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Note on curves in a jacobian

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1. Introduction

1.1. For an abelian variety A over $\mathbb{C}$ and a cycle $\alpha \in CH_d(A)_{\mathbb{Q}}$ we define a subspace Z_α of $CH_d(A)_{\mathbb{Q}}$ by:

$$Z_\alpha := \langle n_* \alpha : n \in \mathbb{Z} \rangle \subset CH_d(A)_{\mathbb{Q}}.$$

Results of Beauville imply that Z_α is finite dimensional (cf. 2.5 below). In case $A = J(C)$, the jacobian of a curve C, Ceresa has shown that the cycle

$$C - C^- := C - (-1)_* C \in Z_C$$

is not algebraically equivalent to zero for generic C of genus $g \geqslant 3$, which implies that for such a curve $\dim_{\mathbb{Q}} Z_C \geqslant 2$.

In this note we investigate the subspace Z_{W_m} of $CH_m(J(C))_{\mathbb{Q}}$, with W_m the image of the m-th symmetric power of C in $J(C)$ (so $W_1 = C$). To simplify matters we will actually work modulo algebraic equivalence (rather than linear equivalence, note that translates of a cycle are algebraically equivalent).

1.2. Let $Z_\alpha/\approx_{\text{alg}}$ be the image of $Z_\alpha \subset CH_d(A)_{\mathbb{Q}}$ in $CH_d(A)_{\mathbb{Q}}/\approx_{\text{alg}}$. A d-cycle α is Abel-Jacobi equivalent to zero, $\alpha \approx_{\text{AJ}} 0$, if α is homologically equivalent to zero and its image in $J_d(A)$, the d-th primitive Intermediate Jacobian of A, is zero. Recall that any curve of genus g is a cover of $\mathbb{P}^1$ for some $d \leqslant (g+3)/2$.

1.3. THEOREM. (1) *For any abelian variety A and any $\alpha \in CH_d(A)$ we have:*

$$\dim_{\mathbb{Q}}(Z_\alpha/\approx_{\text{AJ}}) \leqslant 2.$$

(2) *For any curve of genus g and $1 \leqslant n \leqslant g-1$ we have:*

$$\dim_{\mathbb{Q}}(Z_{W_{g-n}}/\approx_{\text{alg}}) \leqslant n.$$

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(3) *For a curve C which is a d*: 1-*cover of* $\mathbb{P}^1$ *we have*:

$$\dim_{\mathbb{Q}}(Z_C/\approx_{\text{alg}}) \leqslant d - 1.$$

(*We prove* 1.3.1 *in* 2.5, 1.3.2 *in* 2.7 *and* 1.3.3 *in* 3.6.)

1.4. Recall that Ceresa showed that the image of $W_m - W_m^-$ in $J_m(J(C))$ is non-zero for generic C of genus $g \geqslant 3$ and $1 \leqslant m \leqslant g - 2$. Therefore 1.3.1 and 1.3.2 for $n = 2$ are sharp. In case C is hyperelliptic, so C is a 2 : 1 cover of $\mathbb{P}^1$, the cycles W_m and W_m^- are however algebraically equivalent. Therefore 1.3.3 is sharp for hyperelliptic curves ($d = 2$) and generic trigonal curves ($d = 3$). In case C is not hyperelliptic nor trigonal (like the generic curve of genus $g \geqslant 5$), it would be interesting to know if 1.3.3 is actually sharp.

Note that 1.3.1 implies that one cannot use the Intermediate Jacobian anymore to derive new algebraic relations among the cycles in $Z_C/\approx_{\text{alg}}$. Recently M. Nori [N] constructed cycles on complete intersections in $\mathbb{P}^N$ which are Abel-Jacobi equivalent to zero but not algebraically equivalent to zero. There is thus the possibility that similar cycles can be found on the Jacobian of a curve of genus $g \geqslant 5$. A cycle in Z_C, for certain modular curves C, was investigated by B. H. Gross and C. Schoen, [GS], esp. Section 5, see also 2.8.

1.5. The inequalities 1.3.1 and 1.3.2 are consequences of work of Beauville. The main part of the paper deals with the proof of 1.3.3. Recall that on a smooth surface S homological and algebraical equivalence for curves coincide. Thus if we have map $\Phi: S \to J(C)$ and relation

$$a_1[C_1] + \cdots + a_n[C_n] = 0 \quad \text{in } H^2(S, \mathbb{Q}),$$

we get

$$a_1\Phi_* C_1 + \cdots + a_n\Phi_* C_n \approx_{\text{alg}} 0$$

in $J(C)$. We use this remark to obtain our result, the main difficulty is of course to find suitable surfaces, curves in them and maps to $J(C)$ and to determine Neron-Severi groups ($=\text{Im}(CH_1(S) \to H^2(S, \mathbb{Q}))$) of the surfaces involved.

1.6. We are indebted to S. J. Edixhoven and C. Schoen for several helpful discussions.

2. General results

2.1. The effect of n_* and n^* on the Chow groups has been investigated by Beauville ([B1], [B2]), Dehninger and Murre [DM] and Künnemann [K] using the Fourier transform on abelian varieties introduced by Mukai. The main general result is that the diagonal $\Delta \subset A \times A$ can be written, in $CH^*(A \times A)_{\mathbb{Q}}$, as a direct sum of orthogonal projectors corresponding to the Künneth components. It is a conjecture of Murre [M] that such a decomposition should exist for any smooth, projective algebraic variety. Below we summarize some of the results and derive the finite dimensionality of Z_α as well as Theorem 1.3.1 and 1.3.2.

2.2. Let A be a g-dimensional abelian variety, we will view $B := A \times A$ as an abelian scheme over A using the projection on the first factor:

$$\begin{array}{ccc} B & & A \times A \\ \downarrow & = & \downarrow \mathrm{pr}_1 \\ A & & A \end{array}$$

For each integer n, we have an $s_n \in B(A)$ and a cycle $\Gamma_n \in CH^g(B)$:

$$s_n\colon A \to B,\ a \mapsto (a, na),\ \Gamma_n := s_{n*}A \in CH^g(A \times A),$$

and Γ_n is, as the notation suggests, the graph of multiplication by n on A. The cycle Γ_n defines for each i a map on $CH^i(A)$ which is just n_*:

$$n_* = \Gamma_n\colon CH^i(A) \to CH^i(A),\ \alpha \mapsto n_*\alpha = \mathrm{pr}_{2*}(\mathrm{pr}_1^*\alpha \cdot \Gamma_n).$$

Next we introduce a relative version of the Pontryagin product $*$ on the Chow groups of the A-scheme B (cf. [K], (1.2)). Let $m_B\colon B \times_A B \to B$ be the multiplication map, then:

$$\alpha * \beta := m_{B*}(\alpha \times_A \beta),\ \alpha, \beta \in CH^*(B).$$

Since the relative dimension of Γ_n over A is 0, the cycle $\Gamma_n * \Gamma_m$ lies in $CH^g(B)$ and one has ([K], (1.3.4)):

$$\Gamma_n * \Gamma_m = \Gamma_{n+m} \in CH^g(A \times A)_{\mathbb{Q}}, \quad \text{so } \Gamma_n = \Gamma_1^{*n},$$

where we write α^{*n} for the n-fold Pontryagin product of a cycle α with $n > 0$ and we put $\alpha^{*0} := \Gamma_0$. In [K] 1.4.1 a generalization of a theorem of Bloch is proved, which implies:

$$(\Gamma_1 - \Gamma_0)^{*(2g+1)} = 0 \quad (\in CH^g(B)_{\mathbb{Q}}). \tag{2.2.1}$$

Using the ring structure on $CH^g(B)_{\mathbb{Q}}$ with product $*$ one can thus define the following cycles π_i, $0 \leqslant i \leqslant 2g$ in $CH^g(B)_{\mathbb{Q}}$:

$$\pi_i := \frac{1}{(2g-i)!}(\log\Gamma_1)^{*(2g-i)},$$

with:

$$\log\Gamma_1 := (\Gamma_1 - \Gamma_0) - \tfrac{1}{2}(\Gamma_1 - \Gamma_0)^{*2} + \cdots + \tfrac{1}{2g}(\Gamma_1 - \Gamma_0)^{*(2g)}.$$

Let $\Delta = \Gamma_1 \in CH^g(A \times A)_{\mathbb{Q}}$ the class of the diagonal, then:

$$\Delta = \pi_0 + \pi_1 + \cdots + \pi_{2g}, \quad \pi_i\pi_j = \pi_j\pi_i = \begin{cases} \pi_i & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases} \tag{2.2.2}$$

here $\alpha\beta$, for cycles $\alpha, \beta \in CH^g(A \times A)_{\mathbb{Q}}$, is their product as correspondences: $\alpha\beta := p_{13*}(p_{12}^*\alpha \cdot p_{23}^*\beta)$ with the $p_{ij}\colon A^3 \to A^2$ the projection to the i, j factor (the existence of the π_i is proven in [DM], Theorem 3.1, the explicit form of the π_i is given in [K]). Moreover:

$$\Gamma_n\pi_{2g-i} = \pi_{2g-i}\Gamma_n = n^i\pi_{2g-i}, \tag{2.2.3}$$

(one has ${}^t\Gamma_n\pi_i = \pi_i^t\Gamma_n = n^i\pi_i$ and ${}^t\pi_i = \pi_{2g-i}$ ([K], 3.1.1), now take transposes).

2.3. REMARK. We sketch how these results can be obtained from 2. 1. Let $M \subset CH^g(B)_{\mathbb{Q}}$ be the subspace spanned by the Γ_n, $n \in \mathbb{Z}$. Then 2.2.1 implies that $\dim_{\mathbb{Q}} M \leqslant 2g + 1$ (use $\Gamma_i * (\Gamma_1 - \Gamma_0)^{*(2g+1)} = 0$ for all $i \in \mathbb{Z}$). Using the Künneth formula, Poincaré duality, one finds that the cohomology class of Γ_n in

$$H^{2g}(A \times A, \mathbb{Q}) = \oplus H^{2g-1}(A, \mathbb{Q}) \otimes H^i(A, \mathbb{Q}) = \oplus \operatorname{Hom}(H^i(A, \mathbb{Q}), H^i(A, \mathbb{Q}))$$

is given by (note Γ_n induces n_*):

$$[\Gamma_n] = (n^{2g}, n^{2g-1}, \ldots, n, 1) \in \bigoplus_{i=0}^{2g} \operatorname{Hom}(H^i(A, \mathbb{Q}), H^i(A, \mathbb{Q}))$$

(where n^{2g-i} in the i-th component means multiplication by n^{2g-i} on $H^i(A, \mathbb{Q})$). Therefore $\dim_{\mathbb{Q}} M/\sim_{\text{hom}} = 2g + 1$ and $\dim_{\mathbb{Q}} M = 2g + 1$. Thus we have the (surprising) result that homological and linear equivalence coincide in M. We can now define $\pi_i \in M$ by

$$[\pi_i] := (0, \ldots, 0, 1, 0, \ldots, 0) \in \oplus_i \operatorname{Hom}(H^i(A, \mathbb{Q}), H^i(A, \mathbb{Q}))$$

(1 in i-th spot) then 2.2.2 and 2.2.3 follow. To express π_i as combination of $\Gamma_0, \dots, \Gamma_{2g}$, note that the ($\mathbb{Q}$-linear) ring homomorphism:

$$\mathbb{Q}[X] \to M \subset CH^g(B)_{\mathbb{Q}}, \quad X^i \mapsto \Gamma_i = \Gamma_1^{*i}$$

(with $*$ product on $CH^g(B)_{\mathbb{Q}}$) gives an isomorphism $\mathbb{Q}[X]/(X-1)^{2g+1} \cong M$. Since $\Gamma_n \Gamma_m = \Gamma_{nm}$ (product as correspondences), π_{2g-i} corresponds to a polynomial f_{2g-i} with:

$$f_{2g-i}(X^n) = n^i f_{2g-i}(X) \quad \text{so} \quad f_{2g-i}(X) := c_i (\log X)^i \mod (X-1)^{2g+1}$$

and $c_i \in \mathbb{Q}$ can be determined with a little more work.

2.4. Since $\Delta = \Gamma_1 : CH_d(A) \to CH_d(A)$ is the identity, we get:

$$CH_d(A)_{\mathbb{Q}} = \oplus\, CH_d(A)_{(i)}, \quad \text{with } CH_d(A)_{(i)} := \pi_{2g-i} CH_d(A)_{\mathbb{Q}}.$$

and each $CH_d(A)_{(i)}$ is an eigenspace for the multiplication operators:

$$n_* \alpha = n^i \alpha \quad \forall n \in \mathbb{Z}, \quad \forall \alpha \in CH_d(A)_{(i)},$$

a result which was first obtained by Beauville [B2]. Moreover, he proves:

$$CH_d(A)_{(i)} \neq 0 \;\Rightarrow\; d \leqslant i \leqslant d+g \tag{2.4.1}$$

(and gives sharper bounds for some d in Proposition 3 of [B2], it has been conjectured that $CH_d(A)_{(i)} \neq 0 \Leftrightarrow 2d \leqslant i \leqslant d+g$).

2.5. PROPOSITION. *Let A be an abelian variety and et $\alpha \in CH_d(A)_{\mathbb{Q}}$. Then:*

$$\dim_{\mathbb{Q}} Z_\alpha \leqslant g+1 \quad \textit{and} \quad \dim_{\mathbb{Q}}(Z_\alpha / \approx_{\mathrm{AJ}}) \leqslant 2.$$

Moreover, we have $\pi_i Z_\alpha \subset Z_\alpha$ for all i.

Proof. We write α as a sum of weight vectors ($\alpha_{(i)} := \pi_{2g-i}\alpha$):

$$\alpha = \alpha_{(d)} + \alpha_{(d+1)} + \cdots + \alpha_{(g+d)}, \quad \text{so } n_* \alpha_{(i)} = n^i \alpha.$$

Taking g distinct, non-zero, integers n_j, the determinant of the matrix expressing the $n_{i*}\alpha \in Z_\alpha$ in terms of the α_i is a Vandermonde determinant. Thus each $\alpha_{(i)} = \pi_{2g-i}\alpha \in Z_\alpha$ and Z_α is spanned by the $\alpha_{(i)}$.

Since n_* acts as n^{2d} on $H^{2g-2d}(A, \mathbb{Q})$ and as n^{2d+1} on $J_d(A)$, the space $Z_\alpha / \approx_{\mathrm{AJ}}$ is spanned by $\alpha_{(2d)}$ and $\alpha_{(2d+1)}$. □

2.6. In this section, we fix an abel-jacobi map $C \hookrightarrow J(C)$. Note that $W_d = 1/d!\, C^{*d}$ ((Pontryagin product on $J(C)$). Let $\Theta \in CH^1(J(C))$ be a symmetric theta divisor, so Θ is a translate of W_{g-1}. Let $\Theta^d \in CH_{g-d}(J(C))$ be the d-fold intersection of Θ.

2.7. PROPOSITION. *For any d, $1 \leqslant d \leqslant g-1$, we have $\Theta^d \in Z_{W_{g-d}}$, more precisely:*

$$\pi_{2g-i} W_{g-d} \neq 0 \Rightarrow 2(g-d) \leqslant i \leqslant 2g-d, \quad \textit{and} \quad \pi_{2d} W_{g-d} = \frac{1}{d!}\Theta^d.$$

Moreover we have:

$$\dim_{\mathbb{Q}}(Z_{W_{g-d}}/\approx_{\text{alg}}) \leqslant d.$$

Proof. We first prove the case $d = g-1$. Since the map $CH_1(C)_{\mathbb{Q}} \to H^{2g-2}(J(C), \mathbb{Q})$ factors over $\pi_{2g-2} CH_1(J(C))_{\mathbb{Q}}$ (cf. the proof of 2.5 or even better, [B2], p. 650) we know that $\pi_{2g-2} C \neq 0$. Then its Fourier transform

$$F_{CH}(\pi_{2g-2} C) \in \pi_2 CH_{g-1}(J(C))_{\mathbb{Q}} (\cong NS(J(C))_{\mathbb{Q}})$$

(cf. [B2], Prop. 1) is non-zero. For a generic Jacobian $NS_{\mathbb{Q}}$ is one dimensional and thus

$$\pi_2 CH_{g-1}(J(C))_{\mathbb{Q}} = \mathbb{Q}\Theta$$

([B1], Prop. 5 and [B2], Prop. 1). By specializing, we have for all curves that $F_{CH}(\pi_{2g-2} C) \in \mathbb{Q}\Theta$.

From [B1], Proposition 5 we have

$$\Theta = -F_{CH}\left(\frac{1}{(g-1)!}\Theta^{g-1}\right), \text{ so for a nonzero constant } c\text{:}$$

$$F_{CH}(\pi_{2g-2} C) = cF_{CH}\left(\frac{1}{(g-1)!}\Theta^{g-1}\right)$$

and, by using $F_{CH}^2 = (-1)^g \Gamma_{-1}$ and comparing cohomology classes, we get:

$$\pi_{2g-2} C = \frac{1}{(g-1)!}\Theta^{g-1}.$$

Next we recall that $\pi_{2g-1} CH_1(A) = 0$ for any abelian variety A ([B2], Prop.

3), thus:

$$C = C_{(2)} + \cdots + C_{(g+1)} \quad \text{with } C_{(i)} = \pi_{2g-i}C,$$

so $n_* C_{(i)} = n^i C_{(i)}$ and $C_{(2)} = 1/(g-1)!\,\Theta^{g-1}$. Therefore

$$W_{g-d} = \frac{1}{(g-d)!} C^{*(g-d)} = \frac{1}{(g-d)!} C_{(2)}^{*(g-d)} + Y \tag{2.7.1}$$

with Y a sum of cycles $C_{(i_1)} * \cdots * C_{(i_{g-d})}$ with all $i_j \geqslant 2$ and at least one >2. Therefore, using

$$n_*(U * V) = (n_* U) * (n_* V),$$

we get:

$$\pi_{2d} W_{g-d} = \frac{1}{(g-d)} C_{(2)}^{*(g-d)} \quad \text{and} \quad \pi_j W_{g-d} = 0$$

for $j > 2d$.

By [B1], Corollary 2 of Proposition 5, Θ^d lies in the subspace spanned by $(\Theta^{g-1})^{*(g-d)}$, so $C_{(2)}^{*(g-d)} = c\Theta^d$ for a non-zero $c \in \mathbb{Q}$ and taking cohomology classes one finds $c = 1$.

Finally, using 2.4.1 and 2.7.1 we can write

$$W_{g-d} = \alpha_{(2g-2d)} + \cdots + \alpha_{(2g-d)}, \quad \text{with } \alpha_{(i)} := \pi_{2g-i} W_{g-d},$$

and since $\alpha_{(2g-d)} \approx_{\text{alg}} 0$ by [B2], Proposition 4a, we get $\dim_{\mathbb{Q}}(Z_{W_{g-d}}/\approx_{\text{alg}}) \leqslant d$. □

2.8. In the paper [GS] the following cycle (modulo $\approx_{\text{alg}}$) is considered:

$$Z := 3_* C - 3 \cdot 2_* C + 3C = (\Gamma_1 - \Gamma_0)^{*3} C.$$

This cycle is related to Ceresa's cycle in the following way:

2.9. PROPOSITION. *We have:*

$$\pi_{2g-2}(Z) = \pi_{2g-2}(C - C^-) = 0 \quad \textit{and} \quad \pi_{2g-3}(Z) = 3\pi_{2g-3}(C - C^-).$$

In particular, Z is abel-jacobi equivalent to $3(C - C^-)$.

Proof. As we saw before (Proof of 2.7) we can write:

$$C = C_{(2)} + C_{(3)} + \cdots \quad \text{with } C_{(i)} := \pi_{2g-i} C.$$

Since $n_* C_{(i)} = n^i C_{(i)}$ we have:

$$C - C^- = 2(C_{(3)} + 0 + C_{(5)} + \cdots), \quad Z = 6C_{(3)} + 36C_{(4)} + \cdots$$

Therefore $\pi_{2g-2}(C - C^-) = 3\pi_{2g-2}(Z) = 0$, so both cycles are homologically equivalent to zero, and $3\pi_{2g-3}(C - C^-) = \pi_{2g-3}(Z) = 6C_{(3)}$. Since the abel-jacobi map factors over $\pi_{2g-3}CH_1(J(C))$ we have that $Z \approx_{AJ} 3(C - C^-)$. □

3. Proof of 1.3.3

3.1. Let C be a generic d: 1 cover of $\mathbb{P}^1$, and denote by g^1_d the corresponding linear series. For an integer n, $0 < n < d$ we define a curve G_n in the n-fold $C^{(n)} = \mathrm{Sym}^n(C)$:

$$G_n = G_n(g^1_d) = \{x_1 + x_2 + \cdots + x_n \in C^{(n)}: x_1 + x_2 + \cdots + x_n < g^1_d\},$$

see [ACGH], p. 342 for the definition of the scheme-structure on G_n. We define a surface by:

$$S_n = S_n(g^1_d) = G_n \times C.$$

3.2. PROPOSITION. *Let C be a generic d:1-covering of $\mathbb{P}^1$ of genus $g \geqslant 1$. Then S_n is a smooth, irreducible surface and*

$$\dim_{\mathbb{Q}} NS(S_n)_{\mathbb{Q}} = 3.$$

Proof. Since C is generic, the curve G_n is irreducible (consider the monodromy representation) and for a simple covering the smoothness of G_n follows from a local computation. Therefore also S_n is smooth and irreducible.

Note that for a product of 2 curves $S_n = G_n \times C$ we have that

$$NS(S_n)_{\mathbb{Q}} = \mathbb{Q}C \oplus \mathbb{Q}G_n \oplus \mathrm{Hom}(J(C), J(G_n))_{\mathbb{Q}}.$$

Using Hodge structures, the last part are just the Hodge-cycles in $H^1(C, \mathbb{Q}) \otimes H^1(G_n, \mathbb{Q})$. Since $J(C)$ and $\mathrm{Pic}^0(C^{(n)})$ are isogeneous (their $H^1(\mathbb{Q})$'s are isomorphic Hodge structures), composing such an isogeny with the pull-back $\mathrm{Pic}^0(C^{(n)}) \to J(G_n)$, we have a non-trivial map $J(C) \to J(G_n)$. To show that

$\dim_{\mathbb{Q}} \operatorname{Hom}(J(C), J(G_n))_{\mathbb{Q}} \leqslant 1$ we use a degeneration argument. First we recall that the Jacobian of C is simple and has in fact $\operatorname{End}(J(C)) = \mathbb{Z}$.

In case $g = 2$ this is clear since any genus 2 curve can be obtained by deformation from a given $d{:}1$ cover and C is generic. In case $g = 3$ one reasons similarly, taking the necessary care for the hyperelliptic curves. In case $g(C) = g > 3$, assume first that there is an elliptic curve in $J(C)$. Specializing C to a reducible curve with two components C' and C'', both of genus $\geqslant 2$, and themselves generic $d{:}1$-covers we obtain a contradiction by induction. Assume now that there is no elliptic curve in $J(C)$, then we specialize C to $E \times C'$, with E an elliptic curve and C' a generic $d{:}1$-covering of genus $g - 1$. By induction again, $J(C')$ is simple, which contradicts the existence of abelian subvarieties in $J(C)$. We conclude that $J(C)$ is simple. Thus $\operatorname{End}_{\mathbb{Q}}(J(C))$ is a division ring and specializing again we find it must be $\mathbb{Q}$.

We may assume that the g_d^1 exhibits C as a simple cover of $\mathbb{P}^1$, then G_n is a cover of $\mathbb{P}^1$ (of degree $\binom{d}{n}$) with only twofold ramification points. Letting two branch points coincide, we obtain a curve $\bar{C}$ with a node, the normalization C' of $\bar{C}$ has genus $g - 1$ and is again exhibited as a generic $d{:}1$ cover of $\mathbb{P}^1$. The curve G_n acquires $\binom{d-2}{n-1}$ nodes (since twice that number of branch points coincide pairwise) and the normalization of that curve, $\bar{G}_n$, is G'_n, the curve obtained from the g_d^1 on C'. The (generalized) Jacobians of $\bar{C}$, $\bar{G}_n$ are extensions of the abelian varieties (of dim $\geqslant 1$) $J(C')$, $J(G'_n)$ by multiplicative groups.

Since the number of simple factors of $J(G'_n)$ which are isogeneous to $J(C')$ is greater than or equal to the number of simple factors of $J(G_n)$ which are isogeneous to $J(C)$, and $J(C)$, $J(C')$ have $\operatorname{End}_{\mathbb{Q}} = \mathbb{Q}$ it follows:

$$\dim_{\mathbb{Q}} \operatorname{Hom}(J(C), J(G_n))_{\mathbb{Q}} \leqslant \dim_{\mathbb{Q}} \operatorname{Hom}(J(C'), J(G'_n))_{\mathbb{Q}}.$$

Therefore it suffices to show that for a generic elliptic curve $C = E$ and a generic g_d^1 on E we have $\dim_{\mathbb{Q}} \operatorname{Hom}(E, J(E_n))_{\mathbb{Q}} \leqslant 1$ (with $E_n = G_n(g_d^1)$ and $C = E$).

We argue again by induction, using degeneration. Note that for any C, $G_1(g_d^1) \cong G_{d-1}(g_d^1) \cong C$. Since a generic elliptic curve has $\operatorname{Hom}(E, E) = \mathbb{Z}$ the statement is true for $d \leqslant 3$ (and any $0 < n < d$) and since $E_1 \cong E$ it is also true for $n = 1$ (and any $d \geqslant 2$).

We fix E, a generic elliptic curve, but let the g_d^1 acquire a base point $y \in E$. Then $E_n(g_d^1)$, $n \geqslant 2$, becomes a reducible curve $\bar{E}_n$, having two components $E'_n = E_n(g_{d-1}^1)$ and $E'_{n-1} = E_{n-1}(g_{d-1}^1)$ which meet transversally in $N = \binom{d-2}{n-1}$ points. Indeed, let the divisor of the g_{d-1}^1 containing y be $y + y_1 + \cdots + y_{d-2}$, then the points

$$y_{i_1} + \cdots + y_{i_{n-1}} \in E'_{n-1} \quad \text{and} \quad y + y_{i_1} + \cdots + y_{i_{n-1}} \in E'_n$$

are identified.

Thus $J(\bar{E}_n)$ is an extension of $J(E'_n) \times J(E'_{n-1})$ by the multiplicative group $(\mathbb{C}^*)^{N-1}$, in fact there is an exact sequence:

$$1 \to \mathbb{C}^* \xrightarrow{\Delta} (\mathbb{C}^*)^N \to J(\bar{E}_n) \xrightarrow{\pi} J(E'_n) \times J(E'_{n-1}) \to 0,$$

where Δ is the diagonal embedding. We will write $\pi = (\pi_0, \pi_1)$.

We will prove that

$$\mathrm{Hom}(E, J(\bar{E}_n))_{\mathbb{Q}} \to \mathrm{Hom}(E, J(E'_n))_{\mathbb{Q}}, \quad \tilde{\phi} \mapsto \pi_0 \circ \tilde{\phi}$$

is injective. By induction we may assume that

$$\dim \mathrm{Hom}(E, J(E'_n))_{\mathbb{Q}} \leqslant 1,$$

thus also

$$\dim_{\mathbb{Q}} \mathrm{Hom}(E, J(\bar{E}_n))_{\mathbb{Q}} \leqslant 1$$

and since

$$\dim_{\mathbb{Q}} \mathrm{Hom}(E, J(E_n))_{\mathbb{Q}} \leqslant \dim_{\mathbb{Q}} \mathrm{Hom}(E, J(\bar{E}_n))_{\mathbb{Q}}$$

the assertion on the rank of the Neron-Severi group follows.

Assume that $\tilde{\phi} \neq 0$, but $\pi_0 \circ \tilde{\phi} = 0$. Then $\tilde{\phi}(E) \subset \bar{J} := \pi_1^{-1}(J(E'_{n-1}))$. Pulling back this $(\mathbb{C}^*)^{N-1}$-bundle $\bar{J}$ over $J(E'_{n-1})$ to E along $\pi_1 \circ \tilde{\phi}$, we obtain a $(\mathbb{C}^*)^{N-1}$-bundle $\tilde{E}$ over E. The map $\tilde{\phi}: E \to \bar{J}$ gives a section of $\tilde{E} \to E$, thus $\tilde{E}$ is a trivial $(\mathbb{C}^*)^{N-1}$-bundle. We show that this gives the desired contradiction.

$$\begin{array}{ccccccccc} 1 & \to & (\mathbb{C}^*)^{N-1} & \to & \bar{J} & \to & J(E'_{n-1}) & \to 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \pi_1 \tilde{\phi} & \\ 1 & \to & (\mathbb{C}^*)^{N-1} & \to & \tilde{E} & \to & E & \to 0 \end{array}$$

For any distinct $P, Q \in E'_n$ which are in $E'_n \cap E'_{n-1}$, the $(\mathbb{C}^*)^{N-1}$-bundle $\bar{J}$ over $J(E'_{n-1})$ has a quotient $\mathbb{C}^*$-bundle $\bar{J}_{PQ}$ whose extension class is

$$P - Q \in \mathrm{Pic}^0(E'_n) = \mathrm{Ext}^1(J(E'_n), \mathbb{C}^*).$$

By induction, there is a 'unique' map in $\mathrm{Hom}(J(E'_{n-1}), E)$ which must thus be induced by

$$x_1 + \cdots + x_{n-1} \mapsto x_1 + \cdots + x_{n-1} - (n-1)p, \; E'_{n-1} \to E$$

for some $p \in E$. Taking $P = y_1 + \cdots + y_{n-1}$ and $Q = y_2 + \cdots + y_{n-1} + y_n$, the pull-back of $P - Q$ to $E = \mathrm{Pic}^0(E)$ is $y_1 - y_n$. Choosing the degeneration suitably we may assume that $y_1 - y_n$ is not a torsion point on E and thus $\bar{J}_{PQ}$ has a nontrivial pull-back to E, contradicting the fact that $\tilde{E}$, the pull-back of $\bar{J}$ *to* E, is trivial. □

3.3. We define a curve Δ_n in the surface $S_n = G_n \times C$:

$$\Delta_n := \{(x_1 + x_2 + \cdots + x_n, p) \in S_n : p \in \{x_1, \ldots, x_n\}\}$$

and for $n + 1 \leqslant d$, another curve H_n in S_n:

$$H_n = \{(x_1 + x_2 + \cdots + x_n, p) \in S_n : x_1 + x_2 + \cdots + x_n + p < g_d^1\}.$$

Finally we define the map:

$$\Phi_{l,k}^n : S_n \to J(C), \tag{3.3.1}$$

$$(x_1 + x_2 + \cdots + x_n, p) \mapsto l(x_1 + x_2 + \cdots + x_n) + kp - D_{nl+k},$$

where D_{nl+k} is some divisor of degree $nl + k$ on C. Finally we denote by

$$u_n : C^{(n)} \to J(C),\ D \mapsto D_n,$$

with D_n some divisor of degree n, the Abel-Jacobi map on $C^{(n)}$. We simply write C for $u_{1*}C$.

3.4. PROPOSITION. *Let C be a generic* $d:1$ cover of $\mathbb{P}^1$ *of genus* $g \geqslant 1$. *Then:*

$NS(S_n)_{\mathbb{Q}} = \langle C, G_n, \Delta_n \rangle$.

2. *In $NS(S_n)$ we have*

$$H_n = \binom{d}{n} C + dG_n - \Delta_n. \tag{3.4.1}$$

3. *The image of G_n in $J(C)$ is a combination of $C, 2_*C, \ldots, n_*C$:*

$$u_{n*}G_n \approx_{\mathrm{alg}} \binom{d}{n-1} C - \frac{1}{2}\binom{d}{n-2} 2_*C + \cdots + \frac{(-1)^{n-1}}{n}\binom{d}{0} n_*C. \tag{3.4.2}$$

Proof. The first part follows from the previous proposition and from the fact that in the proof of 3.4.1 we will see that H_n can be uniquely expressed as a combination of C, G_n and Δ_n.

The proof of 2 and 3 is by induction. In the case $n = 1$, so $G_1 \cong C$, 3.4.2 is trivial and for 3.4.1 we have $S_1 = C \times C$ and we must show $H_1 = d(C + G_1) - \Delta$. Note the following intersection numbers:

$$H_1 \cdot C = H_1 \cdot G_1 = d - 1, \quad H_1 \cdot \Delta = 2(g + d - 1)$$

(use that the g_d^1 exhibits C as $d{:}1$ cover of $\mathbb{P}^1$ with $2(g + d - 1)$ simple ramification points), which imply the result.

Suppose that 3.4.1 and 3.4.2 are true for all $k < n$. Note that $u_n(G_n) = \Phi_{1,1}^{n-1}(H_{n-1})$ and that map $\Phi_{1,1}^{n-1}$ restricted to H_{n-1} is $n{:}1$ so, using 3.4.1 for $n - 1$, we have in $J(C)$:

$$G_n \approx_{\mathrm{alg}} \frac{1}{n}\left(\binom{d}{n-1} C + dG_{n-1} - (\Phi_{1,1}^{n-1})_* \Delta_{n-1}\right). \tag{3.4.3}$$

where we write G_k for $u_{k*}G_k$. Next we observe that

$$(\Phi_{1,l}^{k})_* \Delta_k = (\Phi_{1,l+1}^{k-1})_* H_{k-1} \quad \text{and} \quad (\Phi_{1,l+1}^{k-1})_* C = (l+1)_* C. \tag{3.4.4}$$

Using 3.4.1 in the cases $n - 2, n - 3, \ldots, 1$ we get:

$$\begin{aligned}
(\Phi_{1,1}^{n-1})_* \Delta_{n-1} &\approx_{\mathrm{alg}} (\Phi_{1,2}^{n-2})_* H_{n-2} \\
&\approx_{\mathrm{alg}} \binom{d}{n-2} 2_* C + dG_{n-2} - (\Phi_{1,3}^{n-3})_* H_{n-3} \\
&\approx_{\mathrm{alg}} \binom{d}{n-2} 2_* C - \binom{d}{n-3} 3_* C + d(G_{n-2} - G_{n-3}) + (\Phi_{1,4}^{n-4})_* H_{n-4} \\
&\approx_{\mathrm{alg}} \binom{d}{n-2} 2_* C - \binom{d}{n-3} 3_* C + \cdots + (-1)^{n-2}\binom{d}{0} n_* C + \\
&\quad + d(G_{n-2} - G_{n-3} + \cdots + (-1)^{n-2} G_2 + (-1)^{n-1} G_1)
\end{aligned} \tag{3.4.5}$$

(note that $(\Phi_{1,n-1}^{1})_* H_1 \approx_{\mathrm{alg}} \binom{d}{1}(n-1)_* C + dG_1 - \binom{d}{0} n_* C$). Substitute the expression for $(\Phi_{1,1}^{n-1})_* \Delta_{n-1}$ from 3.4.5 into 3.4.3:

$$\begin{aligned}
G_n \approx_{\mathrm{alg}} \frac{1}{n}\Bigg(&\binom{d}{n-1} C - \binom{d}{n-2} 2_* C + \cdots + (-1)^{n-2}\binom{d}{1}(n-1)_* C \\
&+ (-1)^{n-1}\binom{d}{0} n_* C + d\Big(G_{n-1} - G_{n-2} + \cdots \\
&+ (-1)^{n-3} G_2 + (-1)^{n-2} G_1\Big)\Bigg).
\end{aligned} \tag{3.4.6}$$

Using the formula 3.4.2 for G_k, $k = 1, 2, \ldots, n-1$ and the relation $\Sigma_{k=0}^{l}(-1)^k\binom{d}{l-k} = \binom{d-1}{l}$, (cf. 4.2.1) we get:

$$G_{n-1} - G_{n-2} + \cdots + (-1)^{n-2}G_1$$
$$\approx_{\mathrm{alg}} \binom{d-1}{n-3}C - \frac{1}{2}\binom{d-1}{n-4}2_*C + \cdots + (-1)^{n-3}\frac{1}{n-1}\binom{d-1}{0}(n-1)_*C. \tag{3.4.7}$$

Substituting this in formula 3.4.6 and using the identity:

$$\frac{1}{n}\left[\binom{d}{n-k} + \frac{d}{k}\binom{d-1}{n-k+1}\right] = \frac{1}{k}\binom{d}{n-k}$$

we obtain 3.4.2.

To obtain 3.4.1, note that by (1), there are $a, b, c \in \mathbb{Q}$ such that:

$$H_n = aC + bG_n + c\Delta_n. \tag{3.4.8}$$

To find them we compute the homology classes of the curves in 3.4.8 in $H_2(J(C), \mathbb{Q})$ and the intersection numbers of H_n with C and G_n.

For the generic curve C, the homology class $[B]$ of a curve B in $J(C)$ is a multiple of the class $[C]$ of C and this multiple is $\frac{1}{g}\Theta \cdot B$. We apply this to $B = G_n$. Using the map $u: C^{(n)} \to J(C)$, we have $u_*(G_n)\cdot\Theta = u_*(G_n\cdot\theta)$ with θ the pull-back of Θ to $C^{(n)}$.

The homology class of G_n in $H_2(C^{(n)}, \mathbb{Q})$ is:

$$G_n = \sum_{k=0}^{n-1}\binom{d-g-1}{k}\frac{x^k\theta^{n-k-1}}{(n-1-k)!}, \quad \text{so} \quad G_n\cdot\theta = \sum_{k=0}^{n-1}\binom{d-g-1}{k}\frac{x^k\theta^{n-k}}{(n-1-k)!}, \tag{3.4.9}$$

cf. formula (3.2) on p. 342 of [ACGH] (here x is the class of the divisor $C^{(n-1)}$ in $C^{(n)}$). From p. 343 of [ACGH] one has: $u_*(x^{n-i}\theta^i) = \binom{g}{i}i![\Theta^g]/g!$ and since $[\Theta^g]/g!$ is the positive generator of $H^{2g}(J(C), \mathbb{Z})$, we find:

$$G_n\cdot\Theta = \sum_{k=0}^{n-1}\binom{d-g-1}{k}\binom{g}{n-k}\frac{(n-k)!}{(n-1-k)!}$$
$$= g\sum_{k=0}^{n-1}\binom{d-g-1}{k}\binom{g-1}{n-k-1}$$
$$= g\binom{d-2}{n-1}. \tag{3.4.10}$$

Therefore:

$$[G_k] = \binom{d-2}{k-1}[C]. \tag{3.4.11}$$

Next we compute the homology class of Δ_n. We use 3.4.4 for $k = n$, $l = 1$: $(\Phi^n_{1,1})_*\Delta_n = (\Phi^{n-1}_{1,2})_* H_{n-1}$. By induction, similar to 3.4.5 and using 3.4.7 (with $n-1$ replaced by n) we have:

$$\begin{aligned}(\Phi^n_{1,1})_*\Delta_n &\approx_{\text{alg}} d(G_{n-1} - G_{n-2} + \cdots + (-1)^{n-3}G_2 + (-1)^{n-2}G_1) + \\ &\quad + \binom{d}{n-1}2_*C - \binom{d}{n-2}3_*C + \cdots + (-1)^{n-1}\binom{d}{0}(n+1)_*C \\ &\approx_{\text{alg}} d\sum_{k=1}^{n-1}(-1)^{k-1}\frac{1}{k}\binom{d-1}{n-1-k}k_*C + \sum_{k=2}^{n+1}(-1)^k\binom{d}{n+1-k}k_*C.\end{aligned} \tag{3.4.12}$$

Taking the homology classes in $J(C)$ we get:

$$[(\Phi^n_{1,1})_*\Delta_n] = \left(d\sum_{k=1}^{n-1}(-1)^{k-1}k\binom{d-1}{n-1-k} + \sum_{k=2}^{n+1}(-1)^k k^2\binom{d-1}{n+1-k}\right)[C],$$

Using 4.2.3:

$$\begin{aligned}\sum_{k=2}^{n+1}(-1)^k\binom{d}{n+1-k}k^2 &= \binom{d}{n} + \sum_{k=0}^{n+1}(-1)^k\binom{d}{n+1-k}k^2 \\ &= \binom{d}{n} + \binom{d-3}{n-1} - \binom{d-3}{n},\end{aligned}$$

we find:

$$[(\Phi^n_{1,1})_*\Delta_n] = \left(d\binom{d-3}{n-2} + \binom{d}{n} + \binom{d-3}{n-1} - \binom{d-3}{n}\right)[C]$$

and because of

$$\binom{d-3}{n-1} - \binom{d-3}{n} = d\binom{d-3}{n-1} - (n+1)\binom{d-2}{n},$$

$$\binom{d-2}{n-1} = \binom{d-3}{n-1} + \binom{d-3}{n-2}$$

we finally obtain:

$$[(\Phi^n_{1,1})_* \Delta_n] = \left(d\binom{d-2}{n-2} - (n+1)\binom{d-2}{n} + \binom{d}{n} \right)[C]. \tag{3.4.13}$$

Applying $(\Phi^n_{1,1})_*$ to 3.4.8 and taking homology classes we get the following equation for a, b, c:

$$(n+1)\binom{d-2}{n} = a + b\binom{d-2}{n-1} + c\left(d\binom{d-2}{n-1} - (n+1)\binom{d-2}{n} + \binom{d}{n} \right).$$

On the other hand, we have the following intersections numbers in S_n:

$$C \cdot C = G_n \cdot G_n = 0,\ C \cdot G_n = 1,\ C \cdot \Delta_n = n,\ G_n \cdot \Delta_n = \binom{d-1}{n-1}$$

and

$$H_n \cdot C = d - n, \quad H_n \cdot G_n = \binom{d-1}{n}.$$

Therefore, by intersecting 3.4.8 with C and G_n respectively, we find two more equations for a, b, c:

$$d - n = b + cn, \quad \binom{d-1}{n} = a + c\binom{d-1}{n-1},$$

and 3.4.1 now follows. □

3.5. With these results it is easy to find many relations between the cycles $n_* C$. It is a little surprising that the ones we find are equivalent to $\pi_i C \approx_{\text{alg}} 0$ for $i < 2g - d$ and $\pi_{2g-1} C \approx_{\text{alg}} 0$.

3.6. PROPOSITION. *Let C be a d:1 covering of $\mathbb{P}^1$. Then:*

$$\pi_{2g-i} C \not\approx_{\text{alg}} 0 \Rightarrow 2 \leqslant i \leqslant d.$$

The cycles $\pi_{2g-2}C, \ldots, \pi_{2g-d}C$ span $Z_C/\approx_{\text{alg}}$ and thus

$$dim_{\mathbb{Q}}(Z_C/\approx_{\text{alg}}) \leqslant d - 1.$$

For any set $\{n_1, \ldots, n_{d-1}\}$ of non-zero distinct integers the cycles $n_{1*}C, \ldots, n_{d-1*}C$ also span $Z_C/\approx_{\text{alg}}$.

Proof. First of all we show that for all $n \in \mathbb{Z}$ we have:

$$P_n C := (n+1)_* C - \binom{d}{d-1} n_* C + \cdots + (-1)^d \binom{d}{0} (n-d+1)_* C + F_d \approx_{\text{alg}} 0 \tag{3.6.1}$$

with a cycle F_d depending on d but not on n:

$$F_d = d\left[-C + \binom{d-1}{d-3} C^- - \frac{1}{2}\binom{d-1}{d-4} 2_* C^- + \cdots + (-1)^{d-1} \frac{1}{d-2} \binom{d-1}{0} (d-2)_* C^- \right].$$

This relation follows from the easily verified identity, for all $n \in \mathbb{Z}$ and $d \geqslant 3$:

$$(\Phi^1_{1,n})_* H_1 = (\Phi^{d-2}_{-1,n-1})_* H_{d-2}$$

(for $d = 2$ see below). Indeed, the l.h.s. is by 3.4.1:

$$(\Phi^1_{1,n})_* H_1 \approx_{\text{alg}} (\Phi^1_{1,n})_* (dC + dG_1 - \Delta_1) \approx_{\text{alg}} dn_* C + dC - (n+1)_* C,$$

while for the r.h.s. we use $(\Phi^k_{-1,l})_* \Delta_k = (\Phi^{k-1}_{-1,l-1})_* H_{k-1}$):

$$\begin{aligned}(\Phi^{d-2}_{-1,n-1})_* H_{d-2} &\approx_{\text{alg}} \binom{d}{d-2}(n-1)_* C + dG^-_{d-2} - (\Phi^{d-3}_{-1,n-2})_* H_{d-3} \\ &\approx_{\text{alg}} \binom{d}{d-2}(n-1)_* C - \binom{d}{d-3}(n-2)_* C \\ &\quad + \cdots + (-1)^{d-2}(n-d+1)_* C + d(G^-_{d-2} - G^-_{d-3} + \\ &\quad + \cdots + (-1)^{d-3} G^-_1)\end{aligned} \tag{3.6.2}$$

(cf. also the Proof of 3.4.2). From this 3.6.1 follows by using 3.4.7.

A more convenient set of relations is obtained from 3.6.1 as follows:

$$\begin{aligned}(P_n - P_{n-1})C &= (n+1)_* C - (d+1) n_* C + \\ &\quad + \binom{d+1}{d-1}(n-1)_* C + \cdots + (-1)^{d+1}(n-d)_* C\end{aligned} \tag{3.6.3}$$

$$= \left(\Gamma_{m+d+1} - \binom{d+1}{1} \Gamma_{m+d} + \binom{d+1}{2} \Gamma_{m+d-1} + \cdots + (-1)^{d+1} \Gamma_m \right) C$$

$$\approx_{\text{alg}} 0. \qquad (3.6.4)$$

where we substituted: $n := m + d$. Relation 3.6.4 can be rewritten as:

$$\Gamma_m * (\Gamma_1 - \Gamma_0)^{*(d+1)} C \approx_{\text{alg}} 0 \quad \forall m \in \mathbb{Z}. \qquad (3.6.5)$$

In case $d = 2$ let R_{n-1}: $(n-1)_* C \approx_{\text{alg}} 2C + 2n_* C - (n+1)_* C$ be the relation obtained by applying $(\Phi^1_{1,n})_*$ to 3.4.1. Then $-R_n + R_{n-1}$ is the relation $\Gamma_n * (\Gamma_1 - \Gamma_0)^{*3} C \approx_{\text{alg}} 0$.

Next we look at the expansion of the π_i's in $\Gamma_1 - \Gamma_0$:

$$\pi_i = \frac{1}{(2g-i)!} (\log \Gamma_1)^{*(2g-i)}$$
$$= (\Gamma_1 - \Gamma_0)^{*(2g-i)} * (a_{2g-i}(\Gamma_1 - \Gamma_0)^{*0} + \cdots + a_{2g}(\Gamma_1 - \Gamma_0)^{*i}).$$

Therefore $\pi_i C \approx_{\text{alg}} 0$ if $2g - i \geqslant d + 1$ so $\pi_{2g-i} C \approx_{\text{alg}} 0$ if $i \geqslant d + 1$. Since $\pi_{2g} = \Gamma_0$ we also have $\pi_{2g} C = 0$, and from [B2], Proposition 3 we know $C_{(1)} = 0$ thus:

$$C \approx_{\text{alg}} C_{(2)} + C_{(3)} + \cdots + C_{(d)} \quad \text{with} \quad C_{(i)} := \pi_{2g-i} C$$

and we conclude that $\dim_{\mathbb{Q}}(Z_C/\approx_{\text{alg}}) \leqslant d - 1$. We can in fact obtain $C_{(1)} \approx_{\text{alg}} 0$ from 3.6.1, because a somewhat tedious computation shows that, for some $c \in \mathbb{Q}$:

$$\Gamma_{-1} \left(\frac{1}{d-1} P_0 - \frac{1}{d} P_{-1} \right) = c\Gamma_0 + \log \Gamma_1 \quad \text{mod} (\Gamma_1 - \Gamma_0)^{*(d+1)} * CH^g(A \times A),$$

here $\Gamma_{-1}\Gamma_n = \Gamma_{-n}$ is the product as correspondences, the equality is in (a quotient of) the ring $CH^g(A \times A)$ with $*$-product. Since Γ_0 acts trivially on one cycles and $P_n C \approx_{\text{alg}} 0$ the statement follows. In fact one shows that both sides are equal to, for some $c_1 \in \mathbb{Q}$:

$$\sum_{n=1}^{d} (-1)^{n-1} \frac{1}{n} \binom{d}{n} \Gamma_n + c_1 \Gamma_0.$$

(Comparing this expression with the one for $u_{d*} G_d$ in 3.4.2 we see that $\pi_{2g-1} C \approx_{\text{alg}} u_{d*} G_d \approx_{\text{alg}} 0$ since $G_d \cong \mathbb{P}^1$ maps to a point in $J(C)$, note that some care must be taken as we defined G_n only for $n < d$).

Finally we observe that since $n_* C_{(i)} = n^i C_{(i)}$, the determinant of the matrix expressing the $n_{i*}C$ in the $C_{(i)}$ is a Vandermonde determinant, which is nonzero. □

3.7. PROPOSITION. (1) *For a curve C with* $\pi_i C \approx_{\mathrm{alg}} 0$ *for* $i \leqslant 2g-4$ (*for example any curve with a* g^1_3) *we have, with* $C_{(i)} := \pi_{2g-i} C$:

$$C_{(2)} \approx_{\mathrm{alg}} \frac{1}{2}(C + C^-)$$

$$C_{(3)} \approx_{\mathrm{alg}} \frac{1}{2}(C - C^-)$$

Furthermore:

$$n_* C \approx_{\mathrm{alg}} \frac{n^3+n^2}{2} C - \frac{n^3-n^2}{2} C^-.$$

(2) *For a curve C with* $\pi_i C \approx_{\mathrm{alg}} 0$ *for* $i \leqslant 2g-5$ (*for example any curve with a* g^1_4) *we have:*

$$C_{(2)} \approx_{\mathrm{alg}} \frac{1}{12}(-2_* C + 12C + 4C^-)$$

$$C_{(3)} \approx_{\mathrm{alg}} \frac{1}{2}(C - C^-)$$

$$C_{(4)} \approx_{\mathrm{alg}} \frac{1}{12}(2_* C - 6C + 2C^-).$$

Proof. We give the proof of the second statement, the first being similar but easier. By assumption we have ($C_{(i)} := \pi_{2g-i} C$):

$$C \approx_{\mathrm{alg}} C_{(2)} + C_{(3)} + C_{(4)} \quad \text{with} \quad n_* C_{(i)} = n^i C_{(i)}.$$

Therefore:

$$\begin{aligned} C &\approx_{\mathrm{alg}} C_{(2)} + C_{(3)} + C_{(4)} \\ C^- &\approx_{\mathrm{alg}} C_{(2)} - C_{(3)} + C_{(4)} \\ 2_* C &\approx_{\mathrm{alg}} 4C_{(2)} + 8C_{(3)} + 16C_{(4)}. \end{aligned}$$

The result follows by straightforward linear algebra. □

3.8. Note that if one specializes a curve C with a g^1_4 to a trigonal curve, then actually $C_{(4)} = \frac{1}{12}(2_* C - 6C + 2C^-) \approx_{\text{alg}} 0$. However, we could not decide whether for a generic 4:1 cover of $\mathbb{P}^1$ we have $C_{(4)} \approx_{\text{alg}} 0$.

4. Appendix

4.1. We recall some facts on binomial coefficients. The binomial coefficients $\binom{n}{k}$ are defined (also for negative $n \in \mathbb{Z}$!) as (cf. [ACGH], VIII.3)

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} \quad (k>0), \binom{n}{0} := 1, \binom{n}{k} := 0 \quad (k<0).$$

With this definition, they are the coefficients in the expansion of $(1+x)^n$:

$$(1+x)^n = 1 + \binom{n}{1}x + \cdots + \binom{n}{k}x^k + \cdots = \sum_{k=0}^{\infty} \binom{n}{k} x^k.$$

Comparing the coefficients of x^k in $(1+x)^n(1+x)^m = (1+x)^{n+m}$ one finds, for all $n, m \in \mathbb{Z}$:

$$\sum_i \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}, \tag{4.1.1}$$

4.2. *Lemma. We have:*

$$\sum_{i=0}^{l} (-1)^i \binom{m}{l-i} = \binom{m-1}{l}, \tag{4.2.1}$$

$$\sum_{i=0}^{l} (-1)^{i-1} i \binom{m}{l-i} = \binom{m-2}{l-1}, \tag{4.2.2}$$

$$\sum_{i=0}^{l} (-1)^i i^2 \binom{m}{l-i} = \binom{m-3}{l-2} - \binom{m-3}{l-1}. \tag{4.2.3}$$

Proof. Using $\binom{-1}{k} = (-1)^k$ and 4.1.1, the first line can be written as:

$$\sum_{i=0}^{l} (-1)^i \binom{m}{l-i} = \sum_{i=0}^{l} \binom{-1}{i}\binom{m}{l-i} = \binom{m-1}{l}.$$

The second line follows in the same way, using $\binom{-2}{k-1} = (-1)^{k-1}k$:

$$\sum_{i=0}^{l} (-1)^{i-1} i \binom{m}{l-i} = \sum_{i=0}^{l} \binom{-2}{i-1}\binom{m}{l-i} = \binom{m-2}{l-1}.$$

For the last line we use $\binom{-3}{k} = (-1)^k(k+1)(k+2)/2$, so:

$$(-1)^k k^2 = 2\binom{-3}{k} - 3\binom{-2}{k} + \binom{-1}{k}.$$

Therefore:

$$\begin{aligned}
&\sum_{i=0}^{i} (-1)^i i^2 \binom{m}{l-1} \\
&= 2\sum_{i=0}^{l}\binom{-3}{i}\binom{m}{l-i} - 3\sum_{i=0}^{l}\binom{-2}{i}\binom{m}{l-i} + \sum_{i=0}^{l}\binom{-1}{i}\binom{m}{l-i} \\
&= 2\binom{m-3}{l} - 3\binom{m-2}{l} + \binom{m-1}{l}.
\end{aligned}$$

Now use that:

$$\binom{m-1}{l} = \binom{m-2}{l-1} + \binom{m-2}{l} = \binom{m-3}{l-2} + 2\binom{m-3}{l-1} + \binom{m-3}{l},$$

and

$$\binom{m-2}{l} = \binom{m-3}{l-1} + \binom{m-3}{l}.$$

□

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