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Intersections of higher weight cycles and modular forms

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Introduction

In 1975 Hirzebruch and Zagier [HZ] computed the pairwise intersection multiplicities for certain families of algebraic cycles $\{T_n^c : n \in \mathbb{N}\}$ in the Hilbert modular surfaces associated to $\mathbb{Q}(\sqrt{p})$, for prime $p \equiv 1 \pmod 4$, and showed that the generating functions

$$f_m(\tau) := \sum_{n=0}^{\infty} (T_m^c \cdot T_n^c) e^{2\pi i n \tau}$$

were elliptic modular forms of weight 2 and Nebentypus for $\Gamma_0(p)$. Soon thereafter Kudla [Kd1] established by different methods the analogous result for compact quotients of (products of) the complex 2-ball, and later Cogdell [Cg1] extended Kudla's methods and results to the corresponding noncompact quotients. In the meanwhile Zagier [Z2] had noticed that if certain weighting factors were inserted into the formula for $(T_m^c \cdot T_n^c)$ then the new modified Fourier series was again an elliptic modular form with the same level and Nebentypus but now of higher weight, and he asked if these weighted intersection numbers might be obtainable as the ordinary geometric intersection multiplicities of some algebraic cycles in some appropriate homology theory for the Hilbert modular surfaces.

The purpose of this paper is to answer Zagier's question for a quaternionic modular surface S, the compact without-cusps analogue of a Hilbert modular surface, using the approach of Kudla and Cogdell. To begin with we consider,

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for each $k \geq 0$, a complex projective variety $\mathcal{A}^{(k)}$ with the structure of a family of $4k$-dimensional abelian varieties parameterized by S, and we construct a natural family of *algebraic* cycles $\{T_n^{(k)} : n \in \mathbb{N}\}$ in $\mathcal{A}^{(k)}$ with the structure of a family of algebraic cycles over the Hirzebruch–Zagier cycles in S. Then for the pairwise intersection multiplicities $(T_m^{(k)} \cdot T_n^{(k)})$ we get an expression which, though not closed, contains the very weighting factors Zagier introduced. We also describe, for $k > 0$, the harmonic differential forms on $\mathcal{A}$ that are Poincare dual to the $T_n^{(k)}$. Finally recent results of Kudla and Millson [Mi3] [KM6] together with a theorem of Eichler and Zagier [EZ] allow us to deduce very quickly that the generating function

$$F_{k,m}(\tau) := \sum_{n \geq 0} (T_m^{(k)} \cdot T_n^{(k)})\, e^{2\pi i n \tau}$$

for the pairwise intersection multiplicities is an elliptic modular form of weight $2k+2$ and Nebentypus on a suitable congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

0.1. To describe our results and the contents of this paper more precisely, let L be a lattice of rank 4 equipped with an indefinite anisotropic integer-valued quadratic form q whose discriminant D is positive and nonsquare; thus, the extension of q to $L \otimes \mathbb{R}$ has signature (2,2). Then canonically associated to (L, q) we have the following data: the level N and quadratic character ε of modulus N, as in [He]; the vector space $V := L \otimes \mathbb{Q}$ with the symmetric bilinear form defined by $(u, v) := q(u+v) - q(u) - q(v)$ which is even and integral on L; the even Clifford algebra $C^+(V)$ of V, which is a totally indefinite division quaternion algebra over the real quadratic field $\mathbb{Q}(\sqrt{D})$; the order $\mathcal{O} := C^+(L)$ in $C^+(V)$; the spin group $G := \mathrm{Spin}(V)$, which here consists of the norm 1 units of $C^+(V)$, and which comes equipped with its vector representation $\psi : G \to \mathrm{SO}(V)$ of kernel $\{\pm 1\}$; the spin representation (σ, W) of G, where W is the underlying vector space of $C^+(V)$ and σ is left multiplication, which is irreducible over $\mathbb{Q}$ since $C^+(V)$ is a simple algebra spanned by G; the lattice $\Lambda \subset W$ underlying the order $\mathcal{O}$; the arithmetic group $\Gamma(L) := G \cap \mathcal{O}$, which preserves both L and $\mathcal{O}$; and the hermitian symmetric domain $\mathfrak{X} \simeq G(\mathbb{R})/K$ for a maximal compact K. We also fix a torsion-free normal subgroup Γ of finite index in $\Gamma(L)$ [Bo2]. Then $S := \Gamma\backslash\mathfrak{X}$, and as a C^∞-manifold

$$\mathcal{A}^{(k)} \cong \Gamma\backslash(\mathfrak{X} \times W(\mathbb{R})^k/\Lambda^k);$$

that $\mathcal{A}^{(k)}$ can be given the structure of an algebraic family of (polarized) abelian varieties over S follows from the work of Kuga [Kg1] [Kg2] [Kg3] and Satake [Sa1] [Sa2] [Sa3] [Sa4], as described in section 2.

Next, for $v \in L$ with $q(v) > 0$, let $L_v := \{u \in L : (u, v) = 0\}$, and let q_v be the restriction of q to L_v. Then all the same data associated to (L, q) may be associated to (L_v, q_v), defined in the same way: V_v, G_v, $\mathfrak{X}_v$, W_v, Λ_v, etc. However, it should be noted that (in spite of the notation) (L_v, q_v) and therefore all the data associated to it depend only on the line $\mathbb{Q}v$ and not on the choice of specific vector $v \in L \cap \mathbb{Q}v$. Thus, the purpose of section 1, in addition to recalling various definitions, is to clarify the relationship between the discriminant of L_v (resp. $C^+(V_v)$, $\mathcal{O}_v$) and that of L (resp. $C^+(V)$, $\mathcal{O}$). The most interesting result is that $((v, v)D_v^{-1}D)^{1/2} =: \mathfrak{n}_L(v)$ is the greatest common divisor of $\{(u, v) : u \in L\}$ (Proposition 1.1.3); this number shows up later as a way of restoring dependence on the choice of specific vector v.

To construct the cycles $T_n^{(k)}$ in section 2, we first construct some subvarieties of $\mathcal{A}^{(k)}$ which are in some sense "too big" and then we trim them down algebraically. The inclusion of L_v in L induces inclusions of the objects associated to L_v into the corresponding objects associated to L, so that if we let $\Gamma_v := G_v \cap \Gamma$ then we get a complex projective curve S_v, an algebraic family of abelian varieties $\mathcal{A}_v^{(k)}$ whose dimension is half that of $\mathcal{A}^{(k)}$, and natural holomorphic (thus algebraic) immersions $i_v : S_v \to S$ and $h_v : \mathcal{A}_v^{(k)} \to \mathcal{A}^{(k)}$ which are compatible with the fibrations, so that the diagram (2.4.3) commutes. However, the $h_v(\mathcal{A}_v^{(k)})$ for $q(v) = n$ are *not* the components of the $T_n^{(k)}$ (unless $k = 0$); in fact, when $k > 0$ the intersection multiplicity of $h_v(\mathcal{A}_v^{(k)})$ with $h_u(\mathcal{A}_u^{(k)})$ is zero! So next we observe, following Kuga [Kg1] and [Go2], that there is an algebraic decomposition of the middle cohomology $H^{4k+2}(\mathcal{A}^{(k)}, \mathbb{Q})$ into subspaces isomorphic to $H^a(\Gamma, E)$, for various $\mathbb{Q}$-representations E of G. In particular, this means that there exists an algebraic cycle $\mathcal{P}$ in the ring of correspondences on $\mathcal{A}^{(k)} \times \mathcal{A}^{(k)}$ which induces a projection from $H^{4k+2}(\mathcal{A}^{(k)}, \mathbb{Q})$ onto the subspace isomorphic to $H^2(\Gamma, E_{2k})$, where E_{2k} is the irreducible constituent of highest weight in $\bigwedge^{4k} W^k$ (see Lemma 2.3.5). This distinguished subspace of $H^{4k+2}(\mathcal{A}^{(k)}, \mathbb{Q})$ will be denoted by $H^{4k+2}(\mathcal{M}, \mathbb{Q})$, where we use this notation because $H^{4k+2}(\mathcal{M}, \mathbb{Q})$ may be thought of as the Betti realization in degree $4k + 2$ of the "motive" $\mathcal{M}$ defined by the variety $\mathcal{A}$ and the projector $\mathcal{P}$ [Gr] [D3]. Now using the correspondence $\mathcal{P}$, we get (for $n > 0$)

$$T_n^{(k)} := \sum_{v \in \Gamma \backslash L(n)} \mathfrak{n}_L(v)^{2k} \mathcal{P}[h_v(\mathcal{A}_v)]$$

where the sum is over representatives for the Γ-equivalence classes of $L(n) := \{v \in L : q(v) = n\}$. Thus $T_n^{(k)}$ is an algebraic cycle on $\mathcal{A}^{(k)}$

representing a class in $H^{4k+2}(\mathcal{M}, \mathbb{Q})$.

Section 3 is devoted to computing the pairwise intersection multiplicities of these algebraic cycles. For a pair of vectors $u, v \in L$, let

$$Q_{2k}(u,v) := \sum_{j=0}^{k} (-1)^j \binom{2k-j}{j} q(u)^j q(v)^j (u,v)^{2k-2j} \, ;$$

up to a convenient renormalization, this is Zagier's weighting factor, the ultraspherical Gegenbauer polynomial associated to the representation E_{2k} above [Vl], Ch. IX. Further let

$$D(u,v) := (u,v)^2 - 4q(u)q(v)$$

denote the discriminant of the sublattice of L generated by u and v, and let

$$\nu(u,v) := \begin{cases} 1 & \text{if } q(u) > 0 \text{ and } D(u,v) < 0, \\ \chi(S_u) & \text{if } q(u) > 0 \text{ and } D(u,v) = 0, \\ \chi(S_v) & \text{if } q(v) > 0 \text{ and } D(u,v) = 0, \\ \frac{1}{2}\chi(S) & \text{if } u = v = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\chi(\cdot)$ is the Euler characteristic [Hi]. Note that since q is anisotropic, $D(u,v) = 0$ if and only if $\mathbb{Q}u = \mathbb{Q}v$.

THEOREM 0.1.1. *For* $m, n \geq 0$,

$$(T_m^{(k)} \cdot T_n^{(k)}) = \sum_{(u,v) \in \Gamma \backslash (L(m) \times L(n))} \nu(u,v) Q_{2k}(u,v),$$

where Γ *acts diagonally on* $L(m) \times L(n)$.

Our first proof of this theorem is geometric. The structure of the $T_n^{(k)}$ is such that when $i_u(S_u)$ and $i_v(S_v)$ meet transversely $(T_m^{(k)} \cdot T_n^{(k)})$ should be a sum over the points of intersection of $i_u(S_u)$ with $i_v(S_v)$ of intersections in the fibers of $\mathcal{A}^{(k)}$ over those points counted with proper multiplicity, and an application of Kudla's methods [Kd1] confirms this, see Lemma 3.5. The harder part of the proof is to show that the intersection multiplicity in a fiber is $Q_{2k}(u,v)$, and for this a lemma of Millson (Lemma 3.8) plays a crucial role. If, on the other hand, $T_m^{(k)}$ and $T_n^{(k)}$ have a component Y in common, we suppose first that $i_u(S_u) = i_v(S_v)$ is nonsingular. Then we are able to show that the top Chern class of the normal bundle of Y factors into the product of the top Chern class of the normal bundle of $i_u(S_u)$ times the top Chern class of of the normal bundle of the fiber of Y. This allows us to write $(Y \cdot Y)$ in the form $\chi(S_v)Q_{2k}(u,v)$, and then the extension to the case where $i_u(S_u)$ may have singularities (normal crossings) is an adaptation of Kudla's method [Kd1], Proposition 5.2.

In section 4 we write down, for $k > 0$, the harmonic differential forms $\omega_n^{(k)}$ on $\mathcal{A}^{(k)}$ that are Poincare dual to the $T_n^{(k)}$, see Theorem 4.4, and give in (4.5) an alternate proof for Theorem 0.1.1, for the case $k > 0$, by integrating $\omega_n^{(k)}$ over $T_m^{(k)}$. It follows from [Kg1] and [MS] that the space of harmonic forms on $\mathcal{A}^{(k)}$ representing classes in $H^{4k+2}(\mathcal{M}, \mathbb{C})$ is canonically isomorphic to a space of cusp forms of weight $2k+2$ for Γ. Then our construction of a suitable cusp form and the verification that it corresponds to the Poincare dual of $T_n^{(k)}$ are based on the constructions in [Z1] [Z2] [Z3].

In the last section of the paper we give a quick proof that

THEOREM 0.1.2. *$F_{k,m}(\tau)$ is an elliptic modular form of weight $2k+2$, level N and character ε; it is a cusp form if $k > 0$.*

The key ingredient of the proof is that

$$\Phi\left(\left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)\right) := \sum_{(u,v)\in\Gamma\backslash(L\times L)} \nu(u,v)\exp\left(\pi i \operatorname{tr}\left(\begin{smallmatrix} (u,u) & (u,v) \\ (v,u) & (v,v) \end{smallmatrix}\right)\left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)\right) \qquad (0.1.3)$$

is a Siegel modular form of genus 2, weight 2, level N and character ε, as follows from [KM4] [KM6] [Mi3]. Then it is proved in [EZ], Theorem 3.1, that the Fourier-Jacobi coefficients $\varphi(\tau, z)$ of Φ with respect to τ' have the property that certain linear combinations of their Taylor coefficients with respect to z --- which coincide with our $F_{k,m}(\tau)$ -- are elliptic modular forms of higher weight, as required.

0.2. As indicated earlier, this paper was motivated by a question of Zagier [Z2] and the approach to intersection numbers as the Fourier coefficients of modular forms of Kudla [Kd1]. In 1979 Tong [T1] took a different approach to Zagier's question: In the cohomology of a Hilbert modular surface X with coefficients in a vector bundle E he associated to a Hirzebruch–Zagier cycle T_m on X a current coming from a section of the restriction of the vector bundle to T_m, and then using the methods of [TT] he verified that pairing two such currents yielded Zagier's weighted intersection numbers. Now it should be remarked that the cohomological home $H^{4k+2}(\mathcal{M}, \mathbb{Q})$ for our cycles is isomorphic to the cohomology of the surface with coefficients in the sheaf of locally constant sections of the vector bundle E, and with this identification, our cycles represent the same cohomology classes as Tong's currents, up to differences having to do with the cusps. The major differences between Tong's methods and ours are that: (i) our cycles are constructed geometrically so as to be algebraic cycles in the cohomology of some variety $\mathcal{A}^{(k)}$ with constant coefficients; (ii) both of our computations of intersection multiplicities are different from Tong's; and (iii) since we are not actually working with the Hilbert modular surfaces that Zagier studied, we need a new proof that the generating function for the intersection numbers is a modular form.

Millson [Mi1] has also looked at special cycles of higher weight coming from sub-torus bundles over totally geodesic cycles, of which $h_v(\mathcal{A}_v^{(k)}) \subset \mathcal{A}^{(k)}$ (viewed as Reimannian manifolds) would be an example, and then used these cycles to get nonvanishing theorems for the cohomology of arithmetic subgroups $\Gamma' \subset \mathrm{O}(n,1)$. In particular, there is the following analogy between his work and ours: $H^{4k+2}(\mathcal{M},\mathbb{R})$ is also isomorphic to $H^2(\Gamma,\mathcal{H}_{2k}(V(\mathbb{R})))$, the cohomology of Γ with coefficients in harmonic polynomials of degree $2k$. Then the nonvanishing of the (first) cohomology of Γ' with coefficients in harmonic polynomials of some degree on its natural representation space, [Mi1] Theorem 3.2, is the $\mathrm{O}(n,1)$ analogue of the fact in this paper that $T_m^{(k)}$ represents a nontrivial cohomology class. Lemma 3.8 below, which was shown to us by Millson, allows us to go beyond nonvanishing to the actual computation of intersection multiplicities.

In another direction, Oda [O] considered middle-dimensional cycles in the locally symmetric varieties associated with $\mathrm{SO}(2,n-2)$, and showed (among other things) that the functions of the form

$$F(\tau;\varphi) := \sum_{n=1}^{\infty} e^{2\pi i n\tau} \sum_{v\in\Gamma\backslash L(n)} \int_{C_v} \varphi(x)\, l(x,v)^{k-(n-2)}\omega(x)$$

were cusp forms of weight $(2k-n+4)/2$ and Nebentypus on a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, where φ is a cusp form of weight k for $\mathrm{SO}(2,n-2)$, and $l(x,v)$ is a (harmonic) polynomial weighting factor, and $\omega(x)$ is a suitable differential. In the context of the present paper, φ would correspond to a differential form on a Kuga variety $\mathcal{A}$ which could be paired with a higher weight cycle in the same variety, and Oda's integral would be what remains after integrating out the fiber variable; this may be compared with our second computation of the intersection multiplicities, where we integrate the Poincare dual form of one cycle over another (4.5). In addition, Oda's work showed that the Hirzebruch–Zagier map $T_m^c \mapsto f_m(\tau)$, and Kudla's and Cogdell's (and our) analogues thereof, could be viewed as geometric formulations of the liftings of automorphic forms determined by a restriction of the Weil representation [We] [LV] to a dual reductive pair [Ho], as in Shintani [Shn].

Since 1982 this last idea has motivated extensive investigation by Kudla and Millson [KM1] [KM2] [KM3] [KM4] [KM5] [KM6] [Mi2] [Mi3], Tong and Wang [TW1] [TW2] [TW3] [TW4] [TW5] [TW6] [TW7] [Wa] [T2], and Cogdell [Cg2]. For example, the case $k = 0$ of our work (no fiber) fits into the Kudla-Millson framework in the following way: Let G' denote either

SL_2 or Sp_4, and let $\mathfrak{X}'$ denote the corresponding symmetric space. Then there is a canonically constructed theta series $\theta(x, x')$, with $x \in \mathfrak{X}$ and $x' \in \mathfrak{X}'$, which defines a closed differential form on $S = \Gamma\backslash\mathfrak{X}$ and a modular form as a function of x'. In case $G' = \mathrm{SL}_2$, we have

$$F_{0,m}(\tau) = \int_{T_m^{(0)}} \theta(x, x'),$$

and conversely, the harmonic Poincare dual of $T_m^{(0)}$ can be obtained by integrating $\theta(x, x')$ against an elliptic modular cusp form. And we actually use the case $G' = \mathrm{Sp}_4$ in that Φ defined by (0.1.3) is given by

$$\Phi(x') = \int_S \theta(x, x').$$

By comparison with this, our approach to the harmonic Poincare dual, Theorem 4.2, and our proof that $F_{k,m}$ is a modular form, Theorem 5.3, in the case $k > 0$ seem terribly *ad hoc*; yet it may be hoped that these results, especially the appearance of [EZ], Theorem 3.1, in the proof of Theorem 5.3, may prove to be suggestive examples when the results of Kudla and Millson and Tong and Wang are extended to higher weight.

0.3. While the mathematical debt this paper owes to the work of Hirzebruch, Zagier, Kudla, Millson and Cogdell may be clear, my personal debt of gratitude to these authors is equally large: To Zagier for suggesting the problem and much helpful encouragement; and to Hirzebruch and Zagier for the opportunity to spend some time at the Max-Planck-Insitut fur Mathematik in Bonn, where an important part of this work was done; and to Kudla, Millson and Cogdell for several lengthy and very valuable conversations. I also benefitted from conversations with Cipra, Wang, Tong and Ozaydn, and from the encouragement of K. Weih. And finally I would like to thank the referee for some helpful suggestions, especially for the quick proof of Corollary 3.3 in the form that it appears here.

The main results of this paper were announced in [Go1].

1. Algebraic preliminaries

This section may be skipped or quickly scanned on a first reading, and refered to as needed. Here we determine the relationship between the discriminants of L and its sublattice L_v, we describe the even Clifford algebras of V and V_v and the relationships between them and between their orders generated by L and L_v, and we recall some facts about the finite-dimensional class one representations of the spin group of V.

1.1. As indicated in the introduction, the basic data which governs this entire paper is a lattice L of rank 4 equipped with an indefinite anisotropic integral-valued quadratic form q whose discriminant is positive and nonsquare. Letting $V := L \otimes \mathbb{Q}$, these assumptions on q and its discriminant imply that $\mathbf{signature}(V_{\mathbb{R}}, q) = (2, 2)$. In addition, we equip V with the symmetric bilinear form even and integral on L defined by

$$(u, v) := q(u + v) - q(u) - q(v), \tag{1.1.1}$$

noting that with this definition $(v, v) = 2q(v)$.

Now let $L^+ := \{ v \in L : q(v) > 0 \}$, and for $v \in L^+$, let

$$V_v := \{v\}^\perp = \{ u \in V : (u, v) = 0 \},$$

let $L_v := V_v \cap L$, and let q_v denote the restriction of q to L_v and V_v. Although we have indexed these objects by v, it may be helpful to note that V_v, L_v and q_v actually depend only on the line $\mathbb{Q}v$.

In order to describe the relationship between the discriminants of L and L_v, recall that the discriminant $D := D(L)$ of L is given by $D(L) := \det(\,(v_i, v_j)\,)$, where $\{v_1, v_2, v_3, v_4\}$ is any $\mathbb{Z}$-basis for L, and $D_v := D(L_v)$ may be defined similarly. Let, for any $v \in L$,

$$\mathfrak{n}_L(v) := \text{g. c. d.}\{ (u, v) : u \in L \}. \tag{1.1.2}$$

Then $\mathfrak{n}_L(mv) = m\mathfrak{n}_L(v)$ for $m \in \mathbb{Z}$, so we see that $\mathfrak{n}_L(v)^{-2}(v, v)$ depends only on L and $\mathbb{Q} \cdot v$, the defining data of L_v, and not on v.

PROPOSITION 1.1.3. $D(L_v) = \mathfrak{n}_L(v)^{-2}(v, v)D(L)$.

Proof. Let $M := \mathbb{Z} \cdot v \oplus L_v$, and $D(M)$ denote the discriminant of the restriction of $(\ ,\ )$ to M. Then $D(M) = [L : M]^2 D(L) = (v, v)D(L_v)$, so it suffices to prove that $[L : M] = (v, v)\mathfrak{n}_L(v)^{-1}$. Therefore we define $f: L \to \mathbb{Z}/(v, v)\mathfrak{n}_L(v)^{-1}\mathbb{Z}$ by $f(u) := (u, v)\mathfrak{n}_L(v)^{-1} \pmod{(v, v)\mathfrak{n}_L(v)^{-1}}$. Then $m \subset \ker f$, for if $u \in M$ then $u = av + bw$ with $a, ab \in \mathbb{Z}$ and $w \in L_v$, whence $f(u) \in (v, v)\mathfrak{n}_L(v)^{-1}\mathbb{Z}$. Conversely, $\ker f \subset M$, for if $f(u) = c(v, v)\mathfrak{n}_L(v)^{-1}$ for some $c \in \mathbb{Z}$, then $u - cv \in L_v$, so $u \in M$. Finally, f is surjective, for by the definition of $\mathfrak{n}_L(v)$ it follows that 1 is in the ideal generated by $\{ (u, v)\mathfrak{n}_L(v)^{-1} : u \in L \}$ and f is $\mathbb{Z}$-linear. □

REMARK. The same proof applies when $\mathbb{Z}$ is replaced by any commutative principal ideal domain R, and L is a free R-module, and $L_v = \{v\}^\perp \cap L$ for some element v such that $(v, v) \neq 0$.

1.2. The even Clifford algebra $C^+(V)$ of V may be defined as the quotient of the even part of the tensor algebra of V, that is the subalgebra generated by products of an even number of elements of V, by the ideal generated by the elements of the form $v \otimes v - q(v) \cdot 1$ [Cas] Chapter 10, [Ch] Chapter II, [E] section 4.5, [A] Chapter V. On $C^+(V)$ there is a canonical involution ι, namely the antiautomorphism induced by $v_1 \otimes \cdots \otimes v_{2r} \mapsto v_{2r} \otimes \cdots \otimes v_1$ in the tensor algebra, as well as a trace map $c \mapsto c + c^\iota$ and a norm map $c \mapsto c \cdot c^\iota$. In particular, for $u, v \in V$,

$$uv + vu = (u, v) \tag{1.2.1}$$

in $C^+(V)$. The even Clifford algebra $C^+(V_v)$ of V_v may be defined similarly, and the inclusion of V_v in V induces an inclusion of $C^+(V_v)$ in $C^+(V)$ such that the canonical involution, trace and norm of $C^+(V_v)$ are the restrictions of those of $C^+(V)$. Alternatively, $C^+(V_v)$ may be characterized as the subalgebra of $C^+(V)$ fixed by the involution $c \mapsto q(v)^{-1}vcv$.

PROPOSITION 1.2.2.

(i) *The even Clifford algebra $C^+(V_v)$ is an indefinite division quaternion algebra over $\mathbb{Q}$ whose canonical involution, trace and norm as even Clifford algebra coincide with its canonical involution, reduced trace and reduced norm as quaternion algebra. The reduced discriminant $d(C^+(V_v))$ of $C^+(V_v)$ as quaternion algebra is the product of those primes at which q_v is anisotropic.*

(ii) *The center of $C^+(V)$ is spanned by 1 and an element ζ which is determined up to homothety as the product of elements in a basis of V and may be normalized so that $\zeta^2 = D$.*

(iii) $C^+(V) = \mathbb{Q}(\zeta) \cdot C^+(V_v)$.

(iv) *The even Clifford algebra $C^+(V)$ is a totally indefinite division quaternion algebra over the real quadratic field $\mathbb{Q}(\zeta)$ whose canonical involution, trace and norm as even Clifford algebra coincide with its canonical involution, reduced trace and reduced norm as quaternion algebra over $\mathbb{Q}(\zeta)$. The reduced discriminant $d(C^+(V))$ of $C^+(V)$ as quaternion algebra over $\mathbb{Q}(\zeta)$ is the product of those rational primes at which q is anisotropic.*

(v) *The rational primes which divide $d(C^+(V))$ split in $\mathbb{Q}(\zeta)$, while the rational primes which divide $d(C^+(V_v))/d(C^+(V))$ are inert or ramified in $\mathbb{Q}(\zeta)$.*

Proof. For (i) see [Cas], for (ii) see [Ch], and for (iii) see [A], (V.6), or [Cas]. Since $\det q$ is not a rational square, $C^+(V)$ is a simple $\mathbb{Q}$-algebra of rank 8, whence by (iii) it is a totally indefinite quaternion algebra over $\mathbb{Q}(\zeta)$ with canonical involution, reduced trace and reduced norm as claimed. To complete the proof of both (iv) and (v), let p be a rational prime and let $\mathfrak{p}$ be a prime of $\mathbb{Q}(\zeta)$ dividing p. Then

$$C^+(V) \otimes_{\mathbb{Q}(\zeta)} \mathbb{Q}(\zeta)_{\mathfrak{p}} \simeq (C^+(V_v) \otimes_{\mathbb{Q}} \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{Q}(\zeta)_{\mathfrak{p}}.$$

Now, if q is anisotropic at p, then so is q_v, and $C^+(V_v) \otimes \mathbb{Q}_p$ is a division algebra. However, q is anisotropic at p if and only if p splits in $\mathbb{Q}(\zeta)$ and the Hasse-Minkowski invariant of q at p is $(-1)^p$ [Cas] (4.2.6). In ths case, $\mathbb{Q}(\zeta)_{\mathfrak{p}} \simeq \mathbb{Q}_p$ and so $C^+(V) \otimes \mathbb{Q}(\zeta)_{\mathfrak{p}}$ is a division algebra. Conversely, if q is isotropic at p, then either q_v is also, in which case $C^+(V_v)$ is unramified at p and $C^+(V)$ is unramified at $\mathfrak{p}$, or else $[\mathbb{Q}(\zeta)_{\mathfrak{p}} : \mathbb{Q}_p] = 2$, in which case $\mathbb{Q}(\zeta)_{\mathfrak{p}}$ splits $C^+(V_v) \otimes \mathbb{Q}_p$. □

It follows from this proposition that whenever F_0 is a field containing $\mathbb{Q}$ and an element $\omega \neq \pm\zeta$ such that $\omega^2 = D$, then

$$\varepsilon_1 := \frac{\omega + \zeta}{2\omega} \qquad \text{and} \qquad \varepsilon_2 := \frac{\omega - \zeta}{2\omega}$$

are orthogonal central idempotents in $C^+(V(F_0)) \cong C^+(V) \otimes_{\mathbb{Q}} F_0$. But then

$$\begin{aligned} C^+(V(F_0)) &= C^+(V(F_0))\varepsilon_1 \oplus C^+(V(F_0))\varepsilon_2 \\ &= C^+(V_v(F_0))\varepsilon_1 \oplus C^+(V_v(F_0))\varepsilon_2, \end{aligned} \tag{1.2.3}$$

so that $C^+(V_v)$ embeds diagonally into $C^+(V)$ over F_0. Moreover, the involution $c \mapsto q(v)^{-1}vcv$, of which $C^+(V_v)$ is the set of fixed points, maps ζ to $-\zeta$, thus interchanging ε_1 and ε_2. Now if $F_1 \supset \mathbb{Q}$ is any splitting field for $C^+(V_v)$, so that

$$C^+(V_v(F_1)) \xrightarrow{\sim} M_2(F_1), \tag{1.2.4}$$

and F is any field containing F_0F_1, then F is a splitting field for $C^+(V)$ over $\mathbb{Q}$. Over such a field F we have the commutative diagram

$$\begin{tikzcd} C^+(V_v(F)) \arrow[r, "\sim"] \arrow[d] & M_2(F) \arrow[d, "\Delta"] \\ C^+(V(F)) \arrow[r, "\sim"] & M_2(F) \oplus M_2(F) \end{tikzcd} \tag{1.2.5}$$

where the left-hand vertical arrow is the natural inclusion and the right-hand vertical arrow is the diagonal map.

1.3. The lattice $L \subset V$ determines the order $\mathcal{O} := \mathcal{O}(L) := C^+(L)$ in $C^+(V)$ generated over $\mathbb{Z}$ by products of an even number of elements of L, and likewise $L_v \subset V_v$ determines $\mathcal{O}_v := \mathcal{O}(L_v) := C^+(L_v)$ in $C^+(V_v)$. From the definition of L_v as $V_v \cap L$ we clearly have $\mathcal{O}(L_v) \subset C^+(V_v) \cap \mathcal{O}(L)$.

LEMMA 1.3.1.
(i) $\mathcal{O}(L_v) = C^+(V_v) \cap \mathcal{O}(L)$.

(ii) *The reduced discriminant of* $\mathcal{O}(L)$ *as an order in the simple* $\mathbb{Q}$*-algebra* $C^+(V)$ *is* $d(\mathcal{O}(L)) = D(L)^2$.

(iii) *The reduced discriminant of* $\mathcal{O}(L_v)$ *is*

$$d(\mathcal{O}(L_v)) = \tfrac{1}{2}D(L_v) = \mathfrak{n}_L(v)^{-2}q(v)D(L).$$

Proof. As a special case of [Cas] Theorem 7.3.1, for any $\mathbb{Z}$-lattice M and any $m_1, \ldots, m_l \in M$, the following are equivalent: (a) there exist $m_{l+1}, \ldots, m_n$ such that $\{m_1, \ldots, m_n\}$ is a $\mathbb{Z}$-basis for M; and (b) if $a_1m_1 + \cdots + a_lm_l \in M$ with $a_1, \ldots, a_l \in \mathbb{Q}$, then $a_i \in \mathbb{Z}$ for $1 \le i \le l$. Thus, ((b) implies (a)) a basis $\{v_1, v_2, v_3\}$ of L_v extends to a basis $\{v_1, v_2, v_3, v_4\}$ of L. Therefore a basis $\{\, 1, v_iv_j : 1 \le i < j \le 3 \,\}$ of $\mathcal{O}(L_v)$ extends to a basis $\{\, 1, v_1v_2v_3v_4, v_iv_j : 1 \le i < j \le 4 \,\}$ of $\mathcal{O}(L)$. Then ((a) implies(b)) yields (i).

Now fix a basis $\{v_1, v_2, v_3, v_4\}$ of L and let

$$\{E_1 = 1,\ E_2 = v_1v_2,\ \ldots,\ E_7 = v_3v_4, E_8 = v_1v_2v_3v_4\}$$

be the $\mathbb{Z}$-basis of $\mathcal{O}(L)$ it generates. Then it is a straightforward, if lengthy, computation, using (1.2.1) repeatedly, to verify that $\det(\mathrm{tr}_{C^+(V)/\mathbb{Q}}(E_iE_j))^{1/2} = D(L)^2$, proving (ii), and (iii) is proved similarly. □

1.4. Recall that the spin group $G := \mathrm{Spin}(V)$ of V is the semisimple algebraic group defined over $\mathbb{Q}$ by

$$\mathrm{Spin}(V) := \{\, g \in C^+(V) : gg^\iota = 1 \text{ and } gug^{-1} \in V \text{ for all } u \in V \,\},$$

where gug^{-1} makes sense as an element of the full Clifford algebra of V. As $\dim V \le 4$ the second condition is redundant; however, it points out that G comes equipped with its vector representation (ψ, V), defined by $\psi(g) \cdot v := gvg^{-1}$, which maps G into $\mathrm{SO}(V)$ with kernel $\{\pm 1\}$ [Cas] [Ch]. The spin representation (σ, W) of G is defined to be (equivalent to) the left regular representation of G on any left ideal of $C^+(V)$; since $C^+(V)$ is a simple $\mathbb{Q}$-algebra, over $\mathbb{Q}$ the spin representation is just the left regular representation of G on $C^+(V)$. The spin group $G_v := \mathrm{Spin}(V_v)$ of V_v with its vector representation (ψ_v, V_v) and its spin representation (σ, W_v) may be defined similarly. Moreover, as with the even Clifford algebras, the inclusion of V_v in V induces an inclusion of G_v into G, so that G_v may also be realized as $C^+(V_v) \cap G$, or as the subgroup of G fixed by $g \mapsto vgv^{-1}$, or as the stabilizer via ψ of V_v.

In order to describe the irreducible finite-dimensional representations of G and G_v, let F_0, F_1 and F be as at the end of (1.2). Then over F_0 the spin representation decomposes into a sum of two inequivalent half-spin representations,

$$(\sigma, W) \sim_{F_0} (\sigma^{(1)}, W^{(1)}) \oplus (\sigma^{(2)}, W^{(2)}), \tag{1.4.1}$$

where $(\sigma^{(i)}, W^{(i)})$ is given by left multiplication of G on $C^+(V(F_0))\varepsilon_i$, as in (1.2.3). Thus

$$G \simeq_{F_0} G^{(1)} \times G^{(2)} \quad \text{and} \quad G^{(i)}(F_0) \simeq G_v(F_0), \tag{1.4.2}$$

where $G^{(i)} := \ker \sigma^{(3-i)}$, while

$$G_v(F_1) \xrightarrow{\sim} \mathrm{SL}_2(F_1). \tag{1.4.3}$$

Therefore, as in (1.2.5) we have a commutative diagram

$$\begin{array}{ccc} G_v(F) & \xrightarrow{\sim} & \mathrm{SL}_2(F) \\ \downarrow & & \downarrow{\scriptstyle\Delta} \\ G(F) & \xrightarrow{\sim} & \mathrm{SL}_2(F) \times \mathrm{SL}_2(F). \end{array} \tag{1.4.4}$$

In particular, over F the half-spin representation $\sigma^{(i)}$ projects G into the i^{th} factor of $\mathrm{SL}_2 \times \mathrm{SL}_2$. .

Now the rational finite-dimensional representations of G and G_v may be described in terms of the more familiar theory for SL_2: Let $(\overline{\rho}_k, \overline{V}_k)$ denote the symmetric tensor representation of SL_2 of degree k and dimension $k+1$, for $k \in \mathrm{N}$. Then every absolutely irreducible finite-dimensional representation of G_v is equivalent over a splitting field to the composition (ρ_k, V_k) of $G_v \to \mathrm{SL}_2$ with $(\overline{\rho}_k, \overline{V}_k)$ for some k. And every absolutely irreducible finite-dimensional representation of G is equivalent over a splitting field to $(\rho_{(k_1,k_2)}, V_{(k_1,k_2)})$ for some $(k_1, k_2) \in \mathrm{N} \times \mathrm{N}$, where

$$(\rho_{(k_1,k_2)}, V_{(k_1,k_2)}) := (\rho_{k_1}^{(1)}, V_{k_1}^{(1)}) \otimes (\rho_{k_2}^{(2)}, V_{k_2}^{(2)}), \tag{1.4.5}$$

and $(\rho_k^{(i)}, V_k^{(i)})$ is the composition of $G \to G^{(i)} \to \mathrm{SL}_2$ with $(\overline{\rho}_k, \overline{V}_k)$. Moreover, $\rho_{(k_1,k_2)}$ (resp. ρ_k) factors through ψ (resp. ψ_v) if and only if $k_1 + k_2$ (resp. k) is even, as this is the condition under which $\rho_{(k_1,k_2)}(-1) = 1$ (resp. $\rho_k(-1) = 1$). We will be particularly interested in the representations of G which contain a G_v-invariant vector, the so-called "class one" representations [Vl]. Since [Sp]

$$\rho_{(k_1,k_2)}\Big|_{G_v} \sim \rho_{k_1+k_2} \oplus \rho_{k_1+k_2-2} \oplus \cdots \oplus \rho_{|k_1-k_2|},$$

these are exactly the $\rho_{(k_1,k_2)}$ with $k_1 = k_2$.

1.4.6. To find the $\mathbb{Q}$-simple representations of G and G_v we need only determine the Galois orbits of the absolutely irreducible ones [Sa3]. Since without loss of generality F_1 may be taken to be quadratic over $\mathbb{Q}$ [Vi], it follows that any $\mathbb{Q}$-irreducible finite-dimensional representation of G_v is equivalent over a splitting field to either ρ_k or $2\rho_k$ for some k. For example, the $\mathbb{Q}$-irreducible spin representation σ_v is equivalent over F_1 to $2\rho_1$, while $\psi_v \sim_{F_1} \rho_2$. Also without loss of generality we may let $F_0 = \mathbb{Q}(\sqrt{D})$, and take $F = F_0F_1$ to be biquadratic over $\mathbb{Q}$. Then the nontrivial Galois automorphism of F_0 interchanges the two half-spin representations, and it follows that any $\mathbb{Q}$-irreducible finite-dimensional representation of G is equivalent over a splitting field to some $\rho_{(k,k)}$ or $\rho_{(k_1,k_2)} \oplus \rho_{(k_2,k_1)}$ or $2(\rho_{(k_1,k_2)} \oplus \rho_{(k_2,k_1)})$. For example, $\sigma \sim_F 2(\rho_{(1,0)} \oplus \rho_{(0,1)})$ and $\psi \sim_F \rho_{(1,1)}$.

1.4.7. We may also use the lattices L and L_v to define arithmetic subgroups of G and G_v respectively. Let $\Gamma(L) := \mathcal{O} \cap G$ and $\Gamma(L_v) := \mathcal{O}_v \cap G_v$. Then $\Gamma(L)$ is the group of norm 1 units of $\mathcal{O}$, implying in particular that it preserves $\mathcal{O}$, and similarly for $\Gamma(L_v)$. Of course $\Gamma(L)$ also preserves L, as it follows from the definitions that $\Gamma(L)$ is normal in the subgroup of G which maps L to itself; and again, a similar statement applies to $\Gamma(L_v)$.

2. Algebraic cycles of higher weight

Before constructing and describing the algebraic cycles which interest us, we must first describe the algebraic variety and the subspace of its cohomology in which they live. This variety, which has the structure of a family of polarized abelian varieties parameterized by an algebraic surface, is essentially determined by L and q.

2.1. Let $\mathfrak{X}$ denote the symmetric space associated to the real Lie group $G(\mathbb{R})$. Then $\mathfrak{X} \simeq G(\mathbb{R})/K$, where K is a maximal compact subgroup of $G(\mathbb{R})$, but also $\mathfrak{X}$ may be identified with the set of negative-definite 2-dimensional subspaces of $V(\mathbb{R})$; in this realization the action of G on $\mathfrak{X}$ factors through ψ. Moreover, these negative 2-planes may be given compatible orientations by further identifying $\mathfrak{X}$ with a fixed one of the two connected components of its inverse image under the natural map from the Grassmannian of oriented 2-dimensional subspaces of $V(\mathbb{R})$ to the Grassmannian of (unoriented) 2-dimensional subspaces of $V(\mathbb{R})$. Once this is done, then $\mathfrak{X}$ inherits a complex struicture via the Borel embedding $\mathfrak{X} \to \mathbb{P}(V(\mathbb{C}))$, defined by mapping a negative 2-plane V^- to the complex line ℓ with the properties that $V^- \otimes_{\mathbb{R}} \mathbb{C} = \ell \oplus \bar{\ell}$ and $q(t) = 0$ and $(\bar{t}, t) < 0$ and $it \wedge \bar{t}$ is positive with respect to the orientation of V^- for all nonzero $t \in \ell$ [Sa4] (A.6.1).

Now let Γ denote a fixed torsion-free normal subgroup of finite index in $\Gamma(L)$ (1.4.6); such groups exist by [Bo2]. Then Γ acts freely and properly discontinuously on $\mathfrak{X}$, and the quotient

$$S := S_\Gamma := \Gamma\backslash\mathfrak{X}$$

is a compact complex manifold [MT] [BHC] which can be embedded as a smooth complex projective surface [K].

REMARK 2.1.1. We have chosen here to work with S_Γ, with Γ torsion-free, normal and of finite index in $\Gamma(L)$ as above, rather than the more canonical

$S_{\Gamma(L)} := \Gamma(L)\backslash\mathfrak{X}$, because the latter may have finite quotient singularities due to noncentral torsion in $\Gamma(L)$. However, since $S_{\Gamma(L)}$ is a rational homology manifold covered by the smooth S_Γ with finite covering group $\Gamma(L)/\Gamma$, all of our results concerning the rational (co)homology of S_Γ could be reformulated for $S_{\Gamma(L)}$ by dividing by $(\Gamma(L):\Gamma)$ in the appropriate places.

2.2. To construct an algebraic family of abelian varieties parameterized by S, let (σ, W) be the spin representation of G obtained by identifying W with $C^+(V)$ and σ with multiplication on the left by G, and let Λ be the lattice in W corresponding to $\mathcal{O}$. Further, let $j(x) \in C^+(V)$ be the ordered product of the elements in any oriented orthonormal basis of $x = V^- \in \mathfrak{X}$; note that this product is independent of the choice of oriented orthonormal basis, and that $j(x) \in G(\mathbb{R})$ with $j(x)^2 = -1$. Then following [Sa2], we may define a nondegenerate skew-symmetric bilinear form on $W(\mathbb{R})$, integer-valued on Λ, by $\beta(c_1, c_2) := \operatorname{tr}_{C^+(V(\mathbb{R}))/\mathbb{R}}(bc_1^\iota c_2)$ for $c_1, c_2 \in W(\mathbb{R})$, where $0 \neq b \in C^+(V)$ is chosen such that $\operatorname{tr}_{C^+(V)/\mathbb{Q}}(bc) \in \mathbb{Z}$ for all $c \in \mathcal{O}$ and $b^\iota = -b$ and $bj(x)$ is positive with respect to the involution $c \mapsto j(x)^{-1}c^\iota j(x)$ for all $c \in C^+(V(\mathbb{R}))$ and $x \in \mathfrak{X}$. Then $\sigma(G(\mathbb{R})) \subset \operatorname{Sp}(W(\mathbb{R}), \beta)$, and moreover, this inclusion induces a holomorphic map from $\mathfrak{X} \simeq G(\mathbb{R})/K$ to the Siegel upper half-plane $\operatorname{Sp}(W(\mathbb{R}), \beta)/K'$, where K' is any maximal compact containing $\sigma(K)$. Therefore, from the theory developed in [Kg1], or see [Sa2] [Sa4] [Mu1], it follows that there exists a unique complex structure on the C^∞ manifold

$$\mathcal{A}_\Gamma^{(1)} := (\Gamma \ltimes \Lambda)\backslash(\mathfrak{X} \times W(\mathbb{R}))$$

such that it becomes a smooth complex projective variety.

Now fix a nonnegative integer k and let

$$\mathcal{A} := \mathcal{A}_\Gamma^{(k)} := \mathcal{A}_\Gamma^{(1)} \times_{S_\Gamma} \cdots \times_{S_\Gamma} \mathcal{A}_\Gamma^{(1)}$$

(k factors) denote the k-fold fiber product. Then $\mathcal{A}$, too, is a smooth complex projective variety. As a C^∞ manifold

$$\mathcal{A} \cong (\Gamma \ltimes \Lambda^k)\backslash(\mathfrak{X} \times W^k(\mathbb{R})) \cong \Gamma\backslash(\mathfrak{X} \times W^k(\mathbb{R})/\Lambda^k), \tag{2.2.1}$$

where (σ^k, W^k) denotes the k-fold direct sum of (σ, W) with itself. For the sake of completeness we will also allow $k = 0$, in which case we will understand that $\mathcal{A} = S$.

2.2.2. To describe the complex structure on $\mathcal{A}$, let $j(x) \in C^+(V)$ be the ordered product of the elements in any oriented orthonormal basis of $x = V^- \in \mathfrak{X}$, as above. Then since $j(x) \in G(\mathbb{R})$ and $j(x)^2 = -1$, it follows that $J(x) := \sigma^k \circ j(x)$ determines a complex structure on $W^k(\mathbb{R})$. Now glueing together the disjoint union of $\{\, (W^k(\mathbb{R}), J(x)) : x \in \mathfrak{X} \,\}$ gives

$\mathfrak{X} \times W^k(\mathbb{R})$ a holomorphic structure as a vector bundle over $\mathfrak{X}$. In particular, the natural map $\varphi : \mathcal{A} \to S$ is holomorphic, and thus algebraic.

Moreover, since β is $\mathbb{Z}$-valued on Λ and $(w_1, w_2) \mapsto \beta(w_1, j(x)w_2)$ is a symmetric positive-definite bilinear form, $A^k(x) = (W^k(\mathbb{R})/\Lambda^k, J(x), \beta^k)$ is a polarized abelian variety for each $x \in \mathfrak{X}$ (whose endomorphism ring tensored with $\mathbb{Q}$ contains $C^+(V)$ [Sa2]), where β^k is the obvious extension of β to W^k. Moreover, for each $\gamma \in \Gamma$ the automorphism $\sigma^k(\gamma)$ of $W^k(\mathbb{R})$ induces an isomorphism from $A^k(x)$ to $A^k(\psi(\gamma)x)$. Therefore we may identify the fiber $A^k(s) := \varphi^{-1}(s)$, for $s \in S$, with $A^k(x)$ for any $x \in \mathfrak{X}$ which maps to s.

REMARK 2.2.3. A variant of the above construction would be to replace W^k with $W \otimes W'$, where W' is a k-dimensional rational vector space on which G acts trivially, and Λ by $\Lambda \otimes \Lambda'$, for some lattice $\Lambda' \subset W'$. Then β could be replaced by $\beta \otimes \beta'$, where β' is a positive-definite bilinear form on W', integral on Λ'. The result would be a family $\mathcal{A}'$ of abelian varieties

$$A'(x) = ((W(\mathbb{R}) \otimes W(\mathbb{R})')/(\Lambda \otimes \Lambda'), j(x) \otimes 1, \beta \otimes \beta')$$

with $A'(x)$ isogenous to $A(x)$. In particular, $\mathcal{A}'$ and $\mathcal{A}$ would have isomorphic rational (co)homology.

REMARK 2.2.4. Similarly to the situation of (2.1.1), we could construct the rational homology manifold $\mathcal{A}_{\Gamma(L)} \cong \Gamma(L)\backslash(\mathfrak{X} \times W^k(\mathbb{R})/\Lambda^k)$ finitely covered by $\mathcal{A}_\Gamma$, and as before reformulate all our results for $\mathcal{A}_{\Gamma(L)}$.

REMARK 2.2.5. It should also be remarked that $\mathcal{A}$ is a family of Hodge type, in the sense of [Mu1] [Mu2], as well as a family of PEL-type, in the sense of Shimura [Shm] (see also [Sa2] [Sa4]). It furthermore follows from Shimura's theory of canonical models [Shm] [D1] [D2] that S_Γ and $\mathcal{A}$ are both defined over some number field F_Γ.

2.3. Next we wish to identify a certain distinguished "algebraically defined" subspace of the middle cohomology of $\mathcal{A}$. Following [Kg3], for $m \in \mathbb{Z}$ let $\theta_m : \mathcal{A} \to \mathcal{A}$ be the endomorphism induced by $(x, w) \mapsto (x, mw)$ for $(x, w) \in \mathfrak{X} \times W^k(\mathbb{R})$. Then the induced endomorphism θ_m^* acts as m^b on the $E_2^{a,b}$ term of the Leray spectral sequence for $\varphi : \mathcal{A} \to S$ and commutes with the d_2-differentials, from which one deduces that this sequence degenerates at the E_2 term ("Lieberman's trick"). Thus

$$H^r(\mathcal{A}, \mathbb{Q}) = \bigoplus_{a+b=r} H^{\langle a,b \rangle}(\mathcal{A}, \mathbb{Q}), \tag{2.3.1}$$

where for $0 \leq a \leq 4$ and $0 \leq b \leq 8k$

$$\begin{aligned} H^{\langle a,b \rangle}(\mathcal{A}, \mathbb{Q}) : &= \{ \xi \in H^{a+b}(\mathcal{A}, \mathbb{Q}) : \theta_m^* \xi = m^b \xi \ \text{ for } m \in \mathbb{Z} \} \\ &\simeq H^a(S, R^b \varphi_* \mathbb{Q}) = E_2^{a,b}. \end{aligned} \tag{2.3.2}$$

Now we fix a base point $x_0 \in \mathfrak{X}$ with image $s_0 \in S$ and identify $\pi_1(S, s_0)$ with Γ by letting $\gamma \in \Gamma$ correspond to the homotopy class of the image in S of a path in $\mathfrak{X}$ joining x_0 to γx_0. Similarly, $H_1(A^k(x_0), \mathbb{Z}) \simeq \Lambda^k$ and thus

$$H^b(A(x_0), \mathbb{Q}) \simeq \bigwedge^b \check{W}^k, \tag{2.3.3}$$

where $\check{W}$ is dual to W. Then with these identifications the monodromy action of $\pi_1(S, s_0)$ on $H^b(A^k(x_0), \mathbb{Q})$ becomes the $\bigwedge^b \check{\sigma}^k$-action of Γ on $\bigwedge^b \check{W}^k$. Thus we have

$$H^{\langle a,b \rangle}(\mathcal{A}, \mathbb{Q}) \simeq H^a(\Gamma, \bigwedge^b \check{W}^k), \tag{2.3.4}$$

since both are isomorphic to $H^a(S, R^b \varphi_* \mathbb{Q})$. Thus each $H^{\langle a,b \rangle}(\mathcal{A}, \mathbb{Q})$ may be further decomposed according as $\bigwedge^b \check{W}^k$ decomposes as a Γ- module. Note that Γ is Zariski-dense in G [Bo1], so that $\bigwedge^b \check{W}^k$ decomposes identically under the action of Γ or of G.

LEMMA 2.3.5. *For each $k \geq 0$ there is among all the absolutely irreducible constituents of $\bigwedge^* (\check{\sigma}^k, \check{W}^k)$ a unique one (π_{2k}, E_{2k}) of maximal dimension. Moreover, this representation occurs in $\bigwedge^* (\check{\sigma}^k, \check{W}^k)$ with multiplicity one, it is contained in $\bigwedge^{4k} (\check{\sigma}^k, \check{W}^k)$, it is equivalent to $(\rho_{(2k,2k)}, V_{(2k,2k)})$, and it is defined over $\mathbb{Q}$.*

Proof. After passing to a field F over which $\check{\sigma}^k \sim_F 2k(\rho_{(1,0)} \oplus \rho_{(0,1)})$, it follows from the Cauchy decomposition formula [L] (1.7.3) and the Jacobi-Trudi formula [McD] (I.3.5) (or see [Kg1] (IV.2.2) or [KS]) that for $0 \leq b \leq 8k$

$$\bigwedge^b \check{\sigma}^k \sim_F \bigoplus_{0 \leq k_1, k_2 \leq b} m(k, b; k_1, k_2) \rho_{(k_1,k_2)},$$

where

$$m(k, b; k_1, k_2) := \sum_{b_1 + b_2 = b} a(2k, b_1, k_1) a(2k, b_2, k_2),$$

with

$$a(n, c, d) := \binom{n}{\frac{c+d}{2}} \binom{n}{\frac{c-d}{2}} - \binom{n}{\frac{c+d+2}{2}} \binom{n}{\frac{c-d-2}{2}},$$

understanding that $\binom{n}{m} = 0$ when $m \notin \mathbb{N}$. Then it is simple to check that the maximum of $(k_1 + 1)(k_2 + 1)$ for which $m(k, b; k_1, k_2) \neq 0$ occurs when $k_1 = k_2 = 2k$ and $b = 4k$, in which case $m(k, b; k_1, k_2) = 1$. Thus $(\rho_{(2k,2k)}, V_{(2k,2k)}) \hookrightarrow \bigwedge^{4k} (\check{\sigma}^k, \check{W}^k)$ is the unique absolutely irreducible constituent of maximal dimension and multiplicity one. Moreover, because it has multiplicity one, it is fixed by $\mathrm{Aut}(F/\mathbb{Q})$, and is therefore defined over $\mathbb{Q}$. □

Now with (π_{2k}, E_{2k}) as in the lemma, let $H^{4k}(\mathcal{E}_{2k}(x), \mathbb{Q})$ denote the subspace of $H^{4k}(A^k(x), \mathbb{Q})$ isomorphic to E_{2k} via (2.3.3), and let $H^{4k+2}(\mathcal{M}, \mathbb{Q})$ denote the subspace of $H^{\langle 2,4k\rangle}(\mathcal{A}, \mathbb{Q})$ isomorphic to $H^2(\Gamma, E_{2k})$ via (2.3.4). Then the following proposition is proved (in much greater generality) in [Go2].

PROPOSITION 2.3.6.

(i) *For each $x \in \mathfrak{X}$, there exists an algebraic cycle $P(x)$ in the ring of correspondences on $A^k(x) \times A^k(x)$ which induces a projection from $H^{4k}(A^k(x), \mathbb{Q})$ to $H^{4k}(\mathcal{E}_{2k}(x), \mathbb{Q})$. In particular, an algebraic class in $H^{4k}(A^k(x), \mathbb{Q})$ projects to an algebraic class in $H^{4k}(\mathcal{E}_{2k}(x), \mathbb{Q})$, and an element of $H^{4k}(\mathcal{E}_{2k}(x), \mathbb{Q})$ is algebraic if and only if it is so as an element of $H^{4k}(A^k(x), \mathbb{Q})$.*

(ii) *There exists an algebraic cycle $\mathcal{P}$ in the ring of correspondences on $\mathcal{A} \times \mathcal{A}$ which induces a projection from $H^{4k+2}(\mathcal{A}, \mathbb{Q})$ to $H^{4k+2}(\mathcal{M}, \mathbb{Q})$. In particular, an algebraic class in $H^{4k+2}(\mathcal{A}, \mathbb{Q})$ projects to an algebraic class in $H^{4k+2}(\mathcal{M}, \mathbb{Q})$, and an element of $H^{4k+2}(\mathcal{M}, \mathbb{Q})$ is algebraic if and only if it is so as an element of $H^{4k+2}(\mathcal{A}, \mathbb{Q})$.* □

Recall that the action of a correspondence C on the cohomology $H^r(Z, \mathbb{Q})$ of a projective variety Z is defined by lifting an element of $H^r(Z, \mathbb{Q})$ to $H^{2\dim Z}(Z \times Z, \mathbb{Q})$ via the first projection from $Z \times Z$ to Z, then taking the cup product with the class in $H^{2\dim Z}(Z \times Z, \mathbb{Q})$ represented by C, and then taking the image in $H^r(Z, \mathbb{Q})$ under the Gysin homomorphism associated to the first projection from $Z \times Z$ to Z. Here, once one sees that a polynomial in θ_m^* induces a projection from $H^{4k+2}(\mathcal{A}, \mathbb{Q})$ to $H^{\langle 2,4k\rangle}(\mathcal{A}, \mathbb{Q})$, then (ii) may be deduced from (i) by showing that when x represents a generic point on S then $P(x)$ may be extended to a correspondence on $\mathcal{A}$ which projects $H^{\langle 2,4k\rangle}(\mathcal{A}, \mathbb{Q})$ to $H^{4k+2}(\mathcal{M}, \mathbb{Q})$. Furthermore, since we are working in the middle cohomology of $A^k(x)$ and $\mathcal{A}$, we may arrange that the projections induced by $P(x)$ and $\mathcal{P}$ are orthogonal with respect to the cup product pairings on these spaces.

REMARK 2.3.7. In the language of [Go2], the cohomology spaces $H^{4k}(\mathcal{E}_{2k}(x), \mathbb{Q})$ and $H^{4k+2}(\mathcal{M}, \mathbb{Q})$ are said to be "algebraically defined," since there are algebraic correspondences which induce projections onto them. Thus they may be thought of as the Betti realizations of Grothendieck motives [Gr] [D3], although the technical distinction is that we have not produced correspondences which induce projections from the full cohomology rings $H^*(\mathcal{A}, \mathbb{Q})$ and $H^*(A^k(x), \mathbb{Q})$. It may also be noted that Proposition 2.3.6 is still true when (π_{2k}, E_{2k}) is replaced by any other $\mathbb{Q}$-rational sub-G-module of $\bigwedge^*(\check{\sigma}^k, \check{W}^k)$ (and the indices are changed accordingly), see [Go2].

2.4. As in section one, let $v \in L^+$ determine $V_v = \{v\}^\perp$ and $L_v = V_v \cap L$, with even Clifford algebra $C^+(V_v)$ containing the order $\mathcal{O}_v = C^+(L_v)$ and spin group G_v. Further, let $\mathfrak{X}_v$ denote the symmetric space associated with the real Lie group $G_v(\mathbb{R})$. Then as in (2.1), $\mathfrak{X}_v$ may be identified with the set of negative 2-planes in $V_v(\mathbb{R})$, so that there is a natural embedding of $\mathfrak{X}_v$ into $\mathfrak{X}$, and moreover $\mathfrak{X}_v$ may be given a complex structure which makes this embedding holomorphic. Now let $\Gamma_v := \Gamma \cap G_v$ and

$$S_v := S_{\Gamma_v} := \Gamma_v \backslash \mathfrak{X}_v.$$

Then S_v is a smooth complex projective curve which comes naturally equipped with a morphism $i_v : S_v \to S$ of degree one induced by the inclusion of $\mathfrak{X}_v$ in $\mathfrak{X}$. The image $C_v := i_v(S_v)$ may intersect itself transversely if there are elements of Γ not in Γ_v which identify points of $\mathfrak{X}_v$, but it will not have any other singularities.

2.4.1. Now, as in (2.2), let (σ_v, W_v) be the spin representation of G_v obtained by identifying W_v with $C^+(V_v)$ and letting G_v act from the left, and let $\Lambda_v \subset W_v$ be the lattice corresponding to $\mathcal{O}_v$. Then the restriction β_v of the nondegenerate skew-symmetric bilinear form β to the subspace $W_v \subset W$ is again a nondegenerate skew-symmetric bilinear form, for the conditions in (2.2) on the nonzero $b \in C^+(V)$ used to define β insure that $(w_1, w_2) \mapsto \beta(w_1, j(x)w_2)$ is a symmetric positive-definite bilinear form for all w_1, $w_2 \in W_v$ and all $x \in \mathfrak{X}_v$. Hence σ_v maps G_v into the symplectic group $\mathrm{Sp}(W_v, \beta_v)$. Thus, as before, there is a unique complex structure on

$$\mathcal{A}_v^{(1)} := (\Gamma_v \ltimes \Lambda_v)\backslash(\mathfrak{X}_v \times W_v(\mathbb{R}))$$

with which this quotient becomes a smooth complex projective variety. Then with k fixed as in (2.2.1), let

$$\mathcal{A}_v := \mathcal{A}_v^{(k)} := \mathcal{A}_v^{(1)} \times_{S_v} \cdots \times_{S_v} \mathcal{A}_v^{(1)}$$

be the k-fold fiber product. The fibers of the natural map $\varphi_v : \mathcal{A}_v \to S_v$ are polarized abelian varieties $A_v^k(x) := (W_v^k(\mathbb{R})/\Lambda_v^k, J_v(x), \beta_v^k)$, where $J_v(x) := \sigma_v^k \circ j(x)$ with j the same as in (2.2), and $x \in \mathfrak{X}_v$, and β_v^k is the obvious extension of β_v to W_v^k. As a C^∞-manifold,

$$\mathcal{A}_v \cong (\Gamma_v \ltimes \Lambda_v^k)\backslash(\mathfrak{X}_v \times W_v^k(\mathbb{R})) \cong \Gamma_v\backslash(\mathfrak{X}_v \times W_v^k(\mathbb{R})/\Lambda^k). \qquad (2.4.2)$$

The natural inclusions $\mathfrak{X}_v \hookrightarrow \mathfrak{X}$ and $W_v \hookrightarrow W$ induce a degree one morphism $h_v : \mathcal{A}_v \to \mathcal{A}$ which is compatible with the fiber structures in the sense that the following diagram commutes.

$$\begin{tikzcd} A_v^k(s_v) \arrow[r, hook, "j_v"'] \arrow[d] & A^k(i_v(s_v)) \arrow[d] \\ \mathcal{A}_v \arrow[r, "h_v"'] \arrow[d, "\varphi_v"'] & \mathcal{A} \arrow[d, "\varphi"] \\ S_v \arrow[r, "i_v"] & S \end{tikzcd} \tag{2.4.3}$$

If $k = 0$ then we take $\mathcal{A}_v = S_v$ and $h_v = i_v$.

DEFINITION 2.4.4. When $k > 0$:

(i) For $v \in L^+$ and $x \in \mathfrak{X}_v$, let $H^{4k}(\mathcal{E}_{2k}(x), \mathbb{Q})$ and $P(x)$ be as in Proposition 2.3.6(i). Then

$$t_v(x) := \mathfrak{n}_L(v)^{2k} P(x) \circ j_v(A_v^k(x))$$

is an algebraic cycle on $A^k(x)$ which represents the projection to $H^{4k}(\mathcal{E}_{2k}(x), \mathbb{Q})$ of the image in $H^{4k}(A^k(x), \mathbb{Q})$ of the fundamental class of $A_v^k(x)$.

(ii) Let $H^{4k+2}(\mathcal{M}, \mathbb{Q})$ and $\mathcal{P}$ be as in Proposition 2.3.6(ii). Then for each $v \in L^+$

$$T_v := \mathfrak{n}_L(v)^{2k} \mathcal{P} \circ h_v(\mathcal{A}_v).$$

is an algebraic cycle "of higher weight" on $\mathcal{A}$ which represents a class in $H^{4k+2}(\mathcal{M}, \mathbb{Q})$.

(iii) Further, for each positive integer n, let

$$L(n) := \{\, v \in L : q(v) = n \,\}.$$

Then we will refer to

$$T_n := T_n^{(k)} := \sum_{v \in \Gamma \backslash L(n)} T_v$$

as an "arithmetic cycle of higher weight", or a "Hirzebruch–Zagier cycle of higher weight".

(iv) Formally we also let

$$T_0 := \begin{cases} \frac{1}{2} c_1(S), & \text{if } k = 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $c_1(S)$ is the first Chern class of S.

(v) When $k = 0$, then $T_v := C_v$ and $T_n =: C_n$ are precisely the *Hirzebruch–Zagier cycles* on S, as in [HZ] [Kd1] [Cg] [HLR].

The picture to have in mind, then, is that T_v is an algebraic cycle with the structure of a family of algebraic cycles $t_v(c)$ parameterized by C_v.

3. Intersection multiplicities

In this section we compute the pairwise intersection multiplicities of the algebraic and arithmetic cycles of higher weight, in the sense of rational homology. The basic outline of the computation is simple: When $C_u \neq C_v$ all their intersections are transverse, and the description (2.4.4) of T_v as a family of algebraic cycles $t_v(c)$ parameterized by C_v and the methods of [Kd1] allow us to describe $(T_u \cdot T_v)$ as a sum over the intersections of C_u with C_v, properly counted, of terms which represent the intersection multiplicities in the fibers over those points; the hard work comes when we evaluate these "fiber intersection multiplicities." Next we consider the case where $C_u = C_v$ and T_u is a rational multiple of T_v, first computing the intersection multiplicity $(T_u \cdot T_v)$ under the assumption that C_v is nonsingular, and then completing the proof by showing as in [Kd1] that the general case may be reduced to the nonsingular one.

3.1. Before stating the main theorem, we need some notation. For $u, v \in V$, let

$$D(u,v) := (u,v)^2 - 4q(u)q(v). \tag{3.1.1}$$

Then $D(u,v)$ is the discriminant of the restriction of q to the lattice generated by u and v; for $u, v \in L^+$, this binary quadratic form is positive definite if and only if $D(u,v) < 0$. On the other hand, as q is anisotropic, $D(u,v) = 0$ if and only if $\mathbb{Q}u = \mathbb{Q}v$, in which case $q(u)q(v)$ is a rational square.

Also for $u, v \in V$, let

$$\begin{aligned} Q_{2k}(u,v) &:= q(u)^k q(v)^k C^1_{2k}((u',v')) && (3.1.2)\\ &= \sum_{j=0}^{k} (-1)^j \binom{2k-j}{j} q(u)^j q(v)^j (u,v)^{2k-2j} \\ &=: P_{2k}((u,v), q(u)q(v)), && (3.1.3) \end{aligned}$$

where $C^1_{2k}(t)$ is the ultraspherical Gegenbauer polynomial [Vl], Ch. IX, (or any text on orthogonal polynomials), and $u' := (u,u)^{-1/2}u \in V(\mathbb{R})$ is the unit vector in the u-direction for $u \neq 0$. In particular, $Q_{2k}(u,v)$ is a spherical function in u and v with respect to q; it is homogeneous of degree $2k$ in both u and v. Alternatively, $P_{2k}(r,n)$ is the coefficient of t^{2k} in $(1 - rt + nt^2)^{-1}$.

Further, for $u, v \in L$ let

$$\nu(u,v) := \begin{cases} 1 & \text{if } q(u) > 0 \text{ and } D(u,v) < 0, \\ \chi(S_u) & \text{if } q(u) > 0 \text{ and } D(u,v) = 0, \\ \chi(S_v) & \text{if } q(v) > 0 \text{ and } D(u,v) = 0, \\ \frac{1}{2}\chi(S) & \text{if } u = v = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1.4}$$

where $\chi(Z)$ denote the Euler characteristic of a nonsingular variety Z.

In the following, $(\cdot)$ will denote the intersection pairing, i.e., intersection multiplicity in the sense of rational homology, of two middle-dimensional algebraic cycles on a variety; implicitly, cycles may be identified with the homology or cohomology classes they represent. Then the main result of this section is the following.

THEOREM 3.2. *For* $u, v \in L^+$,

$$(T_u \cdot T_v) = \sum_{\gamma \in \Gamma_u \backslash \Gamma / \Gamma_v} \nu(u, \gamma v) Q_{2k}(u, \gamma v)$$

Before proving this theorem, we deduce the following corollary.

COROLLARY 3.3.

(i) *For* $u \in L^+$ *and* $n \geq 0$,
$$(T_u \cdot T_n) = \sum_{v \in \Gamma_u \backslash L(n)} \nu(u,v) Q_{2k}(u,v).$$

(ii) *For* $m, n \geq 0$,
$$(T_m \cdot T_n) = \sum_{(u,v) \in \Gamma \backslash (L(m) \times L(n))} \nu(u,v) Q_{2k}(u,v),$$
where Γ *acts diagonally on* $L(m) \times L(n)$.

Proof. For (i),

$$\begin{aligned} (T_u \cdot T_n) &= \sum_{v \in \Gamma \backslash L(n)} (T_u \cdot T_v) \\ &= \sum_{v \in \Gamma \backslash L(n)} \sum_{\gamma \in \Gamma_u \backslash \Gamma / \Gamma_v} \nu(u, \gamma v) Q_{2k}(u, \gamma v) \\ &= \sum_{v \in \Gamma_u \backslash L(n)} \nu(u,v) Q_{2k}(u,v). \end{aligned}$$

Part (ii) follows immediately. □

3.4. Turning toward the proof of Theorem 3.2, when we identify $\mathfrak{X}_u$ and $\mathfrak{X}_v$ with the set of negative 2-planes in $\mathfrak{X}$ orthogonal to u and v respectively, then we readily deduce that

$$\mathfrak{X}_u \cap \mathfrak{X}_v = \begin{cases} x(u,v) & \text{if } D(u,v) < 0 \\ \mathfrak{X}_u = \mathfrak{X}_v & \text{if } D(u,v) = 0 \\ \varnothing & \text{if } D(u,v) > 0, \end{cases} \tag{3.4.1}$$

where $x(u,v) := \{u,v\}^{\perp} \in \mathfrak{X}$. (These $x(u,v)$ where $D(u,v) < 0$ are called "special points" in [HZ]; it is at these points that the fibers $A^k(x(u,v))$ have complex multiplication [Sa2].) Thus, when $D(u,v) < 0$ the intersection of $\mathfrak{X}_u \times W_u^k(\mathbb{R})$ with $\mathfrak{X}_v \times W_v^k(\mathbb{R})$ may be identified with the intersection of $W_u^k(\mathbb{R})$ with $W_v^k(\mathbb{R})$ in $W^k(\mathbb{R})$, when $W^k(\mathbb{R})$ is identified with $\{x(u,v)\} \times W^k(\mathbb{R})$. On the other hand, when $D(u,v) = 0$, then $\mathbb{Q}u = \mathbb{Q}v$ and thus $\mathfrak{X}_u \times W_u^k(\mathbb{R}) = \mathfrak{X}_v \times W_v^k(\mathbb{R})$.

LEMMA 3.5. *If* $C_u \neq C_v$, *then*

$$(T_u \cdot T_v) = \sum_{\substack{\gamma \in \Gamma_u \backslash \Gamma / \Gamma_v \\ D(u,\gamma v) < 0}} (t_u(x(u,\gamma v)) \cdot t_{\gamma v}(x(u,\gamma v))).$$

Proof. The following proof is based on [Kd1], Prop. 5.1. Suppose that $C_u \neq C_v$, i.e. that $\Gamma w \cap \mathbb{Q}v = \varnothing$. Then at every point p in $C_u \cap C_v$ every branch of C_u through p is transverse to every branch of C_v through p, since the intersection of a pair of branches lifts via a local isomorphism to an intersection of $\mathfrak{X}_{\gamma u}$ with $\mathfrak{X}_{\delta v}$ in a neighborhood of $x(\gamma u, \delta v)$ for some $\gamma, \delta \in \Gamma$. Therefore

$$(C_u \cdot C_v) = \sum_{p \in C_u \cap C_v} \#(i_u^{-1}(p)) \cdot \#(i_v^{-1}(p)),$$

where $i_v : S_v \to S$ as in (2.4.3). However, as $C_u \neq C_v$, there is a bijection

$$\bigsqcup_{p \in C_u \cap C_v} i_u^{-1}(p) \times i_v^{-1}(p) \longleftrightarrow \{\gamma \in \Gamma_u \backslash \Gamma / \Gamma_v : D(u,\gamma v) < 0\}$$

given by $\gamma \mapsto (s_u(x), s_v(\gamma^{-1}x))$, where $x \in \mathfrak{X}$ maps to $p \in S$ and $s_u : \mathfrak{X}_u \to S_u$ is the canonical projection. $\square$

PROPOSITION 3.6. *For* $u, v \in L^+$ *with* $D(u,v) \leq 0$,

$$(t_u(x) \cdot t_v(x)) = Q_{2k}(u,v),$$

where $x = x(u,v)$ *if* $D(u,v) < 0$, *and* x *is any point of* $\mathfrak{X}_u = \mathfrak{X}_v$ *if* $D(u,v) = 0$.

Proof. If we fix x as stated and then drop it from the notation, and divide by $(\mathrm{n}_L(u)\mathrm{n}_L(v))^{2k}$, then by (1.3.1) and (3.1.2) what we need to prove is that

$$(P\circ j_u(A_u^k)\cdot P\circ j_v(A_v^k)) = \frac{d(\mathcal{O}_u)^k d(\mathcal{O}_v)^k}{d(\mathcal{O})^k}\, C_{2k}^1(\,(u',v')\,). \qquad (3.6.1)$$

To verify this, identify $H_{4k}(A^k,\mathbb{Q})$ with $\bigwedge^{4k}W^k$, as in (2.3); with this identification the intersection pairing becomes the natural pairing

$$\langle\ ,\ \rangle : \bigwedge^{4k}W^k \times \bigwedge^{4k}W^k \longrightarrow \bigwedge^{8k}W^k \xrightarrow{\sim} \mathbb{Q}, \qquad (3.6.2)$$

normalized by sending the (positively oriented) generator η of $\bigwedge^{8k}\Lambda^k$ to 1. Then the projector P induces a G-equivariant projection $P_* : \bigwedge^{4k}W^k(\mathbb{R}) \to \check{E}_{2k}$, where $\check{E}_{2k}$, the dual to E_{2k}, occurs in $\bigwedge^{4k}W^k(\mathbb{R})$ with multiplicity one (2.3.5). So if we let η_v denote the positive generator of $H_{4k}(A_v^k,\mathbb{Z}) \simeq \bigwedge^{4k}(\Lambda_v^k) \simeq \mathbb{Z}$, then

$$(P\circ j_u(A_u^k)\cdot P\circ j_v(A_v^k)) = \langle P_*\circ j_{u*}(\eta_u), P_*\circ j_{v*}(\eta_v)\rangle. \qquad (3.6.3)$$

Now since $W^k \sim 2k(V_{(1,0)}\oplus V_{(0,1)})$ and $V \sim V_{(1,0)}\otimes V_{(0,1)}$ as G-modules (1.4.6), while $\check{E}_{2k} \sim V_{(2k,2k)}$ (2.3), there is a unique G-submodule of $\bigwedge^{4k}W^k$ which is isomorphic to $V^{\otimes 2k}$, and which in turn contains $\check{E}_{2k}$. So if we identify this submodule with $V^{\otimes 2k}$, then we get the G-equivariant factorization

$$P_* : \bigwedge^{4k}W^k \xrightarrow{P_1} V^{\otimes 2k} \xrightarrow{P_2} \check{E}_{2k}. \qquad (3.6.4)$$

The next two lemmas will complete the proof of the proposition by describing the effect of P_1 and P_2.

LEMMA 3.7.
(i) *The restriction to* $V^{\otimes 2k}$, *via* P_1, *of the intersection pairing (3.6.2) is the bilinear form* $d(\mathcal{O})^{-k}(\ ,\ )^{\otimes 2k}$.
(ii) *For* $v \in L^+$, $P_1\circ j_{v*}(\eta_v) = q(v)^{-k}d(\mathcal{O}_v)^k v^{\otimes 2k}$.

Proof. Let $v \in L^+$. Then the first step is to extend scalars to $\mathbb{R}$ and choose coordinates "adapted to v" for $W^k(\mathbb{R})$ and $V(\mathbb{R})$ which we can use to make the identification of $V^{\otimes 2k}$ with a submodule of $\bigwedge^{4k}W^k$ explicit. Therefore, let us fix an isomorphism (1.2.4) with $F_1 = \mathbb{R}$, which in turn induces diagrams (1.2.5) and (1.4.4) with $F = \mathbb{R}$. Then it is reasonable to put

$$\begin{aligned} &W_v^k(\mathbb{R}) \xrightarrow{\sim} M_{2,2k}(\mathbb{R}) : w \longmapsto (w_{mn})_{1\le m\le 2,\,1\le n\le 2k}, \\ &W^k(\mathbb{R}) \xrightarrow{\sim} M_{2,2k}(\mathbb{R})\oplus M_{2,2k}(\mathbb{R}) : w \longmapsto (w_{mn}^i)_{1\le m\le 2,\,1\le n\le 2k}^{i=1,2} \end{aligned} \qquad (3.7.1)$$

where $M_{2,2k}(\mathbb{R})$ denotes matrices with 2 rows and $2k$ columns, and the two summands of $W^k(\mathbb{R})$ are isotypic for the half-spin representations. Then $G_v(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{R})$ and $G(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ act naturally from the left. Similarly we may define

$$\mu : V(\mathbb{R}) \xrightarrow{\sim} M_2(\mathbb{R}) : u \longmapsto \left(u_{jl}\right)_{1 \leq j,l \leq 2} \tag{3.7.2}$$

by composing $u \mapsto q(v)^{-1/2}vu \in C^+(V(\mathbb{R}))$ with projection onto the first factor of $C^+(V(\mathbb{R})) \xrightarrow{\sim} M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$. Then $q(u) = \det \mu(u)$ and

$$(u_1, u_2) = \mathrm{tr}(\mu(u_1)^\iota \mu(u_2)), \tag{3.7.3}$$

for $u_1, u_2 \in V(\mathbb{R})$, while $g \in G(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ acts as $\psi(g)(u) = g_2\mu(u)g_1^\iota$.

Now let $W_n^{(i)} := \mathrm{Span}_{\mathbb{R}}\{\mathbf{w}_{1n}^i, \mathbf{w}_{2n}^i\}$, for $i = 1, 2$ and $1 \leq n \leq 2k$, where $\mathbf{w}_{mn}^i$ is the $\binom{i}{mn}$- coordinate vector, and let $V^{(n)}(\mathbb{R}) = V(\mathbb{R})$, for $1 \leq n \leq 2k$, with coordinate vectors $\mathbf{u}_{jl}^n$. Then we factor P_1

$$\begin{aligned} \bigwedge^{4k} W^k(\mathbb{R}) &\underset{P_1'}{\longrightarrow} (W_1^{(1)} \wedge W_1^{(2)}) \wedge \cdots \wedge (W_{2k}^{(1)} \wedge W_{2k}^{(2)}) \\ &\underset{P_1''}{\xrightarrow{\sim}} V^{(1)}(\mathbb{R}) \otimes \cdots \otimes V^{(2k)}(\mathbb{R}), \end{aligned}$$

where P_1' is the natural projection onto a subspace, and P_1'' is given explicitly by

$$P_1'' : W_n^{(1)} \wedge W_n^{(2)} \xrightarrow{\sim} V^{(n)}(\mathbb{R}) : \mathbf{w}_{jn}^1 \wedge \mathbf{w}_{ln}^2 \longmapsto (-1)^{3-j}\mathbf{u}_{l,3-j} \tag{3.7.4}$$

for $1 \leq n \leq 2k$.

Now we can calculate $P_1 \circ j_{v*}(\eta_v)$. Since $j_{v*}(\mathbf{w}_{mn}) = \mathbf{w}_{mn}^1 + \mathbf{w}_{mn}^2$, in coordinates adapted to v,

$$\begin{aligned} j_{v*}(\eta_v) &= d(\mathcal{O}_v)^k j_{v*}\Big(\bigwedge_{1 \leq n \leq 2k} (\mathbf{w}_{1n} \wedge \mathbf{w}_{2n})\Big) \\ &= d(\mathcal{O}_v)^k \bigwedge_{1 \leq n \leq 2k} ((\mathbf{w}_{1n}^1 + \mathbf{w}_{1n}^2) \wedge (\mathbf{w}_{2n}^1 + \mathbf{w}_{2n}^2)). \end{aligned}$$

Therefore we have

$$P_1' \circ j_{v*}(\eta_v) = d(\mathcal{O}_v)^k \bigwedge_{1 \leq n \leq 2k} (\mathbf{w}_{1n}^1 \wedge \mathbf{w}_{2n}^2 - \mathbf{w}_{2n}^1 \wedge \mathbf{w}_{1n}^2)$$

$$P_1'' \circ P_1' \circ j_{v*}(\eta_v) = d(\mathcal{O}_v)^k \bigoplus_{1 \leq n \leq 2k} (\mathbf{u}_{11}^n + \mathbf{u}_{22}^n).$$

But $\mathbf{u}_{11} + \mathbf{u}_{11} = \mu(q(v)^{-1/2}v)$, so (ii) is proved.

To prove (i), we observe that a typical element of the image of P_1' can be represented by $\bigwedge_{n=1}^{2k} \big(\sum_{j,l=1}^{2} a_{jl} \mathbf{w}_{jn}^1 \wedge \mathbf{w}_{ln}^2 \big)$. Thus it follows from (3.7.4) and (3.7.3) that the transport of the intersection pairing to the image of P_1 is a scalar multiple of $(\ ,\ )^{\otimes 2k}$. Then the correct scalar is determined by the normalization of the intersection pairing (3.6.2) and the fact that $\eta = d(\mathcal{O})^k \bigwedge_{i,m,n} \mathbf{w}_{mn}^i$. $\square$

The following lemma was very kindly pointed out to us by John Millson.

LEMMA 3.8 (Millson). *Let $P_2 : V^{\otimes 2k} \to \check{E}_{2k}$ as in (3.6.4). Then for $u, v \in L^+$ with $D(u,v) < 0$,*

$$\langle P_2(u^{\otimes 2k}), P_2(v^{\otimes 2k}) \rangle = q(u)^k q(v)^k C_{2k}^1(\,(u', v')\,).$$

Proof. In order to do this computation, we identify $E_{2k}(\mathbb{R})$ with $\mathcal{H}_{2k}(V(\mathbb{R}))$, the space of homogeneous polynomials of degree $2k$ on $V(\mathbb{R})$ which are harmonic with respect to the Laplacian Δ associated to the bilinear form $(\ ,\ )$. In [Vl], Ch. IX, for example, it is shown that every irreducible class-one representation of $\mathrm{SO}(n)$ is equivalent to a representation on a space of homogeneous harmonic polynomials, and then the Weyl unitary trick [Va] can be used to transport this result to the special orthogonal group of an indefinite form. Since the representation of G on E_{2k} factors through (the connected component of) $\mathrm{SO}(V)$, by verifying the equality of their dimensions we get $E_{2k}(\mathbb{R}) \sim \mathcal{H}_{2k}(V(\mathbb{R}))$ as $G(\mathbb{R})$-modules. Note, too, that as these are irreducible G-representations, they carry a unique-up-to-homothety G-invariant bilinear form, which we normalize by restricting the intersection pairing $\langle\ ,\ \rangle$ to $\check{E}_{2k}(\mathbb{R})$, and then with this pairing, identify $E_{2k}(\mathbb{R})$ with both $\check{E}_{2k}(\mathbb{R})$ and $\mathcal{H}_{2k}(V(\mathbb{R}))$ in such a way that the pairings of corresponding elements are equal.

Similarly, the form $(\ ,\ )$ induces an isomorphism $V(\mathbb{R}) \xrightarrow{\sim} \check{V}(\mathbb{R})$, defined by $v \mapsto f_v := (u \mapsto (u,v))$, such that $(u,v) = (f_u, f_v)$. Thus we get the following factorization of P_2:

$$\begin{array}{ccccccccc} V^{\otimes 2k}(\mathbb{R}) & \xrightarrow{\sim} & \check{V}^{\otimes 2k}(\mathbb{R}) & \to & S^{2k}\check{V}(\mathbb{R}) & \xrightarrow[\mathfrak{h}]{} & \mathcal{H}_{2k}(V(\mathbb{R})) \simeq E_{2k}(\mathbb{R}) \simeq \check{E}_{2k}(\mathbb{R}) \\ v^{\otimes 2k} & \mapsto & f_v^{\otimes 2k} & \mapsto & f_v^{2k} & \longmapsto & \mathfrak{h}(f_v^{2k}); \end{array}$$

here $S^{2k}\check{V}$ denotes the space of symmetric tensors of $\check{V}$ of degree $2k$, which may also be thought of as the space of homogeneous polynomials of degree $2k$ on V; and the harmonic projection $\mathfrak{h}$, which exists for the same reasons as in the definite case, see [Vl], Ch. IX, is given explicitly by

$$\mathfrak{h}f(u) := \sum_{j=0}^{k} (-1)^j \frac{(2k-j)!}{j!\,(2k)!} q(u)^j \Delta^j f(u)$$

[Vl], IX.2.5(15).

Now we claim that $\langle \mathfrak{h}(f_u^{2k}), \mathfrak{h}(f_v^{2k})\rangle = \mathfrak{h}(f_v^{2k})(u)$. Since $\mathfrak{h}$ is a projection,

$$\langle \mathfrak{h}(f_u^{2k}), \mathfrak{h}(f_v^{2k})\rangle = \langle f_u^{2k}, \mathfrak{h}(f_v^{2k})\rangle.$$

On the other hand, $\mathcal{H}_{2k}(V(\mathbb{R}))$, indeed $S^{2k}\check{V}(\mathbb{R})$ as well, is generated by monomials of the form $f = f_{v_1}\cdots f_{v_{2k}}$ with $v_1, \ldots v_{2k} \in V(\mathbb{R})$. But then from the definition of f_v determined by the choice of isomorphism $V(\mathbb{R}) \xrightarrow{\sim} \check{V}(\mathbb{R})$ we find that

$$\begin{aligned}\langle f_u^{2k}, f\rangle &= (u \otimes \cdots \otimes u,\ v_1 \otimes \cdots \otimes v_{2k}) \\ &= \prod_{1\leq j\leq 2k} (u, v_j) = f(u).\end{aligned}$$

To complete the proof it remains only to check that

$$\mathfrak{h}(f_v^{2k})(u) = q(u)^k q(v)^k C_{2k}^1(\,(u', v')\,).$$

But this is straightforward to compute from the formula for $\mathfrak{h}$ given above; or one may choose a basis $\{v_1, \ldots, v_4 = v'\}$ for $V(\mathbb{R})$ such that $(v_i, v_j) = (-1)^i\delta_{ij}$, with associated coordinate functions $x_i(u) := (u, v_i)$ and Laplacian $\Delta = \sum_{i=1}^{4}(-1)^i \frac{\partial^2}{\partial x_i{}^2}$, and then apply verbatim the computations of [Vl], Chapter IX, sections 2.3, 2.5 and 3.1. □

This completes the proof of Proposition 3.6. □

PROPOSITION 3.9. *When $u, v \in L^+$, if $C_u = C_v$ and is nonsingular, then*

$$(T_u \cdot T_v) = \chi(S_u) Q_{2k}(u, v).$$

Proof. The idea of the proof is this: Since T_u and T_v are rational multiples of each other, the basic issue is to compute the self-intersection multiplicity of T_v. Let Y be an irreducible subvariety of $\mathcal{A}$ occuring in the support of T_v. Then to evaluate $(Y \cdot Y)$ we will compute the top Chern class $c_{2k+1}(\nu_Y)$ of its normal bundle by putting suitable coordinates on $\mathcal{A}$; what we will find is that the normal bundle is the Whitney sum of two bundles, $\nu_Y = \nu_{C_v} \oplus \nu_y$, where y is the irreducible subvariety of $A^k(x)$ corresponding to Y occuring in the support of $t_v(x)$ [Hi], (I.4.1.4). It will then follow that $c_{2k+1}(\nu_Y) = c_1(\nu_{C_v})c_{2k}(\nu_y)$, from which we will deduce that

$$(T_u \cdot T_v) = (C_u \cdot C_v)(t_u(x) \cdot t_v(x)). \tag{3.9.1}$$

Finally we will check that $(C_u \cdot C_v) = \chi(S_v)$, which together with Proposition 3.6 will complete the proof.

To begin the proof an alternate description of T_v will be useful. First observe that when $x \in \mathfrak{X}_v$, then $A(x)$ is isomorphic to $A_v(x) \times A_v(x)$: They are isomorphic as differentiable tori by (1.3.1(i)) and (1.2.3), and as complex tori because the equivalence $(\sigma, W)|_{G_v} \sim (\sigma_v, W_v) \oplus (\sigma_v, W_v)$ is induced by (1.2.3), so that the complex structures $\sigma(j(x))$ and $\sigma_v(j(x)) \times \sigma_v(j(x))$ are compatible as well (2.2.2). Let $\alpha_v : A_v^{2k}(x) \xrightarrow{\sim} A^k(x)$ denote such an isomorphism, with $x \in \mathfrak{X}_v$. Then $\alpha_v^{-1}(t_v(x))$ is an algebraic cycle $t'_v(x)$ on $A_v^{2k}(x)$; indeed, $\alpha_v^{-1} \circ j_v(A_v^k(x))$ (2.4.3) sits diagonally in $A_v^{2k}(x)$, and $t'_v(x)$ may be obtained by applying the projector $\alpha_v^{-1}(P(x))$ to $\alpha_v^{-1} \circ j_v(A_v^k(x))$. Furthermore, with the assumption that C_v is nonsingular, $i_v : S_v \to S$ is an isomorphism of S_v onto C_v, so we may identify the restriction of $\mathcal{A}$ to C_v, i.e. $\varphi^{-1}(C_v)$, with the pullback of $\mathcal{A}$ over S_v and with $\mathcal{A}_v^{(2k)}$, in the notation of (2.4.1). Then T_v pulls back to an algebraic cycle T'_v on $\mathcal{A}_v^{(2k)}$ which has the structure of a family of algebraic cycles $t'_v(s)$ parameterized by S_v.

Now let s_0 be a generic point of S_v, and write $t'_v(s_0) = \sum a_i y_i(s_0)$ (finite sum) as an algebraic cycle, with the $y_i(s_0)$ irreducible subvarieties of $A_v^{(2k)}(s_0)$ and the $a_i \in \mathbb{Q}$. Then correspondingly $T'_v = \sum a_i Y_i$, where Y_i is the closure with respect to s_0 of $y_i(s_0)$ in $\mathcal{A}_v^{(2k)}$; this reflects the basic relationship between the projectors $P(s_0)$ and $\mathcal{P}$ of (2.3.6) [Go], (2.B). (Equivalently, Y_i is the subvariety of $\mathcal{A}_v^{(2k)}$ whose intersection with any $A_v^{2k}(s)$ is the specialization to s of $y_i(s_0)$.) Letting Y (resp. $y(s)$) stand for any one of the Y_i (resp. $y_i(s)$), we can summarize the discussion so far in the following commutative diagram.

$$\begin{tikzcd}
y(s) \arrow[r, hook, "j'"'] \arrow[d] & A_v^{2k}(s) \arrow[r, "\sim", "\alpha_v"'] \arrow[d] & A^k(i_v(s)) \arrow[d] \\
Y \arrow[r, hook, "h'"] \arrow[d] & \mathcal{A}_v^{(2k)} \arrow[r, hook, "h_v^{(2k)}"] \arrow[d] & \mathcal{A} \arrow[d] \\
S_v \arrow[r, equal] & S_v \arrow[r, hook, "i_v"] & S
\end{tikzcd} \tag{3.9.2}$$

Under the asumption that C_v is nonsingular, $h_v^{(2k)}$ and i_v are embeddings, and Y and $y(s)$ are also nonsingular. Letting $h := h' \circ h_v^{(2k)}$, we wish to compute $(h(Y) \cdot h(Y))$.

To do so we choose explicit coordinates for $\mathcal{A}$ and $\mathcal{A}_v^{(2k)}$, adapted to v as in the proof of Lemma 3.7. Indeed, we may fix an isomorphism (1.2.4) with $F_1 = \mathbb{R}$ and corresponding diagrams (1.2.5) and (1.4.4) as we did there, and again use the coordinates (3.7.1) for $W^k(\mathbb{R})$. From (1.4.4) we also get

$$\begin{aligned} &\mathfrak{X}_v \xrightarrow{\sim} \mathfrak{H} : x \longmapsto \tau \\ &\mathfrak{X} \xrightarrow{\sim} \mathfrak{H} \times \mathfrak{H} : x \longmapsto (\tau_1, \tau_2), \end{aligned} \tag{3.9.3}$$

where $\mathfrak{H}$ denotes the complex upper half-plane, on which $\mathrm{SL}_2(\mathbb{R})$ acts by linear fractional transformation. Putting a complex structure $J(x)$ on $W^k(\mathbb{R})$ as in (2.2), with $x = (\tau_1, \tau_2) \in \mathfrak{X}$, gives us complex coordinates

$$W^k(\mathbb{R}) \xrightarrow{\sim} \mathbb{C}^{2k} \times \mathbb{C}^{2k} : w \longmapsto \left(z^i_n = -w^i_{1n} + \tau_i w^i_{2n}\right)^{i=1,2}_{1 \le n \le 2k} \tag{3.9.4}$$

[Kg1] [KS]. In these coordinates $G(\mathbb{R})$ acts holomorphically on the complex space $\mathfrak{X} \times W^k(\mathbb{R})$ by

$$\begin{aligned} &(\gamma_1, \gamma_2)(\tau_1, \tau_2; z^1, z^2) \\ &\quad = (\gamma_1(\tau_1), \gamma_2(\tau_2); j(\gamma_1, \tau_1)^{-1} z^1, j(\gamma_2, \tau_2)^{-1} z^2) \end{aligned} \tag{3.9.5}$$

where $j(\gamma, \tau) := c\tau + d$ when $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Now following [Hi], Ch. I, we consider the normal bundle ν_Y of Y in $\mathcal{A}$. When we identify S with the zero section of $\mathcal{A}$, then (3.9.5) makes it clear that the tangent bundle θ_S of S is naturally a subbundle of the tangent bundle $\theta_{\mathcal{A}}$ of $\mathcal{A}$; likewise, θ_{S_v} is a subbundle of $\theta_{\mathcal{A}_v^{(2k)}}$ and of θ_Y. It follows that ν_Y contains the normal bundle ν_{S_v} of S_v in S as a subbundle. If we now view the objects in (3.9.2) as oriented differentiable or almost complex manifolds [Hi], (I.4.6) and (I.4.9), rather than as complex manifolds, then there is a splitting $\nu_Y = \nu_{S_v} \oplus \nu_y$ as a Whitney sum [Hi], (I.4.1.4), where ν_y is the normal bundle of y in A^{2k} (at any point $i_v(s)$). It then follows from the formalism of the Chern class [Hi], (I.4.4), (or of the Euler class [Hi], (I.4.11), in the differentiable category,) that $c_{2k+1}(\nu_Y) = c_1(\nu_{S_v}) \cup c_{2k}(\nu_y)$, as these are the Chern classes of highest degree in each case. Indeed, this factorization reflects the factorization $H^{4k+2}(Y, \mathbb{Q}) \simeq H^2(S_v, \mathbb{Q}) \oplus H^{4k}(y, \mathbb{Q})$, which we have because $H^{4k}(y, \mathbb{Q})$ is a trivial Γ_v-module, since the class in $H^{4k}(A_v^{2k}(s_0), \mathbb{Q})$ represented by y is algebraic and therefore G_v-invariant [Kg]. Similarly in $H^0(Y, \mathbb{Q}) \simeq H^0(S_v, \mathbb{Q}) \otimes H^0(y, \mathbb{Q})$ we have $1_Y = 1_{S_v} \cup 1_y$. Therefore, since the pullback h^* of the class in $H^{4k+2}(\mathcal{A}, \mathbb{Q})$ represented by Y is $c_{2k+1}(\nu_Y)$ [Hi], (I.4.11(18)), we have

$$\begin{aligned} (h(Y) \cdot h(Y)) &= c_{2k+1}(\nu_Y) \cup 1_Y \\ &= (c_1(\nu_{S_v}) \cup 1_{S_v})(c_{2k}(\nu_y) \cup 1_y) \\ &= (C_v \cdot C_v)(y \cdot y). \end{aligned}$$

Since $C_u = C_v$ and $T_v = \sum a_i Y_i$ and $t_v = \sum a_i y_i$, and similarly for T_u and t_u, this proves (3.9.1).

Since Proposition 3.6 applies even when $C_u = C_v$, it remains only to check that $(C_v \cdot C_v) = \chi(S_v)$; by [Hi], (I.4.10), (I.4.11), this will be true if $c_1(\nu_{s_v}) = c_1(\theta_{s_v})$, i.e., if $\nu_{s_v} \cong \theta_{s_v}$. (An alternate proof would be to use the adjunction formula [GH].) In the coordinates of (3.9.3) we have

$$\begin{array}{ccc} \mathfrak{H} & \xrightarrow{\Delta} & \mathfrak{H} \times \mathfrak{H} \\ \downarrow{\scriptstyle \Gamma_v} & & \downarrow{\scriptstyle \Gamma} \\ S_v & \xhookrightarrow{i_v} & S \end{array}$$

with $G_v(\mathbb{R})$ embedding diagonally in $G(\mathbb{R})$ (1.4.4). Therefore $\nu_{S_v} = i_v^* \theta_S / \theta_{S_v} \cong \theta_{S_v}$. □

Now Theorem 3.2 will follow from Lemma 3.5, Proposition 3.6 and the following lemma.

LEMMA 3.10. *If* $C_u = C_v$, *then*

$$(T_u \cdot T_v) = \sum_{\gamma \in \Gamma_u \backslash \Gamma / \Gamma_v} \nu(u, \gamma v) Q_{2k}(u, \gamma v).$$

Proof. Since T_u is a rational multiple of T_v when $C_u = C_v$, it suffices to verify the formula for $u = v \in L^+$. The following proof is adapted from [Kd1], Proposition 5.2. Since the issue is to compute $(T_v \cdot T_v)$ when C_v and therefore T_v has singularities, we lift the problem to a normal cover $\tilde{A} \to \tilde{S}$ of $A \to S$ where the cycles $\tilde{C}_v$ and $\tilde{T}_v$ lying over C_v and T_v respectively are nonsingular, and there we apply Proposition (3.9).

To desingularize C_v first observe that all its singularities are normal self-crossings which occur when elements of Γ identify points of $\mathfrak{X}_v$ which are not identified by Γ_v. Let

$$\Delta_v := \{ \gamma \in \Gamma : D(v, \gamma v) < 0 \};$$

then just as in Lemma 3.5, there is a bijection between the set of these transverse crossings, counted with multiplicities, and $\Gamma_v \backslash \Delta_v / \Gamma_v$. Now, since this set is finite, we may choose an integer m such that $\delta v \not\equiv v \pmod{mL}$ for all δ in a set of representatives of $\Gamma_v \backslash \Delta_v / \Gamma_v$. Let

$$\tilde{\Gamma} := \{ \gamma \in \Gamma : \gamma \equiv 1 \pmod{m\mathcal{O}} \},$$

normal and of finite index in Γ, and correspondingly, let $\tilde{S} := \tilde{\Gamma} \backslash \mathfrak{X}$, and $\tilde{\Gamma}_v := G_v \cap \tilde{\Gamma}$, and $\tilde{S}_v := \tilde{\Gamma}_v \backslash \mathfrak{X}_v$; then $\tilde{C}_v$ may be defined equivalently as the image of $\mathfrak{X}_v$ in $\tilde{S}$, or as the image of $\tilde{S}_v$ in $\tilde{S}$, or as the inverse image of C_v with respect to the covering map $\tilde{S} \to S$. Furthermore, $\tilde{C}_v$ is nonsingular. For as before its singularities biject with $\tilde{\Gamma}_v \backslash \tilde{\Delta}_v / \tilde{\Gamma}_v$, where

$\tilde{\Delta}_v := \{\, \gamma \in \tilde{\Gamma} : D(v, \gamma v) < 0 \,\}$. But $\tilde{\Delta}_v$ is empty, for if $\delta \in \tilde{\Delta}_v$, then $\delta \in \Delta_v$ and $\delta v \equiv v \pmod{mL}$, but it follows from the choice of m that $\delta v \not\equiv v \pmod{mL}$ for all $\delta \in \Delta_v$. Therefore, if we let $\tilde{\mathcal{A}} := \tilde{\Gamma} \ltimes \Lambda^k \backslash \mathfrak{X} \times W^k(\mathbb{R})$, as in (2.2.1), then the (algebraic) cycle $\tilde{T}_v$ to which T_v pulls back via $\tilde{\mathcal{A}} \to \mathcal{A}$ is also nonsingular.

Now let $\xi := (\Gamma : \tilde{\Gamma})$ denote the degree of $\tilde{\mathcal{A}} \to \mathcal{A}$ or of $\tilde{S} \to S$, and similarly $\xi_v := (\Gamma_v : \tilde{\Gamma}_v)$ the degree of $\tilde{S}_v \to S_v$. Then $\tilde{T}_v$ consists of $\nu_v := \xi/\xi_v$ distinct cycles T'_{v_i}, where $v_i = \gamma_i v$ as γ_i runs through a set of representatives for $\tilde{\Gamma}\backslash\Gamma/\Gamma_v$, and T'_{v_i} is defined analogously to T_v but on $\tilde{\mathcal{A}}$. Then

$$\begin{aligned}
(T_v \cdot T_v) &= \xi^{-1}(\tilde{T}_v \cdot \tilde{T}_v) = \xi^{-1}\sum_{i,j}(T'_{v_i} \cdot T'_{v_j}) \\
&= \xi^{-1}\sum_{i}(T'_{v_i} \cdot T'_{v_i}) + \xi^{-1}\sum_{i \neq j}(T'_{v_i} \cdot T'_{v_j}).
\end{aligned}$$

Assuming, as we may, that $v_1 = v$, from Proposition 3.9 we get

$$\begin{aligned}
\xi^{-1}\sum_{i}(T'_{v_i} \cdot T'_{v_i}) &= \xi^{-1}\nu_v(T'_v \cdot T'_v) \\
&= \xi_v^{-1}\chi(\tilde{S}_v)Q_{2k}(v, v) \\
&= \chi(S_v)Q_{2k}(v, v).
\end{aligned}$$

On the other hand, using (3.5) and (3.6),

$$\begin{aligned}
\xi^{-1}\sum_{i \neq j}(T'_{v_i} \cdot T'_{v_j}) &= \xi_v^{-1}\sum_{j \neq 1}(T'_v \cdot T'_{v_j}) \\
&= \xi_v^{-1}\sum_{j \neq 1}\sum_{\substack{\gamma \in \tilde{\Gamma}_v\backslash\tilde{\Gamma}/\tilde{\Gamma}_{v_j} \\ D(v, \gamma v_j) < 0}} Q_{2k}(v, \gamma v_j).
\end{aligned}$$

Now recall that $\tilde{\Gamma}$ was chosen so that $\tilde{\Delta}_v = \varnothing$, and that $v_j = \gamma_j v$ for γ_j running over a set of representatives for $\tilde{\Gamma}\backslash\Gamma/\Gamma_v$. Therefore

$$\begin{aligned}
&\bigsqcup_{j \neq 1}\{\, \gamma \in \tilde{\Gamma} : D(v, \gamma v_j) < 0 \,\}/\tilde{\Gamma}_{v_j} \\
= \; &\bigsqcup_{j}\{\, \gamma \in \tilde{\Gamma} : D(v, \gamma\gamma_j v) < 0 \,\}/\gamma_j\Gamma_v\gamma_j^{-1} \\
= \; &\Delta_v/\Gamma_v,
\end{aligned}$$

from which it follows that

$$\bigsqcup_{j \neq 1} \tilde{\Gamma}_v\backslash\{\, \gamma \in \tilde{\Gamma} : D(v, \gamma v_j) < 0 \,\}/\tilde{\Gamma}_{v_j} = \tilde{\Gamma}_v\backslash\Delta_v/\Gamma_v.$$

Combining this with (3.10.1), we have

$$\xi^{-1}\sum_{i\neq j}(T'_{v_i}\cdot T'_{v_j}) = \xi_v^{-1}\sum_{\gamma\in\tilde{\Gamma}_v\backslash\Delta_v/\Gamma_v} Q_{2k}(v,\gamma v). \tag{3.10.2}$$

Next we observe that $\Gamma_v/\tilde{\Gamma}_v$ acts freely on $\tilde{\Gamma}_v\backslash\Delta_v/\Gamma_v$. For if $\gamma\in\Gamma_v$ and $\delta\in\Delta_v$ are such that $\gamma\tilde{\Gamma}_v\delta\Gamma_v = \tilde{\Gamma}_v\delta\Gamma_v$, then there exist $\gamma_1,\gamma_2\in\tilde{\Gamma}_v$ such that $\gamma_1\gamma\gamma_2$ fixes $\delta\Gamma_v$. But then $\gamma_1\gamma\gamma_2$ fixes $x(v,\delta v)$, which implies that it equals 1 and that $\gamma\in\tilde{\Gamma}_v$, since Γ acts freely on $\mathfrak{X}$. Therefore

$$\xi_v^{-1}\sum_{\gamma\in\tilde{\Gamma}_v\backslash\Delta_v/\Gamma_v} Q_{2k}(v,\gamma v) = \sum_{\gamma\in\Gamma_v\backslash\Delta_v/\Gamma_v} Q_{2k}(v,\gamma v),$$

as required. □

4. Harmonic Poincare dual forms

In this section we observe, föllowing [Kg1], that there is an inner product preserving isomorphism from a space of modular forms for Γ to the space $\mathcal{H}^{4k+2}(\mathcal{M},\mathbb{C})$ of harmonic differential forms on $\mathcal{A}$ representing classes in $H^{4k+2}(\mathcal{M},\mathbb{C})$. Then we describe the harmonic differential form on $\mathcal{A}$ Poincare dual to T_v, for $k>0$, and use this form to obtain an alternate proof for Theorem 3.2 in the case $k>0$; for the harmonic Poincare dual form in the case $k=0$, see [KM4].

4.1. Throughout this section we will use the complex coordinates

$$X\times W^k(\mathbb{R})\xrightarrow{\sim}\mathfrak{H}\times\mathfrak{H}\times\mathbb{C}^{2k}\times\mathbb{C}^{2k} : (x,w)\longmapsto(\tau_1,\tau_2;z^1,z^2)$$

described in (3.9.3) and (3.9.4), with the holomorphic action of $G(\mathbb{R})$ as in (3.9.5). We will also identify $V(\mathbb{R})$ with $M_2(\mathbb{R})$ as in (3.7.2) whenever necessary. Recall that all these coordinates depend on a choice of $v\in V(\mathbb{R})$ with $q(v)>0$.

4.1.1. In order to describe the Riemannian metric on $\mathcal{A}$ as a complex projective variety whose fibers over S are polarized abelian varieties, it will be helpful to normalize the polarization chosen in (2.2). Recall that it was defined on $c_1,c_2\in W(\mathbb{R})$ by $\beta(c_1,c_2) := \mathrm{tr}_{C^+(V(\mathbb{R}))/\mathbb{R}}(bc_1^\iota c_2)$, where the nonzero $b\in C^+(V)$ satisfied an integrality condition, a positivity condition, and $b^\iota=-b$. It follows that the characteristic polynomial of b over the center $\mathbb{Q}(\zeta)$ of $C^+(V)$ is $X^2+\mu$ for some totally positive $\mu\in\mathbb{Q}(\zeta)$. Thus if we let $\alpha\mapsto\alpha^{(i)}$ for $i=1,2$ denote the embeddings $\mathbb{Q}(\zeta)\hookrightarrow\mathbb{R}$ induced by (1.2.3), then we may choose b so that (1.2.5) yields the identification

$b \mapsto \left(\begin{pmatrix} & \lambda^{(1)} \\ -\lambda^{(1)} & \end{pmatrix}, \begin{pmatrix} & \lambda^{(2)} \\ -\lambda^{(2)} & \end{pmatrix}\right)$ for some positive $\lambda^{(1)}, \lambda^{(2)} \in \mathbb{R}$, and moreover, this identification is compatible with our coordinatization of $W(\mathbb{R})$. Now let $\tau = x + y\sqrt{-1}$ and $z = \xi + \eta\sqrt{-1}$. Then just as in [Kg1], IV-1(49), the Riemannian metric on $\mathcal{A}$ is

$$ds^2 = \sum_{i=1}^{2} y_i^{-2}\left(|d\tau_i|^2 + \lambda^{(i)} \sum_{n=1}^{4k} |y_i dz_n^i - \eta_n^i d\tau_i|^2\right). \tag{4.1.1}$$

More precisely, this metric on $\mathfrak{H} \times \mathfrak{H} \times \mathbb{C}^{2k} \times \mathbb{C}^{2k}$ is $G(\mathbb{R}) \ltimes W^k(\mathbb{R})$-invariant and therefore descends to a metric on $\mathcal{A}$. Let $\mathcal{H}^*(\mathcal{A}, \mathbb{C})$ denote the space of differential forms on $\mathcal{A}$ which are harmonic with respect to this metric, where we identify differential forms on $\mathcal{A}$ with $\Gamma \ltimes \Lambda^k$-invariant differential forms on $\mathfrak{H} \times \mathfrak{H} \times \mathbb{C}^{2k} \times \mathbb{C}^{2k}$.

4.1.2. Let $\mathfrak{H}^+ := \mathfrak{H}$ and let $\mathfrak{H}^-$ denote the complex lower half-plane, and with s_1 and s_2 designating elements of $\{+, -\}$, let $\mathfrak{H}^{s_1 s_2} := \mathfrak{H}^{s_1} \times \mathfrak{H}^{s_2}$. Then viewing Γ as a discrete arithmetic subgroup of $G(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, let $S_{2k+2}^{s_1 s_2}(\Gamma)$ denote the space of holomorphic modular forms of weight $2k+2$ on $\mathfrak{H}^{s_1 s_2}$ with respect to Γ; that is, $f \in S_{2k+2}^{s_1 s_2}(\Gamma)$ iff f is a holomorphic function on $\mathfrak{H}^{s_1 s_2}$ with the property that for all (τ_1, τ_2) in that domain and all $(\gamma_1, \gamma_2) \in \Gamma$

$$f(\gamma_1\tau_1, \gamma_2\tau_2) = j(\gamma_1, \tau_1)^{2k+2} j(\gamma_2, \tau_2)^{2k+2} f(\tau_1, \tau_2),$$

where $j(\gamma, \tau) = c\tau + d$ when $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Clearly $S_{2k+2}(\Gamma)$ (resp. $S_{2k+2}^{-+}(\Gamma)$) is complex conjugate to $S_{2k+2}^{++}(\Gamma)$ (resp. $S_{2k+2}^{+-}(\Gamma)$), and moreover it is known that all four of the $S_{2k+2}^{s_1 s_2}(\Gamma)$ have the same dimension [MS].

There is an inner product on the space of modular forms: For f and g in the same $S_{2k+2}^{s_1 s_2}(\Gamma)$, their Petersson inner product is defined by

$$(f, g) := \int_{\mathcal{F}^{s_1 s_2}} f(\tau_1, \tau_2)\overline{g(\tau_1, \tau_2)} y_1^{2k+2} y_2^{2k+2}\, dV_1\, dV_2 \tag{4.1.3}$$

where $\mathcal{F}^{s_1 s_2}$ is a fundamental domain for Γ on $\mathfrak{H}^{s_1 s_2}$, and $dV_i := y_i^{-2} dx_i dy_i$ is the $G^{(i)}$-invariant measure on $\mathfrak{H}^{s_i}$, for $i = 1, 2$.

4.1.4. Now in the manner of [Kg1], II-3, we may define a linear map ω from $\sum_{s_1, s_2 \in \{+,-\}} S_{2k+2}^{s_1 s_2}(\Gamma)$ to differential $(4k+2)$-forms on $\mathcal{A}$ by

$$\omega(f) := \begin{cases} 2^{-(2k+1)} D^{-k} \sqrt{-1} f(\tau_1, \tau_2)\, dz^1 dz^2 d\tau_1 d\tau_2, & \text{if } f \in S_{2k+2}^{++}(\Gamma) \\ 2^{-(2k+1)} D^{-k} \sqrt{-1} f(\tau_1, \bar{\tau}_2)\, dz^1 d\bar{z}^2 d\tau_1 d\bar{\tau}_2, & \text{if } f \in S_{2k+2}^{+-}(\Gamma) \end{cases}$$

and $\omega(f) := \overline{\omega(\bar{f})}$ if $f \in S_{2k+2}^{-+}(\Gamma)$ or $S_{2k+2}(\Gamma)$. Here $dz^i := dz_1^i \dots dz_{2k}^i$ for $i = 1, 2$; recall also that $D = d(\mathcal{O})^{1/2}$ (1.3.1(ii)).

PROPOSITION 4.2. *When $k > 0$, then ω maps $\sum_{s_1,s_2 \in \{+,-\}} S_{2k+2}^{s_1 s_2}(\Gamma)$ isomorphically onto $\mathcal{H}^{4k+2}(\mathcal{M}, \mathbb{C})$. If $k = 0$ the image of ω in $\mathcal{H}^2(S, \mathbb{C})$ has codimension two, and its orthogonal complement is generated by $\omega_i := (4\pi\sqrt{-1})^{-1} y_i^{-2} d\tau_i d\bar{\tau}_i$ for $i = 1, 2$. Furthermore, ω is an isometry in the sense that*

$$\int_{\mathcal{A}} \omega(f) \wedge \overline{\omega(g)} = (f, g). \tag{4.2.1}$$

Proof. The first two sentences are implicit in [Kg1] and [MS]. For ω may be factored

$$\omega : \sum_{s_1,s_2 \in \{+,-\}} S_{2k+2}^{s_1 s_2}(\Gamma) \longrightarrow \mathcal{H}^2(S, \tilde{E}_{2k}) \longrightarrow \mathcal{H}^{4k+2}(\mathcal{A}, \mathbb{C}),$$

where $\tilde{E}_{2k}$ is the vector bundle over S associated to E_{2k}. That the first arrow is an isomorphism for $k > 0$ and has an image of codimension two with the indicated orthogonal complement when $k = 0$ is proved in [MS], while the injectivity of the second arrow is proved in [Kg1]; and then it follows that the image is $\mathcal{H}^{4k+2}(\mathcal{M}, \mathbb{C})$. To prove the last sentence, we first observe that

$$\int_{A^k(\tau_1,\tau_2)} dz^1\, d\bar{z}^1\, dz^2\, d\bar{z}^2 = 2^{4k} d(\mathcal{O})^k y_1^{2k} y_2^{2k}. \tag{4.2.2}$$

Then if $f, g \in S_{2k+2}^{+-}(\Gamma)$, for example,

$$\begin{aligned}
&\int_{\mathcal{A}} \omega(f) \wedge \overline{\omega(g)} \\
&= 2^{-(4k+2)} d(\mathcal{O})^{-k} \int_S f(\tau_1, \bar{\tau}_2)\overline{g(\tau_1, \bar{\tau}_2)} \int_{A^k(\tau_1,\tau_2)} dz^1 d\bar{z}^1 dz^2 d\bar{z}^2\; d\tau_1 d\bar{\tau}_2 d\bar{\tau}_1 d\tau_2, \\
&= 4^{-1} \int_{\Gamma\backslash\mathfrak{H}^{++}} f(\tau_1, \bar{\tau}_2)\overline{g(\tau_1, \bar{\tau}_2)} y_1^{2k} y_2^{2k}\; d\tau_1 d\bar{\tau}_1 d\tau_2 d\bar{\tau}_2 \\
&= (f, g),
\end{aligned}$$

and the other cases may be computed similarly. □

4.3. For $u \in L^+$ represented by $\mu(u) = \left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{smallmatrix}\right)$ as in (3.7.2), let

$$b_u(\tau_1, \tau_2) := (u_{21}\tau_1\tau_2 + u_{22}\tau_2 - u_{11}\tau_1 - u_{12})^{-1}.$$

As a special case, since $\mu(v) = q(v)^{1/2}I$ we have

$$b_v(\tau_1, \tau_2) = q(v)^{-1/2}(\tau_2 - \tau_1)^{-1} \tag{4.3.1}$$

In general it may be checked that $b_u(\tau_1, \tau_2)$ is holomorphic on $\mathfrak{H}^{+-}$ and that for $g \in G(\mathbb{R})$

$$b_{\psi(g^\iota)(u)}(\tau_1, \tau_2) = j(g_1, \tau_1)^{-1} j(g_2, \tau_2)^{-1} b_u(g_1\tau_1, g_2\tau_2). \tag{4.3.2}$$

From this and the cocycle relation $j(\gamma\gamma', \tau) = j(\gamma, \gamma'\tau)j(\gamma', \tau)$ it follows that

$$\theta_u(\tau_1, \tau_2) := \sum_{\gamma \in \Gamma/\Gamma_u} b_{\psi(\gamma)(u)}(\tau_1, \tau_2)^{2k+2}$$

is in $S^{+-}_{2k+2}(\Gamma)$ for $k > 0$.

THEOREM 4.4. *Let $k > 0$, let $c_k := (-1)^k 2^{2k}(2k+1)\pi^{-1}$, and let*

$$\omega_u := c_k q(u)^{2k+1}(\omega(\theta_u) + \overline{\omega(\theta_u)}).$$

Then ω_u is the harmonic differential form on $\mathcal{A}$ Poincare dual to the cycle T_u.

Proof. For computational simplicity we may take $u = v$, and show that the cap product of η with T_v equals the cup product of η with ω_v for any $\eta \in \mathcal{H}^*(\mathcal{A}, \mathbb{C})$. However, both T_v and ω_v lie in the dual space to $\mathcal{H}^{4k+2}(\mathcal{M}, \mathbb{C})$, and pair nontrivially only with forms of Hodge type $(2k+1,\ 2k+1)$, so we may assume that η is an element of

$$\mathcal{H}^{4k+2}(\mathcal{M}, \mathbb{C}) \cap \mathcal{H}^{2k+1,2k+1}(\mathcal{A}) = \omega(S^{+-}_{2k+2}(\Gamma) + S^{-+}_{2k+2}(\Gamma)).$$

Therefore without loss of generality we may assume $\eta = \omega(f) + \omega(g)$ with $f \in S^{+-}_{2k+2}(\Gamma)$ and $g \in S^{-+}_{2k+2}(\Gamma)$.

By construction $h_v(\mathcal{A}_v)$ is homologically equivalent to $\mathfrak{n}_L(v)^{-2k}T_v$ plus something orthogonal to $\mathcal{H}^*(\mathcal{M}, \mathbb{C})$. Therefore, recalling that in coordinates adapted to v all three horizontal arrows of (2.4.3) are realized as diagonal maps,

$$\begin{aligned}
\eta \cap T_v &= \mathfrak{n}_L(v)^{2k} \int_{h_v(\mathcal{A}_v)} \omega(f) + \omega(g) \\
&= \frac{\mathfrak{n}_L(v)^{2k}\sqrt{-1}}{2^{2k+1}D^k} \int_{S_v} (f(\tau, \bar{\tau}) + g(\bar{\tau}, \tau)) \int_{A_v^k(\tau)} dz\, d\bar{z}\, d\tau\, d\bar{\tau} \\
&= q(v)^k \int_{S_v} (f(\tau, \bar{\tau}) + g(\bar{\tau}, \tau)) y^{2k+2}\, dV, \qquad (4.4.1)
\end{aligned}$$

where $dz = dz_1 \dots dz_{2k}$. Since similarly to (4.2.2) the inner integral on the second line equals $2^{2k} d(\mathcal{O}_v)^k y^{2k}$, the last line follows from (1.3.1(ii)).

On the other hand, from the definition of ω_v and (4.2.1)

$$\int_{\mathcal{A}} (\omega(f) + \omega(g)) \wedge \omega_v = c_k q(v)^{2k+1}\big((f, \theta_v) + (g, \overline{\theta_v}) \big).$$

Therefore what we need to show is that

$$(f, \theta_v) = c_k^{-1} q(v)^{-(k+1)} \int_{S_v} f(\tau, \bar{\tau}) y^{2k+2}\, dV, \tag{4.4.2}$$

and similarly for $(g, \overline{\theta_v})$. However, this may be proved in exactly the same way as [Z2], Theorem 6: Since $\mathrm{Im}(\gamma(\tau)) = \mathrm{Im}(\tau)|j(\gamma, \tau)|^{-2}$, from (4.3.2) and the definition of θ_v we get

$$\begin{aligned}
&(f, \theta_v) \\
&= \int_{\Gamma\backslash\mathfrak{H}^{+-}} \sum_{\gamma \in \Gamma/\Gamma_v} f(\gamma_1\tau_1, \gamma_2\tau_2)\big(\overline{b_v(\gamma_1\tau_1, \gamma_2\tau_2)}\big)^{2k+2} \\
&\qquad\qquad \times \mathrm{Im}(\gamma_1\tau_1)^{2k+2}\mathrm{Im}(\gamma_2\tau_2)^{2k+2}\, dV_1\, dV_2 \\
&= q(v)^{-(k+1)} \int_{S_v} \left(\int_{\mathfrak{H}^-} f(\tau_1, \tau_2)(\bar{\tau}_2 - \bar{\tau}_1)^{-(2k+2)} y_2^{2k+2}\, dV_2 \right) y_1^{2k+2}\, dV_1,
\end{aligned}$$

where after interchanging the order of summation and integration we have chosen a fundamental domain for Γ_v in $\mathfrak{H}^{+-}$ of the form $\mathcal{F}_v \times \mathfrak{H}^-$. But now it follows from [Z3], p. 46, (note that here we are integrating over $\mathfrak{H}^-$) that the inner integral equals $c_k^{-1} f(\tau_1, \bar{\tau}_1)$, and this completes the proof. $\square$

REMARK 4.4.3. Presumeably the definition of θ_u and the result of Theorem 4.4 could be extended to the case $k = 0$ by considering

$$\theta_u(\tau_1, \tau_2; s) := \sum_{\gamma \in \Gamma/\Gamma_u} b_{\psi(\gamma)(u)}(\tau_1, \tau_2)^2 |b_{\psi(\gamma)(u)}(\tau_1, \tau_2)|^{2s}$$

as in [Z1] appendix 1; see also [KM4].

4.5. We can now sketch an alternate proof of Theorem 3.2, for the case $k > 0$, by evaluating the cap product $\omega_u \cap T_v$. The actual computations reduce to those in the proof of the Eichler-Selberg trace formula for $\mathrm{SL}_2(\mathbb{Z})$ in [Z3], reflecting the close analogy between the cycles T_n and the Hecke correspondences for a modular curve (cf. [HZ]), and the fact that $\mathrm{SL}_2(\mathbb{Q}) \times \mathrm{SL}_2(\mathbb{Q})$ and G (as well as $\mathrm{SL}_2(\mathbb{Q}(\sqrt{D}))$ for nonsquare $D > 0$, [HZ]) are both (all) rational forms of the same real group, making them indistinguishable when scalars are extended to $\mathbb{R}$ and C^∞-coordinates are chosen.

Computing as before with coordinates adapted to v,

$$\begin{aligned}
(T_u \cdot T_v) &= \omega_u \cap T_v \\
&= \mathfrak{n}_L(v)^{2k} \int_{\mathcal{A}_v} h_v^* \omega_u \\
&= 2^{-(2k+1)} \sqrt{-1}\, c_k D^{-k} \mathfrak{n}_L(v)^{2k} q(u)^{2k+1} \\
&\qquad \times \int_{\mathcal{A}_v} (\theta_u(\tau, \bar\tau) + \theta_u(\bar\tau, \tau))\, dz\, d\bar z\, d\tau\, d\bar\tau \\
&= c_k q(v)^k q(u)^{2k+1} \int_{S_v} (\theta_u(\tau, \bar\tau) + \theta_u(\bar\tau, \tau)) y^{2k+2}\, dV \\
&= c_k q(v)^k q(u)^{2k+1} \int_{\Gamma_v \backslash \mathfrak{H}} \sum_{\gamma \in \Gamma/\Gamma_u} (R(\tau, \psi(\gamma)(u)) + R(\bar\tau, \psi(\gamma)(u)))\, dV,
\end{aligned} \tag{4.5.1}$$

where $R(\tau, u) := (b_u(\tau, \bar\tau)\mathrm{Im}(\tau))^{2k+2}$ as in (4.3). In particular, it follows from (4.3.2) that

$$R(\tau, \psi(\gamma^\iota)(u)) = R(\gamma\tau, u) \tag{4.5.2}$$

for $\gamma \in \Gamma_v$, since Γ_v sits diagonally in Γ.

Now we observe that any coset $\gamma\Gamma_u \in \Gamma/\Gamma_u$ belongs to a unique double coset $\Gamma_v \backslash \delta/\Gamma_u \in \Gamma_v \backslash \Gamma/\Gamma_u$; and if we let $\varepsilon := \gamma\delta^{-1} \in \Gamma_v$, then replacing γ by $\gamma\gamma'$ with $\gamma' \in \Gamma_u$ replaces ε by $\varepsilon(\delta\gamma'\delta^{-1})$ with $(\delta\gamma'\delta^{-1}) \in \Gamma_v \cap \Gamma_{\psi(\delta)(u)}$. In view of (4.5.2), then

$$\sum_{\gamma \in \Gamma/\Gamma_u} R(\tau, \psi(\gamma)(u)) = \sum_{\substack{\gamma \in \Gamma_v \backslash \Gamma/\Gamma_u \\ \varepsilon \in \Gamma_v/\Gamma_v \cap \Gamma_{\psi(\gamma)(u)}}} R(\varepsilon\tau, \psi(\gamma)(u)). \tag{4.5.3}$$

Furthermore, when $k > 0$ this sum converges uniformly on compact sets. Thus we may substitute (4.5.3) into (4.5.1) and interchange the order of summation and integration to get

$$\begin{aligned}
(T_u \cdot T_v) = \sum_{\gamma \in \Gamma_v \backslash \Gamma/\Gamma_u} c_k q(v)^k q(u)^{2k+1} \\
\times \int_{(\Gamma_v \cap \Gamma_{\psi(\gamma)(u)}) \backslash \mathfrak{H}} (R(\tau, \psi(\gamma)(u)) + R(\bar\tau, \psi(\gamma)(u)))\, dV.
\end{aligned} \tag{4.5.4}$$

Next we observe that unless $\mathbb{Q}u = \mathbb{Q}v$, in which case $\Gamma_v = \Gamma_u$, then $\Gamma_u \cap \Gamma_v$ is a torsion-free subgroup of the group of integral units in

$$C^+(V_u) \cap C^+(V_v) = C^+(\{u,v\}^\perp) \simeq \mathbb{Q}(\sqrt{D(u,v)D}).$$

Therefore

$$\Gamma_v \cap \Gamma_u = \begin{cases} \{1\} & \text{if } D(u,v) < 0 \\ \Gamma_u = \Gamma_v & \text{if } D(u,v) = 0 \\ \langle \varepsilon \rangle & \text{if } D(u,v) > 0 \end{cases} \tag{4.5.5}$$

for a unit ε of infinite order in the real quadratic field $\mathbb{Q}(\sqrt{D(u,v)D})$. Hence it is natural to divide the sum in (4.5.4) into three pieces, corresponding to the cases of (4.5.5), and evaluate the integral in each case.

To do this, let $\tilde{v} := q(v)^{-1/2}v$, represented in coordinates by the 2 by 2 identity matrix, let u be represented by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and let $t = (u, \tilde{v}) = a + d$ (3.7.3). Then

$$R(\tau, u) = \frac{y^{2k+2}}{(c(x^2+y^2) + (d-a)x - b - ty\sqrt{-1})^{2k+2}}.$$

Hence it follows immediately from the computations in [Z3], Theorem 2, cases 1 and 4, that

$$c_k q(v)^k q(u)^{2k+1} \int\limits_{(\Gamma_v \cap \Gamma_u)\backslash \mathfrak{H}} (R(\tau,u) + R(\bar{\tau},u))\, dV \tag{4.5.6}$$

$$= \begin{cases} Q_{2k}(u,v) & \text{if } D(u,v) < 0 \\ 0 & \text{if } D(u,v) > 0. \end{cases}$$

On the other hand, if $D(u,v) = 0$, then $u = q(u)^{1/2}\tilde{v}$, whence

$$R(\tau,u) = R(\bar{\tau},u) = (-1)^{k+1} 2^{-(2k+2)} q(u)^{-(k+1)}.$$

However, in this case $Q_{2k}(u,v) = (2k+1)q(u)^k q(v)^k$. Hence

$$c_k q(v)^k q(u)^{2k+1} \int\limits_{\Gamma_v \backslash \mathfrak{H}} (R(\tau,u) + R(\bar{\tau},u))\, dV = \chi(S_v) Q_{2k}(u,v), \tag{4.5.7}$$

since

$$\chi(S_v) = \int\limits_{S_v} \frac{-1}{2\pi} \frac{dx\,dy}{y^2}.$$

Now Theorem 3.2 follows by putting (4.5.6) and (4.5.7) into (4.5.4).

5. Modular Forms of Nebentypus

In this section we apply the results of Kudla and Millson [Mi3] [KM6] and Eichler and Zagier [EZ] to deduce that the Fourier series generating function for the intersection multiplicities of the T_n with a fixed T_m is an elliptic modular form of weight $2k+2$ with the level and character of the lattice L.

5.1. Following Hecke [He], recall that the *level* of L is the least positive integer N such that NB^{-1} is again an even integral form on L, where B is the matrix representing the bilinear form (,) on L with respect to some $\mathbb{Z}$-basis. Note that the level N and the discriminant D are divisible by the same primes. Recall also that the *character* ε associated to q and L is the quadratic character of modulus N induced by the character $\left(\frac{\operatorname{disc}(\mathbb{Q}(\sqrt{D}))}{\bullet}\right)$ associated to the field $\mathbb{Q}(\sqrt{D})$.

5.1.1. As usual, we let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Then an *elliptic modular form* of *weight* m, *level* N and *character* ε is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that

(i) $f(\gamma(\tau)) = \varepsilon(d) j(\gamma, \tau)^m f(\tau)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$; and

(ii) $\lim\limits_{\tau \to i\infty} \varepsilon(d)^{-1} j(\gamma, \tau)^{-m} f(\alpha(\tau))$ is finite, for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,

where $j(\gamma, \tau)$ is as in (4.1.2). An elliptic modular form is of *Nebentypus* if ε is nontrivial (otherwise of *Haupttypus*) [He]. If the limit in (ii) is 0 for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, then f is said to be a *cusp form*. As usual, we will denote the spece of modular forms of weight m, level N and character ε by $M_m(\Gamma_0(N), \varepsilon)$ and the subspace of cusp forms by $S_m(\Gamma_0(N), \varepsilon)$.

5.1.2. Recall also that the *Siegel upper half-plane of genus 2* is the set $\mathfrak{H}_2$ of symmetric $Z \in M_2(\mathbb{C})$ with positive definite imaginary part, on which an element $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{R})$ acts by $Z \mapsto (AZ + B)(CZ + D)^{-1}$. Let

$$\Gamma_0^{(2)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) : C \equiv 0 \pmod{N} \right\}.$$

Then a *Siegel modular form* of genus 2, *weight* m, *level* N and *character* ε is a holomorphic function $\Phi : \mathfrak{H}_2 \to \mathbb{C}$ such that

$$\Phi(\gamma(Z)) = \varepsilon(\det D) \det(CZ + D)^m \Phi(Z)$$

for $\gamma \in \Gamma_0^{(2)}(N)$. By Kocher's theorem [Car] [F], a Siegel modular form will have a Fourier series of the form $\Phi(Z) = \sum_{A \geq 0} r(A) \exp(\pi i \operatorname{tr} AZ)$, where the sum is taken only over positive semidefinite even $A \in M_2(\mathbb{Z})$.

PROPOSITION 5.2. *For $u, v \in L$, let $A(u,v) := \begin{pmatrix} (u,u) & (u,v) \\ (v,u) & (v,v) \end{pmatrix}$, and let $\nu(u,v)$ be as in* (3.1.4). *Then*

$$\Phi(Z) := \sum_{(u,v)\in\Gamma\backslash(L\times L)} \nu(u,v)\exp(\pi i \operatorname{tr} A(u,v)Z)$$

is a Siegel modular form of genus 2, weight 2, level N and character ε, where N and ε are the level and character of L.

Proof. Let $\{e_1, e_2, e_3, e_4\}$ be a basis of $V(\mathbb{R})$ such that

$$(e_i, e_j) = \begin{cases} 1, & \text{if } i = j = 1, 2 \\ -1, & \text{if } i = j = 3, 4 \\ 0, & \text{otherwise,} \end{cases}$$

let $(\ ,\ )'$ be the positive definite form on $V(\mathbb{R})$ defined by $(e_i, e_j)' := \delta_{ij}$ and let $A'(u,v) := \begin{pmatrix} (u,u)' & (u,v)' \\ (v,u)' & (v,v)' \end{pmatrix}$. Then it follows from [Mi3] [KM6] that $\Phi(Z)$ is a Siegel modular form of genus 2 and weight 2, and from [KM4] that it has the same level and character as

$$\Phi'(Z) := \sum_{(u,v)\in L\times L} \exp(\pi i \operatorname{tr} A'(u,v)Z).$$

However, the level and character of Φ' are computed in [LV], Theorem 2.6.22(a), to be those of $(L, (\ ,\ )')$, which equal those of $(L, (\ ,\ ))$. □

Observe that Φ' is the theta function for representations of binary quadratic forms by the positive definite lattice $(L, (\ ,\ )')$ [F], and similarly Φ is the theta function for Γ-inequivalent representations of positive semidefinite binary quadratic forms by the indefinite lattice $(L, (\ ,\ ))$. The proof of the next theorem shows that Φ has somehow encoded within it all the intersection multiplicities for all the arithmetic cycles of all weights $k \geq 0$.

THEOREM 5.3. *Let $0 \leq m, k \in \mathbb{Z}$ and let N and ε be the level and character of L. Then*

$$F_{k,m}(\tau) := \sum_{n\geq 0} (T_m^{(k)} \cdot T_n^{(k)})\, e^{2\pi i n\tau}$$

is an elliptic modular form of weight $2k+2$, level N and character ε. Moreover, if $k > 0$ then $F_{k,m}$ is a cusp form.

Proof. Let $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathfrak{H}_2$, with $\tau, \tau' \in \mathfrak{H}$ and $z \in \mathbb{C}$, and consider the Fourier-Jacobi expansion of Φ [PS] [EZ], Theorem 6.1, with respect to τ',

$$\Phi(Z) = \sum_{m=0}^{\infty} \varphi_m(\tau, z)\, e^{2\pi i m \tau'}.$$

Then $\varphi_m(\tau, z)$ is a Jacobi form of weight 2 and index m, with level N and character ε in the sense of [EZ], meaning that $\varphi : \mathfrak{H} \times \mathbb{C} \to \mathbb{C}$ is a holomorphic function such that:

(i) $$\varphi_m\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = \varepsilon(d) \exp\Big(2\pi i \frac{cmz^2}{c\tau + d}\Big)(c\tau + d)^2 \varphi_m(\tau, z)$$
for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$;

(ii) $$\varphi_m(\tau, z + \lambda\tau + \mu) = \exp(-2\pi i m(\lambda^2\tau + 2\lambda z))\, \varphi_m(\tau, z)$$
for $(\lambda, \mu) \in \mathbb{Z}^2$;

(iii) $$\varepsilon(d)^{-1}\mathbf{e}\Big(\frac{-cmz^2}{c\tau + d}\Big)^{-1}(c\tau + d)^{-2} \varphi_m\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big)$$
has a Fourier expansion of the form
$$\sum_{\substack{n \geq 0 \\ r^2 \leq 4mn}} c(n, r)\, \mathbf{e}(n\tau + rz)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. From the definition of φ in Proposition 5.2 we have

$$\varphi_m(\tau, z) = \sum_{\substack{n \geq 0 \\ r^2 \leq 4mn}} c_m(n, r)\, \mathbf{e}(n\tau + rz)$$

with

$$c_m(n, r) := \sum_{\substack{(u,v) \in \Gamma \backslash L(m) \times L(n) \\ (u,v) = r}} \nu(u, v).$$

In particular $F_{0,m}(\tau) = \varphi_m(\tau, 0) \in M_2(\Gamma_0(N), \varepsilon)$, which proves the theorem when $k = 0$. In general

$$\begin{aligned} F_{k,m}(\tau) &= \sum_{\substack{n \geq 0 \\ (u,v) \in \Gamma \backslash L(m) \times L(n)}} Q_{2k}(u, v)\nu(u, v)\mathbf{e}(n\tau) \\ &= \sum_{n \geq 0} \Big(\sum_{r^2 \leq 4mn} P_{2k}(r, mn) c_m(n, r) \Big)\, \mathbf{e}(n\tau) \end{aligned}$$

using Corollary 3.3(ii) and the definitions of $Q_{2k}(u, v)$ and $P_{2k}(r, mn)$ in (3.1.2) and (3.1.3) respectively. But now Theorem 5.3 follows as a special case of [EZ], Theorem 3.1, where it is shown more generally that this

"k^{th} development coefficient" of a Jacobi form of weight 2 is an elliptic modular form of weight $2k + 2$, and a cusp form if $k > 0$; although [EZ] consider only Jacobi forms of Haupttypus, their same proofs work for forms of Nebentypus. □

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