this time in $\mathbb{Q}$LSc, we finally get a fibration or a divisorial contraction. Since $R_2 \cdot E > 0$ on the final step, this contraction is not a contraction of E, so E survives on the final variety X_1 and $|-K_{X_1}| = L_{X_1} + (r-a)E_{X_1}$ splits.

3.3. VERSION. $K + cL$ is neither terminal nor canonical for any choice of $c < 1$ and $\kappa_{\text{num}}(Y/S, K_Y + cL_Y) = -\infty$. So we can find a terminal model over X for $K + cL$ and then repeat the same procedure. In the absolute case, i.e. when $S = \text{point}$, we can also do it for $K + L$ since $K_Y + L_Y = \Sigma(a_j - r_j)E_j$ will be not quasieffective divisor on Y, so $\kappa_{\text{num}}(Y, K_Y + L) = -\infty$.

3.4. REMARK. We can carry out our construction also in the case when $K + 1/mL$ is not canonical, where $L \subset |-mK_X|$ is any linear subsystem with no base components. In [R3] this is called "a system with imposed adjunction".

4. Applications: case of Q-Fanos

4.1. CONSTRUCTION. Let X be a Q-Fano. Consider the complete linear system $L = |-K_X|$. By Th. 2.18 it is movable. If a g.e. of L has worse than Du Val singularities, make the double projection 3.1. As a result we get one of the following:

   (i) $\pi_1 \colon X_1 \to S_1 = pt.$ is another Q-Fano.
   (ii) $\pi_1 \colon X_1 \to S$ is a Del Pezzo fibration or a conic bundle.

4.2. TERMINATION. If in Case 4.1(i) a g.e. has again bad singularities, repeat the procedure and so on. I claim that case 4.1(i) cannot occur infinitely many times.

Indeed, by Construction 3.1 we get the splitting of the anticanonical system

$$L_{X_1} + E \subset |-K_{X_1}|.$$

By Th. 2.18 the anticanonical system on a Q-Fano has no fixed components, hence

$$h^0(-K_{X_1}) > h^0(L_{X_1}) = h^0(L_X) = h^0(-K_X).$$

All Q-Fanos form a bounded family by [Kaw1], so this process will eventually stop.   □

4.3. THEOREM. (*Existence of a good model*). *Let X be a Q-Fano such that* $\dim \phi_{|-K|}(X) = 3$. *Then X is birationally equivalent to a Q-Fano X_1 such that* $\dim \phi_{|-K_{X_1}|}(X_1) = 3$ *and a g.e. of $|-K_{X_1}|$ has only Du Val singularities.*

Proof. By Termination 4.2, after a finite number of applications of Construction 3.1 we get either what we need or the Case 4.1(ii). Now by the construction $K_{X_1} + L_{X_1}$ is π-negative and, since $\dim \phi_{|L_{X_1}|} = 3$, the restriction of L_{X_1} on a generic fiber of π_1 is not trivial. This implies that $-K_{X_1}$ on a generic fiber is divisible and one of the following holds:

4.3.1. π is a $\mathbb{P}^2$- or a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over a curve C.

4.3.2. π_1 is a conic bundle over a surface S with a birational section L_{X_1}.

Let us show that in both cases X_1 is rational (and so is X). Indeed, in Case 4.3.1 by [Ma, 29.4] X_1 is birationally equivalent to $C \times \mathbb{P}^2$. Now $h^1(\mathcal{O}_X) = 0$ (by

the vanishing Lemma 2.8) and X has rational singularities. This implies that $C = \mathbb{P}^1$.

Let us consider the second case. It is known that, since the conic bundle π_1 has a section, X_1 is birationally equivalent to $S \times \mathbb{P}^1$. Now a g.e. of L has singularities worse than Du Val, therefore is a ruled surface, and so is S. As in the previous case for a resolution of singularities $T \to S$ one has $h^1(\mathcal{O}_T) = 0$, so T is rational. Finally, note that $\mathbb{P}^3$ is also a Q-Fano with a good g.e. $\square$

Gorenstein model

4.4. The following application is inspired by the paper of Conte and Murre [CM]. This article reproduces using modern techniques, results of Fano [Fa] about a projective 3-fold $X \subset \mathbb{P}$ whose general hyperplane section is an Enriques surface. This property implies that X is a Q-Fano of index two, i.e. $-2K_X$ is a Cartier divisor and $\mathcal{O}(-2K_X) = \mathcal{O}_X(-2)$. Their main result is that on a "general" such X there exists a linear system $|\varphi| = |-K_X|$ which gives a birational morphism to a variety $Z \subset \mathbf{P}^N$ whose hyperplane section are $K3$ surfaces, i.e. a Fano variety in the classical sense. (Conte and Murre do not say explicitly that $|\varphi| = |-K_X|$).

Another interesting fact is that [Fa] and [CM] show that a "general" such variety X has exactly 8 singular points that are cones over the Veronese surface, i.e. cyclic singularities of type $\frac{1}{2}(1, 1, 1)$.

This is easy to explain using the Riemann-Roch formula 2.2. Indeed, $-K_X$ is not linearly equivalent to a hyperplane section (otherwise this section would be a $K3$ surface), but $2(-K_X) \sim 2H$. Therefore, the Weil divisor $D \sim K_X + H$ is a 2-torsion element, so $h^0(D) = 0$ and by the vanishing Lemma 2.8 $h^i(D) = 0$ for $i > 0$, hence $\chi(D) = 0$.

The Riemann-Roch formula gives

$$\chi(D) = 1 + \sum_P c_P(D)$$

and each singular point P contributes $-\frac{1}{8}$ to the right-hand side of this equality. Therefore X should have exactly 8 singular points.

4.5. THEOREM. *Let* X *be a* Q-*Fano variety, and assume that a g.e. of* $L = |-K_X|$ *has Du Val singularities. Then there is a partial resolution* $f: Y \to X$ *with the following properties:*

   (i) $f: Y \to X$ *is crepant for* $K + L$, *i.e.*

$$K_Y + L_Y = f^*(K + L) = 0;$$

(ii) *a g.e. of L_Y is a nonsingular (minimal) K3 surface;*
(iii) *Y has terminal factorial Gorenstein singularities;*
(iv) *L_Y is free.*

Proof. Let $f: Y \to X$ be a terminal model for $K + L$. Then (i) is obviously satisfied and $K_Y + L_Y$ is terminal. Now by Lemma 1.22, L can have at most isolated nonsingular base points.

Also, $L_Y = |-K_Y|$, so Y cannot have non-Gorenstein points.

Finally, let us prove (iv). Suppose that

$$\mathrm{Bs}\, L_Y = \{P_i\} \neq \emptyset.$$

Then $L_Y^3 > 0$, so L_Y is big and, as we just have shown, nef. Therefore for a g.e. $D \in L_Y$, the (surjective) restriction $L_Y|_{D_Y}$ is big, nef and has a finite number of base points and it is well known ([SD]) that this is impossible for a smooth K3 surface. $\qquad\square$

4.6. COROLLARY. *Suppose that a g.e. of $|-K_X|$ has only Du Val singularities and that $\dim_{\varphi|-K|}(X) = 3$, where $\varphi_{|-K|}$ is the rational map, defined by $|-K|$. Then X is birationally equivalent to a Gorenstein Fano variety Z with canonical singularities and free anticanonical system $|-K_Z|$.*

Proof. Let $\varphi_{L_Y}: Y \to Z'$ be the morphism, defined by the free linear system L_Y and let

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \psi\ } & Z \\
& \searrow{\scriptstyle \varphi_{L_Y}} & \downarrow \\
& & Z'
\end{array}
$$

be the Stein factorization of φ_{L_Y}. Then $L_Y = \varphi^*|H|$, where $|H|$ is an ample free linear system on Z and, since $L_Y \sim -K_Y, H \sim -K_Z$ and Z is a Gorenstein Fano variety. Since $K_Y = \psi^* K_Z$, Z has canonical singularities. $\qquad\square$

4.7. REMARK. It is not clear to the author whether $\rho(Z) = 1$ or not.

4.8. COROLLARY. Any $\mathbb{Q}$-Fano such that $\dim_{\varphi|-K|}(X) = 3$ is birationally equivalent to a Fano variety Z with Gorenstein canonical singularities and free anticanonical system $|-K_Z|$.

Proof. Combine Theorem 4.3 and Corollary 4.6. $\qquad\square$

4.9. REMARK. Another possible application of the material is the following. In the classification of smooth Fano varieties ([I1], [I2]) the first step is to restrict the anticanonical linear system $|-K_X|$ on a g.e. $S \in |-K_X|$ which is a smooth K3 surface, so well-known results about ample linear systems on K3 surfaces give information about $|-K_X|$.

One can try to apply this method to ℚ-Fanos. However, not much is known about ample linear systems $|D|$ on $K3$ surfaces with Du Val singularities in the case when D is not a Cartier divisor. In [A], [N], [U] the question when $|D|$ can have multiple base curves is studied.

Version

4.10. THEOREM. *Let* X *be a* ℚ-Fano *with* $h^0(-K_X) \geqslant 3$ *(for this it is sufficient that* $d = (-K_X)^3 \geqslant 6$ *by Lemma 2.13). Then* X *is birationally equivalent either to a* ℚ-*Fano* X_1 *such that a g.e. of* $|-K_{X_1}|$ *has only Du Val singularities or to a conic bundle.*

Proof. The same as the proof of Theorem 4.3 and Theorem 2.18.   □

5. Application: Gorenstein resolution of terminal singularities

5.1. Let $X \in P$ be a 3-dimensional terminal singularity. In this section we construct special partial resolutions of (X, P). The idea is to consider the rational map given by the (infinite dimensional) linear system $|-K_X|$.

5.2. THEOREM. *Let* (X, P) *be a 3-dimensional terminal singularity. Then there is birational morphism* $f_G : (Y_G, E) \to (X, P)$ *such that the following properties hold:*

(i) Y_G *has* ℚ-*factorial Gorenstein terminal singularities;*

(ii) $\mathcal{O}_{Y_G}(-K_{Y_G})$ *is* f_G-*free;*

(iii) $f_{G*}\mathcal{O}_{Y_G}(-K_{Y_G}) = \mathcal{O}_X(-K_X)$.

Proof. Although the result is purely local let us for simplicity suppose (as we may) that X is a projective variety and P is the only singular point of X. And let $L = |-K + d|$, where $d \in \mathrm{Div}(X)$ is sufficiently ample Cartier divisor.

One has:

(5.2.1) $\mathcal{O}_X(-K + d)$ is generated by global sections.

(5.2.2) For d sufficiently ample the base locus of L does not depend on d, i.e. the situation "stabilizes".

(5.2.3) P is the only base point of L.

By [R2] $|-K|$ locally in the neighborhood of the point P has Du Val singularity and so does L. By Lemma 1.2.2 $K + L$ is canonical and let $f_G : Y_G \to X$ be a terminal model for $K + L$ over X. Then by Lemma 1.22 M_G can have only finite set of base points which are nonsingular for Y_G. Let us show that for a sufficiently ample d this set is empty. Let $\tilde{D} \in |\bar{M}_G|$ be a fixed surface. Then $\tilde{D}$ is nonsingular and the exceptional fiber C of $f : \tilde{D} \to D$ is a tree of nonsingular rational (-2)-curves that form one of the weighted graphs

A_n, D_n, E_n with the structure of the fundamental cycle. By Lemma 5.3 below for a sufficiently ample d the restriction of M_G to any of these rational curves is surjective and hence L is base point free.

So, we proved that Y satisfies the conditions (i) and (ii). Finally, (iii) follows from (5.2.1). $\square$

5.3. LEMMA. *Let S be a surface with a Du Val singularity, $f: \tilde{S} \to S$ be the minimal desingularization and M be a nef divisor on S. Then for any exceptional curve E of f*

$$R^1 f_*(\tilde{S}, M - E) = 0$$

Proof. Since the singularity of S is rational, by [Ar] there is a sequence of divisors on $\tilde{S}$ with a support in the exceptional set $C_1 = E, \ldots, C_{i+1} = C_i + E_i$ such that $C_i \cdot E_i = 1$ and the last divisor C_k is the fundamental cycle of the singularity, in particular, $-C_k$ is f-nef. Now

$$0 \to \mathcal{O}(M - C_{i+1}) \to (M - C_i) \to \mathcal{O}_{E_i}(M - C_i) \to 0$$

gives $R^1 f_*(M - C_{i+1}) \to > R^1 f_*(M - C_i)$ since

$$R^1 f_*(\mathcal{O}_{E_i}(M - C_i)) = H^1(\mathcal{O}_{P^1}(d - 1)) = 0, \quad \text{where } d = M \cdot E_i \geqslant 0$$

Therefore $R^1 f_*(M - C_k) \to > R^1 f_*(M - E)$ and the former is zero by the standard vanishing 2.7 since $M - C_k - K_{\tilde{S}}$ is f-nef. $\square$

5.4. THEOREM. *Let (X, P) be a 3-dimensional terminal singularity. Then there is a birational morphism $F_G: (X_G, E) \to (X, P)$ such that the following properties hold:*

(i) *X_G has Gorenstein canonical singularities;*
(ii) *$\mathcal{O}_{X_G}(-K_{X_G})$ is F_G-ample and F_G-free;*
(iii) *$F_{G*}\mathcal{O}_{X_G}(-K_{X_G}) = \mathcal{O}_X(-K_X)$.*

Proof. We get X_G from Y_G applying the Base Point Free theorem ([KMM], 3-1-1) to a sheaf $\mathcal{O}_{Y_G}(-NK_{Y_G})$, $N \gg 0$. $\square$

of the terms, used in this paper, were introduced by M. Reid in [R2], [R3] (among them "general elephant" and "Mori fiber space").

References

[A]   V. Alexeev, Ample Weil divisors on K3 surfaces with Du Val singularities, *Duke Math. J.* 64 (1991), 617–624.

[Ar]   M. Artin, On isolated rational singularities of surfaces, *Am. J. Math.* 88 (1966), 129–137.

[CM]   A. Conte-J. P. Murre, Algebraic varieties of dimension three whose hyperplane sections are Enriques surfaces, *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV Ser.* 12 (1985), 43–80.

[CT]   J.-L. Colliot-Thélène, Arithmétique des variétés rationelles et problèmes birationelles, In: *"Proc. Int. Conf. Math.",* 1986, 641–653.

[Co]   A. Corti, Thesis, in preparation.

[Fa]   G. Fano, Sulle varietà algebriche a tre dimensioni le cui sezioni iperpiane sono superficie di genere zero e bigenere uno, *Memoire società dei XL*, 24 (1938), 41–66.

[Fl1]   A. R. Fletcher, Letter to V. A. Iskovskih, 1987.

[Fl2]   A. R. Fletcher, Working with weighted complete intersections, *Max-Plank Institute preprint*, 1989.

[I1]   V. A. Iskovskih, Fano threefolds I, II, *Izv. Akad. Nauk SSSR*, 41 (1977), 516–562, resp. 42 (1978), 506–549.

[I2]   V. A. Iskovskih, *Lectures on Algebraic Threefolds. Fano Varieties,* Publ. by Moscow University, 1988.

[Kaw1] Y. Kawamata, Boundedness of *Q*-Fano threefolds, preprint, 1989.

[Kaw2] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, *Ann. of Math.* 127 (1988), 93–163.

[Kaw3] Y. Kawamata, The minimal discrepancy coefficients of terminal singularities in dimension 3. Appendix to [Sh3].

[Kaw4] Y. Kawamata, Termination of log flips for algebraic 3-folds, preprint, 1991.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the Minimal Model Program, In: *"Algebraic Geometry, Sendai", Adv. Stud. Pure Math.* 10 (1987), 283–360.

[Kd]   K. Kodaira, *On Compact Analytic Surfaces, I. II, III,* In "Collected works", v. 3, Iwanami Shoten, Publishers and Princeton University Press, 1975.

[Ko1]   J. Kollár, Flops, *Nagoya Math. J.*, 113 (1989), 14–36.

[Ko2]   J. Kollár, Flips, flops, minimal models etc, preprint, 1990.

[Ma]   Yu. I. Manin, Cubic forms.

[M1]   S. Mori, Threefolds whose canonical bundles are not numerically effective, *Ann. of Math.* 116 (1982), 133–176.

[M2]   S. Mori, *Classification of Higher Dimensional Varieties,* In: "Algebraic geometry Bowdoin 1985", *Proc. Symp. Pure Math.* 46 (1987), 269–332.

[M3]   S. Mori, *Lectures on Fano 3-folds,* University of Utah, 1988.

[N]   V. V. Nikulin, Linear systems on singular K3 surfaces, preprint, 1990.

[R1]   M. Reid, Projective morphisms according to Kawamata, preprint, 1983.

[R2]   M. Reid, *Young Person's Guide to Canonical Singularities,* In: "Algebraic geometry Bowdoin 1985", *Proc. Symp. Pure Math.* 46 AMS, 1987, 345–416.

[R3]   M. Reid, Birational geometry of 3-folds according to Sarkisov, *unpublished lecture notes*, 1991.

[Sh1]   V. V. Shokurov, Smoothness of a general anticanonical divisor on a Fano variety, *Izv. Akad. Nauk SSSR*, 14 (1980), 395–405.

[Sh2]   V. V. Shokurov, The nonvanishing theorem, *Izv. Akad. Nauk SSSR*, 49 (1985), 635–651.

[Sh3]   V. V. Shokurov, 3-fold log flips, preprint, 1991 (translated by M. Reid).

[U]   T. Urabe, Fixed components of linear systems on K3 surfaces, preprint, 1990.

[W1]   P. M. H. Wilson, Varieties, *Bull. London Math. Soc.,* 19 (1987), 1–48.

[W2]   P. M. H. Wilson, Toward a birational classification of algebraic varieties, *Bull. London Math. Soc.,* 19 (1987), 1–48.

[KoMo]   J. Kollár and S. Mori, Classification of three-dimensional flips, *J. Amer. Math. Soc.,* 5 (1992), 533–703.