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Abelian varieties in $W_d^r(C)$ and points of bounded degree on algebraic curves

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1. Introduction

The purpose of this work is to answer some questions raised by Abramovich and Harris in [AH] and [A1]. In particular, we give in 5.17 counterexamples to their main conjecture: for each $d \geqslant 4$, we construct a curve C defined over a number field K, that has infinitely many points p such that $[K(p):K] \leqslant d$, but that nevertheless admits no maps of degree d or less onto $\mathbf{P}^1$ or an elliptic curve. It was proved in [AH] that there are no such curves for $d = 2$ or 3, and no such curves of genus $\neq 7$ for $d = 4$. We give two different constructions: in the first one, the genus of C is $d(d-1)/2 + 1$, and C does have a morphism of degree $(d+1)$ onto an elliptic curve. In the second one, d is even $\geqslant 8$, the genus of C is $d^2/4 + 1$, and C has no morphisms onto a non-rational curve. For $d = nm$, with $n \geqslant 2$ and $m \geqslant 4$, there are examples with C of arbitrarily large genus.

As explained in [AH] and [A1], this problem is closely related to the study of abelian varieties in the loci $W_d(C)$ in the Jacobian of a curve C. We start off in this direction, by examining in section 3 the validity of the following statement from *loc. cit.* (suitably modified to avoid trivial counterexamples):

STATEMENT $A(d, h, g)$. *Let C be a complex projective curve of genus g, and assume that for some $d < g$, the locus $W_d(C)$ contains a maximal abelian variety A of dimension h. Then C is the image of a curve C' that admits a map of degree at most d/h onto a curve of genus h.*

This statement is easy to check for $h = 1$, and, when $g > d(d-1)/2 + 1$, holds for $d \leqslant 7$ or d prime ([A1], theorem 11). On the other hand, the Prym construction gives counterexamples to $A(2h, h, 2h+1)$ for any $h \geqslant 4$ (cf. remark 3.7). Note that the statement implies that $W_d(C)$ cannot contain an abelian variety of dimension $> d/2$ for $d < g$. We prove this in proposition 3.3, in a slightly more general form. Using ideas from [AH], we also prove statement

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$A(2h,h,g)$ for $g > 3h$ (corollary 3.6) and statement $A(d,h,g)$ for $h > d/4$ and $g > 6d$ (proposition 3.8). Cases where h is small with respect to d remain very much open.

In the next sections, which are independent from section 3, we study the following statement from [AH]:

STATEMENT $S(d, h, g)$. *Suppose $C' \to C$ is a surjective map of complex projective smooth curves with C of genus g. If C′ admits a map of degree d or less onto a curve of degree h or less, so does C.*

Abramovich proved in [A1] statements $S(2,h,g)$ for $g > 2h$ and $S(3,h,g)$ for $g > 3h+1$, and, with Harris ([AH]), statements $S(2,1,g)$, $S(3,1,g)$ and, for $g \neq 7$, statement $S(4,1,g)$. They also gave a counterexample to $S(3,2,5)$ in *loc. cit.* As explained in (5.16), this implies that, for any $n \geqslant 2$, statement $S(3n,2,g)$ does not hold for infinitely many values of g.

We give in (5.5) counterexamples to $S(d,1,d(d-1)/2+1)$ for any $d \geqslant 4$, and to $S(d,2,d^2/4+1)$ for d even $\geqslant 8$. This disproves in particular $S(4,1,7)$, the missing case in [AH]. It follows again that for any $n \geqslant 2$, $d \geqslant 4$, statements $S(nd,1,g)$ and $S(2nd,2,g)$ do not hold for infinitely many values of g.

A word of warning about [A1] and [AH]: those articles contain incomplete proofs which were later amended in [A2]. However, there are still some gaps, and lemma 6, the second part of lemma 8 and corollary 1 in [AH], as well as the corresponding statements in [A1], should be considered unproved at the moment. Theorem 2 of [AH], although its proof relied on those statements, has been since proved in a different way by Abramovich (with the extra hypothesis added in [A2]). We will quote it here, although our results do not depend on it.

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2. Notation

Unless otherwise specified, the ground field is the field of complex numbers. Let C be a smooth (connected projective algebraic) curve. For any integer d, we write $\mathrm{Pic}^d(C)$ for the scheme parametrizing isomorphism classes of line bundles of degree d on C, and $J(C)$, the Jacobian of C, for $\mathrm{Pic}^0(C)$. For any point z in $\mathrm{Pic}^d(C)$, we write L_z for a line bundle of degree d on C associated to z. For any non-negative integers d and r, we set $W^r_d(C) = \{z \in \mathrm{Pic}^d(C) | h^0(C, L_z) > r\}$, endowed with its usual scheme structure, and $W_d(C) = W^0_d(C)$.

3. Abelian varieties in $W^r_d(C)$

Let C be a smooth complex curve. We show that any abelian variety contained in $W^r_d(C)$ has dimension $\leqslant d/2 - r$, and study when equality holds.

LEMMA 3.1. *Let C be a smooth curve of genus g and let Θ be a theta divisor of $J(C)$. Assume that Θ contains a subvariety Z stable by translation by an abelian subvariety A of $J(C)$. Then:*

$$\dim(Z) + \dim(A) \leqslant g - 1.$$

Proof. We may assume Z to be irreducible. Moreover, replacing Z by $Z - W_r(C) + W_r(C)$, where $(r + 1)$ is the multiplicity on Θ of a generic point of Z, we may assume that Z meets the set Θ_{reg} of smooth points of Θ. Let $G: \Theta_{\text{reg}} \to \mathbf{P}T^*_0 J(C)$ be the Gauss map. For any point x of $Z \cap \Theta_{\text{reg}}$, the hyperplane $T_x\Theta$ of $T_x J(C)$ contains $x + T_0 A$, hence $G(Z \cap \Theta_{\text{reg}}) \subset \mathbf{P}T^*_0(J(C)/A)$. The conclusion follows from the fact that on a Jacobian, the map G has *finite fibers* ([ACGH], p. 246). □

REMARK 3.2. Lemma 3.1 does not hold in a general abelian variety of dimension $\geqslant 4$: there are abelian varieties of any given dimension such that their theta divisor contains an abelian subvariety as a divisor.

PROPOSITION 3.3. *Let C be a smooth curve of genus g such that $W^r_d(C)$ contains a subvariety Z stable by translation by an abelian subvariety A of $J(C)$. Then, if $d \leqslant g - 1 + r$, one has:*

$$\dim(Z) + \dim(A) \leqslant d - 2r.$$

Proof. Apply lemma 3.1 to the subvariety $Z - W_r(C) + W_{g-1-d+r}(C)$ of $W_{g-1}(C)$ (isomorphic to Θ). One gets:

$$\dim(Z) + r + g - 1 - d + r + \dim(A) \leqslant g - 1.$$ □

The next proposition shows exactly when there is equality in proposition 3.3, under a stronger assumption on d. We begin with a lemma.

LEMMA 3.4. *Let C be a smooth curve of genus g such that $W_d(C)$ contains a subvariety Z stable by translation by a non-zero abelian subvariety A of $J(C)$. Assume that:*

$$\dim(Z) + \dim(A) = d.$$

Then, if $d + \dim(Z) \leqslant g - 1$, there exist a curve B of genus $h = \dim(A)$ and a morphism $p: C \to B$ of degree 2 such that $A = p^*J(B)$ and $Z = p^*\mathrm{Pic}^h(B) + W_{d-2h}(C)$.

Proof. We follow ideas from [AH]. Let Z_2 be the image of Z under the addition map $W_d(C) \times W_d(C) \to W_{2d}(C)$. As in lemma 1 of [AH], the maximal integer r such that Z_2 is contained in $W^r_{2d}(C)$ satisfies $r \geqslant h$ and

$2\dim(Z) - r \leqslant \dim(Z_2)$. Since $2d \leqslant g - 1 + h$, proposition 3.3 applies to Z_2 and gives $\dim(Z_2) + h \leqslant 2d - 2r$. It follows that $r = h$.

Proposition 3.3 implies that Z is not contained in $W^1_d(C)$, hence, for z generic, we may write D_z for the unique element of the linear system $|L_z|$. For z and z' generic in Z, and u generic in A, the divisor $D_{z+u} + D_{z'-u}$ is in $|L_z \otimes L_{z'}|$. Since $\dim(A) = \dim|L_z \otimes L_{z'}|$, we get a generic divisor of $|L_z \otimes L_{z'}|$ in this way. Let P_z be the greatest common divisor of the D_{z+u}'s as u varies in A. The fixed part of $|L_z \otimes L_{z'}|$ is then $P_z + P_{z'}$. Write $E_z = D_z - P_z$ and $e = \deg(E_z)$. The map that sends u to E_{z+u} induces an embedding of A into $W_e(C)$ with image A_z. Let $\phi_{z,z'}: C \to \mathbf{P}^h$ be the morphism associated with $|E_z + E_{z'}|$. The rational map from A onto $|E_z + E_{z'}|$ that takes u to $E_{z+u} + E_{z'-u}$ factorizes through the quotient of A by the involution $\iota(u) = z' - z - u$.

Assume first that the resulting rational map $\alpha: A/\iota \dashrightarrow |E_z + E_{z'}|$ has degree 1. Since A/ι is not rational for $h > 1$, we have $h = 1$ and a general point of C appears in a single E_{z+u}. It follows that $\phi_{z,z'}$ factors through a morphism $p: C \to A$ of degree e.

Assume now that α has degree > 1. For u generic in A, there exists v in A such that $E_{z+u} + E_{z'-u} = E_{z+v} + E_{z'-v}$ and $z + v$ is different from $z + u$ and from $z' - u$. Therefore, fixing z, z' and u, for w generic in A, there exists $v(w)$ in A such that $E_{z+u} + E_{z'+w-u} = E_{z+v(w)} + E_{z'+w-v(w)}$. It follows that E_{z+u} decomposes as $E'_{z+u} + E''_{z+u}$, with $0 < E'_{z+u} < E_{z+v(w)}$ and $0 < E''_{z+u} < E_{z'+w-v(w)}$. Let A' be the closure of $\{E_{z+v(w)} - E'_{z+u} \mid w \in A\}$, let A'' be the closure of $\{E_{z'+w-v(w)} - E''_{z+u} \mid w \in A\}$, and let h' (resp. h'') be the dimension of A' (resp. A''). For any $D' = E_{z+v(w)} - E'_{z+u}$ in A' and $D'' = E_{z'+w'-v(w')} - E''_{z+u}$ in A'', we have:

$$D' + D'' = E_{z'+w-u} - E_{z'+w-v(w)} + E_{z'+w'-v(w')} \equiv E_{z'+w'+v(w)-v(w')-u},$$

hence $A_{z'}$ is the image of $A' \times A''$ by the addition map. This implies $h = h' + h''$, since $A_{z'}$ is not contained in $W^1_e(C)$. Furthermore, we have $h^0(E_z + E_{z'} - D' - D'') > 0$ for any D'' in A'', hence $h^0(E_z + E_{z'} - D') > h'' = h - h'$. It follows that the elements of A' form an h'-dimensional family of divisors, whose images by $\phi_{z,z'}$ each span at most an $(h' - 1)$-plane. By lemma 4 of [AH], either $\phi_{z,z'}$ is not birational, or the elements of A' have degree $\leq h'$, hence $A' = C^{(h')}$, and similarly $A'' = C^{(h'')}$. In that case, $A = W_h(C)$ hence $h \geqslant g$ since A is an abelian variety. This contradicts the hypothesis.

Therefore $\phi_{z,z'}$ is not birational for generic z and z'. Let B be the normalization of its image. If B is rational, since the linear series that defines $\phi_{z,z'}$ is complete, the image of $\phi_{z,z'}$ is a rational normal curve in $\mathbf{P}^h$. Any $E_{z+u} + E_{z'-u}$, hence also any $E_{z+u} + E_{z'}$, is then h times an element of $W^1_{2e/h}(C)$. This yields an embedding of A into $W^1_{2e/h}(C)$. The pull-back $\tilde{A}$ in $C^{(2e/h)}$ of the image of this embedding has dimension $\geqslant h + 1$, the image of $\tilde{A}^h$ in $C^{(2e)}$ has dimension $\geqslant h(h + 1)$ and dominates $A_z + E_{z'}$, which is in $W^h_{2e}(C)$, but not in $W^{h+1}_{2e}(C)$ for z and z' generic. Hence $h \geqslant (h + 1)h - h = h^2$ and $h = 1$. This contradicts $h = h' + h'' > 1$.

It follows that whatever the degree of α, the morphisms $\phi_{z,z'}$ factor through a fixed morphism $p: C \to B$ of degree $n > 1$, where B is a non-rational curve. As in lemma 3 of [AH], the E_{z+u}'s are pullbacks of divisors on B, hence A embeds into $p^*W_{e/n}(B)$. It follows that $Z \subset W_{d-e}(C) + p^*W_{e/n}(B)$. Since Z is $(d-h)$-dimensional, we get $d - h \leqslant d - e + e/n$. We know that A embeds into $W_e(C)$, hence $h \leqslant e/2$ by proposition 3.3. It follows that $n = 2$ and $h = e/2$, that the above inclusion is an equality and that $A = p^*W_h(B)$. In particular, the genus of B is h and the lemma is proved. □

PROPOSITION 3.5. *Let C be a smooth curve of genus g such that $W_d^r(C)$ contains a subvariety Z stable by translation by a non-zero abelian subvariety A of $J(C)$. Assume that:*

$$\dim(Z) + \dim(A) = d - 2r.$$

Then, if $d + \dim(Z) \leqslant g - 1$, there exist a curve B of genus $h = \dim(A)$ and a morphism $p: C \to B$ of degree 2 such that $A = p^*J(B)$ and

$$Z = p^* \operatorname{Pic}^{h+r}(B) + W_{d-2r-2h}(C).$$

Proof. The subvariety $Z' = Z - W_r(C)$ of $W_{d-r}(C)$ is stable by translation by A and satisfies $\dim(Z') + \dim(A) = d - r$. Since $(d - r) + \dim(Z') \leqslant g - 1$, one can apply lemma 3.4 to Z'. Therefore, there exist a curve B of genus h and a morphism $p: C \to B$ of degree 2 such that $Z' = p^* \operatorname{Pic}^h(B) + W_{d-r-2h}(C)$. It follows that the linear system associated to any point of Z contains an effective divisor of the form $p^*D + E$, where E does not contain any fiber of p and $\deg(D) \geqslant h$. Let $2s$ be the number of ramification points of p. One has:

$$\begin{aligned}\deg(D + p_*E) - s &= d - \deg(D) - (g - (2h - 1)) \\ &\leqslant g - 1 - \dim(Z) - h - g + 2h - 1 \\ &= h - \dim(Z) - 2 < 0.\end{aligned}$$

It follows from [Mu] that $h^0(p^*D + E) = h^0(D) > r$, since Z is contained in $W_d^r(C)$. But Z is stable by translation by $A = p^*J(B)$, hence $\deg(D) \geqslant h + r$. It follows that $Z \subset p^* \operatorname{Pic}^{h+r}(B) + W_{d-2r-2h}(C)$. Since both sets have the same dimension, they are equal. □

The following immediate consequence of proposition 3.5 proves a stronger form of the statement $A(2h, h, g)$ mentioned in the introduction, for $g > 3h$.

COROLLARY 3.6. *Let C be a smooth curve of genus g such that $W_d^r(C)$ contains an abelian variety A. Assume that $d \leqslant g - 1 + r$. Then $\dim(A) \leqslant d/2 - r$. When $d \leqslant 2/3(g - 1 + r)$, equality holds if and only if d is even and there exist a curve B of genus $(d/2 - r)$ and a morphism $p: C \to B$ of degree 2 such that $A = p^* \operatorname{Pic}^{d/2}(B)$.*

REMARK 3.7. The Prym construction gives counterexamples to $A(2h, h, 2h + 1)$ for $h \geqslant 4$, hence *a fortiori* to the second part of the proposition when $d = g - 1$ is even and $r = 0$: let D be a genus-$(h+1)$ curve and let $\pi: C \to D$ be an étale covering of degree 2. The genus of C is $g = 2h + 1$ and $W_{g-1}(C)$ contains a copy of the Prym variety A of π, an abelian variety of dimension h. We claim that for D general and $h \geqslant 4$, there does not exist a diagram:

$$\begin{array}{ccc} C' & \xrightarrow{q} & C \\ {\scriptstyle p}\big\downarrow & & \\ B & & \end{array},$$

with q onto, p of degree 2 and B of genus h, hence contradicting $A(2h, h, 2h + 1)$. Assume such a diagram exists. Then $q(p^*J(B))$ is an abelian subvariety of $J(C)$ of dimension $\leqslant h$. But $J(C)$ is isogeneous to $A \times J(D)$. For D general, both A and $J(D)$ are simple and, when $h \geqslant 4$, the abelian variety A is not isogeneous to a Jacobian. It follows that $q(p^*J(B))$ must be a point, which is clearly impossible. One can also show that the construction in section 5 of [AH] gives counterexamples to $A(4, 2, 5)$.

The following proposition proves the statement $A(d, h, g)$ for $h > d/4$ and $g > 6d$.

PROPOSITION 3.8. *Let C be a smooth curve of genus g. Let d be an integer and suppose that $W_d(C)$ contains an abelian variety A, assumed to be maximal, of dimension $h > d/4$. Then, if $g > 6d$, there exist a curve B of genus h, a morphism $p: C \to B$ of degree $n = 2$ or 3 and a point D of $W_{d-hn}(C)$, such that $A = D + p^* \operatorname{Pic}^h(B)$.*

REMARK 3.9. The proposition also holds for $g > d(d - 1)/2 + 1$ (use theorem 2 of [AH]). This bound is better for small values of d.

Proof of Proposition 3.8. Since the case $h = 1$, $d \leqslant 3$ was treated in [AH], we will assume $h > 1$. Subtracting if necessary from A the sum of sufficiently many points of C, we may assume that A is not contained in $W^1_d(C)$. Subtracting then the common fixed parts of the linear systems corresponding to the points of A, we may also assume that A is not contained in any $x + W_{d-1}(C)$. These operations only make d smaller, so that the inequalities $h > d/4$ and $g > 6d$ are still valid.

First, we make the extra assumption that A is not contained in the big diagonal of $W_d(C)$, so that we can apply the results of [A1] and [AH].

For any positive integer n, let A_n be the image of A under the addition map $W_d(C) \times \cdots \times W_d(C) \to W_{nd}(C)$. Let $r(n)$ be the maximal integer such that A_n is contained in $W^{r(n)}_{nd}(C)$. Assume that the morphism $C \to \mathbf{P}^{r(2)}$ associated to a generic element of A_2 is birational. Then the same holds for the morphisms associated to a generic element of A_n for any $n \geqslant 2$. Lemma 5 of [AH] gives $r(2) \geqslant h + 1$. We need the following result from [ACGH]:

LEMMA 3.10. *Let r and d be two integers with* $d \leqslant g + 1$ *and let L be a base-point-free* g^r_d *on C such that the morphism* $C \to \mathbf{P}^r$ *associated to L is birational. Then the dimension of* $W^r_d(C)$ *at the point corresponding to L is less than or equal to* $h^0(L^2) - 3r$. *If* $d < g$ *and* $L^2 \neq K_C$, *this dimension is also less than or equal to* $d - 3r$.

Since $2d \leqslant g + 1$, the lemma yields $h \leqslant r(4) + 1 - 3r(2)$, hence $r(4) \geqslant 4h + 2$. The first part of lemma 8 from [AH] gives $r(6) \geqslant r(4) + \min(r(4), 2d)$. Using proposition 3.3, we get $r(6) \geqslant 8h + 2$. Since $6d < g$, we can apply the second part of lemma 3.10 to a generic point in A_6, to get $h \leqslant 6d - 3r(6)$, hence $h \leqslant (6d - 6)/25 \leqslant d/4$, which contradicts the hypothesis.

Therefore, the morphism associated to a generic element of A_2 is not birational. Since $h > 1$, lemma 14 of [A1] implies that there exist a curve B and a morphism $p: C \to B$ of degree $n \geqslant 2$ such that n divides d and $A \subset p^* W_{d/n}(B)$. Since $h > d/4 \geqslant d/2n$, corollary 3.6 implies $d/n \geqslant g(B)$. Therefore, $p^* \operatorname{Pic}^{d/n}(B)$ is contained into $W_d(C)$. Since A is maximal in $W_d(C)$, it is equal to $p^* \operatorname{Pic}^{d/n}(B)$ and $h = g(B) \leqslant d/n$. Since A is not contained in $W^1_d(C)$, one has $h = d/n$. This finishes the proof of the proposition in that case.

If all points of A have multiplicities, one can remove them. The first part of the proof then shows that $A = mp^* \operatorname{Pic}^{d/2m}(B)$, for some integer $m \geqslant 2$. But A then embeds into $W_{d/m}(C)$, and that contradicts proposition 3.3 since $h > d/4$. Thus, this case does not occur and the proposition is proved. □

4. Two constructions

(4.1) Let E be a complex elliptic curve and let $E^{(2)}$ be its second symmetric product. Let $p: E \times E \to E$ be the first projection, let $q: E \times E \to E^{(2)}$ be the quotient map and let $s: E^{(2)} \to E$ be the sum map.

We fix a point $\mathbf{o}$ on E, making E into a commutative group with unit $\mathbf{o}$. To avoid confusion between addition of divisors and addition of points of E, we will write (x) for the divisor defined by a point x of E. There exists a unique locally free rank 2 sheaf $\mathscr{E}$ on E that is a non-trivial extension:

$$0 \to \mathcal{O} \to \mathscr{E} \to \mathcal{O}((\mathbf{o})) \to 0.$$

The sheaf $\mathscr{E}$ defines a $\mathbf{P}^1$-bundle $\mathbf{P}\mathscr{E} \to E$ and an invertible sheaf $\mathcal{O}(1)$ on $\mathbf{P}\mathscr{E}$.

There exists a commutative diagram:

$$\begin{array}{ccc} E^{(2)} & \xrightarrow{u} & \mathbf{P}\mathscr{E} \\ & {\scriptstyle s}\searrow \quad \nearrow & \\ & E & \end{array},$$

where u is an isomorphism. Furthermore, $u^*\mathcal{O}(1)$ is isomorphic to $\mathcal{O}(H)$, where H is the unique element of the linear system $|q_* p^*(\mathbf{o})|$. For any point x of E, we write H_x for the only element of the linear system $|q_* p^*(x)|$, we write F_x for the fiber $s^{-1}(x)$ and C_x for the curve $\{(y) + (y + x) | y \in E\}$ in $E^{(2)}$. Finally, let $E[2]'$ be the set of non-zero points of order two of E. The following facts are classical or elementary:

(i) the Picard group of $E^{(2)}$ is isomorphic to $\mathbf{Z}[H] \oplus s^* \operatorname{Pic}(E)$.
(ii) the curve H_x is linearly equivalent to $H + F_x - F_{\mathbf{o}}$.
(iii) the linear system $|4H - F - F_x|$ is empty when $2x \neq 0$, and is a pencil if and only if $x = 0$.
(iv) when $x \in E[2]'$, the curve C_x is the only element of $|2H - F_x|$; when $x \notin E[2]'$, the linear system $|2H - F_x|$ is empty.

PROPOSITION 4.2. *Let x be a point on E. For $n > 3$, the linear system $|nH - F_x|$ is base-point-free and has projective dimension $(n - 2)(n + 1)/2$. It is very ample for $n > 4$. The linear system $|3H - F_x|$ is a pencil with three distinct simple base points, hence contains a smooth irreducible curve.*

Proof. For any point ε of $E[2]'$, the linear system $|3H - F_x|$ has degree 1 on the elliptic curve C_ε (cf. fact (iv) above). It follows that it has at least one base point on this curve. Using fact (iv) again, it is easy to see, by restricting to the curve H_p, that the linear system $|3H - F_x|$ has no base point on H_p if $x + p$ does not belong to $E[2]'$. Hence the base points of the linear system $|3H - F_x|$ are $(\varepsilon - x) + (\varepsilon' - x)$, for any ε and ε' distinct in $E[2]'$. They are simple since $(3H - F_x)^2 = 3$.

The rest of the proposition follows easily from Reider's main theorem ([R1]). □

It follows from proposition 4.2 that for $d \geqslant 2$ and for any point x of E, the linear system $|(d + 1)H - F_x|$ contains a smooth irreducible curve C, whose genus is $d(d - 1)/2 + 1$.

Since $d > 1$, the curve H_x is not contained in C and sending a point x of E to the class of the divisor $H_x.C$ defines a morphism ψ from E into $C^{(d)}$. This morphism has the property that *it is not induced by a morphism from C to E*. In fact, let x be any point of E and let $a_i = x + x_i$, $i = 1, \ldots, d$ be the d points of the support of the divisor $\psi(x)$. Then $\psi(x)$ and $\psi(x_1)$ have a point in common, to wit a_1. Since x and x_1 are distinct in general, ψ cannot be induced by a morphism.

Let ϕ be the morphism $E \to W_d(C)$ induced by ψ. Since C is ample on $E^{(2)}$, the restriction map $\mathrm{Pic}^0(E^{(2)}) \to \mathrm{Pic}^0(C)$ is injective, hence so is ϕ. Note that s induces a morphism from C onto E of degree $(d + 1)$ and that the induced morphism from E into $W_{d+1}(C)$ is a translate of ϕ.

We will use this construction in section 5 to illustrate and complement some points of [AH].

(4.3) For the second construction, we consider a smooth genus-2 curve B, its Jacobian $(J(B), \Theta)$ and a smooth curve C in $|e\Theta|$ $(e \geqslant 2)$. We will always assume Θ to be symmetric. Sending a point a of $J(B)$ to the divisor $(\Theta + a)$. C defines a morphism ψ from $J(B)$ into $C^{(2e)}$, which again is not induced by a morphism from C to B. Indeed, if $\psi(a) = x_1 + \cdots + x_{2e}$, then $x_1 - a$, hence also $a - x_1$, are in Θ. It follows that the divisors $\psi(a)$ and $\psi(a + 2x_1)$ have a point in common, although a and $a + 2x_1$ are distinct in general. We will denote by ϕ the morphism $J(B) \to W_{2e}(C)$ induced by ψ. The induced map $\mathrm{Pic}^0(J(B)) \to \mathrm{Pic}^0(C)$ being injective, so is ϕ.

5. Discussion of some results from [AH]

(5.1) The first item we want to discuss is theorem 2 in [AH]. Let C be a smooth curve such that $W_d(C)$ contains an abelian variety A. As before, let A_2 be the subset of $W_{2d}(C)$ which consists of the sums of any two elements of A. This theorem says that if the morphism associated to a general point of A_2 is *birational* onto its image, then $g(C) \leqslant d(d - 1)/2 + 1$. If A is an elliptic curve, one has to assume further that the inclusion of A in $W_d(C)$ does not come from a morphism (as mentioned in [A2]). We show that this bound is sharp when A is an elliptic curve. With the notation of (4.1), for any smooth curve C in $|dH|$, the scheme $W_d(C)$ contains a copy of the elliptic curve E, and elements of E_2 induce the linear systems $|H + H_x|$ on C.

PROPOSITION 5.2. *Let $d \geqslant 3$. A generic curve C in $|dH|$ has genus $d(d - 1)/2 + 1$, and the morphism κ induced by $|2H|$ on C is birational.*

REMARKS 5.3. (a) With the notation of the proof of proposition 3.8, one has $r(k) = k(k + 1)/2 - 1$ for $k \leqslant d$.

(2) The proposition also holds for a generic curve in $|(d + 1)H - F_x|$, for $d \geqslant 3$. This gives another example for which the bound in theorem 2 [AH] is sharp.

Proof of proposition 5.2. It is enough to find a divisor D in $|dH|$ and a component D' of D such that D is generically reduced on D', the restriction of κ to D' is birational onto its image and $\kappa(D - D')$ does not contain $\kappa(D')$. Note that $\kappa(H_x)$ (resp. $\kappa(C_\varepsilon)$) is a line for any point x of E (resp. any point ε of $E[2]'$). On the other hand, for x not in $E[2]'$, the restriction of κ to F_x is birational onto a smooth conic. Pick a point ε in $E[2]'$ and set:

$$D = C_\varepsilon + (d-3)H + H_\varepsilon + F_0 \quad \text{and} \quad D' = F_0.$$

The curve $\kappa(D')$ is the only smooth conic of $\kappa(D)$ and the restriction of κ to D' is birational onto its image. This finishes the proof of the proposition. □

What happens when the abelian variety A contained in $W_d(C)$ is not an elliptic curve? It is likely that the bound on the genus of C from [AH], theorem 2, is not sharp in that case and that there should be a better bound involving the dimension of A. Here is an example where A is a surface, and for which we think that the genus of C is maximal.

PROPOSITION 5.4. *Let $d = 2e \geqslant 6$. A generic curve C in $|e\Theta|$ has genus $d^2/4 + 1$, and the morphism κ induced by $|2\Theta|$ on C is birational.*

Proof. It is enough to find one element D of $|e\Theta|$ that is not invariant under the involution of A that takes x to $-x$. Take any 3 non-zero points x, y and z on A such that $x + y + z = 0$ and set $D = (e-3)\Theta + \Theta_x + \Theta_y + \Theta_z$. □

(5.5) We will now give counterexamples to some of the statements $S(d, 1, g)$ and $S(d, 2, g)$ from [AH] mentioned in the introduction.

Keeping the notation of (4.1), let C be a smooth curve in $|(d+1)H - F_x|$ and let C' be its inverse image in $E \times E$. Then the degree of either projections from C' onto E is d. We want to show that for $d \geqslant 4$, the curve C has no morphisms of degree d or less onto rational or elliptic curves, contradicting $S(d, 1, g(C))$. We first deal with pencils on C, using the following result from [R2] (corollary 1.40, proposition 2.10 and remark 2.11.1; our D is his E_1):

THEOREM 5.6. (I. Reider). *Let L be a nef line bundle on a smooth projective surface S and let C be a smooth curve in $|L|$. Assume that C has a base-point-free pencil of degree $d < L^2/4$. Then, there exists a divisor D on S such that:*

(i) $h^0(S, D) \geqslant 2$.
(ii) $C \cdot D < 2d$.
(iii) $(C - D) \cdot D \leqslant d$.

We prove:

PROPOSITION 5.7. *Let $d \geqslant 4$ and let x be a point on E. Then, a general curve in $|(d+1)H - F_x|$ has no pencils of degree d or less.*

REMARK 5.8. The same conclusion holds for smooth curves in $|(d+m)H - s^*D|$, where D is a divisor of degree m on E, and $d \geqslant 4$ and $0 < m < d/2$.

Proof of proposition 5.7. Let C be a smooth curve in $|(d+1)H - F_x|$ and assume it has a base-point-free pencil M of degree $d' \leqslant d$.

We first assume $d \geqslant 5$, from which it follows that $C^2 = d^2 - 1 > 4d \geqslant 4d'$. Theorem 5.6 then implies that there exists a divisor D on $E^{(2)}$ such that:

$$h^0(E^{(2)}, D) \geqslant 2 \tag{5.9}$$

$$C \cdot D < 2d' \leqslant 2d. \tag{5.10}$$

$$(C - D) \cdot D \leqslant d' \leqslant d. \tag{5.11}$$

Write $aH - bF$ for the numerical equivalence class of D. We get from (5.10):

$$2d \geqslant 2d' > C \cdot D = ad - b(d+1)$$

hence $(a - b)d < 2d + b$.

Note also that since $|D|$ is non-empty, one has:

$$0 \leqslant D \cdot F = a$$

$$0 \leqslant D \cdot C_\varepsilon = a - 2b \quad (\text{since } C_\varepsilon^2 = 0).$$

Case 1: $b < 0$. Then $0 < (a - b)d < 2d$ hence $a - b = 1$. Since $a \geqslant 0$, the only possibility is $D \sim F$, which contradicts (5.9).

Case 2: $b \geqslant 0$. Then $(a - b)d < 2d + bd/2$ hence $(a - 2b) + b/2 < 2$. We have:

either $a = 2b$. Then $(C - D) \cdot D = b(d - 1)$ and (5.11) implies $b = 1$, which contradicts (5.9),

or $a = 2b + 1$, in which case $(C - D) \cdot D = b(d - 3) + d - 1$ and (5.11), plus our assumption that $d \geqslant 5$, imply $b = 0$, which contradicts (5.9).

Note that C does not need to be general in the above argument.

We now turn to the case $d = 4$. The above method gives $d' = 4$. As in [R2] section 2, there exist a rank 2 vector bundle T and a zero cycle Z of degree 4 on $E^{(2)}$, that fit into the following exact sequences (where $\mathscr{I}_Z$ is the ideal sheaf of Z):

$$\begin{array}{c}
0 \\
\downarrow \\
\mathcal{O}_{E^{(2)}} \oplus \mathcal{O}_{E^{(2)}} \\
\downarrow \\
0 \to \mathcal{O}_{E^{(2)}} \to T \to \mathscr{I}_Z(C) \to 0 \\
\downarrow \\
\mathcal{O}_C(C - M) \\
\downarrow \\
0
\end{array} \tag{5.12}$$

Since Z has degree 4, proposition 4.2 gives $h^0(E^{(2)}, \mathscr{I}_Z(C)) \geqslant 9-4=5$. and the horizontal exact sequence gives $h^0(E^{(2)}, T) \geqslant 5+1-h^1(E^{(2)}, \mathcal{O}_{E^{(2)}}) = 5$. Since $h^0(E^{(2)}, \Lambda^2 T) = h^0(E^{(2)}, C) = 9$, there exist two independent sections s and t of T such that $s \wedge t = 0$. Let D be the largest effective (or zero) divisor along which s vanishes. The induced map $\mathcal{O}_{E^{(2)}}(D) \to T$ vanishes on a finite (or empty) subscheme Z' of $E^{(2)}$, and, as in (2.12) in [GL], one gets an exact sequence:

$$0 \to \mathcal{O}_{E^{(2)}}(D) \to T \to \mathcal{O}_{E^{(2)}}(C-D) \to \mathcal{O}_{Z'}(C-D) \to 0 \tag{5.13}$$

It follows that:

$$0 \to H^0(E^{(2)}, D) \to H^0(E^{(2)}, T) \to G^0(E^{(2)}, \mathscr{I}_{Z'}(C-D))$$

is exact, where the rightmost map is given by $u \to s \wedge u$. Both s and t are in its kernel, hence $h^0(E^{(2)}, D) \geqslant 2$. On the other hand, by tensoring the vertical sequence in (5.12) by $\mathcal{O}_{E^{(2)}}(-D)$, we see that $h^0(C, C-M-D) \geqslant 1$.

Finally, since the second Chern class of T is 4 by (5.12), exact sequence (5.13) and formula (0.3) in [GL] give $D.(C-D) \leqslant 4$. A case by case analysis shows that there are only two cases compatible with the 3 inequalities $h^0(E^{(2)}, D) \geqslant 2$, $h^0(C, C-M-D) \geqslant 1$ and $D.(C-D) \leqslant 4$, which are $D \sim 2H$ and $D \sim 3H - F$.

In the first case, $C - D \equiv 3H - F_y$ is, by proposition 4.2, a pencil on $E^{(2)}$ with 3 distinct base points a_y, b_y and c_y. The linear system $|5H - F_x|$ is very amp'e on $E^{(2)}$ (proposition 4.2). Therefore, the set of curves C that contain these three points has codimension $\geqslant 2$. It follows that a general C does not contain the whole set $\{a_y, b_y, c_y\}$ for any y. In that case, $|C-D|$ restricts to a pencil on C whose moving part has degree >4. Since $h^0(C, C-M-D) \geqslant 1$, this moving part must be M, which is a contradiction.

In the second case, $C - D \equiv H + H_y$ has no base point and induces a 4:1 morphism κ_y onto $\mathbf{P}^2$, which maps C birationally (proposition 5.2) onto a curve of degree 8. The pencil M must therefore be given by $C - D - G$, where G is a fiber of κ_y contained in C. Let ε be an element of $E[2]'$. The restriction of κ_y to C_ε is 2:1 onto a line. The image of C_ε under the map $\phi\colon E^{(2)} \to \mathbf{P}^8$ associated with $|5H - F_x|$ is a cubic contained in a plane. The projection from this plane induces the embedding $E^{(2)} \to \mathbf{P}^5$ associated with the very ample linear system $|2H + H_{\varepsilon - x}|$. Therefore, the projective span of any four points of $\phi(E^{(2)})$, such that two are on $\phi(C_\varepsilon)$, has dimension 3. In particular, the projective span of the image under ϕ of any fiber of any κ_y over any point of $\kappa_y(C_\varepsilon)$, has dimension 3. Hence, in the 3-dimensional space of all fibers of the κ_y's, those whose image under ϕ does *not* span a $\mathbf{P}^3$ has dimension $\leqslant 1$. It follows that a general curve C in $|5H - F_x|$ does not contain any fiber of any κ_y, hence cannot have a pencil of degree 4.

This finishes the proof of the proposition. □

We now turn to morphisms onto elliptic curves.

PROPOSITION 5.14. *Let $d \geqslant 4$ and let x be a point on E. Then, a general curve in $|(d+1)H - F_x|$ does not have a morphism of degree d or less onto an elliptic curve.*

Proof. Let C be a general curve in $|(d+1)H - F_x|$. It follows from [M1], corollary 5.2, which can be applied thanks to proposition 4.2, that the endomorphism ring of $J(C)/E$ is isomorphic to $\mathbf{Z}$. It follows that $J(C)/E$ does not contain any elliptic curve, hence that any morphism from C onto an elliptic curve must factor through the degree-$(d+1)$ restriction of p to C. This proves the proposition. □

We now consider the construction in (4.3) and set $A = J(B)$. Note that there exists a map $B \times B \to J(B)$ that is finite of degree two on the inverse image C' of C. The degree of either projection from C' onto B is $2e$. It turns out that for $e \geqslant 4$ and sufficiently general B, the curve C itself has no morphisms of degree $2e$ or less onto a curve of genus 2 or less, thereby contradicting $S(2e, 2, g(C))$. More precisely, we have:

PROPOSITION 5.15. *Let (A, Θ) be a principally polarized abelian surface whose Néron-Severi group has rank* 1 *and let C be a general curve in $|e\Theta|$. Then, if $e \geqslant 4$, the curve C has no pencils of degree $2e$ or less and no morphisms onto non-rational curves.*

Proof. We first rule out the existence of pencils of degree $\leqslant 2e$. Assume C has a base-point-free pencil M of degree $d' \leqslant 2e$. Suppose first that $e > 4$. We have $C^2 = 2e^2 > 8e \geqslant 4d'$ hence, by theorem 5.6, there exists a divisor D on A such that:

$$h^0(A, D) \geqslant 2$$
$$C \cdot D < 2d' \leqslant 4e.$$

If $a\Theta$ is the numerical equivalence class of D, we get the contradiction $a \geqslant 2$ and $a < 2$. Suppose now $e = 4$. The same argument rules out the existence of pencils of degree <8, so we have $d' = 8$. We follow the proof of proposition 5.7, keeping its notation. We have $h^0(A, \mathscr{I}_Z(C)) \geqslant 16 - d' = 8$ and $h^0(A, T) \geqslant 8 + 1 - h^1(A, \mathscr{O}_A) = 7$. Since $h^0(A, \Lambda^2 T) = h^0(A, 4\Theta) = 16$, there exist two independent sections s and t of T such that $s \wedge t = 0$. Again, there exists a divisor D in A such that $h^0(A, D) \geqslant 2$ and $h^0(C, C - M - D) \geqslant 1$, from which follows that $D \equiv \Theta + \Theta_a$ for a point a in A. Let N be a (degree 8) element of $|(C - D)_{|C} - M|$ and let $\kappa\colon A \to \mathbf{P}^3$ be the map associated with the linear system $|\Theta + \Theta_{-a}|$. Then $\kappa(N)$ is contained in a line l, and, since $\kappa(A)$ is a quartic,

N is the cycle $\kappa^*(l)$. The cohomology sequence of the exact sequence:

$$0 \to \mathcal{O}_A \to \mathcal{O}_A(2\Theta) \oplus \mathcal{O}_A(2\Theta) \to \mathcal{I}_N(4\Theta) \to 0$$

gives $\dim|\mathcal{I}_N(4\Theta)| = 8$. The set of possible D's is 2-dimensional; for each D, the set of possible N's is 4-dimensional. This gives a bad set of C's of dimension $8 + 2 + 4 = 14$. Since $\dim|4\Theta| = 15$, we may assume that C does not contain any of these divisors N, hence has no pencils of degree 8.

Now, assume that for a general curve C in $|e\Theta|$, there is a surjective morphism $p: C \to C'$ onto a non-rational curve. As above, corollary 5.2 from [M1] shows that $J(C)/A$ is simple, hence the map $p^*: J(C') \to J(C)$ has to factor through ϕ. Since p^* has finite kernel, $J(C')$ is isogeneous to A, hence the curve C' cannot change as C varies in $|e\Theta|$. Letting C degenerate to a union of e copies of B, we see that $C' = B$ and that p has degree $\leqslant e$. But this gives a pencil of degree $\leqslant 2e$ on C, which we just saw does not exist. Therefore, a general curve C has no morphisms onto a non-rational curve. This finishes the proof of proposition 5.15. □

(5.16) We have now constructed counterexamples to $S(d, 1, d(d-1)/2 + 1)$ for any $d \geqslant 4$, and to $S(2e, 2, e^2 + 1)$ for any $e \geqslant 4$. Once one gets a hold of one counterexample C to $S(d, h, g)$, it is easy to construct, for any given $n > 1$, counterexamples to $S(nd, h, g')$ for infinitely many values of g'. Take a cyclic cover $\pi: C^\# \to C$ of degree $n > 1$ ramified at $2r$ points. Assume there is a curve $B^\#$ of genus h or less and a morphism $C^\# \to B^\#$ of degree $< nd$. Pick an embedding of $B^\#$ of degree $2h$ into $\mathbf{P}^3$. One checks easily that for $r > 2hnd$, the composition $C^\# \to \mathbf{P}^3$ factorizes through π, hence C itself has a morphism of degree $< d$ onto $B^\#$, which does not hold.

Therefore, by taking r large enough, we get, for $n \geqslant 2$ and $d \geqslant 4$, families of counterexamples to $S(nd, 1, g)$ and $S(2nd, 2, g)$, both for infinitely many different g's.

(5.17) We now turn our attention to the main conjecture in [AH] mentioned in the introduction.

CONJECTURE (Abramovich-Harris). *If C is a curve defined over a number field K, then C admits a map of degree d or less onto $\mathbf{P}^1$ or an elliptic curve if and only if there exists a finite extension L of K such that C has infinitely many points defined over extensions of degree d or less of L.*

The "only if" direction follows from the fact that for any abelian variety A defined over K, there exists a finite extension of K over which A has positive rank. Assume conversely that C has no maps of degree d or less onto $\mathbf{P}^1$ or an elliptic curve. We may also assume that C has a point defined over K. Then C has infinitely many points defined over extensions of degree d or less of L if and only if the symmetric product $C^{(d)}$ has infinitely many points defined over L. But $C^{(d)}$ is isomorphic to $W_d(C)$ hence, by Faltings' results [F], the conjecture will hold for C if and only if $W_d(C)$ contains no abelian varieties.

We start from an elliptic curve E defined over $\mathbf{Q}$. Our previous construction yields a curve C defined over a number field K, that has no maps of degree d or less onto $\mathbf{P}^1$ or an elliptic curve. We may assume that $E(K)$ is infinite. Then the inclusion $E \subset W_d(C)$ and the discussion above imply that C has infinitely many points defined over extensions of degree d or less of K. This gives counter-examples to the conjecture for $d \geqslant 4$ and C of genus $d(d-1)/2+1$, and for $d = nm$ with $n \geqslant 2$ and $m \geqslant 4$ and infinitely many different genera.

Another series of counterexamples is given by the construction in (4.3): by [M2], there exists a smooth genus 2 curve B defined over $\mathbf{Q}$ such that the Néron-Severi group (over $\mathbf{C}$) of its Jacobian $(J(B), \Theta)$ is generated by the class of Θ. Our construction yields a curve C defined over a number field K, such that $W_{2e}(C)$ contains $J(B)$. We may assume that $J(B)(K)$ is infinite. It follows that C has infinitely many points defined over extensions of degree $2e$ or less of K. However, according to proposition 5.15, the curve C has no morphisms of degree $2e$ or less onto a curve of degree one or less. This gives other counterexamples to the conjecture for d even $\geqslant 8$ and C of genus $g = d^2/4 + 1$.

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