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Kähler manifolds with numerically effective Ricci class

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Introduction

Compact Kähler manifolds with semipositive Ricci curvature have been investigated by various authors. S. Kobayashi [Ko61] first proved the simple connectedness of Fano manifolds, namely manifolds with positive Ricci curvature or equivalently, with ample anticanonical line bundle $-K_X$. Later on, generalizing results of Y. Matsushima [Ma69], A. Lichnerowicz [Li71, 72] proved the following interesting fibration theorem: if X is a compact Kähler manifold with semipositive Ricci class, then X is a smooth fibration over its Albanese torus and there is a group of analytic automorphisms of X lying over the group of torus translations (see also Section 2 for another proof of these facts based on the solution of Calabi's conjecture and on Bochner's technique). Finally, there were extensive works in the last decades to study the structure and classification of Ricci flat Kähler manifolds, see e.g. [Ca57], [Bo74a,b], [Be83] and [Kr86]; of special interest for physicists is the subclass of so-called Calabi-Yau manifolds, i.e. Ricci flat compact Kähler manifolds with finite fundamental group, which appear as a natural generalization of K3 surfaces.

To make things precise, one says that X has semipositive Ricci class $c_1(X)$ if $c_1(X)$ contains a smooth semipositive closed (1,1)-form, or equivalently if $-K_X$ carries a smooth hermitian metric with semipositive curvature. By the Aubin–Calabi–Yau theorem, this is equivalent to X having a Kähler metric with semipositive Ricci curvature. On the other hand, recent developments of algebraic geometry (especially those related to Mori's minimal model program) have shown the importance of the notion of numerical effectivity, which generalizes hermitian semipositivity but is much more flexible. It would thus be important to extend the above mentioned results to the case when $-K_X$ is numerically effective. The purpose of this paper is to contribute to the following two conjectures.

CONJECTURE 1. *Let X be a compact Kähler manifold with numerically effective anticanonical bundle $-K_X$. Then the fundamental group $\pi_1(X)$ has polynomial growth.*

CONJECTURE 2. *Let X be a compact Kähler manifold with $-K_X$ numerically effective. Then the Albanese map $\alpha: X \to \mathrm{Alb}(X)$ is surjective.*

Before we state the results, let us recall the definition of a numerically effective line bundle L on a compact complex manifold (see [DPS91] for more details). The abbreviation "nef" will be used for "numerically effective".

DEFINITION. *Let X be a compact complex manifold with a fixed hermitian metric ω. A holomorphic line bundle L over X is* nef *if for every $\epsilon > 0$ there exists a smooth hermitian metric h_ϵ on L such that the curvature satisfies*

$$\Theta_{h_\epsilon} \geqslant -\epsilon\omega.$$

Of course this notion does not depend on the choice of ω. If X is projective, L is nef precisely if $L \cdot C \geqslant 0$ for all curves $C \subset X$. Our main contribution to Conjecture 1 is

THEOREM 1. *Let X be a compact Kähler manifold with $-K_X$* nef. *Then $\pi_1(X)$ is a group of subexponential growth.*

The main tool to prove this result is the solution of the Calabi conjecture by Aubin [Au76] and Yau [Y77], combined with volume bounds for geodesic balls due to Bishop [Bi63] and Gage [Ga80] (see Section 1 for details). In fact, the volume of a geodesic ball of radius R in the universal covering of X essentially counts the number of words of $\pi_1(X)$ of length $\leqslant R$. The difficulty is that we have to deal with a sequence of metrics with Ricci curvature closer and closer to being semipositive, but nevertheless slightly negative in some points, and moreover the diameter of X need not remain uniformly bounded; this difficulty is solved by observing that a large fraction of the volume of X remains at bounded distance without being disconnected (Lemma 1.3). A by-product of our proof is that Conjecture 1 holds in the semipositive case. This was in fact already known since a long time in the context of riemannian manifolds (cf. e.g. [HK78]); our method is then nothing more than the usual riemannian geometry proof combined with the Aubin–Calabi–Yau theorem. In this way we get:

THEOREM 2. *Let X be a compact Kähler manifold with $-K_X$ hermitian semipositive. Then $\pi_1(X)$ has polynomial growth of degree* $\leqslant 2 \dim X$, *in particular $h^1(X, \mathcal{O}_X) \leqslant \dim X$.*

Note that there are simple examples of compact Kähler manifolds X with $-K_X$ nef but not hermitian semipositive, e.g. some ruled surfaces over elliptic curves (see examples 1.7 and 3.5 in [DPS91]). Also, to give a more precise idea of what Conjecture 1 means, let us recall Gromov's well-known result [Gr81]: a finitely generated group has polynomial growth if and only if it contains a nilpotent subgroup of finite index. Much more might be perhaps expected in the present situation:

QUESTION. *Let X be a compact Kähler manifold with $-K_X$* nef. *Does there exist a finite étale covering $\tilde{X}$ of X such that Albanese map $\tilde{X} \to \mathrm{Alb}(\tilde{X})$ induces an isomorphism of fundamental groups?*

If this would be the case, $\pi_1(X)$ would always be an extension of a finite group by a free abelian group of even rank. Concerning Conjecture 2, the following result will be proved in Section 2:

THEOREM 3. *Let X be a n-dimensional compact Kähler manifold such that $-K_X$ is* nef. *Then*

(i) *The Albanese map $\alpha: X \to \mathrm{Alb}(X)$ is surjective as soon as the Albanese dimension $p = \dim \alpha(X)$ is 0, 1 or n, and also for $p = n - 1$ if X is projective.*
(ii) *If X is projective and if the generic fiber F of α has $-K_F$ big, then α is surjective.*

The case $p = 1$ in (i) is a straightforward consequence of Theorem 1, as pointed out to us by F. Campana, if one observes that the growth of the fundamental group of a curve of genus $\geqslant 2$ is of exponential type. The other interesting case $p = n - 1$ is obtained as a consequence of point (ii), which is itself a rather simple consequence of the Kawamata-Viehweg vanishing theorem. Theorem 3 settles Conjecture 2 for projective 3-folds. In that case, we can also obtain a direct algebraic proof of the Albanese surjectivity in most cases by an examination of the structure of Mori contractions of X. When the contraction is not a modification, we give the description of the fibration structure of X. This is done in Section 3.

To conclude let us mention that the first theorem was used in the classification of compact Kähler manifolds with nef tangent bundles [DPS91] in a crucial way.

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1. Subexponential growth of the fundamental group

If G is a finitely generated group with generators $g_1, \dots, g_p$, we denote by $N(k)$ the number of elements $\gamma \in G$ which can be written as words

$$\gamma = g_{i1}^{\epsilon_1} \dots g_{ik}^{\epsilon_k}, \qquad \epsilon_j = 0, 1 \text{ or } -1$$

of length $\leqslant k$ in terms of the generators. The group G is said to have *subexponential growth* if for every $\epsilon > 0$ there is a constant $C(\epsilon)$ such that

$$N(k) \leqslant C(\epsilon) e^{\epsilon k} \quad \text{for } k \geqslant 0.$$

This notion is independent of the choice of generators. In the free group with two generators, we have

$$N(k) = 1 + 4(1 + 3 + 3^2 + \dots + 3^{k-1}) = 2 \cdot 3^k - 1.$$

It follows immediately that a group with subexponential growth cannot contain a non abelian free subgroup. The main goal of this section is to prove

THEOREM 1.1. *Let X be a compact Kähler manifold such that K_X^{-1} is* nef. *Then $\pi_1(X)$ has subexponential growth.*

Proof. The first step consists in the construction of suitable Kähler metrics by means of the Aubin-Calabi-Yau theorem. Let ω be a fixed Kähler metric on X. Since K_X^{-1} is nef, for every $\epsilon > 0$ there exists a smooth hermitian metric h_ϵ on K_X^{-1} such that

$$u_\epsilon = \Theta_{h_\epsilon}(K_X^{-1}) \geqslant -\epsilon\omega.$$

By [Au76] and [Y77, 78] there exists a unique Kähler metric ω_ϵ in the cohomology class $\{\omega\}$ such that

$$\text{Ricci}(\omega_\epsilon) = -\epsilon\omega_\epsilon + \epsilon\omega + u_\epsilon. \qquad (+)$$

In fact u_ϵ belongs to the Ricci class $c_1(K_X^{-1}) = c_1(X)$, hence so does the right hand side $-\epsilon\omega_\epsilon + \epsilon\omega + u_\epsilon$. In particular there exists a function f_ϵ such that

$$u_\epsilon = \mathrm{Ricci}(\omega) + i\partial\bar\partial f_\epsilon.$$

If we set $\omega_\epsilon = \omega + i\partial\bar\partial\varphi$ (where φ depends on ϵ), equation (+) is equivalent to the Monge–Ampère equation

$$\frac{(\omega + i\partial\bar\partial\varphi)^n}{\omega^n} = e^{\epsilon\varphi - f_\epsilon} \tag{++}$$

because

$$\begin{aligned} i\partial\bar\partial \log(\omega + i\partial\bar\partial\varphi)^n/\omega^n &= \mathrm{Ricci}(\omega) - \mathrm{Ricci}(\omega_\epsilon) \\ &= \epsilon(\omega_\epsilon - \omega) + \mathrm{Ricci}(\omega) - u_\epsilon \\ &= i\partial\bar\partial(\epsilon\varphi - f_\epsilon). \end{aligned}$$

It follows from the results of [Au76] that (++) has a unique solution φ, thanks to the fact the right hand side of (++) is increasing in φ. Since $u_\epsilon \geqslant -\epsilon\omega$, equation (+) implies in particular that $\mathrm{Ricci}(\omega_\epsilon) \geqslant -\epsilon\omega_\epsilon$.

We now recall a well-known differential geometric technique used to get bounds for $N(k)$ (this technique has been explained to us in a very efficient way by S. Gallot). Let (M, g) be a compact Riemannian m-fold and let $E \subset \tilde M$ be a fundamental domain for the action of $\pi_1(M)$ on the universal covering $\tilde M$. Fix $a \in E$ and set $\beta = \operatorname{diam} E$. Since $\pi_1(M)$ acts isometrically on $\tilde M$ with respect to the pull-back metric $\tilde g$, we infer that

$$E_k = \bigcup_{\gamma \in \pi_1(M),\ \mathrm{length}(\gamma) \leqslant k} \gamma(E)$$

has volume equal to $N(k)\,\mathrm{Vol}(M)$ and is contained in the geodesic ball $B(a, \alpha k + \beta)$, where α is the maximum of the length of loops representing the generators g_j. Therefore

$$N(k) \leqslant \frac{\mathrm{Vol}(B(a, \alpha k + \beta))}{\mathrm{Vol}(M)} \tag{$\star$}$$

and it is enough to bound the volume of geodesic balls in $\tilde M$. For this we use the following fundamental inequality due to R. Bishop [Bi63], Heintze–Karcher [HK78] and M. Gage [Ga80].

LEMMA 1.2. *Let*

$$\Phi : T_{\tilde{M},a} \to \tilde{M}, \qquad \Phi(\zeta) = \exp_a(\zeta)$$

be the (geodesic) exponential map. Denote by

$$\Phi^\star \, \mathrm{d}V_g = a(t, \zeta)\, \mathrm{d}t\, \mathrm{d}\sigma(\zeta)$$

the expression of the volume element in spherical coordinates with $t \in \mathbb{R}_+$ *and* $\zeta \in S_a(1) =$ *unit sphere in* $T_{\tilde{M},a}$. *Suppose that* $a(t, \zeta)$ *does not vanish for* $t \in]0, \tau(\zeta)[$, *with* $\tau(\zeta) = +\infty$ *or* $a(\tau(\zeta), \zeta) = 0$. *Then* $b(t, \zeta) = a(t, \zeta)^{1/(m-1)}$ *satisfies on* $]0, \tau(\zeta)[$ *the inequality*

$$\frac{\partial^2}{\partial t^2} b(t, \zeta) + \frac{1}{m-1} \mathrm{Ricci}_g(v(t, \zeta), v(t, \zeta)) b(t, \zeta) \leqslant 0$$

where

$$v(t, \zeta) = \frac{\mathrm{d}}{\mathrm{d}t} \exp_a(t\zeta) \in S_{\Phi(t\zeta)}(1) \subset T_{\tilde{M}, \Phi(t\zeta)}. \qquad \square$$

If $\mathrm{Ricci}_g \geqslant -\epsilon g$, we infer in particular

$$\frac{\partial^2 b}{\partial t^2} - \frac{\epsilon}{m-1} b \leqslant 0$$

and therefore $b(t, \zeta) \leqslant \alpha^{-1} \sinh(\alpha t)$ with $\alpha = \sqrt{\epsilon/(m-1)}$ (to check this, observe that $b(t, \zeta) = t + o(t)$ at 0 and that $\sinh(\alpha t)\partial b/\partial t - \alpha \cosh(\alpha t) b$ has a negative derivative). Now, every point $x \in B(a, r)$ can be joined to a by a minimal geodesic arc of length $<r$. Such a geodesic arc cannot contain any focal point (i.e. any critical value of Φ), except possibly at the end point x. It follows that $B(a, r)$ is the image by Φ of the open set

$$\Omega(r) = \{(t, \zeta) \in [0, r[\times S_a(1); t < \tau(\zeta)\}.$$

Therefore

$$\mathrm{Vol}_g(B(a, r)) \leqslant \int_{\Omega(r)} \Phi^\star \, \mathrm{d}V_g = \int_{\Omega(r)} b(t, \zeta)^{m-1} \, \mathrm{d}t\, \mathrm{d}\sigma(\zeta).$$

As $\alpha^{-1} \sinh(\alpha t) \leqslant t\, \mathrm{e}^{\alpha t}$, we get

$$\mathrm{Vol}_g(B(a,r)) \leqslant \int_{S_a(1)} \mathrm{d}\sigma(\zeta) \int_0^r t^{m-1}\,\mathrm{e}^{(m-1)\alpha t}\,\mathrm{d}t \leqslant v_m r^m\,\mathrm{e}^{\sqrt{(m-1)\epsilon}\,r} \qquad (\bigstar\bigstar)$$

where v_m is the volume of the unit ball in $\mathbb{R}^m$. □

In our application, the difficulty is that the metric $g = \omega_\epsilon$ varies with ϵ as well as the constants $\alpha = \alpha_\epsilon$, $\beta = \beta_\epsilon$ in ($\bigstar$), and $\alpha_\epsilon\sqrt{(m-1)\epsilon}$ need not converge to 0 as ϵ tends to 0. We overcome this difficulty by the following lemma, which shows that an arbitrary large fraction of the volume of $\tilde{X}$ remains at bounded distance without being disconnected.

LEMMA 1.3. *Let U_1, U_2 be compact subsets of $\tilde{X}$. Then for every $\delta > 0$, there are closed subsets $U_{1,\epsilon,\delta} \subset U_1$ and $U_{2,\epsilon,\delta} \subset U_2$ with $\mathrm{Vol}_\omega(U_j \backslash U_{j,\epsilon,\delta}) < \delta$, such that any two points $x_1 \in U_{1,\epsilon,\delta}$, $x_2 \in U_{2,\epsilon,\delta}$ can be joined by a path of length $\leqslant C\delta^{-1/2}$ with respect to ω_ϵ, where C is a constant independent of ϵ and δ.*

Proof. The basic observation is that

$$\int_X \omega_\epsilon \wedge \omega^{n-1} = \int_X \omega^n$$

does not depend on ϵ, therefore $\|\omega_\epsilon\|_{L^1(X)}$ is uniformly bounded. First suppose that $U_1 = U_2 = K$ where K is a compact convex set in some coordinate open set Ω of $\tilde{X}$. We simply join $x_1 \in K$, $x_2 \in K$ by the segment $[x_1, x_2] \subset K$ and compute the average ω_ϵ-length of this segment when x_1, x_2 vary (the average being computed in L^2 norm with respect to the Lebesgue measure of Ω). By Fubini and the Cauchy–Schwarz inequality we get

$$\int_{K\times K}\left(\int_0^1 \sqrt{\omega_\epsilon((1-t)x_1 + tx_2)(x_2 - x_1)}\,\mathrm{d}t\right)^2 \mathrm{d}x_1\,\mathrm{d}x_2$$

$$\leqslant |x_2 - x_1| \int_0^1 \mathrm{d}t \int_{K\times K} \omega_\epsilon((1-t)x_1 + tx_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

$$\leqslant 2^{2n}\,\mathrm{diam}\,K \cdot \mathrm{Vol}(K) \cdot \|\omega_\epsilon\|_{L^1(K)} \leqslant C_1$$

where C_1 is independent of ϵ; the last inequality is obtained by integrating first with respect to $y = (1-t)x_1$ when $t \leqslant \frac{1}{2}$, resp. $y = tx_2$ when $t \geqslant \frac{1}{2}$ (observe that $\mathrm{d}x_j \leqslant 2^{2n}\,\mathrm{d}y$ in both cases).

It follows that the set S of pairs $(x_1, x_2) \in K \times K$ such that $\mathrm{length}_{\omega_\epsilon}([x_1, x_2])$ exceeds $(C_1/\delta)^{1/2}$ has measure $<\delta$ in $K \times K$. By Fubini, the set Q

of points $x_1 \in K$ such that the slice $S(x_1) = \{x_2 \in K;\ (x_1, x_2) \in S\}$ has volume $\mathrm{Vol}(S(x_1)) \geq \frac{1}{2}\mathrm{Vol}(K)$ itself has volume $\mathrm{Vol}(Q) < 2\delta/\mathrm{Vol}(K)$. Now for x_1, $x_2 \in K\backslash Q$ we have by definition $\mathrm{Vol}(S(x_j)) < \frac{1}{2}\mathrm{Vol}(K)$, therefore

$$K\backslash(S(x_1) \cup (K\backslash S(x_2)) \neq \emptyset.$$

If y is a point in this set, then $(x_1, y) \notin S$ and $(x_2, y) \notin S$, hence

$$\mathrm{length}_{\omega_\epsilon}([x_1, y] \cup [y, x_2]) \leq 2(C_1/\delta)^{1/2}.$$

By continuity, a similar estimate still holds for any two points x_1, $x_2 \in \overline{K\backslash Q}$, with some $y \in K$. When $U_1 = U_2 = K$, the lemma is thus proved with $U_{j,\epsilon,\delta} = \overline{K\backslash Q}$: note that

$$\mathrm{Vol}_\omega(U_j\backslash U_{j,\epsilon,\delta}) \leq \mathrm{Vol}_\omega(Q) \leq C_2\,\mathrm{Vol}(Q) < 2C_2\delta/\mathrm{Vol}(K)$$

and replace δ by $c\delta$ with $c = \mathrm{Vol}(K)/(2C_2)$ to get the desired bound δ for the volume of $U_j\backslash U_{j,\epsilon,\delta}$.

If U_1, U_2 are isomorphic to compact convex subsets in $\mathbb{C}^n$, we find a chain of such sets $V_1, \ldots, V_N$ with $V_1 = U_1$, $V_N = U_2$ and $V_j^\circ \cap V_{j+1}^\circ \neq \emptyset$. By the first case, there exists for each $j = 1, \ldots, N$ a subset $V_{j,\epsilon,\delta} \subset V_j$ with $\mathrm{Vol}_\omega(V_j\backslash V_{j,\epsilon,\delta}) < \delta$ such that any pair of points in $V_{j,\epsilon,\delta}$ can be joined by a path of length $\leq C_3\delta^{-1/2}$. If we take $\delta < \frac{1}{2}\mathrm{Vol}_\omega(V_j \cap V_{j+1})$ for every j, then $(V_j\backslash V_{j,\epsilon,\delta}) \cup (V_{j+1}\backslash V_{j+1,\epsilon,\delta})$ cannot contain $V_j \cap V_{j+1}$ and therefore $V_{j,\epsilon,\delta} \cap V_{j+1,\epsilon,\delta} \neq \emptyset$. This implies that any $x \in U_{1,\epsilon,\delta} := V_{1,\epsilon,\delta}$ can be joined to any $y \in U_{2,\epsilon,\delta} := V_{N,\epsilon,\delta}$ by a piecewise linear path of length $\leq NC_3\delta^{-1/2}$. The case when U_1, U_2 are arbitrary is obtained by covering these sets with finitely many compact convex coordinate patches. □

We take U to be a compact set containing the fundamental domain E, so large that $U^\circ \cap g_j(U^\circ) \neq \emptyset$ for each generator g_j. We apply Lemma 1.3 with $U_1 = U_2 = U$ and $\delta > 0$ fixed such that

$$\delta < \tfrac{1}{2}\mathrm{Vol}_\omega(E), \qquad \delta < \tfrac{1}{2}\mathrm{Vol}_\omega(U \cap g_j(U)).$$

We get $U_{\epsilon,\delta} \subset U$ with $\mathrm{Vol}_\omega(U\backslash U_{\epsilon,\delta}) < \delta$ and $\mathrm{diam}_{\omega_\epsilon}(U_{\epsilon,\delta}) \leq C\delta^{-1/2}$. The inequalities on volumes imply that $\mathrm{Vol}_\omega(U_{\epsilon,\delta} \cap E) \geq \frac{1}{2}\mathrm{Vol}_\omega(E)$ and $U_{\epsilon,\delta} \cap g_j(U_{\epsilon,\delta}) \neq \emptyset$ for every j (note that all g_j preserve volumes). It is then clear that

$$W_{k,\epsilon,\delta} := \bigcup_{\gamma\in\pi_1(X),\ \mathrm{length}(\gamma)\leqslant k} \gamma(U_{\epsilon,\delta})$$

satisfies

$$\mathrm{Vol}_\omega(W_{k,\epsilon,\delta}) \geqslant N(k)\,\mathrm{Vol}_\omega(U_{\epsilon,\delta}\cap E) \geqslant N(k)\tfrac{1}{2}\mathrm{Vol}_\omega(E) \qquad \text{and}$$

$$\mathrm{diam}_{\omega_\epsilon}(W_{k,\epsilon,\delta}) \leqslant k\,\mathrm{diam}_{\omega_\epsilon} U_{\epsilon,\delta} \leqslant kC\delta^{-1/2}.$$

Since $m = \dim_{\mathbb{R}} X = 2n$, inequality (★★) implies

$$\mathrm{Vol}_{\omega_\epsilon}(W_{k,\epsilon,\delta}) \leqslant \mathrm{Vol}_{\omega_\epsilon}(B(a, kC\delta^{-1/2})) \leqslant C_4 k^{2n}\,\mathrm{e}^{C_5\sqrt{\epsilon}k}.$$

Now X is compact, so there is a constant $C(\epsilon) > 0$ such that $\omega^n \leqslant C(\epsilon)\omega_\epsilon^n$. We conclude that

$$N(k) \leqslant \frac{2\,\mathrm{Vol}_\omega(W_{k,\epsilon,\delta})}{\mathrm{Vol}_\omega(E)} \leqslant C_6 C(\epsilon) k^{2n}\,\mathrm{e}^{C_5\sqrt{\epsilon}k}.$$

The proof of Theorem 1.1 is complete. □

REMARK 1.4. In the non-Kähler case, one might try instead to use hermitian metrics ω_ϵ in the same conformal class as ω, such that $\int_X \omega_\epsilon^n = \int_X \omega^n$ and $\Theta_{\omega_\epsilon}(K_X^{-1}) = u_\epsilon \geqslant -\epsilon\omega$. Then Lemma 1.3 still holds. The major difficulty is that the first Chern form $\Theta_{\omega_\epsilon}(K_X^{-1})$ differs from the Riemannian Ricci tensor Ricci (ω_ϵ) and there is no known analogue of Inequality 1.2 in that case. The fact that we control $\Theta_{\omega_\epsilon}(K_X^{-1})$ by $-\epsilon\omega$ instead of $-\epsilon\omega_\epsilon$ could be also a source of difficulties.

REMARK 1.5. It is well known and easy to check that the equation (++) implies

$$C(\epsilon) = \max \frac{\omega^n}{\omega_\epsilon^n} \leqslant \exp(\max_X f_\epsilon - \min_X f_\epsilon).$$

In fact, this follows from the observation that $i\partial\bar\partial\varphi \geqslant 0$ at a minimum point, thus $(\omega + i\partial\bar\partial\varphi)^n/\omega^n \geqslant 1$ and (++) implies $\epsilon \min \varphi \geqslant \min f_\epsilon$. Similarly we have $\epsilon \max \varphi \leqslant \max f_\epsilon$. Since f_ϵ is a potential of $\Theta_{h_\epsilon}(K_X^{-1}) - \mathrm{Ricci}(\omega)$ and converges to an almost plurisubharmonic function as ϵ tends to 0, it is reasonable to expect that $C(\epsilon)$ has polynomial growth in ϵ^{-1}; this would imply that $\pi_1(X)$ has polynomial growth by taking $\epsilon = k^{-2}$. When K_X^{-1} has a semipositive metric, we can even take $\epsilon = 0$ and find a metric ω_0 with $\mathrm{Ricci}(\omega_0) = u_0 \geqslant 0$. This gives:

THEOREM 1.6. *If X is Kähler and K_X^{-1} is hermitian semipositive (e.g. if K_X^{-m} is generated by sections for some m) then $\pi_1(X)$ has polynomial growth of degree $\leqslant 2n$. In particular*

$$q(X) = h^1(X, \mathcal{O}) = \tfrac{1}{2}\operatorname{rank}_{\mathbb{Z}} H_1(X, \mathbb{Z}) \leqslant n.$$

REMARK 1.7. If X is a Fano manifold, i.e. if K_X^{-1} is ample, the above techniques can be used to show that X is simply connected, as observed long ago by S. Kobayashi [Ko61]. In fact the Aubin–Calabi–Yau theorem provides a Kähler metric ω with $\mathrm{Ricci}(\omega) = u > 0$, say $u \geqslant \epsilon\omega$. Then Lemma 1.2 implies $\partial^2 b/\partial t^2 + \epsilon/(2n-1) b(t, \zeta) \leqslant 0$, thus $b(t, \zeta) \leqslant \alpha^{-1} \sin(\alpha t)$ with $\alpha = \sqrt{\epsilon/(2n-1)}$. In particular $\tau(\zeta) \leqslant \pi/\alpha$, hence the universal covering $\tilde{X}$ is compact of diameter $\leqslant \pi/\alpha$ and $\pi_1(X)$ is finite (all this was already settled by S. Myers [My41] for arbitrary Riemannian manifolds). The Hirzebruch-Riemann-Roch formula implies

$$\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = k\chi(X, \mathcal{O}_X) \qquad \text{with } k = \#\pi_1(X).$$

Moreover, the Kodaira vanishing theorem applied to the ample line bundle $L = K_{\tilde{X}}^{-1}$ gives

$$H^q(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^q(\tilde{X}, K_{\tilde{X}} \otimes L) = 0 \qquad \text{for } q \geqslant 1,$$

hence $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 1$ and $k = 1$. □

2. Surjectivity of the Albanese map

Let X be a compact Kähler manifold and let $q(X) = h^1(X, \mathcal{O}_X)$ be its irregularity. Recall that the Albanese map is a holomorphic map from X to the Albanese torus

$$\mathrm{Alb}(X) := H^0(X, \Omega_X)^\star / H_1(X, \mathbb{Z}), \qquad \dim \mathrm{Alb}(X) = q(X),$$

defined by

$$\alpha(x)(h) := \int_{x_0}^{x} h, \qquad h \in H^0(X, \Omega_X^1);$$

the path from x_0 to x in the integral is taken modulo an arbitrary loop at x_0, i.e. modulo $H_1(X, \mathbb{Z})$. We first reprove Lichnerowicz' fibration theorem [Li71] by a simpler method based on the Bochner technique (of course Lichnerowicz had somehow to circumvent the Aubin-Calabi-Yau theorem, which was not available at that time). Our starting point is the following basic formula.

FORMULA 2.1. *Let $\#$ be the conjugate linear C^∞-isomorphism $T_X \to \Omega_X^1$, $v \mapsto i\bar{v} \lrcorner\, \omega$, given by a Kähler metric ω. Denote also by $\# : \Lambda^p T_X \to \Omega_X^p$ the induced C^∞ isomorphism from p-vectors to p-forms. Then for an arbitrary smooth section v of $\Lambda^p T_X$ we have*

$$\int_X \|\bar{\partial}(\# v)\|^2 \, \mathrm{d}V_\omega = \int_X \|\bar{\partial} v\|^2 \, \mathrm{d}V_\omega + \int_X \langle \mathcal{R}(v), v \rangle \, \mathrm{d}V_\omega$$

where $\mathrm{d}V_\omega$ *is the Kähler element of volume and* $\mathcal{R}$ *is the hermitian operator*

$$v = \sum_{|I|=p} v_I \frac{\partial}{\partial z_I} \mapsto \mathcal{R}(v) = \sum_{|I|=p} \left(\sum_{k \in I} \rho_k \right) v_I \frac{\partial}{\partial z_I}$$

associated to the Ricci curvature form: ρ_k *denotes the eigenvalues of* $\mathrm{Ricci}(\omega)$ *in an ω-orthonormal frame* $(\partial / \partial z_k)$.

Proof. We first make a pointwise calculation of $\bar{\partial}^\star \bar{\partial} v$ and $\bar{\partial}^\star \bar{\partial}(\# v)$ in a normal coordinate system for the Kähler metric ω. In such coordinates we can write

$$\omega = i \sum_{1 \leqslant m \leqslant n} \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m - i \sum_{1 \leqslant j,k,l,m \leqslant n} c_{jklm} z_j \bar{z}_k \, \mathrm{d}z_l \wedge \mathrm{d}\bar{z}_m + O(|z|^3)$$

where (c_{jklm}) is the curvature tensor of T_X with respect to ω. The Kähler property shows that we have the symmetry relations $c_{jklm} = c_{lkjm} = c_{jmlk}$, and the Ricci tensor $R = \sum R_{lm} \, \mathrm{d}z_l \wedge \mathrm{d}\bar{z}_m$ is obtained as the trace: $R_{lm} = \sum_j c_{jjlm}$. Since ω is tangent of order 2 to a flat metric at the center x_0 of the chart, we easily see that the first order operator $\bar{\partial}^\star$ has the same formal expression at x_0 as in the case of the flat metric on $\mathbb{C}^n$: if w is a smooth $(0, q)$-form with values in a holomorphic vector bundle E trivialized locally by a holomorphic frame (e_λ) such that $(e_\lambda(x_0))$ is orthonormal and $De_\lambda(x_0) = 0$, we have at x_0 the formula

$$w = \sum_{\lambda, |J|=q} w_{\lambda,J} \, e_\lambda \otimes \mathrm{d}\bar{z}_J, \qquad \bar{\partial}^\star w = - \sum_{\lambda, |J|=q, k} \frac{\partial w_{\lambda,J}}{\partial z_k} e_\lambda \otimes \left(\frac{\partial}{\partial \bar{z}_k} \lrcorner\, \mathrm{d}\bar{z}_J \right).$$

This applies of course to the case of sections of $\Lambda^p T_X$ or Ω_X^p expressed in terms of the frames $\partial/\partial z_I$ and $\mathrm{d}z_I$, $|I| = p$. From this, we immediately find that for any smooth sections $v = \Sigma\, v_I \partial/\partial z_I$ and $w = \Sigma\, w_I \,\mathrm{d}z_I$ we have

$$\bar\partial^\star \bar\partial v = -\sum_{I,k} \frac{\partial^2 v_I}{\partial z_k \partial \bar z_k} \frac{\partial}{\partial z_I}, \qquad \bar\partial^\star \bar\partial w = -\sum_{I,k} \frac{\partial^2 w_I}{\partial z_k \partial \bar z_k} \mathrm{d}z_I$$

at the point x_0. Now, we get

$$\# \frac{\partial}{\partial z_m} = i \frac{\partial}{\partial \bar z_m} \lrcorner\, \omega = \mathrm{d}z_m - \sum_{j,k,l} c_{jklm} z_j \bar z_k \,\mathrm{d}z_l + O(|z|^3),$$

$$\# v = \sum_I v_I \,\mathrm{d}z_I - \sum_{I,j,k,l,m} v_I c_{jklm} z_j \bar z_k \,\mathrm{d}z_l \wedge \left(\frac{\partial}{\partial z_m} \lrcorner\, \mathrm{d}z_I\right) + O(|z|^3).$$

Computing $\bar\partial^\star \bar\partial(\# v)$ at x_0 we obtain

$$\begin{aligned}\bar\partial^\star \bar\partial(\# v) &= -\sum_{I,k} \frac{\partial^2 v_I}{\partial z_k \partial \bar z_k} \mathrm{d}z_I + \sum_{I,k,l,m} v_I c_{kklm} \,\mathrm{d}z_l \wedge \left(\frac{\partial}{\partial z_m} \lrcorner\, \mathrm{d}z_I\right) \\ &= \#(\bar\partial^\star \bar\partial v) + \sum_{I,l,m} v_I R_{lm} \,\mathrm{d}z_l \wedge \left(\frac{\partial}{\partial z_m} \lrcorner\, \mathrm{d}z_I\right) = \#(\bar\partial^\star \bar\partial v) + \#\mathcal{R}(v).\end{aligned}$$

Formula 2.1 then follows from this identity by writing

$$\int_X \|\bar\partial(\# v)\|^2 \,\mathrm{d}V_\omega = \int_X \langle \bar\partial^\star \bar\partial(\# v), \# v\rangle \,\mathrm{d}V_\omega = \int_X \langle \bar\partial^\star \bar\partial v + \mathcal{R}(v), v\rangle \,\mathrm{d}V_\omega. \quad \square$$

We easily deduce from this the fibration theorem of Lichnerowicz [Li71, 72], as well as analogous results of [Li67] in the case $\mathrm{Ricci}(\omega) \leqslant 0$.

THEOREM 2.2. *Let (X, ω) be a compact Kähler manifold. Consider the natural contraction pairing*

$$\Psi : H^0(X, \Lambda^p T_X) \times H^0(X, \Omega_X^p) \to \mathbb{C}, \qquad 0 \leqslant p \leqslant n = \dim X.$$

(i) *If* $\mathrm{Ricci}(\omega) \geqslant 0$, *then Ψ has zero kernel in $H^0(X, \Omega_X^p)$. In that case, the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is a submersion and every holomorphic vector field of* $\mathrm{Alb}(X)$ *admits a lifting to X. Therefore, there is a group of analytic automorphisms of X lying over the group of translations of* $\mathrm{Alb}(X)$.

(ii) *If* Ricci$(\omega) \leqslant 0$, *then* Ψ *has zero kernel in* $H^0(X, \Lambda^p T_X)$. *In that case the identity component* $\mathrm{Aut}(X)^\circ$ *of* $\mathrm{Aut}(X)$ *is abelian and leaves invariant all global holomorphic p-forms or p-vector fields.*

Proof. Let v be a smooth section of $\Lambda^p T_X$ and let $w = \# v$ be the associated smooth $(p, 0)$-form. By definition of $\#$ we have $v \lrcorner w = \|v\|^2$. Now, when Ricci$(\omega) \geqslant 0$, Formula 2.1 shows that $\int_X \|\bar\partial w\|^2 \, dV_\omega \geqslant \int_X \|\bar\partial v\|^2 \, dV_\omega$, thus v is holomorphic as soon as w is. Therefore we get an injective conjugate linear map

$$\#^{-1} : H^0(X, \Omega_X^p) \to H^0(X, \Lambda^p T_X)$$

with the property that $(\#^{-1} w) \lrcorner w$ is a non zero constant for $w \neq 0$. This shows that the kernel of Ψ in $H^0(X, \Omega_X^p)$ is zero. On the other hand, when Ricci$(\omega) \leqslant 0$, the inequality is reversed and we get an injection

$$\# : H^0(X, \Lambda^p T_X) \to H^0(X, \Omega_X^p).$$

Hence in that case the kernel of Ψ in $H^0(X, \Lambda^p T_X)$ is zero.

(i) By the above with $p = 1$, every holomorphic 1-form h which is not identically zero does not vanish at all, because there is a vector field v such that $v \lrcorner h = 1$. Let $(h_1, \ldots, h_q)$ be a basis of $H^0(X, \Omega_X^1)$. Then for each point $x \in X$ the 1-forms $h_1(x), \ldots, h_q(x)$ must be independent in $T^\star_{X,x}$. In the basis of $T_{\mathrm{Alb}(X)}$ provided by the h_j's, we have $d\alpha(x) = (h_1(x), \ldots, h_q(x))$ and so $d\alpha(x)$ is surjective. Now, there are holomorphic vector fields $v_1, \ldots, v_q$ on X such that $v_i \lrcorner h_j = \delta_{ij}$. These vector fields clearly generate a subgroup G of $\mathrm{Aut}(X)^\circ$ which lies over the group of translations of $\mathrm{Alb}(X)$.

(ii) Let $v_1, \ldots, v_q$ be a basis of the Lie algebra of $\mathrm{Aut}(X)^\circ$. Then all Lie brackets $[v_i, v_j]$ vanish, because we have

$$[v_i, v_j] \lrcorner h = v_i \cdot (v_j \lrcorner h) - v_j \cdot (v_i \lrcorner h) = 0$$

for every holomorphic 1-form h (just observe that $v_i \lrcorner h$ and $v_j \lrcorner h$ are constant functions). Thus $\mathrm{Aut}(X)^\circ$ is abelian. Moreover, for any holomorphic p-form w, the Lie derivative $\mathscr{L}_{v_i}(w)$ vanishes:

$$\mathscr{L}_{v_i}(w) = d(v_i \lrcorner w) + v_i \lrcorner (dw) = 0,$$

because all holomorphic forms on a compact Kähler manifold are d-closed. Hence w is invariant under $\mathrm{Aut}(X)^\circ$. By duality, we easily conclude that the holomorphic p-vectors are also kept invariant. □

We now discuss Conjecture 2 for compact Kähler manifolds X with $-K_X$

being only nef. The proof of the following theorem has been communicated to us by F. Campana.

THEOREM 2.3. *Let X be a compact Kähler manifold with* $-K_X$ nef. *Let* $\alpha: X \to \mathrm{Alb}(X)$ *be the Albanese map. If* $\dim \alpha(X) = 1$, *then α is surjective.*

Proof. $\alpha(X)$ is a smooth curve C. Assume that C has genus $g \geqslant 2$. Then $\pi_1(C)$ has exponential growth; in fact it contains a free group with $2g-1$ generators. Because of the exact sequence $\pi_1(X) \xrightarrow{\alpha_\star} \pi_1(C) \to \pi_0(F)$, the image of $\alpha_\star$ has finite index in $\pi_1(C)$, hence $\pi_1(X)$ is of exponential growth, contradicting Theorem 1.1. □

First suppose $\dim \alpha(X) = \dim X$. If $\alpha(X) \neq \mathrm{Alb}(X)$, there would be at least two independent sections of K_X coming from $H^0(\Omega^n_{\mathrm{Alb}(X)})$; since $-K_X$ is nef, these sections would not vanish and so $K_X = \mathcal{O}_X$, contradiction. The next interesting case is $\dim \alpha(X) = \dim X - 1$, which we treat next. We first prove a more general statement.

THEOREM 2.4. *Let X be a compact Kähler manifold with* $-K_X$ nef. *Then there is no holomorphic surjective map $\varphi: X \to Y$ to a projective variety Y with* $\kappa(Y) > 0$ *such that $-K_F$ is big for the general fiber F of φ.*

By definition the Kodaira dimension $\kappa(Y)$ is the Kodaira dimension of a desingularisation.

Proof. Assume there is a map φ as above. We may assume that Y is normal by passing to the normalization, and moreover that the fibers are connected by taking the Stein factorization. Choose a very ample divisor H on Y. Letting $m = \dim Y$, we pick a curve

$$C = H_1 \cap \cdots \cap H_{m-1}$$

with $H_i \in |H|$ in general position. Then C is smooth as well as $X_C = \varphi^{-1}(C)$ by Bertini's lemma. Moreover

$$c_1(\omega_Y) \cdot C > 0, \tag{1}$$

since $C \cap \mathrm{Sing}(Y) = \emptyset$ and since $\omega_Y = \pi_\star(\omega_{\tilde{Y}})$ on $Y \backslash \mathrm{Sing}(Y)$ for every desingularisation $\pi: \tilde{Y} \to Y$.

Let $f = \varphi_{|X_C}$. We claim:

$$\omega_{X_C/C}^{-1} \quad \text{is big and nef.} \tag{2}$$

In fact,

$$\omega_{X_C/C}^{-1} = \omega_{X/Y|X_C}^{-1} = \omega_{X|X_C}^{-1} \otimes f^\star(\omega_{Y|C})$$

which is nef because of (1). Letting $p = \dim X_C$ and taking p-th powers, we observe that $c_1(f^\star \omega_{Y|C}) \geqslant c_1(\mathcal{O}(F))$ by (1), F being a generic fibre, thus

$$c_1(\omega_{X_C/C}^{-1})^p \geqslant c_1(\omega_{X|X_C}^{-1})^{p-1} \cdot c_1(f^\star \omega_{Y|C}) \geqslant c_1(\omega_{X|X_C}^{-1})^{p-1} \cdot F = c_1(\omega_F^{-1})^{p-1},$$

which is positive by our assumption that $-K_F$ is big. Now we can apply Kawamata–Viehweg's vanishing theorem [Ka82, Vi82] to obtain

$$H^1(X_C, \omega_{X_C} \otimes \omega_{X_C/C}^{-1}) = 0$$

But $\omega_{X_C} \otimes \omega_{X_C/C}^{-1} = f^\star(\omega_C)$, so via the Leray spectral sequence we conclude $H^1(C, \omega_C) = 0$, which is absurd. □

COROLLARY 2.5. *Let X be a n-dimensional projective (or Moishezon) manifold with $-K_X$* nef. *Assume that the Albanese map α has $(n-1)$-dimensional image. Then α is surjective.*

Proof. If α is not surjective, the image $Y = \alpha(X)$ automatically has $\kappa(Y) > 0$ since we get at least two independent holomorphic forms of maximum degree from the Albanese torus. We may thus assume $\kappa(Y) > 0$. Let F be the general fiber of α, which is a curve. Since $-K_F = -K_{X|F}$ is nef, F is rational or elliptic. In the first case, α is surjective by the previous theorem. If F is elliptic, then $\kappa(X) = 0$, so $h^0(X, mK_X) \neq 0$ for some m and consequently $mK_X = \mathcal{O}_X$. Therefore α is onto by Theorem 2.2 (i). □

The last part of the proof shows more generally that conjecture 2 holds if $\kappa(X) = 0$. A different proof of Corollary 2.5 has been given by F. Campana.

3. Threefolds whose anticanonical bundles are nef

In this section we want to study the structure of projective 3-folds X with $-K_X$ nef in more detail. In particular we prove Conjecture 2 in dimension 3 with purely algebraic methods, except in one very special case. In fact, we prove that the Albanese map is surjective except possibly when all extremal contractions of X are of type (B), defined in Proposition 3.3(2). For the structure of surfaces with $-K_X$ nef we refer to [CP91].

Let always X denote a smooth projective 3-fold with $-K_X$ nef and let $\alpha: X \to \mathrm{Alb}(X)$ be the Albanese map. By the last words of Section 2, the structure of X is clear if $\kappa(X) = 0$; so we will assume $\kappa(X) = -\infty$; note that K_X is not nef in this case. Then there exists an extremal ray on X ([Mo82], [KMM87]); let $\varphi : X \to W$ be the associated contraction. We want to analyze the structure of φ.

PROPOSITION 3.1. *If* $\dim W \leqslant 2$, *α is surjective. More precisely*:

(1) *If W is a point, X is Fano with $b_2 = 1$, in particular X is simply connected.*

(2) *If W is a (smooth) curve, then $g(W) \leqslant 1$. In case $g(W) = 1$, we have $\alpha = \varphi$; if $g(W) = 0$, we have $q(X) = 0$. In all cases φ has the structure of a del Pezzo fibration.*

(3) *If W is a (smooth) surface, then either*
   - (a) *φ is a $\mathbb{P}_1$-bundle and $-K_W$ is* nef
   - (b) *φ is a proper conic bundle with discriminant locus Δ such that $-(4K_W + \Delta)$ is* nef, *and $q(W) \leqslant 1$.*

Proof. (1) If $\dim W = 0$, then $q(X) = 0$, hence our claim is obvious.

(2) Let $\dim W = 1$. Since $R^q \varphi_\star(\mathcal{O}_X) = 0$ for $q > 0$, either φ is the Albanese map and then we must show that $q(W) = 1$ or $q(W) = 0$. So assume $q(W) \geqslant 2$. Then the canonical bundle K_W is ample. Let $K_{X/W}$ be the relative canonical bundle, so

$$K_{X/W} = K_X - \varphi^\star(K_W).$$

Since the Picard number $\rho(X) = \rho(W) + 1 = 2$ (see e.g. [KMM87]), and since $-K_X$ is nef and φ-ample ([KMM87]), $-K_{X/W}$ is ample. Hence by Kodaira vanishing:

$$H^1(X, \mathcal{O}_X(-K_{X/W}) \otimes \mathcal{O}_X(K_X)) = 0,$$

so $H^1(X, \varphi^\star(\mathcal{O}_W(K_W)) = 0$ and $H^1(W, \mathcal{O}_W(K_W)) = 0$, which is absurd.

(3) Now assume $\dim W = 2$. Then W is smooth and φ is a $\mathbb{P}_1$-bundle or a conic bundle ([Mo82]). Since $q(X) = q(W)$, we have a diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & \mathrm{Alb}(X) \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle\gamma}\downarrow \\ W & \xrightarrow{\beta} & \mathrm{Alb}(W) \end{array}$$

with β being the Albanese map of W and γ being finite.

(3a) Assume φ to be a $\mathbb{P}_1$-bundle. We will prove that $-K_W$ is nef, hence β is onto and so is α. Take any irreducible curve $C \subset W$. Since $\varphi^{-1}(C) = \mathbb{P}(E_C)$ with a rank 2-bundle E_C on C, we have (after possibly passing to the normalization of C):

$$-K_{X|\mathbb{P}(E_C)} = -K_{\mathbb{P}(E_C)} + \varphi^\star(N_{C/W})$$

by the adjunction formula. Since

$$-K_{\mathbb{P}(E_C)} = \mathcal{O}_{\mathbb{P}\left(E_C \otimes \frac{\det E_C^\star}{2} \otimes \frac{(-K_C)}{2}\right)}(2),$$

we have

$$-K_{X|\mathbb{P}(E_C)} = \mathcal{O}_{\mathbb{P}\left(E_C \otimes \frac{\det E_C^\star}{2} \otimes \frac{(-K_{W/C})}{2}\right)}(2).$$

Since $-K_X$ is nef, we conclude that

$$E_C \otimes \frac{\det E_C^\star}{2} \otimes \frac{(-K_{W|C})}{2} \text{ is nef.}$$

Now $c_1(E_C \otimes (\det E_C^\star/2) = 0$, hence $-K_{W|C}$ must be nef and $-K_W$ itself is nef.

(3b) Next assume φ to be a proper conic bundle. Let $\Delta \subset W$ be the discriminant locus, i.e.

$$\Delta = \{w \in W;\ X_w \text{ not smooth}\}.$$

From the well-known formula (see e.g. [Mi81])

$$K_X^2 \cdot \varphi^{-1}(C) = -(4K_W + \Delta) \cdot C$$

for every curve $C \subset W$, we deduce from the nefness of $-K_X$ that $-(4K_W + \Delta)$ is nef.

Now we conclude by means of the following:

LEMMA 3.2. *Let W be a smooth projective surface, $\Delta \subset W$ be a (possibly reducible) curve. Assume that $-(4K_W + \Delta)$ is* nef. *Then $q(W) \leqslant 1$.*

Proof. Obviously $\kappa(W) = -\infty$. We can easily reduce the problem to the case of W being minimal. If $W \neq \mathbb{P}_2$, then W is ruled over a curve C. Now it is an easy exercise using [Ha77, V.2] to prove that $g(C) \leqslant 1$. □

PROPOSITION 3.3. *Assume* $\dim W = 3$. *Let E be the exceptional set of φ.*

(1) *If* $\dim \varphi(E) = 0$, *then $-K_W$ is big and nef and $q(X) = 0$.*

(2) *If* $\dim \varphi(E) = 1$, *then W is smooth, φ is the blow-up of the smooth curve $C_0 = \varphi(E)$ and $-K_W$ is again* nef *except for the following special cases:*

C_0 is rational and moreover either

(A) $N_{C_0/W} \simeq \mathcal{O}(-2) \oplus \mathcal{O}(-2)$ *or*
(B) $N_{C_0/W} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-2)$.

Proof. By [Mo82] E is always an irreducible divisor and if $\dim \varphi(E) = 1$, W is smooth and φ is the blow-up of a smooth curve. We may always assume $K_X^3 = 0$, otherwise $q(X) = 0$ by Kawamata–Viehweg vanishing.

(1) We have the following formula of $\mathbb{Q}$-divisors:

$$K_X = \varphi^\star(K_W) + \vartheta E$$

with some $\vartheta \in \mathbb{Q}_+$ ([Mo82], in fact either $E \simeq \mathbb{P}_2$ or E is a normal quadric and $\vartheta = 2$, 1 or 1/2). Hence $-K_W$ is nef. Furthermore:

$$K_X^3 = K_W^3 + \vartheta^3 E^3$$

and since $E^3 > 0$ (E has always negative normal bundle, [Mo82]), we conclude from $K_X^3 = 0$ that $K_W^3 < 0$, so $-K_W$ is big and nef (observe that W might be singular). Now a "singular" version of the Kawamata-Viehweg vanishing theorem ([KMM87, 1.2.5, 1.2.6] applied to $D = 0$) yields

$$H^1(W, \mathcal{O}_W) = 0.$$

Since $R^q\varphi_\star(\mathcal{O}_X) = 0$ for $q > 0$, we get $q(X) = 0$.

(2) From the formula $K_X = \varphi^\star(K_W) + E$, we immediately see that

$$-K_W \cdot C \geqslant 0$$

for every curve $C \neq C_0$.

Let $N_{C_0/W} = N$ denote the normal bundle of C_0. Let $V = N^\star \otimes \mathcal{L}$ with $\mathcal{L} \in \mathrm{Pic}(C_0)$ be its normalization, i.e. $H^0(V) \neq 0$, but $H^0(V \otimes \mathcal{G}) = 0$ for all line bundles $\mathcal{G}$ with $\deg \mathcal{G} < 0$. Let $\mu = \deg \mathcal{L}$. Then E can be written as $E = \mathbb{P}(N^\star) = \mathbb{P}(V)$. The "tautological" line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ has a "canonical" section C_1 satisfying $C_1^2 = -e = c_1(V)$ (see [Ha77, V.2]). In this terminology

$$(K_X \cdot C_1) = (K_W \cdot C_0) + (E \cdot C_1).$$

Let F be a fiber of $\varphi_{|E}$. Write for numerical equivalence

$$-K_{X|E} \equiv aC_1 + bF.$$

Since $(K_X \cdot F) = -1$, we have $a = 1$. Moreover

$$N_{E|X} \equiv -C_1 + \mu F$$

just by definition of μ and by the fact that $N^{\star}_{E|X} = \mathcal{O}_{\mathbb{P}(N^{\star})}(1)$. Hence

$$(K_W \cdot C_0) = (K_X \cdot C_1) - (E \cdot C_1) = -b - \mu;$$

so $-K_W$ is nef if

$$b + \mu \geqslant 0. \tag{$\star$}$$

Letting g be the genus of C_0, we have $K_E^2 = 8(1-g)$; on the other hand we compute by adjunction:

$$K_E^2 = (K_X + E)^2_{|E} = (-2C_1 + (\mu - b)F)^2.$$

Thus $8(1-g) = 4b - 4e - 4\mu$, and consequently

$$b + \mu = 2b - e + 2(g-1). \tag{$\star\star$}$$

Since $-K_{X|E}$ is nef, we have $K_X^2 \cdot E \geqslant 0$, which is equivalent to $b \geqslant e/2$. Combining this with ($\star\star$) shows that $b + \mu \geqslant 0$ at least if $g \geqslant 1$. Therefore, if $g \geqslant 1$, $-K_W$ is nef.

Now assume $g = 0$. Then ($\star$) is equivalent to

$$b \geqslant \frac{e}{2} + 1. \tag{$\star'$}$$

Again by nefness of $-K_{X|E}$ we get

$$0 \leqslant (-K_X \cdot C_1) = (C_1 + bF \cdot C_1) = b - e,$$

so $b \geqslant e$. This settles already the case $e \geqslant 2$.

Since $e \geqslant 0$, we are left with $e = 0$ and $e = 1$ and additionally $e \leqslant b < e/2 + 1$. This leads to (A) and (B). □

REMARK 3.4. Assume that $-K_X$ is big and nef. Then X is "almost Fano" in the following sense. By the Base Point Free Theorem [KMM87] we have a surjective map $\varphi: X \to Y$, given by the base point free system $|-mK_X|$ for some suitable m, to a normal projective variety Y. This variety Y carries an ample line bundle L such that $\varphi^{\star}(L) = -mK_X$. The map φ being a modification, we conclude that $L = -mK_Y$. Thus Y is $\mathbb{Q}$-Gorenstein with at most canonical singularities and the $\mathbb{Q}$-Cartier divisor $-K_Y$ is ample. We say that

Y is "$\mathbb{Q}$-Fano". In particular Y has irregularity 0 ($\mathbb{Q}$-Fano varieties are even expected to be simply connected and so X would be simply connected. This is true in dimension three by a recent result of Kollár, Miyaoka and Mori.). □

We are now interested in those X which are not "almost Fano", i.e. such that $(-K_X)^3 = 0$.

PROPOSITION 3.5. *Assume* $-K_X$ nef *and* $K_X^3 = 0$. *If* φ *is a contraction of type* (B), *then* $q(X) = 0$ *and moreover* X *is birational to a* $\mathbb{Q}$*-Fano variety. In particular* $\mathrm{Alb}(X) = 0$.

Proof. Let $\varphi: X \to W$ be the contraction, which is the blow-up of $C_0 \subset W$ such that $C_0 \simeq \mathbb{P}_1$ and

$$N_{C_0/W} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-2).$$

Let $E \subset X$ be the exceptional divisor. We have

$$E \simeq \mathbb{P}(N^\star_{C_0/W}) = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(2)) = \Sigma_1$$

(Σ_1 = Hirzebruch surface of index 1). Let $C \subset E$ be the unique section with $C^2 = -1$. Let $\pi: X^+ \to X$ be the blow-up of C. Since $N_{C/E} = \mathcal{O}(-1)$ and

$$N_{E/X} = \mathcal{O}_{\mathbb{P}(N^\star_{C_0/X})}(-1) = \mathcal{O}_{\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1))}(-1) \otimes \varphi^\star \mathcal{O}(-2) = \mathcal{O}(-C - 2F_\varphi),$$

we get $N_{E/X|C} = \mathcal{O}(-1)$ and obtain from

$$0 \to N_{C/E} \to N_{C/X} \to N_{E/X|C} \to 0$$

that $N_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Hence the exceptional divisor $D = \pi^{-1}(C)$ is $\mathbb{P}_1 \times \mathbb{P}_1$ and $N_{D/X^+} = \mathcal{O}_{\mathbb{P}(N^\star_{C/X})}(-1) = \mathcal{O}(-1) \boxtimes \mathcal{O}(-1)$. Therefore D can be blown down along the other ruling. Let $\sigma: X^+ \to X^-$ be this blowing down.

CLAIM 3.6. *The anticanonical divisor* $-K_{X^-}$ *is* nef.

Proof. Let $A^- \subset X^-$ be an arbitrary curve not in the center of σ, A^+ the strict transform in X^+ and A the image in X. As $K_{X^+} = \sigma^\star(K_{X^-}) + D$, we have

$$(-K_{X^+} \cdot A^+) = (-K_{X^-} \cdot A^-) - (D \cdot A^+)$$

and

$$(-K_{X^+} \cdot A^+) = (-K_X \cdot A) - (D \cdot A^+).$$

Hence $(-K_{X^-} \cdot A^-) = (-K_X \cdot A) \geqslant 0$. Since the center B of σ is rational with $N_{B/X^+} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, we have $K_{X^-} \cdot B = 0$ and hence $-K_{X^-}$ is nef. $\square$

Let E^+ be the strict transform of E in X^+ and $E^- = \sigma(E^+)$. We have $E = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(2)) = \Sigma_1$, $E^+ \simeq E$, and $E^- \simeq \mathbb{P}_2$ because the (-1)-curve of E^+ gets contracted by σ.

CLAIM 3.7. *We have* $N_{E^-/X^-} = \mathcal{O}(-2)$.

Proof. We first compute E^3, $(E^+)^3$ and $(E^-)^3$. We have

$$E^3 = c_1(N_{E/X})^2 = (-C - 2F_\varphi)^2_{|E} = C^2 + 4C \cdot F_\varphi = -1 + 4 = 3.$$

As π is the blow-up of a curve in E, we get $\pi^\star(E) = E^+ + D$. Hence

$$(E^+)^3 = E^3 - 3\pi^\star(E)^2 \cdot D + 3\pi^\star(E) \cdot D^2 - D^3.$$

Moreover

$$D^3 = c_1(N_{D/X^+})^2 = 2, \qquad E^+ \cdot D^2 = (\mathbb{P}_1 \times \{0\}) \cdot D = -1,$$

$$\pi^\star(E) \cdot D^2 = (E^+ + D) \cdot D^2 = 1, \qquad (E^+)^3 = 3 - 0 + 3 - 2 = 4;$$

note that $\pi^\star(E)^2 \cdot D = 0$ since π projects D to a curve. We finally have $\sigma^\star(E^-) = E^+$ because σ is a blowing down along ruling lines of D which intersect E^+ only in one point. Therefore $(E^-)^3 = (E^+)^3 = 4$. We must have $N_{E^-/X^-} = \mathcal{O}(k)$ for some $k < 0$ (since E^- is exceptional) and

$$(E^-)^3 = c_1(N_{E^-/X^-})^2 = k^2,$$

so $k = -2$ as desired. $\square$

Let $\psi: X^- \to Z$ be the blowing down of E^-. Then Z has only one rational singularity which is in fact terminal. The nefness of $-K_Z$ follows from the nefness of $-K_{X^-}$. A well-known calculation (see [Mo82]) yields

$$K_{X^-} = \psi^\star(K_Z) + \tfrac{1}{2}E^-,$$

hence

$$0 \leqslant (-K_{X^-})^3 = (-K_Z)^3 - \tfrac{1}{8}(E^-)^3 = (-K_Z)^3 - \tfrac{1}{2}.$$

Therefore $(-K_Z)^3 \geqslant 1/2$ and $-K_Z$ is big, i.e. Z is birational to a $\mathbb{Q}$-Fano manifold (see Remark 3.4). The singular Kawamata-Viehweg vanishing theorem (cf. [KMM], 1.2.6) applied to $-K_Z$ gives

$$H^1(Z, \mathcal{O}_Z) = 0,$$

therefore $q(X) = q(Z) = 0$. □

We have thus shown that case (B) does not occur when $q(X) > 0$. Therefore, putting everything together, we have proved:

THEOREM 3.8. *Let X be a smooth projective 3-fold with $-K_X$* nef *and $\kappa(X) = -\infty$. Let $\varphi: X \to W$ be a contraction of an extremal ray. Then $-K_W$ is* nef *expect possibly for the following cases*:
(a) *φ is the blow-up of a smooth rational curve C such that*

$$N_{C/W} \simeq \begin{cases} \mathcal{O}(-1) \oplus \mathcal{O}(-2) & \textit{or} \\ \mathcal{O}(-2) \oplus \mathcal{O}(-2). \end{cases}$$

In the first case X has irregularity 0 *and is birational to a $\mathbb{Q}$-Fano variety.*
(b) *φ is a proper conic bundle over a surface W with $-(4K_W + \Delta)$* nef, *Δ being the discriminant locus.*

REMARK 3.9. Let X be a smooth projective 3-fold with $-K_X$ nef. The above algebraic considerations again show that the Albanese map $\alpha: X \to \mathrm{Alb}(X)$ is surjective, except possibly if all contractions $X \to W$ are of type (A) or if this situation occurs after finitely many blowing-downs.

Proof. We may assume K_X not nef. Let $\varphi: X \to W$ be the contraction of an extremal ray. If $\dim W \leqslant 2$, α is already surjective by Prop. 3.1. If $\dim W = 3$, φ must be either the blow-up of a point, hence $(-K_W)^3 > 0$ and $q(X) = q(W) = 0$ (except possibly for case (A)), or φ is the blow-up of a smooth curve and $-K_W$ is again nef with W smooth. Then we proceed by induction on $b_2(W)$. □

REMARKS 3.10.
(1) In 3.8(a) consider the morphism $\psi = \Phi_{|-mK_X|}$ with suitable m. In case $N_{C/W} = \mathcal{O}(-1) \oplus \mathcal{O}(-2)$, ψ contracts the exceptional curve of $E \simeq \Sigma_1$, in the other case φ contracts all curves in $E \simeq \mathbb{P}_1 \times \mathbb{P}_1$ which are ruling lines not contracted by φ. It would be interesting to know whether 3.8(a) can really occur.
(2) If $\varphi: X \to W$ is a proper conic bundle with $-K_X$ nef, then $(-K_W \cdot C) \geqslant 0$ if $C \not\subset \Delta$ or $C \subset \Delta$ but a multiple of C moves. So $-K_W$ is "almost nef".

It would be interesting to have a rough classification of conic bundles X with $-K_X$ nef.

(3) For contractions of type (A), we have in fact the following additional information:

PROPOSITION 3.11. *Assume that* $-K_X$ *is* nef, $K_X^3 = 0$ *and that* $\varphi: X \to W$ *is of type* (A). *Then* $K_X^2 = 0$.

Proof. Let $\psi: W \to W'$ be the blow-down of $C_0 = \varphi(E)$. Let $\sigma = \psi \circ \varphi$. Let $N^1(Z) = \mathrm{Pic}(Z) \otimes_{\mathbb{Z}} \mathbb{R}/\equiv$ for any Z. $\sigma^\star(N^1(W'))$ is a linear subspace of codimension 2 in $N^1(X)$, in fact $\psi^\star(Z^1(W'))$ is of codimension 1 in $N^1(W)$, and $\varphi^\star(N^1(W))$ is of codimension 1 in $N^1(X)$, as one checks immediately from Mori theory:

Assume $K_X^2 \neq 0$. Then $(K_X^2)^\perp = \{L \in N^1(X);\ L \cdot K_X^2 = 0\}$ is of codimension 1 in $N^1(X)$. Hence:

$$(K_X^2)^\perp = \sigma^\star(N^1(W')) \oplus \mathbb{R} \cdot K_X$$

because $K_X^3 = 0$. Since $K_X^2 \cdot E = 0$, E is in $(K_X^2)^\perp$, so

$$E = \mu K_X + \sigma^\star(H)$$

with $H \in N^1(W')$. Cutting by a fiber of φ yields $\mu = 1$. Since $K_X = \varphi^\star(K_W) + E$, we conclude $\varphi^\star(K_W) = -\sigma^\star(H)$, i.e. $K_W = -\psi^\star(H)$, which is false. □

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