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Minimal deformation of a non-full minimal surface in $S^4(1)$

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Dedicated to Professor Tadasi Nagano on his sixtieth birthday

1. Introduction

Let $\bar{M}$ be a compact Riemann surface and g a holomorphic map of $\bar{M}$ onto $S^2(1)$. We denote by Δ_g the Laplacian for the branched metric induced by g. Then we call the number of eigenfunctions with eigenvalues smaller than 2 *index of g*, the space of eigenfunctions with eigenvalue 2 *null space of g* and the dimension of the null space *nullity of g*. Note that the index is same as the Morse index with respect to the area functional of a complete minimal surface of finite curvature whose (extended) Gauss map is g (see [F]) and the element of the nullity space corresponds to a *bounded* Jacobi field of the minimal surface. Since the pull back functions of coordinate functions on $S^2(1) \subset R^3$ (which are called linear functions) are eigenfunctions with eigenvalue 2, the nullity is not less than 3. We call the eigenfunction of eigenvalue 2 other than coordinate functions an *extra eigenfunction.* In [E-K1] and [M-R], an algebraic constructing method of extra eigenfunction is given as an application of a characterization of extra eigenfunctions.

THEOREM. ([E-K1] and [M-R]). *Let g be a holomorphic map of a compact Riemann surface $\bar{M}$ onto $S^2(1)$. Then the nullity of $g \geqslant 4$ if and only if there exists a complete, finitely branched, minimal surface with planar ends and finite total curvature whose (extended) Gauss map is g.*

As another point of view, we would like to consider *why extra eigenfunctions occur*. Note that a holomorphic map of $\bar{M}$ onto $S^2(1)$ is a non-full branched minimal immersion in $S^3(1)$. The index of g is equivalent to the index of the Jacobi operator of the minimal surface in $S^3(1)$ and the null space is taken as the space of the Jacobi fields. So an extra eigenfunction corresponds to a non-Killing Jacobi field. We can determine all Jacobi fields of the non-full minimal surface in $S^3(1)$ [E-K1] and [M-R]. Thus an extra eigenfunction is closely related to minimal deformations of a non-full minimal surface. Generally, we do not know the existence of a minimal deformation for a given Jacobian field. Note that we have a local minimal deformation for a given Jacobi field [L]. When there exists a minimal deformation such that full minimal surfaces converge to a non-full minimal surface in $S^3(1)$, there exists an extra eigenfunction. Precisely, we have the following. Let g be a holomorphic map of $\bar{M}$ onto $S^2(1)$. Let ψ_t be a smooth 1-parameter family of weakly conformal full harmonic maps in $S^k(1)$ $(k \geqslant 3)$ except $t = 0$ and $\psi_0 = g$. Then we consider an operator $\Delta + 2e(\psi_t)$ for a fixed Riemannian metric compatible with the conformal structure of $\bar{M}$, where Δ is the Laplacian for the metric and $e(\psi_t)$ is the energy function of ψ_t. Note that f satisfies $\Delta f + 2e(\psi_t)f = 0$ if and only if f satisfies $\Delta_{\psi_t} f + 2f = 0$. Since $\Delta_{\psi_t}\psi_t + 2\psi_t = 0$. $\dim\{f : \Delta f + 2e(\psi_t)f = 0\} \geqslant k + 1$. Since $e(\psi_t)$ is smooth on t, the spectrum for $\Delta + 2e(\psi_t)$ are continuous on t by [K-S] and hence the nullity of $g \geqslant k + 1$.

It is natural to consider a problem whether all extra eigenfunction comes from as above. That is, if a holomorphic map of $\bar{M}$ onto $S^2(1)$ admits an extra eigenfunction, then, is g a limit of a one parameter family of full minimal surfaces in $S^N(1)(N \geqq 3)$? In the case of the genus of $\bar{M} = 0$, we should consider $N \geqq 4$, because there does not exist a full minimal surface of genus 0 in $S^3(1)$. In this paper, we give a positive answer that if the genus of $\bar{M} = 0$, then the above observation is generically yes.

First of all, we give a relation between minimal surfaces in R^3 and the twistor theory. *Fix a horizontal line P^1 in the 3-dimensional complex projective space P^3 of holomorphic sectional curvatures 1. Let g be a holomorphic map of a simply connected domain D in C into P^1 without ramification locus. Then a minimal surface with the Gauss map g induces an infinitesimal horizontal deformations of g in P^3 and the converse is also true.* Note that this is a local correspondence. As a global result, we get the following.

THEOREM A. *Let g be a holomorphic map of $\bar{M}$ onto P^1. Then g admits a non-linear infinitesimal horizontal (holomorphic) deformation in P^3 if and only if there exist a complete, finitely branched minimal surface of planar ends and finite total curvature whose (extended) Gauss map is g and it's conjugate minimal surface exists.*

Using Theorem A and the results on the moduli space of harmonic maps of S^2 into $S^4(1)$ by Loo [Loo], we obtain a formula to calculate the nullity of a holomorphic map of S^2 onto $S^2(1)$. Let $G(2, d + 1)$ be the Grassmannian of planes in C^{d+1}. Let $[P \wedge Q]$ denote the plane spanned by two vectors $P = (a_d, \ldots, a_0)$ and $Q = (b_d, \ldots, b_0)$. Then we get a point $[\alpha_{2d-2}, \ldots, \alpha_0]$ of P^{2d-2} such that

$$Q(z)P(z)' - P(z)Q(z)' = \alpha_{2d-2}z^{2d-2} + \cdots + \alpha_0,$$

where $P(z) = a_d z^d + \cdots + a_0$ and $Q(z) = b_d z^d + \cdots + b_0$. Let Ψ_d be the map of $G(2, d+1)$ onto P^{2d-2} defined by

$$\Psi_d([P \wedge Q]) = [\alpha_{2d-2}, \ldots, \alpha_0].$$

Then Ψ_d is a branched covering map of $G(2, d+1)$ onto P^{2d-2}. Let f be a holomorphic map of P^1 onto P^1 of degree d. Then f is given by a rational function of the form $Q(z)/P(z)$, where

$$P(z) = a_d d^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

and

$$Q(z) = b_d z^d + b_{d-1} z^{d-1} + \cdots + b_1 z + b_0$$

such that $a_i,\ b_i \in C$. Note that $\max\{\text{degree of } P, \text{ degree of } Q\} = d$ and the resultant of $P(z)$ and $Q(z)$ is not zero. Let P, Q denote vectors $(a_d, a_{d-1}, \ldots, a_1, a_0)$, $(b_d, b_{d-1}, \ldots, b_1, b_0) \in C^{d+1}$, respectively. Then we obtain the following.

THEOREM B. *Let* $P(z)/Q(z)$ *be a holomorphic map of* S^2 *onto* $S^2(1)$ *of degree* d *defined by polynomials* $P(z)$ *and* $Q(z)$. *Then*

$$\textit{the nullity of } \frac{P(z)}{Q(z)} = 3 + 2 \dim_C(\ker \Psi^d{}_{d*} \textit{ at } [P \wedge Q]).$$

Let M_d be the space of meromorphic functions of degree d on P^1, which is $P(C^{d+1} \times C^{d+1} - \mathscr{R}) \subset P^{2d+2}$, where $\mathscr{R}$ is the irreducible resultant divisor. $A \in \mathrm{GL}(2, C)$ acts on $C^{d+1} \times C^{d+1}$:

$$\text{for } A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad A \cdot (P, Q) = (\alpha P + \beta Q, \gamma P + \delta Q).$$

Thus we obtain an action of $\mathrm{PSL}(2, C)$ on $P(C^{d+1} \times C^{d+1} - \Delta)$, where $\Delta = \{(P, Q) \colon P \wedge Q = 0\}$. The orbit space is $G(2, d+1)$. So M_d is the total space of a $\mathrm{PSL}(2, C)$-bundle on $G(2, d+1)$ – the image of $\mathscr{R}$. Note that the action of

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

for $P(z)/Q(z)$ is given by

$$A\cdot\frac{P(z)}{Q(z)}=\frac{\alpha P(z)+\beta Q(z)}{\gamma P(z)+\delta Q(z)}.$$

Theorem B states that holomorphic maps of S^2 onto $S^2(1)$ of degree d with an extra eigenfunction is the total space of a PSL(2, C)-principal bundle on $\mathfrak{R}$ [the ramification locus of Ψ_d–the resultant]. Note that $\mathfrak{R}$ is a hypersurface with singularities (see, for example, [Nam]) in $G(2, d+1)$. Furthermore Theorem B is a positive answer for a problem posed in [M-R] and [E-K2].

THEOREM C. *Let g be a holomorphic map of S^2 onto $S^2(1)$ of degree d. Then, if g is an element of* PSL(2, C)*-principal bundle on the regular part of $\mathfrak{R}$, then g has exactly two extra eigenfunctions and admits ψ_t of S^2 into $S^4(1)$ such that $\psi_0 = g$ and ψ_t $(t \neq 0)$ gives a full branched minimal surface of genus 0 in $S^4(1)$. Furthermore a holomorphic map with extra eigenfunctions is a limit of these holomorphic maps.*

2. Infinitesimal horizontal deformation

Let P^3 be the 3-dimensional complex projective space with constant holomorphic sectional curvature 1. Then P^3 is the twistor space of $S^4(1)$. In fact, we consider P^3 = the reductive homogeneous space $SO(5)/1\times U(2)$ and $S^4(1) = SO(5)/1\times SO(4)$. Let p be the projection of $S(5)1\times U(2)$ on $SO(5)/1\times SO(4)$ defined by $p(a(1\times U(2)) = a(1\times SO(4))$. Then p is a Riemannian submersion, which is called the Penrose map. Let H and V denote the projections of the tangent space of P^3 onto the subspaces of horizontal and vertical vectors, respectively. For a point $a(1\times U(2))$ in P^3, the tangent space is identified with the space τ, which is given by

$$\begin{pmatrix} 0 & \mu & \nu \\ -t_{\mu} & A & C \\ -t_{\nu} & C & -A \end{pmatrix} \quad A \text{ and } C \text{ are skew symmetric } 2\times 2 \text{ matrices, } \mu,\ \nu\in R^2.$$

The metric and the complex structure J are given by

$$\left\| \begin{matrix} 0 & \mu & \nu \\ -{}^t\mu & A & C \\ -{}^t\nu & C & -A \end{matrix} \right\|^2 = |\mu|^2+|\nu|^2+2|A|^2+2|C|^2$$

and

$$J\begin{pmatrix} 0 & \mu & \nu \\ -{}^t\mu & A & C \\ -{}^t\nu & C & -A \end{pmatrix} = \begin{pmatrix} 0 & -\nu & \mu \\ {}^t\mu & -C & A \\ -{}^t\mu & A & C \end{pmatrix}.$$

Horizontal vectors are given by

$$\begin{pmatrix} 0 & \mu & \nu \\ -{}^t\mu & 0 & 0 \\ -{}^t\nu & 0 & 0 \end{pmatrix} \quad \text{and}$$

vertical vectors are given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & A & C \\ 0 & C & -A \end{pmatrix}.$$

O'Neill [O'N] defined two tensor fields $\mathscr{T}$ and $\mathscr{A}$ for a Riemannian submersion. For the Penrose map p, since vertical fibres are totally geodesic, $\mathscr{T} = 0$ holds. Let X and Y be vectors at $q = p(a(1 \times U(2))$ and $\tilde{X}$ and $\tilde{Y}$ are horizontal lifts at $a(1 \times U(2))$, respectively. Then

$$\mathscr{A}_{\tilde{X}}\tilde{Y} = \tfrac{1}{2}\tau\text{-component for matrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\langle R_Y e_j, e_k\rangle) & \\ 0 & & \end{pmatrix}$$

where $a = (q, e_1, e_2, e_3, e_4)$, and R is the curvature tensor of $S^4(1)$. Let $S^2(1)$ be a fixed totally geodesic surface in $S^4(1)$. Then we have a horizontal line P^1 in P^3 such that $p(P^1) = S^2(1)$. We have a horizontal line P^1 in P^3 such that $p(P^1) = S^2(1)$. Then we obtain three vector bundles $\mathscr{V}$, $\mathscr{N}$, $\mathscr{T}$ on P^1 such that $\mathscr{V}$ is spanned by vertical vectors, $\mathscr{N}$ is spanned by horizontal, normal vectors, $\mathscr{T}$ is spanned by horizontal, tengential vectors. It is easy to see that these are J-invariant. $\mathscr{V}^C$, $\mathscr{N}^C$, $\mathscr{T}^C$ are complexified vector bundles of $\mathscr{V}$, $\mathscr{N}$, $\mathscr{T}$. Furthermore let $\mathscr{V}^{1,0}$ and $\mathscr{V}^{0,1}$ are line subbundles spanned by type (1, 0)-

vectors and type (0, 1)-vectors, respectively. Similarly we can define line bundles $\mathcal{N}^{1,0}$, $\mathcal{N}^{0,1}$, $\mathcal{T}^{1,0}$, $\mathcal{T}^{0,1}$. Since the normal bundle of $S^2(1)$ in $S^4(1)$ is a trivial bundle, there exist two orthonormal vector fields for the normal bundle. Thus we obtain a vector field E_2 of $\mathcal{N}^{0,1}$ with square length 2 globally defined on P^1, i.e.,

$$JE_2 = -iE_2 \quad \text{and} \quad |E_2|^2 = 2.$$

Let $\tilde{\nabla}$ and $\tilde{\mathcal{R}}$ be the covariant differentiation and the curvature tensor of P^3, respectively. Using the formula for $\mathcal{A}$ [O'N], we obtain the following.

LEMMA 2.1 (see, for example, [E1] and [E-S]).
Let E_1 be a tangent vector field locally defined of $\mathcal{T}^{0,1}$ such that

$$JE_1 = -iE_1 \quad \textit{and} \quad |E_1|^2 = 2.$$

Then $\tilde{\nabla}_{E_1}E_2$ is a vertical vector field of type (0, 1) *and square length* 2, *which is denoted by E_3. Furthermore*

$$\tilde{\nabla}_{E_1}\overline{E_2} = 0, \quad (\tilde{\nabla}_{E_1}E_3)^H = 0, \quad (\tilde{\nabla}_{E_1}\overline{E_3})^H = -\overline{E_2}$$

hold. Let ω be the connection form for $\mathcal{V}$ given by

$$(\tilde{\nabla}_X E_3)^V = -i\omega(X)E_3$$

and $\rho^2|dz|^2$ be the metric for a complex coordinate $z = x + iy$ of P^1. We set

$$E_1 = \frac{2}{\rho}\frac{\partial}{\partial \bar{z}}.$$

Then

$$(\tilde{\nabla}_{E_1}E_3)^V = \frac{2\rho_{\bar{z}}}{\rho^2}E_3.$$

Proof. It is enough to prove the last statement.

$$\omega(E_1) = -\frac{i}{2}\langle\tilde{\nabla}_{E_1}\overline{E_3}, E_3\rangle = -\frac{i}{2}\langle\tilde{\nabla}_{E_1}\tilde{\nabla}_{\overline{E_1}}E_2, E_3\rangle$$

$$= -\frac{i}{2}\left\langle\tilde{\nabla}_{2\partial/\rho\partial\bar{z}}\frac{2}{\rho}\tilde{\nabla}_{\partial/\partial z}\overline{E_2}, E_3\right\rangle$$

$$= -\frac{i}{\rho}\left(\frac{\partial}{\partial \bar{z}}\frac{2}{\rho}\right)\langle\tilde{\nabla}_{\partial/\partial z}\overline{E_2}, E_3\rangle - \frac{i2}{\rho^2}\langle\tilde{\nabla}_{\partial/\partial \bar{z}}\tilde{\nabla}_{\partial/\partial z}\overline{E_2}, E_3\rangle$$

$$= 2i\frac{\rho_{\bar{z}}}{\rho^2} - \frac{4}{\rho^2}\langle\tilde{R}_{\partial/\partial \bar{z}\ \partial/\partial z}\overline{E_2}, E_3\rangle = 2i\frac{\rho_{\bar{z}}}{\rho^2}.$$

Q.E.D.

Let $\bar{M}$ be a compact Riemann surface and g a holomorphic map of $\bar{M}$ onto $S^2(1)$. Then we may consider that g is a map of $\bar{M}$ onto P^1 (into P^3). Let $\Gamma(g^*(\mathcal{N} + \mathcal{V})$. For $V \in \Gamma(g^*(\mathcal{N} + \mathcal{V}))$, V is given by

$$V = fE_2 + \overline{fE_2} + \xi + \bar{\xi},$$

where f is a function on $\bar{M}$ and $\xi \in \Gamma(g^*(\mathcal{V}^{1,0}))$. We consider the condition where V is an infinitesimal horizontal deformation of g. Let ϕ_t be a variation of g such that the variational vector field at $t = 0$ is V. Then V is an infinitesimal horizontal deformation of g if and only if, for any vertical field U, we have

$$\left.\frac{d}{dt}\right|_{t=0}\langle\phi_{t*}(X), U\rangle = 0,$$

where X is a tangent vector field. Let $z = x + iy$ be a complex coordinate of $\bar{M}$. Then

$$\left.\frac{d}{dt}\right|_{t=0}\left\langle\phi_{t*}\left(\frac{\partial}{\partial z}\right), U\right\rangle = \langle\tilde{\nabla}_{\partial/\partial z}V, U\rangle + \left\langle\phi_*\left(\frac{\partial}{\partial z}\right), \tilde{\nabla}_V U\right\rangle.$$

Since $\tilde{\nabla}_{V^v}U$ is vertical, $\langle\phi_*(\partial/\partial z), \tilde{\nabla}_{V^v}U\rangle = 0$. We get

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\left\langle\phi_{t*}\left(\frac{\partial}{\partial z}\right), U\right\rangle &= \langle\tilde{\nabla}_{\partial/\partial z}V, U\rangle + \left\langle\phi_*\left(\frac{\partial}{\partial z}\right), \mathcal{A}_{V^H}U\right\rangle \\ &= \langle\tilde{\nabla}_{\partial/\partial z}V, U\rangle - \left\langle\mathcal{A}_{V^H}\phi_*\left(\frac{\partial}{\partial z}\right), U\right\rangle \\ &= \langle\tilde{\nabla}_{\partial/\partial z}V, U\rangle + \langle\mathcal{A}_{\phi_*(\partial/\partial z)}V^H, U\rangle,\end{aligned}$$

which implies that V is a horizontal deformation of g if and only if

$$(2\tilde{\nabla}_X V^H + \tilde{\nabla}_X V^V)^V = 0.$$

Thus we obtain the following.

LEMMA 2.2. *$V = fE_2 + \overline{fE_2} + \xi + \bar{\xi}$ is an infinitesimal horizontal deformation of g if and only if*

$$\xi \text{ is a holomorphic section of } g^*(\mathscr{V}^{1,0}), \tag{2.1}$$

$$2f(\tilde{\nabla}_{\partial/\partial\bar{z}} E_2\rangle^V + (\tilde{\nabla}_{\partial/\partial\bar{z}}\bar{\xi})^V = 0. \tag{2.2}$$

Proof. Since the ramification locus of g are isolated points, it is enough to prove (2.1) and (2.2) on the points except ramification locus. Using a complex coordinate z, we put

$$E_1 = \frac{1}{\left|g_*\left(\frac{\partial}{\partial\bar{z}}\right)\right|} g_*\left(\frac{\partial}{\partial\bar{z}}\right)$$

and $\tilde{\nabla}_{E_1} E_2 = E_3$. Then there exists a function h such that $\bar{\xi} = hE_3$. By Lemma 2.1, we get

$$(\tilde{\nabla}_{E_1} V^H)^V = (\tilde{\nabla}_{E_1}(hE_2 + \overline{hE_2}))^V = hE_3,$$

$$(\tilde{\nabla}_{E_1} V^V)^V = (E_1 h)E_3 + (E_1\bar{h})\overline{E_3} - ih\omega(E_1)E_3 + i\bar{h}\omega(E_1)\overline{E_3}.$$

So we get

$$2fE_3 + (\tilde{\nabla}_{E_1}(hE_3))^V = 0 \quad \text{and} \quad (\tilde{\nabla}_{E_1}(\overline{hE_3}))^V = 0.$$

So the first equation, together with $\tilde{\nabla}_{E_1} E_2 = E_3$, implies (2.2). The second equation says that ξ $(=\overline{hE_3})$ is holomorphic in $g^*(\mathscr{V}^{1,0})$. Q.E.D.

Next we consider the condition that V is an infinitesimal holomorphic deformation of g. V is holomorphic if and only if the $(1,0)$-component $V^{1,0}$ of V satisfies

$$\tilde{\nabla}_{\partial/\partial\bar{z}} V^{1,0} = 0.$$

Since $V^{1,0} = \overline{fE_2} + \xi$,

$$\tilde{\nabla}_{\partial/\partial\bar{z}}(\overline{fE_2 + \xi}) = (\tilde{\nabla}_{\partial/\partial\bar{z}})^V + (\tilde{\nabla}_{\partial/\partial\bar{z}}\xi)^H + \frac{\partial}{\partial\bar{z}}\overline{fE_2}.$$

Thus V is infinitesimal homomorphic if and only if ξ is holomorphic in $g^*(\mathscr{V}^{1,0})$ and

$$\tilde{\nabla}_{\partial/\partial\bar{z}}\overline{fE_2} + (\tilde{\nabla}_{\partial/\partial\bar{z}}\xi)^H = 0.$$

This completes the following.

LEMMA 2.3. *$V = fE_2 + \overline{fE_2} + \xi + \bar{\xi}$ is an infinitesimal holomorphic deformation if and only if*

$$\xi \text{ is a holomorphic section of } \mathscr{V}^{1,0}, \tag{2.3}$$

$$\frac{\partial f}{\partial z} = -\frac{1}{2}\langle\tilde{\nabla}_{\partial/\partial z}\bar{\xi}, \overline{E_2}\rangle. \tag{2.4}$$

We prove that an infinitesimal horizontal deformation is also an infinitesimal holomorphic deformation.

LEMMA 2.4. *Let V be an infinitesimal horizontal deformation of g. Then V is an infinitesimal holomorphic deformation.*

Proof. Since ξ is holomorphic, zero points of ξ is isolated. So it is enough to give a proof for points except ramification locus of g and zero points of ξ. Let $V = fE_2 + \overline{fE_2} + \xi + \bar{\xi}$ be an infinitesimal horizontal deformation. Then ξ is holomorphic in $g^*(\mathscr{V}^{1,0})$ and

$$2f(\tilde{\nabla}_{\partial/\partial\bar{z}}E_2)^V + (\tilde{\nabla}_{\partial/\partial\bar{z}}\bar{\xi}) = 0.$$

We get a covariant differentiation of both sides by $\tilde{\nabla}_{\partial/\partial z}$.

$$2\frac{\partial f}{\partial z}(\tilde{\nabla}_{\partial/\partial\bar{z}}E_2)^V + 2f\tilde{\nabla}_{\partial/\partial z}(\tilde{\nabla}_{\partial/\partial\bar{z}}E_2)^V + \tilde{\nabla}_{\partial/\partial z}(\tilde{\nabla}_{\partial/\partial\bar{z}}\bar{\xi})^V = 0.$$

So we obtain

$$\left(2\frac{\partial f}{\partial z} + \langle\tilde{\nabla}_{\partial/\partial z}\bar{\xi}, \overline{E_2}\rangle\right)(\tilde{\nabla}_{\partial/\partial\bar{z}}E_2)^V = \langle\tilde{\nabla}_{\partial/\partial z}\bar{\xi}, \overline{E_2}\rangle(\tilde{\nabla}_{\partial/\partial\bar{z}}E_2)^V$$
$$-2f(\tilde{\nabla}_{\partial/\partial z}(\tilde{\nabla}_{\partial/\partial\bar{z}}E_2)^V)^V - (\tilde{\nabla}_{\partial/\partial z}(\tilde{\nabla}_{\partial/\partial\bar{z}}\bar{\xi})^V)^V.$$

First of all, we calculate the second part of the right hand side.

$$
\begin{aligned}
(\tilde{\nabla}_{\partial/\partial z}(\tilde{\nabla}_{\partial/\partial \bar{z}}E_2)^V)^V &= \left(\tilde{\nabla}_{\partial/\partial z}\left\{\frac{1}{|\xi|^2}\langle\tilde{\nabla}_{\partial/\partial \bar{z}}E_2,\ \xi\rangle\bar{\xi}\right\}\right)^V \\
&= -\frac{1}{|\xi|^4}\langle\bar{\xi},\ \tilde{\nabla}_{\partial/\partial z}E_2\rangle\langle\tilde{\nabla}_{\partial/\partial \bar{z}}E_2,\ \xi\rangle\bar{\xi} \\
&\quad + \frac{1}{|\xi|^2}\langle\tilde{\nabla}_{\partial/\partial z}\tilde{\nabla}_{\partial/\partial \bar{z}}E_2,\ \xi\rangle\bar{\xi} + \frac{1}{|\xi|^2}\langle\tilde{\nabla}_{\partial/\partial \bar{z}}E_2,\ \tilde{\nabla}_{\partial/\partial z}\xi\rangle\bar{\xi} \\
&= -\frac{1}{|\xi|^2}\langle\tilde{R}_{\partial/\partial z\, \partial/\partial \bar{z}}E_2,\ \xi\rangle\bar{\xi} = 0.
\end{aligned}
$$

Next we calculate the third part.

$$
\begin{aligned}
(\tilde{\nabla}_{\partial/\partial \bar{z}}(\tilde{\nabla}_{\partial/\partial \bar{z}}\bar{\xi})^V)^V &= (\tilde{\nabla}_{\partial/\partial z}\tilde{\nabla}_{\partial/\partial \bar{z}}\bar{\xi})^V \\
&= (-\tilde{R}_{\partial/\partial z\, \partial/\partial \bar{z}}\bar{\xi})^V + (\tilde{\nabla}_{\partial/\partial \bar{z}}(\tilde{\nabla}_{\partial/\partial z}\tfrac{1}{2}\langle\bar{\xi},\ \overline{E_3}\rangle E_3)^H)^V \\
&= (-\tilde{R}_{\partial/\partial z\, \partial/\partial \bar{z}}\bar{\xi})^V + \tfrac{1}{2}\langle\bar{\xi},\ \overline{E_3}\rangle\left(\tilde{\nabla}_{\partial/\partial \bar{z}}\left(\frac{\rho}{2}\,\tilde{\nabla}_{\overline{E_1}}E_3\right)^H\right)^V \\
&= (-\tilde{R}_{\partial/\partial z\, \partial/\partial \bar{z}}\bar{\xi})^V + \frac{\rho}{4}\langle\bar{\xi},\ \overline{E_3}\rangle(-\tilde{\nabla}_{\partial/\partial \bar{z}}E_2)^V \\
&= (-\tilde{R}_{\partial/\partial z\, \partial/\partial \bar{z}}\bar{\xi})^V - \frac{\rho^2}{8}\langle\bar{\xi},\ \overline{E_3}\rangle(\tilde{\nabla}_{E_1}E_2)^V \\
&= (-\tilde{R}_{\partial/\partial z\, \partial/\partial \bar{z}}\bar{\xi})^V - \frac{\rho^2}{4}\bar{\xi}.
\end{aligned}
$$

On the other hand, since P^3 has holomorphic sectional curvature 1, we obtain

$$
(\tilde{R}_{\partial/\partial z\, \partial/\partial \bar{z}}\bar{\xi})^V = \frac{\rho^2}{4}\bar{\xi},
$$

which implies

$$
(\tilde{\nabla}_{\partial/\partial z}(\tilde{\nabla}_{\partial/\partial \bar{z}}\bar{\xi})^V)^V = -\frac{\rho^2}{2}\bar{\xi}.
$$

Furthermore we calculate the first part.

$$\langle \tilde{\nabla}_{\partial/\partial z}\bar{\xi}, \overline{E_2}\rangle (\tilde{\nabla}_{\partial/\partial \bar{z}} E_2)^V = \langle \tilde{\nabla}_{\partial/\partial z}\bar{\xi}, \overline{E_2}\rangle \frac{1}{|\xi|^2} \langle \tilde{\nabla}_{\partial/\partial \bar{z}} E_2, \xi\rangle \bar{\xi}$$

$$= -\frac{\rho^2}{4|\xi|^2} \langle \tilde{\nabla}_{\overline{E_1}} \overline{E_2}, \bar{\xi}\rangle \langle \tilde{\nabla}_{E_1} E_2, \xi\rangle \bar{\xi} = -\frac{\rho^2}{2}\bar{\xi}.$$

Thus we obtain

$$\left(2\frac{\partial f}{\partial z} + \langle \tilde{\nabla}_{\partial/\partial z}\bar{\xi}, \overline{E_2}\rangle\right)(\tilde{\nabla}_{\partial/\partial \bar{z}} E_2)^V = 0.$$

Since $\tilde{\nabla}_{\partial/\partial \bar{z}} E_2$ is non-zero, we get a proof. Q.E.D.

Now we consider the condition for f such that $V = fE_2 + \overline{fE_2} + \xi + \bar{\xi} \in \Gamma(g^*(\mathcal{N} + \mathcal{V}))$ is an infinitesimal horizontal deformation. Note that V must be an infinitesimal holomorphic deformation and hence satisfies (2.4). Thus ξ is determined by f as

$$\xi = 2\overline{\left(\frac{\partial f}{\partial z}\right)} \frac{1}{\langle \tilde{\nabla}_{\partial/\partial \bar{z}} E_2, \overline{E_3}\rangle} \overline{E_3} \tag{2.5}$$

for a point where $\langle \tilde{\nabla}_{\partial/\partial z} E_2, \overline{E_3}\rangle$ does not vanish. We denote it by ξ_f. Since $\rho^2|dz|^2$ is the metric induced by g. Then

$$\rho^2 = 2\left\langle \frac{\partial g}{\partial z}, \overline{\frac{\partial g}{\partial z}}\right\rangle.$$

Set

$$E_1 = \frac{2}{\rho}\frac{\partial g}{\partial z} \quad \text{if } \rho \neq 0$$

and choose E_3 defined by $\tilde{\nabla}_{E_1} E_2 = E_3$. Thus we obtain a local expression of ξ_f by

$$\xi_f = 2\overline{\left(\frac{\partial f}{\partial z}\right)} \frac{1}{\rho} \overline{E_3}. \tag{2.6}$$

Thus the singularity of ξ_f holds only at zeros of s, that is, a point of the ramification locus of g.

LEMMA 2.5. *$fE_2 + \overline{fE_2} + \xi_f + \bar{\xi}_f$ satisfies* (2.2) *if and only if*

$$\frac{\partial^2 f}{\partial z\,\partial\bar{z}} + \frac{\rho^2}{2} f = 0 \quad (\Delta_g f + 2f = 0). \tag{2.7}$$

Proof. Since

$$(\tilde{\nabla}_{\partial/\partial\bar{z}} E_2)^V = \frac{\rho}{2} E_3$$

and

$$(\tilde{\nabla}_{\partial/\partial\bar{z}}\bar{\xi})^V = \left(\tilde{\nabla}_{\partial/\partial\bar{z}} \frac{2}{\rho}\frac{\partial f}{\partial z} E_3\right)^V$$
$$= \frac{2}{\rho}\frac{\partial^2 f}{\partial\bar{z}\,\partial z} E_3 - \frac{2\rho_{\bar{z}}}{\rho^2}\frac{\partial f}{\partial z} E_3 - i\frac{2}{\rho}\frac{\partial f}{\partial z}\,\omega\left(\frac{\partial}{\partial\bar{z}}\right) E_3,$$

by Lemma 2.1, we obtain

$$2f(\tilde{\nabla}_{\partial/\partial\bar{z}} E_2)^V + (\tilde{\nabla}_{\partial/\partial\bar{z}}\bar{\xi})^V = \left(\frac{2}{\rho}\frac{\partial^2 f}{\partial z\,\partial\bar{z}} + \rho f\right) E_3. \qquad \text{Q.E.D.}$$

ξ_f for f which satisfies (2.7) is not generally holomorphic in $\mathscr{V}^{1,0}$. We obtain another condition that ξ_f is holomorphic.

LEMMA 2.6. *ξ_f is a holomorphic section of $\mathscr{V}^{1,0}$ if and only if*

$$\frac{\partial^2 f}{\partial z^2} - 2\frac{\partial f}{\partial z}\frac{\rho_z}{\rho} = 0. \tag{2.8}$$

Proof.

$$(\tilde{\nabla}_{\partial/\partial\bar{z}}\xi_f)^V = \left(\tilde{\nabla}_{\partial/\partial\bar{z}} \frac{2}{\rho}\overline{\frac{\partial f}{\partial z}}\,\overline{E_3}\right)^V$$
$$= \frac{2}{\rho}\overline{\frac{\partial^2 f}{\partial z^2}}\,\overline{E_3} - \frac{2\rho_{\bar{z}}}{\rho^2}\overline{\frac{\partial f}{\partial z}}\,\overline{E_3} + i\frac{2}{\rho}\overline{\frac{\partial f}{\partial z}}\,\omega\left(\frac{\partial}{\partial\bar{z}}\right)\overline{E_3}.$$

By Lemma 2.1, we obtain

$$(\tilde{\nabla}_{\partial/\partial\bar{z}}\xi_f)^V = \left(\frac{2}{\rho}\overline{\frac{\partial^2 f}{\partial z^2}} - \frac{4\rho_{\bar{z}}}{\rho^2}\overline{\frac{\partial f}{\partial z}}\right)\overline{E_3}. \qquad \text{Q.E.D.}$$

Remark that (2.8) implies Hess $f(\partial/\delta z, \partial/\partial z) = 0$. Note that $V = fE_2 + \overline{fE_2} + \xi_f + \overline{\xi_f}$ is an infinitesimal holomorphic deformation if and only if f satisfies (2.8). Furthermore, the condition that V becomes an infinitesimal horizontal deformation requires (2.7) instead of (2.8). So we need to investigate a complex valued function on $\bar{M}$ which satisfies

$$\Delta_g f = -2f \quad \text{and Hess } f\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = 0. \tag{2.9}$$

First of all, we study local solutions of (2.9). Let U be a simply connected open set of $R^2(=C)$ and z a complex coordinate on U. Let χ_1 be a branched minimal immersion of U into R^3. Then χ_1 is given by

$$\mathrm{Re}\left\{\int \Phi \, \mathrm{d}z\right\} + c_1,$$

where $\Phi = (\partial/\partial z)\chi_1$ and $c_1 \in R^3$. Since U is simply connected, there exists a conjugate branched minimal surface whose branched immersion χ_2 is given by

$$\mathrm{Re}\left\{\int i\Phi \, \mathrm{d}z\right\} + c_2.$$

Let g be the Gauss map of U into $S^2(1)$. Then we can define a function f by $\langle \chi_1 + i\chi_2, g\rangle$. Since

$$\frac{\partial}{\partial z}(\chi_1 + i\chi_2) = \Phi + i(i\Phi) = 0,$$

we obtain

$$\frac{\partial f}{\partial z} = \left\langle \chi_1 + i\chi_2, \frac{\partial g}{\partial z}\right\rangle \quad \text{and} \quad \frac{\partial^2 f}{\partial^2 z} = \left\langle \chi_1 + i\chi_2, \frac{\partial^2 g}{\partial^2 z}\right\rangle.$$

$$\langle g, g\rangle = 1, \quad \left\langle g, \frac{\partial g}{\partial z}\right\rangle = \left\langle \frac{\partial g}{\partial z}, \frac{\partial g}{\partial z}\right\rangle = 0.$$

So

$$\left\langle \frac{\partial^2 g}{\partial z^2}, g\right\rangle = -\left\langle \frac{\partial g}{\partial z}, \frac{\partial g}{\partial z}\right\rangle = 0$$

and

$$\left\langle \frac{\partial^2 g}{\partial z^2}, \frac{\partial g}{\partial g}\right\rangle = 0$$

and hence

$$\begin{aligned}\frac{\partial^2 g}{\partial z^2} &= \left\langle \frac{\partial^2 g}{\partial z^2}, g\right\rangle g + \frac{1}{\left|\frac{\partial g}{\partial z}\right|^2}\left(\left\langle \frac{\partial^2 g}{\partial z^2}, \frac{\partial g}{\partial z}\right\rangle \overline{\frac{\partial g}{\partial z}} + \left\langle \frac{\partial^2 g}{\partial z^2}, \overline{\frac{\partial g}{\partial z}}\right\rangle \frac{\partial g}{\partial z}\right)\\ &= 2\,\frac{\rho_z}{\rho}\frac{\partial g}{\partial z}.\end{aligned}$$

Thus f satisfies (2.9). Conversely let g be a non-constant holomorphic map of U into $S^2(1)$ without branched points. Let $f = f_1 + if_2$ be a function on U satisfying (2.9). Then f_1 satisfies (2.7). Generally we have the following.

LEMMA 2.7 (see, for example, [E3]). *Let h be a real-valued function on U which satisfies* (2.7) *and* grad h *the gradient vector fields on U for the metric induced by g. We identify* grad h *with* $g_*(\text{grad}\, h)$ *and may consider that* grad h *is a vector field in R^3. Then the map χ_h of U into R^3 by*

$$\chi_h = hg + \text{grad}\, h$$

is a branched minimal immersion of U into R^3 or a constant map. χ_h is a constant map if and only if h satisfies (2.8).

When χ_{f_1} is a constant, f_1 is a linear function and satisfies (2.8). Since f satisfies (2.9), f_2 also satisfies (2.9) and hence f_2 is a linear function. Thus f is a linear function. Assumed that χ_{f_1} is not a constant. Then χ_{f_1} gives a branched minimal surface. Since U is simply connected, we obtain a conjugate minimal surface χ_2 defined on U. So $\langle \chi_{f_1} + i\chi_2, g\rangle$ satisfies (2.9). $\tilde{f} =$

$i(f - \langle \chi_{f_1} + i\chi_2, g\rangle)$ is a real valued function satisfying (2.9), which implies $f = \langle \chi_{f_1} + i\chi_2, g\rangle + a$ linear function. By a parallel translation of χ_{f_1} and χ_2, we know $f = \langle \chi_{f_1} + i\chi_2, g\rangle$.

Next we study the global solutions of (2.9). For a global solution of (2.7), we have the following.

THEOREM 2.3 ([E-K1], [M-R]). *Let g be a holomorphic map of* $\bar{M}$ *onto* $S^2(1)$ *and h a solution of* (2.7). *Without loss of generality, we may consider that h is not a linear function. Then* $\chi_h = hg + \operatorname{grad} h$ *gives a complete, finitely branched minimal surface of planar ends and finite total curvature whose map is defined on* $\bar{M} - \{$*finite points*) *and* (*extended*) *Gauss map is g* ([E-K1], [M-R]). *If* $\bar{M}$ *has the genus* 0, *then* χ_h *has a conjugate minimal surface.*

Generally, there is no conjugate minimal surface of the above minimal surface for non-zero genus. But if there exists a solution of (2.9), we have two real solutions of (2.7) which are the real part and imaginary part. These solutions give two complete, finitely branched minimal surface of planar ends whose extended Gauss map is g. One of them is a conjugate minimal surface of the other by (2.8). Conversely we obtain the following.

PROPOSITION 2.2. *Let f be a function satisfying* (2.9) *on* $\bar{M}$. *Then f is constructed by a complete, finitely branched minimal surface of planar ends and its conjugate minimal surface.*

Let f be a function satisfying (2.9). Now we investigate the behaviour of ξ_f at a branch point q of g. Let z be a complex coordinate such that $z(q) = 0$. Then there exists n such that $\partial g/\partial z = z^n n(z)$, where $n(0) \neq 0$. We set $E_1 = \overline{n(z)/|n(z)|}$ and $E_3 = \tilde{\nabla}_{E_1} E_2$. Then ξ_f is given by

$$\overline{\frac{\partial f}{\partial z}} \frac{1}{\bar{z}^n |n|} \overline{E_3}.$$

By Proposition 2.2, we have a complete, finitely branched minimal surface χ_1 of planar ends and finite total curvature and its conjugate minimal surface χ_2 (whose extended Gauss maps are g) such that $f = \langle \chi_1 + i\chi_2, g\rangle$. Thus we have

$$\frac{\partial f}{\partial z} = \left\langle \chi_1 + i\chi_2, \frac{\partial g}{\partial z}\right\rangle.$$

For $(\partial/\partial z)\chi_1 = \Phi$, we obtain

$$\frac{\partial}{\partial \bar{z}}(\chi_1 + i\chi_2) = \bar{\Phi} \quad \text{and} \quad \frac{\partial}{\partial z}(\chi_1 + i\chi_2) = 0.$$

Let

$$\frac{a_{-k}}{z^k} + \cdots + \frac{a_{-2}}{z^2} + a_0 + a_1 z + \cdots$$

be the Laurent expansion of Φ at 0, where $a_{-k}, \ldots \in C^3$. We denote by $l(z)$ $z^k\Phi$. Then l is holomorphic and $\langle g, l\rangle = \langle g, \bar{l}\rangle = \langle l, l\rangle = 0$ holds. We have p_1 and p_2 such that

$$\frac{\partial}{\partial z} l = p_1 l + p_2 g.$$

Taking the differentiation of both sides by $\partial/\partial\bar{z}$, we note that

$$\frac{\partial g}{\partial z} = -\frac{p_2}{|l|^2}\bar{l},$$

$$\frac{\partial}{\partial \bar{z}} p_2 = 0, \quad \frac{\partial}{\partial \bar{z}} p_1 = \frac{|p_2|^2}{|l|^2}.$$

For a planar end, we have the following.

LEMMA 2.8 ([E-K1] and [M-R]).

$$k \leqslant n + 1.$$

Since $g(0)$, $l(0)$ and $\overline{l(0)}$ is a basis of C^3, we obtain

$$\left\langle \chi_1 + i\chi_2, \frac{\partial g}{\partial z}\right\rangle = \langle \chi_1 + i\chi_2, g(0)\rangle \left\langle g(0), \frac{\partial g}{\partial z}\right\rangle + \langle \overline{\chi_1 + i\chi_2}, l(0)\rangle \left\langle l(0), \frac{\partial g}{\partial z}\right\rangle \frac{1}{|l(0)|^2} + \langle \chi_1 + i\chi_2, \overline{l(0)}\rangle \left\langle l(0), \frac{\partial g}{\partial z}\right\rangle \frac{1}{|l(0)|^2}.$$

It is easy to show the following.

LEMMA 2.9.

$$\langle l(z),\ g(0)\rangle = z^{n+1}\#$$
$$\langle l(z),\ l(0)\rangle = z^{2n+2}\#,$$

where, # means non-zero functions.

Since

$$\frac{\partial}{\partial \bar{z}}\langle \chi_1 + i\chi_2, g(0)\rangle = \frac{1}{\bar{z}^k}\langle \overline{l(z)}, g(0)\rangle = \bar{z}^{n+1-k}\#,$$

$\overline{\langle \chi_1 + i\chi_2,\ g(0)\rangle}$ is holomorphic. By

$$\left\langle \overline{l(0)}, \frac{\partial g}{\partial z}\right\rangle = \left\langle \overline{l(0)},\ -\frac{p_2}{|l|^2}\overline{l(z)}\right\rangle = -\frac{p_2}{|l|^2}\bar{z}^{2n+2}$$

$$\langle \chi_1 + i\chi_2, l(0)\rangle = \frac{1}{\bar{z}^{k-1}}\#,$$

we obtain

$$\langle \chi_1 + i\chi_2, l(0)\rangle \left\langle \overline{l(0)}, \frac{\partial g}{\partial z}\right\rangle = \bar{z}^{2n+2-k+1}\#.$$

Similarly we obtain

$$\langle \chi_1 + i\chi_2, \overline{l(0)}\rangle = \bar{z}^{2n+2-k+1}\#,$$
$$\left\langle l(0), \frac{\partial g}{\partial z}\right\rangle = \left\langle l(0),\ -\frac{p_2}{|l|^2}\bar{l}\right\rangle.$$

Using these estimates, we have $|\xi_f| <$ a constant and hence ξ_f is holomorphic at p.

LEMMA 2.10. *Let f be a function satisfying* (2.9) *on* $\bar{M}$. *Then*

$$fE_2 + \overline{fE_2} + \xi_f + \overline{\xi_f}$$

is an infinitesimal horizontal (*holomorphic*) *deformation.*

By Theorem 2.3 and Lemma 2.10, we obtain Theorem A.

REMARK. We can see that our argument is closely related to the theory of holomorphic null maps [Br]. In fact, we can give a relation between meromorphic sections of $\mathscr{V}^{1,0}$ and holomorphic null maps of $\bar{M}$ − {isolated points} into C^3. Precise result in a forthcoming paper.

3. Infinitesimal contact deformation

In this section, we use Loo' notations in [Loo].

Recall the diagram:

$$\begin{array}{ccccc} P^3 & \xleftarrow{\beta} & X & \xrightarrow{\psi} & Y \\ p\downarrow & & \tilde{\pi}\downarrow & & \downarrow\pi \\ S^4 & & P^1\times P^1 & = & P^1\times P^1 \end{array}$$

Let N and S denote the north and south poles of $S^4(1)$, respectively. Consider the two lines $L_1 = p^{-1}(N)$ and $L_2 = p^{-1}(S)$. Then we get $L_1 = \{[0, 0, z_2, z_3] \mid [z_2, z_3] \in P^1\}$ and $L_2 = \{[z_0, z_1, 0, 0] \mid [z_0, z_1] \in P^1\}$. Let X be a blow up of P^3 along L_1 and L_2. Then X is given by $X := \{([z_0, z_1, z_2, z_3], [y_0, y_1], [y_2, y_3]) \mid z_0 y_1 = z_1 y_0, z_2 y_3 = z_3 y_2\}$. Let Y be the projective cotangent bundle $PT^*(P^1 \times P^1)$ on $P^1 \times P^1$. Let $\psi: X \to Y$ be defined by

$$\begin{aligned} &\psi([z_0, z_1, z_2, z_3], [y_0, y_1], [y_2, y_3]) \\ &\quad = ([y_0, y_1], [y_2, y_3], [z_0\, dy_1 - z_1\, dy_0, z_2\, dy_3 - z_3\, dy_2]). \end{aligned}$$

Let σ_1 and σ_2 be $\beta^{-1}(L_1)$ and $\beta^{-1}(L_2)$, respectively. Then we observe that $\psi(\sigma_1) = \{([y_0, y_1], [y_2, y_3], [0, 1]) \colon [y_0, y_1] \in P^1$ and $[y_2, y_3] \in P^1\}$ and $\psi(\sigma_2) = \{([y_0, y_1], [y_2, y_3], [1, 0]) \colon [y_0, y_1] \in P^1$ and $[y_2, y_3] \in P^1\}$. Let $[q]$ be a point of Y. Then we have a tangent line $l_{[q]}$ of $T(P^1 \times P^1)$ such q annihilate $l_{[q]}$, and the plane $\mathscr{H}_{[q]}$ such that $X \in T_{[q]}Y$ satisfies $\pi_*(X) \in l_{[q]}$. Hence we obtain a plane field $\mathscr{H}$ on Y, which is called a contact plane field. For a holomorphic map $F: \Sigma \to P^1 \times P^1$, we have a canonical Gauss lift $\tilde{F}$ to Y defined by

$$\tilde{F}(z) = \left(F(z), \left[\text{the annihilator of } F_*\left(\frac{\partial}{\partial z}\right)\right]\right).$$

Note that $\tilde{F}_*(\partial/\partial z) \in \mathscr{H}_{\tilde{F}(z)}$. That is, $\tilde{F}$ is a contact curve. We know the relation between the horizontal plane field on P^3 and the contact plane field on Y.

LEMMA 3.1 ([Loo]). *ψ is contact map, i.e. ψ_* sends the horizontal plane field $\mathscr{H}$ in X to the contact plane field $\mathscr{K}$ in Y.*

Let Σ be a horizontal curve in P^3 which does not intersect L_1 and L_2. Then Σ is also a horizontal curve in X and hence $\psi(\Sigma)$ is a contact curve in Y. Conversely, if a contact curve does not intersect $\psi(\sigma_1)$ and $\psi(\sigma_|)$, then there exists a horizontal curve $\tilde{\Sigma}$ such that $\tilde{\Sigma}$ does not intersect L_1 and L_2, $\psi(\tilde{\Sigma})$ is a finite covering of the contact curve. Furthermore, if V is an infinitesimal horizontal (holomorphic) deformation of Σ, then V gives an infinitesimal contact (holomorphic) deformation U of $\psi(\Sigma)$.

Let (s, t, w) be a complex coordinate of Y corresponding $s = y_1/y_0$, $t = y_3/y_2$ and $\mathrm{d}s + w\,\mathrm{d}t$. Then we consider a line $s \to (s, s, -1)$ as $\psi(P^1)$. Thus we may identify a holomorphic map of P^1 onto P^1 in P^3 with a holomorphic map of P^1 onto $\psi(P^1)$ in Y. Let $g(z)$ be $Q(z)/P(z)$, where $P(z)$ and $Q(z)$ are polynomials such that $\max(\deg P(z), \deg Q(z)) = \deg g$. Since U is an infinitesimal holomorphic deformation of g in Y, we have three meromorphic functions $v^1(z)$, $v^2(z)$ and $v^3(z)$ on P^1 such that

$$U^{1,0} = v^1 \frac{\partial}{\partial s} + v^2 \frac{\partial}{\partial t} + v^3 \frac{\partial}{\partial w}.$$

v^1 and v^2 may be considered as infinitesimal holomorphic deformations of a holomorphic map g of P^1 onto P^1. It is easy to find two one parameter families of holomorphic maps h_t^1 and h_t^2 of P^1 onto P^1 such that

$$v^1(z) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} h_t^1(z), \quad v^2(z) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} h_t^2(z),$$

where we consider that h_t^1 and h_t^2 are two one parameter families of meromorphic functions.

LEMMA 3.2. *U is an infinitesimal contact deformation if and only if*

$$\frac{\partial v^1}{\partial z} - \frac{\partial v^2}{\partial z} + v^3 \frac{\partial g}{\partial z} = 0.$$

Proof. Let ϕ_t be a smooth deformation of g in Y such that the variation vector field at $t = 0$ is U. Then, since U is an infinitesimal contact deformation, we get

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathrm{d}s + w\,\mathrm{d}t)\left(\left(\phi_{t*}\left(\frac{\partial}{\partial z}\right)\right)^{1,0}\right) = 0,$$

which implies a proof. Q.E.D.

We set

$$h_t^1(z) = \frac{a_d z^d + \cdots + a_0}{b_d z^d + \cdots + b_0} = \frac{P_t(z)}{Q_t(z)}$$

$$h_t^2(z) = \frac{\tilde{a}_d z^d + \cdots + \tilde{a}_0}{\tilde{b}_d z^d + \cdots + \tilde{b}_0} = \frac{\tilde{P}_t(z)}{\tilde{Q}_t(z)},$$

where $a_d, \ldots, \tilde{b}_0$ are smooth functions for t.

By the definition of v_1 and v_2, we get

$$v_1 = \frac{1}{Q^2}(P'Q - PQ') \quad \text{and} \quad v_2 = \frac{1}{Q^2}(\tilde{P}'Q - P\tilde{Q}'),$$

where

$$P' = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} P_t(z), \quad Q' = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} Q_t(z),$$

$$\tilde{P}' = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tilde{P}_t(t) \quad \text{and} \quad \tilde{Q}' = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tilde{Q}_t(z).$$

Therefore

$$\begin{aligned} \frac{\partial v_1}{\partial z} &= \frac{1}{Q^2}\left(\frac{\partial P'}{\partial z} Q - P' \frac{\partial Q}{\partial z} - \frac{\partial P}{\partial z} Q' - P \frac{\partial Q'}{\partial z}\right) + 2\frac{\partial Q}{\partial z}\frac{PQ'}{Q^3}, \\ \frac{\partial v_2}{\partial z} &= \frac{1}{Q^2}\left(\frac{\partial \tilde{P}'}{\partial z} Q - \tilde{P}' \frac{\partial Q}{\partial z} - \frac{\partial P}{\partial z} \tilde{Q}' - P \frac{\partial \tilde{Q}'}{\partial z}\right) + 2\frac{\partial Q}{\partial z}\frac{P\tilde{Q}'}{Q^3}. \end{aligned} \tag{3.1}$$

Set

$$P(t) = (a_d, \ldots, a_0), \quad Q(t) = (b_d, \ldots, b_0)$$

$$\tilde{P}(t) = (\tilde{a}_d, \ldots, \tilde{a}_0), \quad \tilde{Q}(t) = (\tilde{b}_d, \ldots, \tilde{b}_0).$$

Then we have two curves $\gamma(t)$ and $\tilde{\gamma}(t)$ in P^{2d-2} defined by

$$\Psi_d[P_t \wedge Q_t] \quad \text{and} \quad \Psi_d[\tilde{P}_t \wedge \tilde{Q}_t],$$

where Ψ_d is defined in the introduction. Let $\bar{\gamma}$ and $\bar{\tilde{\gamma}}$ be two curves in C^{2d-1} given by

$$\text{coeff}\left\{Q_t(z)\,\frac{\partial P_t(z)}{\partial z} - P_t(z)\,\frac{\partial Q_t(z)}{\partial z}\right\}$$

$$\text{coeff}\left\{\tilde{Q}_t(z)\,\frac{\partial \tilde{P}_t(z)}{\partial z} - P_t(z)\,\frac{\partial \tilde{Q}_t(z)}{\partial z}\right\}.$$

Then we get $\gamma(t) = [\bar{\gamma}(t)]$ and $\tilde{\gamma}(t) = [\bar{\tilde{\gamma}}(t)]$.

LEMMA 3.3.

$$\frac{\mathrm{d}\bar{\gamma}}{\mathrm{d}t}(0) - \frac{\mathrm{d}\bar{\tilde{\gamma}}}{\mathrm{d}t}(0) = \text{coeff}\left\{\left(v^{12} + 2\,\frac{Q' - \tilde{Q}'}{Q}\right)\left(-Q^2\,\frac{\partial g}{\partial z}\right)\right\}.$$

Proof. Using (3.1), we get the following:

$$Q^2\left(\frac{\partial v^1}{\partial z} - \frac{\partial v^2}{\partial z}\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tilde{\gamma} - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \bar{\tilde{\gamma}} - 2\,\frac{Q' - \tilde{Q}'}{Q}\left(Q\,\frac{\partial P}{\partial z} - P\,\frac{\partial Q}{\partial z}\right).$$

On the other hand, since

$$\frac{\partial v^1}{\partial z} - \frac{\partial v^2}{\partial z} = -v^3\,\frac{\partial g}{\partial z},$$

we get

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \bar{\gamma} - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \bar{\tilde{\gamma}} = \left(v^3 + 2\,\frac{Q' - \tilde{Q}'}{Q}\right)\left(-Q^2\,\frac{\partial g}{\partial z}\right). \tag{3.2}$$

This completes a proof. Q.E.D.

Without loss of generality, we may assume that Q and $\partial Q/\partial z$ has no common divisor and degree of $Q^2(\partial g/\partial z)$ is $2d - 2$.

Using $w = -1$ for $\psi(P^1)$, we obtain

LEMMA 3.4.

$$v^3(z) = \frac{\beta}{\alpha}$$

where α and β are polynomials. Furthermore zeros of α are contained in zeros of Q.

Proof. If z_0 is not zero point of $Q(z)$, then $Q_t(z_0)$ is non-zero and hence $P(z_0)/Q(z_0)$ is finite, which implies $v^3(z_0)$ is finite, because w-coordinate of $\psi(P^1)$ in Y is -1, and hence z_0 is not a point of zeros of $\alpha(z)$. Q.E.D.

Since the left hand side of (3.2) is degree $2d - 2$,

$$\text{degree}\left(\left(\frac{\beta}{\alpha} + 2\frac{Q' - \tilde{Q}'}{Q}\right)\left(-Q^2\frac{\partial g}{\partial z}\right)\right) \leqslant 2d - 2,$$

Since $Q^2(\partial g/\partial z)$ is a polynomial $P'Q - PQ'$ of degree $2d - 2$ which does not have common divisor with Q, we obtain $\beta/\alpha + 2(Q' - \tilde{Q}'/Q)$ is constant by Lemma 3.5. Using Lemma 3.4, we get $\gamma_*(0) = \tilde{\gamma}_*(0)$. That is,

$$\Psi_d([P_t \wedge Q_t])_*(0) = \Psi_d([\tilde{P}_t \wedge \tilde{Q}_t])_*(0).$$

Now we consider a linear map Ω of the space $\mathscr{E}$ of infinitesimal contact deformations into $\ker \Psi_{d*}$ at $[P \wedge Q]$ defined by

$$X \to X_1 - X_2$$

where X_1 is the tangent vector of $[P_t \wedge Q_t]$ and X_2 is the tangent vector of $[\tilde{P}_t \wedge \tilde{Q}_t]$. We already proved $X_1 - X_2 \in \ker \Psi_{d*}$.

LEMMA 3.5. *Ω is onto.*

Proof. Let $\xi \in \ker \Psi_{d*}$. Then there is a curve $P_t(z)/Q_t(z)$ such that ξ is the tangent vector of $[P_t \wedge Q_t]$ at $t = 0$. We define a section η of the bundle $g^*(TPT^*(P^1 \times P^1))$

$$\frac{P'Q - PQ'}{Q^2}\frac{\partial}{\partial s} + 2\frac{Q'}{Q}\frac{\partial}{\partial w}.$$

Of course, since we use the coordinate (s, t, z) of $PT^*(P^1 \times P^1)$, for $P/Q = \infty$, we should prove that η is well-defined. And it is easy. Using $\xi \in \ker \Psi_{d*}$, we obtain

$$\frac{\partial P'}{\partial z} Q + \frac{\partial P}{\partial z} Q' - \frac{\partial Q'}{\partial z} P - \frac{\partial Q}{\partial z} P' = 0,$$

which implies that η is an infinitesimal contact deformation and $\Omega(\eta) = \xi$.

Q.E.D.

Next we investigate the kernel (ker Ω) of Ω.

Let P_t/Q_t be a one parameter family of holomorphic maps such that $g = P_0/Q_0$. Then $p \cdot P_t/Q_t$ is a map of P^1 onto $S^2(1)$ in $S^4(1)$, which gives a minimal deformation of $p \cdot g$ and hence horizontal deformation in P^3 and contact deformation in Y. Its infinitesimal contact deformation is called an infinitesimal contact deformation of type 1. Let I_t be a one parameter subgroup of isometries of $S^4(1)$. Then $I_t \cdot p \cdot g$ is a minimal deformation of $p \cdot g$. Since I_t is a one parameter subgroup of isometries of P^3 which preserves $\mathscr{H}$, $I_t \cdot g$ is a horizontal deformation of g and hence gives a contact deformation of g in Y. We call the infinitesimal contact deformation an infinitesimal contact deformation of type 2.

LEMMA 3.6. *Infinitesimal contact deformations of type* 1 *and type* 2 *are in* $\ker \Omega$. *Conversely, let* W *be in* $\operatorname{Ker} \Omega$. *Then* W *is the sum of infinitesimal contact deformations of type* 1 *and type* 2.

Proof. Let P_t/Q_t be a one parameter family of holomorphic maps of P^1 onto P^1 such that $P_0/Q_0 = P/Q$. Then P_t/Q_t gives an horizontal deformation of g in P^3, which fix P^1 as the image, and hence a contact deformation of g in Y. It is given by the canonical Gauss lifts of maps

$$\left(\frac{P_t}{Q_t}, \frac{P_t}{Q_t}\right).$$

Its infinitesimal contact deformation is

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \frac{P_t}{Q_t}\right) \frac{\partial}{\partial s} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \frac{P_t}{Q_t}\right) \frac{\partial}{\partial t},$$

which is contained in Ker Ω. Conversely, for an infinitesimal contact deformation of this form $v(\partial/\partial s) + v(\partial/\partial t)$, we have a one parameter family of holomorphic maps P_t/Q_t of P^1 onto P^1 such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \frac{P_t}{Q_t} = v.$$

Next let I_t be a one parameter family of isometries of $S^4(1)$. Then I_t is also a one parameter family of isometries of P^3 which preserves $\mathscr{H}$. Thus $I_t \cdot g$ is a horizontal deformation of g and gives a contact deformation of g in Y. It is easy to see ([Loo]) that it is

$$\left(A_t \cdot \frac{P}{Q},\ B_t \cdot \frac{P}{Q},\ -\frac{\mathrm{d}\left(A_t \cdot \dfrac{P}{Q}\right)}{\mathrm{d}\left(B_t \cdot \dfrac{P}{Q}\right)}\right),$$

where A_t, B_t in $\mathrm{GL}(2, C)$. Since $A_t \cdot P/Q$, $B_t \cdot P/Q$ give a point $[P \wedge Q]$ in $G(2, d+1)$, the infinitesimal contact deformation of this contact deformation is in Ker Ω. These deformations always give non-full branched surface in $S^4(1)$.

Conversely, let V be in Ker Ω. We set

$$V^{1,0} = v^1 \frac{\partial}{\partial s} + v^2 \frac{\partial}{\partial t} + v^3 \frac{\partial}{\partial w}.$$

Then we have one parameter families P_t/Q_t, $\tilde{P}_t/\tilde{Q}_t$ whose variation vector fields v^1, v^2. Since $V \in \mathrm{Ker}\ \Omega$ holds, we get

$$[P_t \wedge Q_t]_*(0) = [\tilde{P}_t \wedge \tilde{Q}_t]_*(0).$$

So there exists one parameter subgroup A_t of $\mathrm{GL}(2, C)$ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} A_t \cdot \frac{P_t}{Q_t} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \frac{\tilde{P}_t}{\tilde{Q}_t}.$$

We define a one parameter family of holomorphic maps of P^1 into $P^1 \times P^1$ by

$$\left(\frac{P_t}{Q_t},\ A_t \cdot \frac{P_t}{Q_t}\right).$$

The canonical Gauss lift gives a one parameter family of contact curves, whose infinitesimal contact deformation is V by Lemma 3.2. So V is a sum of infinitesimal contact deformation of type 1 and type 2. Q.E.D.

Let $g(z) = P(z)/Q(z)$ be a holomorphic map of P^1 onto P^1. Then, identifying the space of infinitesimal horizontal deformations of g in P^3 with $\mathscr{E}$ for g in Y, we get a linear map, which is also denoted by Ω, of $\mathscr{E}$ into Ker ψ_{d*} at $[P \wedge Q]$. Let $\mathscr{D}$ be the quotient space by subspace of type 1. By Lemma 3.6, we can define a linear map of $\mathscr{D}$ into Ker ψ_{d*}. It is clear that the kernal is induced by one parameter isometries of $S^4(1)$ which fix $S^2(1)$. So $\dim \mathscr{D} - 3 = \dim_R \operatorname{Ker} \psi_{d*}$. Since $\mathscr{D}$ is identified with the null space of g by Theorem A, we get a proof of Theorem B.

Generally, let Θ be a branched covering map of an m-dimensional complex manifold M into an m-dimensional complex manifold N. Then the ramification locus of Θ is a complex hypersurface $\mathfrak{R}$ with singularities. For a regular point p of $\mathfrak{R}$, there exists a positive integer d and coordinate neighborhoods $(z^1, \ldots, z^m : U_1)$ at p and $(w^1, \ldots, w^m : U_2)$ at $\Theta(p)$ such that

$$w^1 \cdot \Theta(z^1, \ldots., z^m) = (z^1)^d, \quad w^k \cdot \Theta(z^1, \ldots, z^m) = z^k \quad (k \neq 1).$$

Using this fact (see, for example, [Nam]), we note that, for $g \in$ the regular part of $\mathfrak{R}$, the dimension of the null space of g is 5. Furthermore, there exists two smooth curves $r_1(s)$, $r_2(s)$ in $G(2, d+1)$ such that $r_1(0) = r_2(0) = g$, $r_1(s) \neq r_2(s)$ for $S \neq 0$ and $\Psi_d(r_1(s)) = \Psi_d(r_2(s))$. So we get two one parameter families of holomorphic maps $P_t(z)/Q_t(z)$ and $\tilde{P}_t(z)/\tilde{Q}_t(z)$ of P^1 onto P^1 such that

$$g(z) = \frac{P_0(z)}{Q_0(z)} = \frac{\tilde{P}_0(z)}{\tilde{Q}_0(z)},$$

$[P_t \wedge Q_t] \neq [\tilde{P}_t \wedge \tilde{Q}_t]$ and $\psi_d[P_t \wedge Q_t] = \psi_d[\tilde{P}_t \wedge \tilde{Q}_t]$ for $t \neq 0$. Then the canonical Gauss lift of $(P_t(z)/Q_t(z), \tilde{P}_t(z)/\tilde{Q}_t(z))$ gives a contact deformation of g in Y and hence horizontal deformation of g in P^3. Since these maps are full for $t \neq 0$, we get Theorem C.

COROLLARY 3.1. *In the space of holomorphic maps of S^2 onto $S^2(1)$ of degree d, where nullity is not less than 5, generically, the nullity is 5.*

4. A problem

We give a problem on the index for a holomorphic map of S^2 onto $S^2(1)$.

(4.1) PROBLEM. *Let g be a holomorphic map of S^2 onto $S^2(1)$ of degree d. Then,*
If the nullity of $g = 3 + 2v$, then
is the index $= 2d - 1 - v$?

If [E2], we study equivariant minimal branched immersion of S^2 into $S^{2m}(1)$ with type $(m_1,\dots,m_m)$. In particular, for $m = 2$, we obtain some limits of equivariant minimal branched immersion of S^2 into $S^4(1)$ with type (m_1, m_2) as in [E-K2]. That is, the limit holomorphic map is given by

$$g(z) = z^{m_2} \frac{z^{m_1} - \dfrac{m_2 + m_1}{m_2 - m_1}\alpha}{z^{m_1} - \alpha}, \tag{4.2}$$

where $\alpha \in C^*$.

In [Nay], if $m_1 = 1$, then the index $= 2(m_2 + 1) - 2$ and the nullity $= 5$ hold. By [E-K2], this one parameter family of branched minimal surfaces preserves the multiplicity of eigenvalue 2 and equivariant branched minimal immersions have the multiplicity 5 for eigenvalue 2. By the continuity of spectrum, equivariant branched minimal immersion of type $(1, m_2)$ has $2(m_2 + 1) - 2$ eigenfunctions of eigenvalues smaller than 2. Since $m_1 = 1$ means that the branched minimal immersion has no branched point (see, for example, [E2]), we get the following.

The number of eigenfunctions of eigenvalues smaller than 2 *for an equivariant minimal immersion of* S^2 *into* $S^4(1)$ *of area* $4\pi d$ *is equal to* $2d - 2$.

By Theorem B, we see that $g(z)$ as in (4.2) has the nullity 5. So the index of $g =$ the number of eigenfunctions with eigenvalues smaller than 2 for equivariant branched minimal immersion of S^2 into $S^4(1)$ of type (m_1, m_2). If the problem (4.1) is true, then we can obtain that the number of eigenfunctions with eigenvalues smaller than 2 for equivariant minimal branched surfaces of genus o with area $4\pi d$ in $S^4(1)$ is $2d - 2$. Thus we may consider that the problem (4.1) is useful to calculate such number for minimal branched surfaces of genus o in $S^{2m}(1)$.

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