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On the classification of primitive ideals for complex classical Lie algebras, III

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Introduction

This paper is the continuation of "On the Classification of Primitive Ideals for Complex Classical Lie Algebras", parts I and II ([3] and [4]). References to items, the first of whose three digits is the numeral 1 or 2, are references to items contained in one of those papers. Unexplained notation refers implicitly to parts I and II, as well. We also give an alphabetical index of notation, covering all three papers, at the end of the present paper.

The first aim of this series of papers, as explained in the introduction to part I, is to classify the primitive ideals in the enveloping algebra of a complex semisimple Lie algebra of classical type by determining the fibres of the Duflo map ([2]), that is, determining explicitly when two irreducible highest weight modules have the same annihilator. Joseph, [8] and [9], first accomplished this for $\mathfrak{g}$ of type A_{n-1}, using the Robinson-Schensted algorithm. In part I, the existence of an analogous algorithm, called A, for the Weyl groups of types B_n, C_n, and D_n, was demonstrated. It produces a pair of domino tableaux. Another algorithm, **S** (involving the notion of cycles, peculiar to the domino situation) was also defined, which, given a domino tableau, produces one in a special shape (corresponding to Lusztig's notion of special irreducible Weyl group representation [11]).

When $\mathfrak{g}$ is of type B_n or C_n this first aim is achieved in Theorem 3.5.11 (when λ is integral, otherwise see remark 3.5.13) by showing that $L(w_1\lambda)$ has the same annihilator as $L(w_2\lambda)$ precisely when $\mathbf{S}(\mathbf{A}(\delta(w_1))) = \mathbf{S}(\mathbf{A}(\delta(w_2)))$ (for notation, see section 5). The formulation of this theorem is the same when $\mathfrak{g}$ is of type D_n, and is postponed until a projected Part IV. (The proof for this type requires one more significant ingredient compared to the material treated in parts I through III since the generalized τ-invariant, cf. 3.4.1, is no longer a complete invariant — thus the use of $T_{\alpha\beta}$'s must be supplemented by a new operator: cf. the discussion in [6].)

The second aim of this series of papers is the proof of Vogan's conjecture (or

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the appropriate correction thereof for type D_n) on the generalized τ-invariant of a primitive ideal. This is achieved for types B_n and C_n in Theorem 3.5.9. As remarked previously, this result is essential for the author's further work on annihilators of irreducible Harish-Chandra modules for types B_n, C_n, and D_n, generalizing [5]. The generalized τ-invariant is defined using wall-crossings, which can be computed both in category O and in the Harish-Chandra category, and thus can be used to relate (annihilators of) modules of the different categories.

Except for Section 3 which is stated in complete generality, the results of this paper are stated for $\mathfrak{g}$ of type C_n and when modifications for type B_n are necessary they are given in the remarks. Otherwise the results are equally valid for $\mathfrak{g}$ of type B_n as well. The appropriate analogues for type D_n of these results will be included in Part IV of this series of papers.

This paper is organized as follows: In Section 1 we describe the interaction of $T_{\alpha\beta}$ with cycles. In Section 2 we characterize the set of pairs of tableaux which can be mutually connected by sequences of $T_{\alpha\beta}$'s. The main result is Theorem 3.2.2, see also 3.5.2 in this regard where it is recapitulated in a more complete form.

In Section 4 we recall, in Definition 3.4.1, the definition of the generalized τ-invariant (cf. Vogan, [12]) and give a tableau-theoretic characterization of the equivalence classes generated by the relation of having the same generalized τ-invariant, in Theorem 3.4.17, using the algorithm S. A key step in the proof of this requires a careful analysis of the properties of cycles. We divide cycles into two sorts, called up cycles and down cycles (defined in 3.3.9 and 3.3.12). These results are collected in Section 3. The results up through 3.3.13 are basic, and will assume a greater importance, and require further study, in Part IV of this series of papers, when we analyze the D_n case.

Finally in Section 5 we put all this together. The main theorems of Section 2 and 4 are collected with the results of Part II and used to prove the two main results on primitive ideals: firstly, Theorem 3.5.9 on the generalized τ-invariant, and secondly the classification theorem, 3.5.11 (see also the remark, 3.5.13).

Section 1

3.1.1. NOTATION. (1) Suppose $\mathbf{T} \in \mathcal{T}_K(M)$ where $K = B, C$, or D, and let $u = \inf M$. If $K = B$ or C let $\mathrm{OC}^*(\mathbf{T}) = \mathrm{OC}(\mathbf{T}) \backslash \{c(u, \mathbf{T})\}$; if $K = D$ let $\mathrm{OC}^*(\mathbf{T}) = \mathrm{OC}(\mathbf{T})$.

(2) Suppose $c \in \mathrm{OC}(\mathbf{T}) \backslash \mathrm{OC}^*(\mathbf{T})$. If $K = B$ we write $S(c) = S_b(c)$, if $K = C$ we write $S(c) = S_f(c)$.

We now describe the interaction of $T_{\alpha\beta}$'s with cycles.

3.1.2. PROPOSITION. *Suppose* $\mathbf{T} \in D_{\alpha\beta}(\mathscr{T}_C(M))$ *where* $\{\alpha, \beta\} = \{\alpha_i, \alpha_{i+1}\}$ *with* $i \geqslant 2$. *Let* $\mathbf{T}_1 = T_{\alpha\beta}(\mathbf{T})$.

(1) *Let*

$$G_1 = \{(i-1, S_{j,k}), (i-1, S_{j,k+1}), (i, S_{j+1,k}), (i, S_{j+1,k+1}), (i+1, S_{j,k+2})\}$$

and let

$$G_2 = \{(i, S_{j,k}), (i, S_{j,k+1}), (i+1, S_{j+1,k}), (i+1, S_{j+1,k+1}), (i-1, S_{j+1,k-1})\}.$$

Suppose $G_1 \subseteq \mathbf{T}$ *(respectively* ${}^tG_1 \subseteq \mathbf{T}$*) and suppose* $S_{j,k}$ *(respectively* $S_{k,j}$*) is* ϕ_C-*fixed. Then* $\{i-1, i\}$ *is a closed cycle in* $\mathbf{T}$ *and*

$$c(i+1, \mathbf{T}_1) = c(i+1, \mathbf{T}) \cup \{i-1, i\}.$$

Suppose $G_2 \subseteq \mathbf{T}$ *(respectively* ${}^tG_2 \subseteq \mathbf{T}$*) and suppose* $S_{j,k}$ *(respectively* $S_{k,j}$*) is* ϕ_C-*fixed. Then* $\{i, i+1\}$ *is a closed cycle in* $\mathbf{T}$ *and*

$$c(i-1, \mathbf{T}_1) = c(i-1, \mathbf{T}) \cup \{i, i+1\}.$$

In any of the above situations, if $k \in M$, $k \notin \{i-1, i, i+1\}$ *then* $P'(k, \mathbf{T}_1) = P'(k, \mathbf{T})$. *In particular, if* c *is a cycle in* $\mathbf{T}$ *and* $c \cap \{i-1, i, i+1\} = \varnothing$ *then* c *is a cycle in* $\mathbf{T}_1$. *If neither* $\mathbf{T}$ *nor* $\mathbf{T}_1$ *is in one of the situations described above, then we have* $\mathbf{T}_1 = \mathrm{In}(a, a+1; \mathbf{T})$ *for some* $a \in \{i-1, i\}$, $P'(a, \mathbf{T}_1) = P'(a+1, \mathbf{T})$, $P'(a+1, \mathbf{T}_1) = P'(a, \mathbf{T})$, *and for* $k \notin \{a, a+1\}$ *we have* $P'(k, \mathbf{T}_1) = P'(k, \mathbf{T})$. *In particular if* c *is a cycle in* $\mathbf{T}$ *and either* $c \cap \{a, a+1\} = \varnothing$ *or* $\{a, a+1\} \subseteq c$ *then* c *is a cycle in* $\mathbf{T}_1$. *If* $a \notin c(a+1, \mathbf{T})$ *then*

$$c(a, \mathbf{T}_1) = (c(a+1, \mathbf{T}) \backslash \{a+1\}) \cup \{a\}$$

and

$$c(a+1, \mathbf{T}_1) = (c(a, \mathbf{T}) \backslash \{a\}) \cup \{a+1\}.$$

(2) *There is a c.s.p.b.* μ: $\mathrm{OC}(\mathbf{T}) \to \mathrm{OC}(\mathbf{T}_1)$.

3.1.3. PROPOSITION. *Let* $\mathbf{T} \in \mathscr{T}_C(M)$ *and suppose* $F_1 \subseteq \mathbf{T}$ *(respectively* $F_2 \subseteq \mathbf{T}$*). Let* $\mathbf{T}' = (\mathbf{T} \backslash F_1) \cup F_2$ *(respectively* $\mathbf{T}' = (\mathbf{T} \backslash F_2) \cup F_1$*).*

(1) *If* $k \in M$, $k \notin \{1, 2\}$ *then* $P'(k, \mathbf{T}') = P'(k, \mathbf{T})$. *In particular, any cycle in* $\mathbf{T}$ *which is not equal to either* $c(1, \mathbf{T})$ *or* $c(2, \mathbf{T})$ *is also a cycle in* $\mathbf{T}'$.

(2) *The cycle* $c(2, \mathbf{T})$ *is closed if and only if* $1 \in c(2, \mathbf{T}')$. *If* $c(2, \mathbf{T})$ *is closed then* $c(2, \mathbf{T}') = c(2, \mathbf{T}) \cup c(1, \mathbf{T})$.

(3) *If* $c(2,\mathbf{T})$ *is closed or* $1 \in c(2,\mathbf{T})$ *then there is a c.s.p.b.* $\mu\colon \mathrm{OC}(\mathbf{T}) \to \mathrm{OC}(\mathbf{T}')$.

(4) *If* $c(2,\mathbf{T})$ *is open and* $1 \notin c(2,\mathbf{T})$ *then*

$$\mathrm{OC}(\mathbf{T})\backslash\{c(1,\mathbf{T}), c(2,\mathbf{T})\} = \mathrm{OC}(\mathbf{T}')\backslash\{c(1,\mathbf{T}'), c(2,\mathbf{T}')\},$$

and this equality gives a c.s.p.b. We have $S_b(c(2,\mathbf{T})) = S_b(c(2,\mathbf{T}'))$ *and* $S_f(c(2,\mathbf{T})) = S_f(c(1,\mathbf{T}'))$.

REMARK. The above proposition holds with $\mathbf{T} \in \mathscr{T}_B(M)$, except that we change the last statement of 4) to state that we have $S_f(c(2,\mathbf{T})) = S_f(c(2,\mathbf{T}'))$ and $S_b(c(2,\mathbf{T})) = S_b(c(1,\mathbf{T}'))$.

3.1.4. PROPOSITION. *Let* $\mathbf{T} \in \mathscr{T}_C(n)$ *and let* c *be a cycle in* $\mathbf{T}$ *such that either* $c \in \mathrm{OC}^*(\mathbf{T})$ *or both* c *is closed and for all* $1 \leqslant a \leqslant n-1$, $c \neq \{a, a+1\}$. *Then* $\tau(\mathbf{E}(\mathbf{T},c)) = \tau(\mathbf{T})$.

Proof. Set $\mathbf{T}' = \mathbf{E}(\mathbf{T},c)$. We will show that $\alpha_i \in \tau(\mathbf{T})$ implies that $\alpha_i \in \tau(\mathbf{T}')$ (since $\mathbf{T} = \mathbf{E}(\mathbf{T}',c)$ this suffices). Since $c(1,\mathbf{T})$ is the open cycle in $\mathbf{T}$ which is not in $\mathrm{OC}^*(\mathbf{T})$ we have $P(1,\mathbf{T}') = P(1,\mathbf{T})$, so $\alpha_1 \in \tau(\mathbf{T})$ implies that $\alpha_1 \in \tau(\mathbf{T}')$. Now let $i \geqslant 2$ and suppose that $\alpha_i \in \tau(\mathbf{T})$, that is, that $\rho^1(i,\mathbf{T}) > \rho^2(i-1,\mathbf{T})$. We have $\rho^2(i-1,\mathbf{T}') \leqslant \rho^2(i-1,\mathbf{T}) + 1$ and $\rho^1(i,\mathbf{T}') \geqslant \rho^1(i,\mathbf{T}) - 1$. So clearly either $\rho^1(i,\mathbf{T}') > \rho^2(i-1,\mathbf{T}')$ (that is, $\alpha_i \in \tau(\mathbf{T}')$) or one of the following hold:

(1) $\rho^1(i,\mathbf{T}) = \rho^2(i-1,\mathbf{T}) + 1$ and $i-1 \in c$ and $\rho^2(i-1,\mathbf{T}') = \rho^2(i-1,\mathbf{T}) + 1$

(2) $\rho^1(i,\mathbf{T}) = \rho^2(i-1,\mathbf{T}) + 1$ and $i \in c$ and $\rho^1(i,\mathbf{T}') = \rho^1(i,\mathbf{T}) - 1$

(3) $\rho^1(i,\mathbf{T}) = \rho^2(i-1,\mathbf{T}) + 2$ and $i-1 \in c$ and $\rho^2(i-1,\mathbf{T}') = \rho^2(i-1,\mathbf{T}) + 1$ and $i \in c$ and $\rho^1(i,\mathbf{T}') = \rho^1(i,\mathbf{T}) - 1$.

We will derive a contradiction from each of (1), (2), and (3).

Assume first (1). Then we have (for some j and k) $P(i-1,\mathbf{T}') = \{S_{jk}, S_{j+1,k}\}$ and either $P(i-1,\mathbf{T}) = \{S_{j-1,k}, S_{jk}\}$ or $P(i-1,\mathbf{T}) = \{S_{jk}, S_{j,k+1}\}$. Let $b = N_{\mathbf{T}}(S_{j+1,k+1})$. Now we are assuming that $\rho^1(i,\mathbf{T}) = j+1$, and thus we have $b = i$. Now, since $S_{jk} \in P(i-1,\mathbf{T})$, $S_{j+1,k} \notin P(i-1,\mathbf{T})$, and $S_{j+1,k+1} \in P(i,\mathbf{T})$, condition (4) of Definition 1.1.8 implies that $S_{j+1,k} \in P(i,\mathbf{T})$, that is, that $P(i,\mathbf{T}) = \{S_{j+1,k}, S_{j+1,k+1}\}$. Similarly we have $S_{j,k+1} \in P(i-1,\mathbf{T})$. It follows that $P'(i,\mathbf{T}) = \{S_{j,k+1}, S_{j+1,k+1}\}$. Thus $c(i-1,\mathbf{T}) = \{i-1, i\}$ is a closed cycle in $\mathbf{T}$, contradicting our hypothesis on c.

Assume next (2). Then an argument analogous to the previous one arrives at the same contradiction. Next assume (3). Let $j = \rho^2(i-1,\mathbf{T})$. We must have for some k and l that $P(i-1,\mathbf{T}') = \{S_{jk}, S_{j+1,k}\}$ and $P(i,\mathbf{T}') = \{S_{j+1,l}, S_{j+2,l}\}$. Again by condition (4) of Definition 1.1.8 (applied to $\mathbf{T}'$) we have $l = k+1$. Let $b = N_{\mathbf{T}'}(S_{j,k+1})$. By condition (4) of Definition 1.1.8 we must have $i-1 < b < i$, a contradiction. □

3.1.5. PROPOSITION. *Let* $\{\alpha, \beta\} = \{\alpha_i, \alpha_{i+1}\}$ *with* $i \geqslant 2$ *and let* $\mathbf{T} \in D_{\alpha\beta}(\mathscr{T}_C(n))$.

(1) *Suppose* $c \in \mathrm{OC}^*(\mathbf{T})$. *Let* $\mu\colon \mathrm{OC}(\mathbf{T}) \to \mathrm{OC}(T_{\alpha\beta}(\mathbf{T}))$ *be the c.s.p.b. of Proposition* 3.1.2. *Then* $T_{\alpha\beta}(\mathbf{E}(\mathbf{T},c)) = \mathbf{E}(T_{\alpha\beta}(\mathbf{T}), \mu(c))$.

(2) *Let c be a closed cycle in* $\mathbf{T}$ *and suppose there is a $k \in c$ such that $k \notin \{i-1, i, i+1\}$. Then* $T_{\alpha\beta}(\mathbf{E}(\mathbf{T}, c)) = \mathbf{E}(T_{\alpha\beta}(\mathbf{T}), c(k, T_{\alpha\beta}(\mathbf{T})))$.

Proof. We note first that by Proposition 3.1.4 we have $\mathbf{E}(\mathbf{T}, c) \in D_{\alpha\beta}(\mathscr{T}_C(n))$. The proposition is clearly true if $c \cap \{i-1, i, i+1\} = \varnothing$, so assume not. Assume first that either $\mathbf{T}$ or $T_{\alpha\beta}(\mathbf{T})$ is in one of the distinguished situations described in Proposition 3.1.2. In this case our proposition can be verified by inspection. Henceforth assume the contrary, that is, assume that $T_{\alpha\beta}(\mathbf{T}) = \mathrm{In}(a, a+1; \mathbf{T})$ (for $a = i-1$ or $a = i$), and that $a+1$ does not occupy in $\mathbf{T}$ the square whose occupant determines $P'(a, \mathbf{T})$ (and vice versa). Without loss of generality we may assume that $\beta = \alpha_i$. Let $\mathbf{T}' = \mathbf{E}(\mathbf{T}, c)$ and let $\mathbf{T}_1 = T_{\alpha\beta}(\mathbf{T})$. If c is open set $c_1 = \mu(c)$, if c is closed set $c_1 = c(k, \mathbf{T}_1)$. Our hypothesis on $\mathbf{T}$ says that $P'(a, \mathbf{T}_1) = P'(a+1, \mathbf{T})$, $P'(a+1, \mathbf{T}_1) = P'(a, \mathbf{T})$, and that for $l \notin \{a, a+1\}$, $P'(l, \mathbf{T}_1) = P'(l, \mathbf{T})$, and hence that for $l \notin \{a, a+1\}$, $l \in c_1$ if and only if $l \in c$, and that $a \in c_1$ (respectively $a+1 \in c_1$) if and only if $a+1 \in c$ (respectively $a \in c$). We want to show that $\mathbf{E}(\mathbf{T}_1, c_1) = T_{\alpha\beta}(\mathbf{T}')$. But what we have said already it suffices to show that $T_{\alpha\beta}(\mathbf{T}') = \mathrm{In}(a, a+1; \mathbf{T}')$. Assume for simplicity that $a = i-1$ (the argument is entirely similar if $a = i$). Now since $T_{\alpha\beta}(\mathbf{T}) = \mathrm{In}(i-1, i; \mathbf{T})$ we have $\rho^2(i-1, \mathbf{T}) < \rho^1(i+1, \mathbf{T})$. To show that $T_{\alpha\beta}(\mathbf{T}') = \mathrm{In}(a, a+1; \mathbf{T}')$ it suffices to show that $\rho^2(i-1, \mathbf{T}') < \rho^1(i+1, \mathbf{T}')$ (we also need to show that $\tilde{F}_1(i-1; r, s) \nsubseteq \mathbf{T}'$, but this follows from our hypothesis on $\mathbf{T}$). Now an argument similar to that given in the proof of Proposition 3.1.4 shows that this could only fail to happen if $c = \{i-1, i+1\}$ and is a closed cycle, which contradicts our hypothesis on c. □

3.1.6. PROPOSITION. *Let $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$ and suppose*

$$(\mathbf{T}_1, \mathbf{T}_2) \in D^L_{\alpha\beta}(\mathscr{T}_C(M_1, M_2)).$$

Suppose c is an extended cycle in $\mathbf{T}_1$ relative to $\mathbf{T}_2$ such that $1 \notin c$.

(1) *If $2 \notin c$ then* $T^L_{\alpha\beta}(\mathbf{E}((\mathbf{T}_1, \mathbf{T}_2), c, L)) = \{\mathbf{E}(Y, c, L) \mid Y \in T^L_{\alpha\beta}((\mathbf{T}_1, \mathbf{T}_2))\}$.

(2) *If $c = ec(2, \mathbf{T}_1; \mathbf{T}_2)$ then $T^L_{\alpha\beta}((\mathbf{T}_1, \mathbf{T}_2))$ consists of a single element and we have* $T^L_{\alpha\beta}(\mathbf{E}((\mathbf{T}_1, \mathbf{T}_2), c, L)) = T^L_{\alpha\beta}((\mathbf{T}_1, \mathbf{T}_2))$.

We add here some useful results of the type given at the end of section 5 of [3].

3.1.7. PROPOSITION. *Let $\mathbf{T} \in \mathscr{T}_K(M)$ for $K = B, C$, or D, and let U be a set of cycles in* $\mathbf{T}$. *Set* $\mathbf{T}' = \mathbf{E}(\mathbf{T}; U)$. *Then* $\mathrm{OC}^*(\mathbf{T}') = \mathrm{OC}^*(\mathbf{T})$. *If $c \in \mathrm{OC}^*(\mathbf{T})$ and $c \notin U$ then* $S_b(c, \mathbf{T}') = S_b(c, \mathbf{T})$ *and* $S_f(c, \mathbf{T}') = S_f(c, \mathbf{T})$. *If $c \in \mathrm{OC}^*(\mathbf{T})$ and $c \in U$ then* $S_b(c, \mathbf{T}') = S_f(c, \mathbf{T})$ *and* $S_f(c, \mathbf{T}') = S_b(c, \mathbf{T})$.

Proof. This follows from Propositions 1.5.28 and 1.5.20. □

3.1.8. PROPOSITION. *Let $\mathbf{T} \in \mathscr{T}_K(M)$ for $K = B, C$, or D, and let U and U' be sets of cycles in* $\mathbf{T}$. *Set* $\mathbf{T}' = \mathbf{E}(\mathbf{T}; U)$ *and* $\mathbf{T}'' = \mathbf{E}(\mathbf{T}'; U')$ *(this latter makes sense by Corollary 1.5.29). Then* $\mathbf{T}'' = \mathbf{E}(\mathbf{T}; U'')$ *where* $U'' = U \cup U' \backslash (U \cap U')$. *In particular* $\mathbf{E}((\mathbf{E}; U); U) = \mathbf{T}$.

Proof. See the proof of Proposition 1.5.31. □

3.1.9. PROPOSITION. *Let* $\mathbf{T}, \mathbf{T}' \in \mathcal{T}_K(M)$ *for* $K = B$, C, *or* D. *Then* $\mathbf{S}(\mathbf{T}') = \mathbf{S}(\mathbf{T})$ *if and only if there is a* $U \subseteq \mathrm{OC}^*(\mathbf{T})$ *such that* $\mathbf{T}' = \mathbf{E}(\mathbf{T}; U)$.

Proof. Let $V \subseteq \mathrm{OC}^*(\mathbf{T})$ (respectively $V' \subseteq \mathrm{OC}^*(\mathbf{T}')$) be the set of open unboxed cycles in $\mathbf{T}$ (respectively $\mathbf{T}'$), so that $\mathbf{S}(\mathbf{T}) = \mathbf{E}(\mathbf{T}; V)$ (respectively $\mathbf{S}(\mathbf{T}') = \mathbf{E}(\mathbf{T}'; V')$). Assume first that $\mathbf{S}(\mathbf{T}') = \mathbf{S}(\mathbf{T})$. Note that by Proposition 3.1.7 we have $\mathrm{OC}^*(\mathbf{T}') = \mathrm{OC}^*(\mathbf{S}(\mathbf{T}')) = \mathrm{OC}^*(\mathbf{T})$. Now by Proposition 3.1.8 we have first that $\mathbf{T}' = \mathbf{E}(\mathbf{S}(\mathbf{T}');\ V')$, and then that $\mathbf{T}' = \mathbf{E}(\mathbf{T}; U)$ where $U = V \cup V' \backslash (V \cap V') \subseteq \mathrm{OC}^*(\mathbf{T})$.

Now assume $\mathbf{T}' = \mathbf{E}(\mathbf{T}; U)$ for some $U \subseteq \mathrm{OC}^*(\mathbf{T})$. Then (by Remark 5.3.2 and Proposition 3.1.7) we have $V' = V \cup U \backslash (V \cap U)$. Now by Proposition 3.1.8 we have $\mathbf{T} = \mathbf{E}(\mathbf{T}';\ U)$, so again using Proposition 3.1.8 we have

$$\mathbf{S}(\mathbf{T}) = \mathbf{E}(\mathbf{T}; V) = \mathbf{E}(\mathbf{E}(\mathbf{T}'; U); V) = \mathbf{E}(\mathbf{T}'; V') = \mathbf{S}(\mathbf{T}'). \qquad \square$$

We also remark the following:

3.1.10. PROPOSITION. *Let* $\mathbf{T} \in \mathcal{T}_K(M)$ for $K = B$ or C. *Then* ${}^t\mathbf{T}$ *is special if and only if* $\mathbf{T}$ *is special.*

Proof. Suppose first $\mathbf{T} \in \mathcal{T}_C(M)$. Then it is easy to see that $\mathbf{T}$ is special if and only if for all $i \in \mathbb{N}^*$ either both $\rho_{2i-1}(\mathbf{T})$ and $\rho_{2i}(\mathbf{T})$ are even or $\rho_{2i-1}(\mathbf{T}) = \rho_{2i}(\mathbf{T})$. Similarly, $\mathbf{T}$ is special if and only if for all $i \in \mathbb{N}^*$ either both $\kappa_{2i-1}(\mathbf{T})$ and $\kappa_{2i}(\mathbf{T})$ are even or $\kappa_{2i-1}(\mathbf{T}) = \kappa_{2i}(\mathbf{T})$. If instead $\mathbf{T} \in \mathcal{T}_B(M)$ then we have that $\mathbf{T}$ is special if and only if $\rho_1(\mathbf{T})$ is odd and for all $i \in \mathbb{N}^*$ either both $\rho_{2i}(\mathbf{T})$ and $\rho_{2i+1}(\mathbf{T})$ are odd or $\rho_{2i}(\mathbf{T}) = \rho_{2i+1}(\mathbf{T})$; and also that $\mathbf{T}$ is special if and only if $\kappa_1(\mathbf{T})$ is odd and for all $i \in \mathbb{N}^*$ either both $\kappa_{2i}(\mathbf{T})$ and $\kappa_{2i+1}(\mathbf{T})$ are odd or $\kappa_{2i}(\mathbf{T}) = \kappa_{2i+1}(\mathbf{T})$. The proposition follows from this. □

REMARK. The above proposition does not hold for $\mathcal{T}_D(M)$, e.g. for $|M| = 1$ and $\mathbf{T} \in \mathcal{T}_D(M)$ we have $\mathbf{T}$ is special if and only if ${}^t\mathbf{T}$ is not special.

Section 2

3.2.1. DEFINITION. Let Σ be a finite sequence of pairs of adjacent simple roots, $\Sigma = (\alpha^1, \beta^1), \ldots, (\alpha^k, \beta^k)$. We call such a Σ a sequence for Π, and write $|\Sigma| = k$. If $\Pi' \subseteq \Pi$ and if we have $\{\alpha^l, \beta^l\} \subseteq \Pi'$ for all $1 \leqslant l \leqslant k$ then we say that Σ is a sequence for Π'. We set $\Sigma^{-1} = (\alpha^k, \beta^k), \ldots, (\alpha^1, \beta^1)$. If Σ' is another sequence for Π, say $\Sigma' = (\gamma^1, \delta^1), \ldots, (\gamma^l, \delta^l)$, we define $\Sigma\Sigma'$ to be the sequence $(\alpha^1, \beta^1), \ldots, (\alpha^k, \beta^k), (\gamma^1, \delta^1), \ldots, (\gamma^l, \delta^l)$.

If X is one of the sets W, $\mathcal{S}(n, n)$, or $\mathcal{T}_C(n, n)$, and if $U \subseteq X$ we define $T_\Sigma^L(U)$ as follows:

(1) If $k = 0$ then $T_\Sigma^L(U) = U$.

(2) If $k = 1$ and α^1 and β^1 have the same length then

$$T_\Sigma^L(U) = \{T_{\alpha^1\beta^1}^L(x) \mid x \in U \cap D_{\alpha^1\beta^1}^L(X)\}.$$

(3) If $k = 1$ and $\{\alpha^1, \beta^1\} = \{\alpha_1, \alpha_2\}$ then $T_\Sigma^L(U) = \bigcup T_{\alpha^1\beta^1}^L(x)$ where this union is taken over $x \in U \cap D_{\alpha^1\beta^1}^L(X)$.

(4) If $k > 1$ set $\Sigma' = (\alpha^1, \beta^1), \ldots, (\alpha^{k-1}, \beta^{k-1})$ and $\Sigma^k = (\alpha^k, \beta^k)$ and define $T_\Sigma^L(U) = T_{\Sigma^k}^L(T_{\Sigma'}^L(U))$.

For $x \in X$ we will write $T_\Sigma^L(x)$ for $T_\Sigma^L(\{x\})$. If Σ is a sequence for $\Pi \backslash \{\alpha_1\}$ then $T_\Sigma^L(x)$ consists of at most one object. When (for such a Σ) $T_\Sigma^L(x) = \{y\}$ we will write $T_\Sigma^L(x) = y$.

We define similarly T_Σ^R. Also, for $U \subseteq \mathscr{T}_K(n)$ (with $K = B. C$, or D) and Σ a sequence for $\Pi \backslash \{\alpha_1\}$ we define $T_\Sigma(U)$, analogously to the above.

REMARK. If $x \in X$ and $y \in T_\Sigma^L(x)$ then $x \in T_{\Sigma^{-1}}^L(y)$.

The main theorem of this section is the following:

3.2.2. THEOREM. *Suppose* $(\mathbf{T}_1, \mathbf{T}_2)$, $(\mathbf{T}'_1, \mathbf{T}'_2) \in \mathscr{T}_C(M_1, M_2)$ *with* $M_1 = \{1, \ldots, n\}$ *and suppose* $\mathbf{S}(\mathbf{T}_2) = \mathbf{S}(\mathbf{T}'_2)$. *Then there is a sequence* Σ for Π such that $(\mathbf{T}'_1, \mathbf{T}'_2) \in T_\Sigma^L((\mathbf{T}_1, \mathbf{T}_2))$.

REMARK. We have also the analogous statement with left and right interchanged.

We first prove the easy converse of Theorem 3.2.2.

3.2.3. PROPOSITION. *Let* $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_C(M_1, M_2)$ with $M_1 = \{1, \ldots, n\}$, *let* Σ *be a sequence for* Π, *and suppose* $(\mathbf{T}'_1, \mathbf{T}'_2) \in T_\Sigma^L((\mathbf{T}_1, \mathbf{T}_2))$. *Then* $\mathbf{S}(\mathbf{T}'_2) = \mathbf{S}(\mathbf{T}_2)$.

Proof. The proof is by induction on $|\Sigma|$. It then follows easily from the definitions and Propositions 3.1.8 and 3.1.9. □

We will prove Theorem 3.2.2 in two steps. The first step is:

3.2.4. PROPOSITION. *Suppose* $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_C(M_1, M_2)$ *with* $M_1 = \{1, \ldots, n\}$. *Then there is a sequence* Σ *for* Π *and a* $(\mathbf{T}'_1, \mathbf{T}'_2) \in T_\Sigma^L((\mathbf{T}_1, \mathbf{T}_2))$ *such that* $\mathbf{T}'_2$ *is special.*

The proofs of both Theorem 3.2.2 and Proposition 3.2.4 use induction on n. For these arguments we will use Lemmas 3.2.6–8.

3.2.5. DEFINITION. Let $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_C(M_1, M_2)$ and let $e = \sup M_1$. Let $\mathbf{T}'_1 = \mathbf{T}_1 - e$ and let $(\mathbf{T}'_2, v, \varepsilon) = \beta((\mathbf{T}_2, P(e, \mathbf{T}_1)))$. We define $(\mathbf{T}_1, \mathbf{T}_2) - L = (\mathbf{T}'_1, \mathbf{T}'_2)$. We define similarly $(\mathbf{T}_1, \mathbf{T}_2) - R$.

3.2.6. LEMMA. *Let* $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_C(M_1, M_2)$ *with* $M_1 = \{1, \ldots, n\}$ *and suppose* Σ

is a sequence for $\Pi\backslash\{\alpha_n\}$. *Then*

$$T^L_\Sigma((\mathbf{T}_1, \mathbf{T}_2) - L) = \{(\mathbf{T}'_1, \mathbf{T}'_2) - L \mid (\mathbf{T}'_1, \mathbf{T}'_2) \in T^L_\Sigma((\mathbf{T}_1, \mathbf{T}_2))\}.$$

Proof. We prove the lemma when $|\Sigma| = 1$: the general result then follows easily using induction on $|\Sigma|$. So assume $\Sigma = (\alpha, \beta)$. If $\alpha_1 \notin \{\alpha, \beta\}$ then the statement is obvious, so assume $\alpha_1 \in \{\alpha, \beta\}$. Then the lemma is a consequence of Proposition 2.3.3.

3.2.7. LEMMA. *Let* $(\mathbf{T}_1, \mathbf{T}_2) \in \mathcal{T}_C(M_1, M_2)$ *with* $M_1 = \{1, \ldots, n\}$, *suppose* Σ *is a sequence for* $\Pi\backslash\{\alpha_n\}$, *and suppose* $(\mathbf{T}'_1, \mathbf{T}'_2) \in T^L_\Sigma((\mathbf{T}_1, \mathbf{T}_2))$.

(1) *We have* $P(n, \mathbf{T}'_1) \in \{P(n, \mathbf{T}_1), P'(n, \mathbf{T}_1)\}$.

(2) *If* $P(n, \mathbf{T}'_1) = P'(n, \mathbf{T}_1)$ *then one of the following hold:*

(i) *there is a* $k \in M_1\backslash\{n\}$ *with* $P(k, \mathbf{T}'_1) \cap P(n, \mathbf{T}_1) \neq \varnothing$.

(ii) *there is a* $k \in M_1\backslash\{n\}$ *and there are cycles* $c^1_1, c^2_1 \in \mathrm{OC}^*(\mathbf{T}'_1)$ (*where possibly* $c^1_1 = c^2_1$) *and* $c_2 \in \mathrm{OC}^*(\mathbf{T}'_2)$ *with* $S_b(c_2, \mathbf{T}'_2) = S_b(c^2_1, \mathbf{T}'_1) \in P(k, \mathbf{T}'_1)$ *and* $S_f(c_2, \mathbf{T}'_2) = S_f(c^1_1, \mathbf{T}'_1) \in P(n, \mathbf{T}_1)$.

(3) *If* $\mathrm{Shape}(\mathbf{T}'_1 - n) = \mathrm{Shape}(\mathbf{T}_1 - n)$ *then* $P(n, \mathbf{T}'_1) = P(n, \mathbf{T}_1)$.

(4) *Suppose* $P(n, \mathbf{T}_1)$ *is boxed and* $\mathbf{T}'_1 - n$ *is special. Then* $P(n, \mathbf{T}'_1) = P(n, \mathbf{T}_1)$; *in particular* $\mathbf{T}'_1$ *is special.*

Proof. We prove statement (1) by induction on $|\Sigma|$: the case $|\Sigma| = 0$ is trivial and the case $|\Sigma| = 1$ is clear from the definitions. Write $\Sigma = \Sigma_1\Sigma_2$ with $|\Sigma_2| = 1$, and let $(\mathbf{T}^1_1, \mathbf{T}^1_2) \in T^L_{\Sigma_1}((\mathbf{T}_1, \mathbf{T}_2))$ be such that $(\mathbf{T}'_1, \mathbf{T}'_2) \in T^L_{\Sigma_2}((\mathbf{T}^1_1, \mathbf{T}^1_2))$. Then $P(n, \mathbf{T}'_1) \in \{P(n, \mathbf{T}^1_1), P'(n, \mathbf{T}^1_1)\}$. By induction we have $P(n, \mathbf{T}^1_1) \in \{P(n, \mathbf{T}_1), P'(n, \mathbf{T}_1)\}$. On the other hand, it is easy to see, (using Proposition 1.5.33) that the set of ϕ_C-fixed squares in $\mathrm{Shape}(\mathbf{T}^1_1)$ coincides with the set of ϕ_C-fixed squares in $\mathrm{Shape}(\mathbf{T}_1)$. It then is easy to see from Definition 1.5.8 (since $n = \sup M_1$) that if $P(n, \mathbf{T}^1_1) = P(n, \mathbf{T}_1)$ then $P'(n, \mathbf{T}^1_1) = P'(n, \mathbf{T}_1)$, and if $P(n, \mathbf{T}^1_1) = P'(n, \mathbf{T}_1)$ then $P'(n, \mathbf{T}^1_1) = P'(n, \mathbf{T}_1)$. This completes the proof of statement (1).

To prove (2), we again use induction on $|\Sigma|$, the case $|\Sigma| = 0$ being vacuously true. Let Σ_1, Σ_2, and $(\mathbf{T}^1_1, \mathbf{T}^1_2)$ be as in the proof of part (1). By induction part (2) of the lemma is true with Σ_1 and $(\mathbf{T}^1_1, \mathbf{T}^1_2)$ in place of Σ and $(\mathbf{T}'_1, \mathbf{T}'_2)$. Write $\Sigma_2 = (\alpha, \beta)$. If $\{\alpha, \beta\} \neq \{\alpha_1, \alpha_2\}$ the desired conclusion is obvious, so assume $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$. If $P(n, \mathbf{T}^1_1) = P(n, \mathbf{T}_1)$ then (in light of the last statement in the proof of part (1)) the desired conclusion follows from the definition of $T^L_{\alpha\beta}$ and of extended cycle. Finally, assume $P(n, \mathbf{T}^1_1) = P'(n, \mathbf{T}_1)$. Then the number k given by the lemma (and induction) is contained in $ec(n, \mathbf{T}^1_1; \mathbf{T}^1_2)$. It follows that, if the conclusion of our statement does not hold (with the same k) then $P(n, \mathbf{T}'_1) = P'(n, \mathbf{T}^1_1) = P(n, \mathbf{T}_1)$.

Statement (3) is a direct consequence of statement (2), that is, clearly both (i) and (ii) imply that $\mathrm{Shape}(\mathbf{T}'_1 - n) \neq \mathrm{Shape}(\mathbf{T}_1 - n)$ (in the case that (ii) holds, we note that

$$S_b(c_2, \mathbf{T}'_2) = S_b(c_1^2, \mathbf{T}'_1) \in \text{Shape}(\mathbf{T}'_1 - n);$$

on the other hand c_2 is also a cycle in $\mathbf{T}_2$ (by Propositions 3.2.3, 3.1.9, and 3.1.7) and since $S_f(c_2, \mathbf{T}'_2) \in \text{Shape}(\mathbf{T}_1) = \text{Shape}(\mathbf{T}_2)$ we have $S_f(c_2, \mathbf{T}'_2) = S_b(c_2, \mathbf{T}_2)$), and so $S_b(c_2, \mathbf{T}'_2) \notin \text{Shape}(\mathbf{T}_2) = \text{Shape}(\mathbf{T}_1)$).

Statement (4) also follows from statement (2): to see this assume that $P(n, \mathbf{T}'_1) = P'(n, \mathbf{T}_1)$. If (i) holds then the square in $P(k, \mathbf{T}'_1) \cap P(n, \mathbf{T}_1)$ is a filled corner in Shape$(\mathbf{T}'_1 - n)$. If (ii) holds then since $P(n, \mathbf{T}'_1) = P'(n, \mathbf{T}_1)$ is unboxed and $n \in c_1^1$ we have that c_1^1 is an unboxed cycle in $\mathbf{T}'_1$, hence that c_2 is an unboxed cycle in $\mathbf{T}'_2$, hence that c_1^2 is an unboxed cycle in $\mathbf{T}'_1$. It follows that $S_b(c_1^2, \mathbf{T}'_1)$ is a filled corner in Shape$(\mathbf{T}'_1 - n)$. So in each case we find a contradiction with the hypothesis that $\mathbf{T}'_1 - n$ is special. □

3.2.8. LEMMA. *Let* $\mathbf{T} \in \mathcal{T}_C(M)$ *and let* P *be an extremal position in* $\mathbf{T}$. *Let* $e = \sup M$. *Then there is a tableau* $\mathbf{T}_1 \in \mathcal{T}_C(M)$ *such that* Shape$(\mathbf{T}_1)$ = Shape$(\mathbf{T})$ *and* $P(e, \mathbf{T}_1) = P$.

Proof. If $P = P(e, \mathbf{T})$ then we may take $\mathbf{T}_1 = \mathbf{T}$ so assume not. Write $M = \{k_1, \ldots, k_m\}$ with $k_1 < \cdots < k_m = e$. Let $(\mathbf{T}', v, \varepsilon) = \beta((\mathbf{T}, P))$ and let j be such that $v = k_j$. Let $\mathbf{T}_2$ be the tableau obtained from $\mathbf{T}'$ by replacing k_{j+1} with k_j, k_{j+2} with k_{j+1}, etc., and let $\mathbf{T}_1 = \text{Adj}(\mathbf{T}_2, P, e)$. Then $\mathbf{T}_1 \in \mathcal{T}_C(M)$, Shape$(\mathbf{T}_1)$ = Shape$(\mathbf{T})$ and $P(e, \mathbf{T}_1) = P$, as was to be shown. □

We can now prove Proposition 3.2.4 and Theorem 3.2.2.

PROOF OF PROPOSITION 3.2.4. The proof uses induction on n. We assume by induction that both Theorem 3.2.2 and Proposition 3.2.4 are true when $M_1 = \{1, \ldots, n-1\}$ (the case $n = 0$ being trivial). Let $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2) = (\mathbf{T}_1, \mathbf{T}_2) - L$. By induction there is a sequence Σ_1 for $\Pi \backslash \{\alpha_n\}$ and a $(\bar{\mathbf{T}}_1^1, \bar{\mathbf{T}}_2^1) \in T_{\Sigma_1}^L((\mathbf{T}_1, \mathbf{T}_2))$ with $\bar{\mathbf{T}}_2^1$ special. By Lemma 3.2.6 there is a $(\mathbf{T}_1^1, \mathbf{T}_2^1) \in T_{\Sigma_1}^L((\mathbf{T}_1, \mathbf{T}_2))$ such that $(\bar{\mathbf{T}}_1^1, \bar{\mathbf{T}}_2^1) = (\mathbf{T}_1^1, \mathbf{T}_2^1) - L$. Now clearly either $\mathbf{T}_2^1$ is special or $P(n, \mathbf{T}_1^1)$ is unboxed and $\{n\}$ is an extended cycle in $\mathbf{T}_1^1$ relative to $\mathbf{T}_2^1$. If the former we are done, so assume the latter. We will assume also that $P(n, \mathbf{T}_1^1)$ is horizontal: the case where $P(n, \mathbf{T}_1^1)$ is vertical follows from this case using Remark 2.1.12–2) and Propositions 2.3.6 and 3.1.10. Let $P(n, \mathbf{T}_1^1) = \{S_{ij}, S_{i,j+1}\}$. Then $S_{i,j+1}$ is a filled corner in $\mathbf{T}_1^1$ and $S_{i+1,j}$ is an empty hole in $\mathbf{T}_1^1$, in particular $S_{i+1,j-1} \in \text{Shape}(\mathbf{T}_1^1)$ and $\phi_C(S_{ij}) = Y$.

There are several cases. In the first three cases we assume that $i > 1$. Then since $\phi_C(S_{1,1}) = X$ we have that i is odd, in particular $i \geqslant 3$. We set $r = \rho_{i-2}(\mathbf{T}_1^1)$ and $s = \rho_{i-1}(\mathbf{T}_1^1)$.

Case 1. Here $r > s$. Then (since $\mathbf{T}_1^1 - n = \bar{\mathbf{T}}_1^1$ is special) we have $\phi_C(S_{i-2,r}) = Y$ and $\phi_C(S_{i-1,s}) = W$; in particular $r \geqslant s + 2$ and $s \geqslant j + 2$. Let $P_1 = \{S_{i-2,r-1}, S_{i-2,r}\}$ and $P_2 = \{S_{i-1,s-1}, S_{i-1,s}\}$.

Case 2. Here $r = s > j + 1$. Set $P_1 = \{S_{i-2,r}, S_{i-1,r})$ and $P_2 = \{S_{i-2,r-1}, S_{i-1,r-1}\}$.

Case 3. Here $r = s = j + 1$. Let P_1 and P_2 be as in case 2.

In these three cases, by induction on Theorem 3.2.2 and using Lemma 3.2.8 (twice) there is a sequence Σ_2 for $\Pi\backslash\{\alpha_n\}$ and a $(\bar{\mathbf{T}}_1^2, \bar{\mathbf{T}}_2^2) \in T_{\Sigma_2}^L((\bar{\mathbf{T}}_1^1, \bar{\mathbf{T}}_2^1))$ such that $\bar{\mathbf{T}}_2^2 = \bar{\mathbf{T}}_2^1$, $P(n-1, \bar{\mathbf{T}}_1^2) = P_1$ and $P(n-2, \bar{\mathbf{T}}_1^2) = P_2$. Using Lemma 3.2.6 let $(\mathbf{T}_1^2, \mathbf{T}_2^2) \in T_{\Sigma_2}^L((\mathbf{T}_1^1, \mathbf{T}_2^1))$ be such that $(\bar{\mathbf{T}}_1^2, \bar{\mathbf{T}}_2^2) = (\mathbf{T}_1^2, \mathbf{T}_2^2) - L$. Set $(\mathbf{T}_1^3, \mathbf{T}_2^3) = T_{\alpha_{n-1}\alpha_n}^L((\mathbf{T}_1^2, \mathbf{T}_2^2))$ and let $(\bar{\mathbf{T}}_1^3, \bar{\mathbf{T}}_2^3) = (\mathbf{T}_1^3, \mathbf{T}_2^3) - L$. In cases 1 and 2 we have $\mathbf{T}_1^3 = \mathrm{In}(n-1, n; \mathbf{T}_1^2)$; in case 3 we have

$$\mathbf{T}_1^3 = (\mathbf{T}_1^2 \backslash \tilde{F}_2(n-2; i-2, j)) \cup F_2(n-2; i-2, j).$$

Let $\Sigma_3 = (\alpha_{n-1}, \alpha_n)$. By induction we have a sequence Σ_4 for $\Pi\backslash\{\alpha_n\}$ and a $(\bar{\mathbf{T}}_1^4, \bar{\mathbf{T}}_2^4) \in T_{\Sigma_4}^L((\bar{\mathbf{T}}_1^3, \bar{\mathbf{T}}_2^3))$ such that $\bar{\mathbf{T}}_2^4$ is special. Let $(\mathbf{T}_1^4, \mathbf{T}_2^4)$ be such that $(\mathbf{T}_1^4, \mathbf{T}_2^4) \in T_{\Sigma_4}^L((\mathbf{T}_1^3, \mathbf{T}_2^3))$ and $(\bar{\mathbf{T}}_1^4, \bar{\mathbf{T}}_2^4) = (\mathbf{T}_1^4, \mathbf{T}_2^4) - L$. Then $\mathbf{T}_1^4$ is special: in cases 1 and 2 this follows from by Lemma 3.2.7–4 (in case 1 we note that since $\bar{\mathbf{T}}_1^1$ is special P_1 is boxed), in case 3 we have that $\bar{\mathbf{T}}_1^4$ special implies that

$$\mathrm{Shape}(\bar{\mathbf{T}}_1^4) = (\mathrm{Shape}(\bar{\mathbf{T}}_1^3)\backslash\{S_{i-2,j+1}\}) \cup \{S_{i+1,j}\},$$

so

$$P(n, \mathbf{T}_1^4) = P'(n, \mathbf{T}_1^3) = \{S_{i-2,j+1}, S_{i-1,j+1}\}.$$

So, setting $\Sigma = \Sigma_1\Sigma_2\Sigma_3\Sigma_4$ and $(\mathbf{T}'_1, \mathbf{T}'_2) = (\mathbf{T}_1^4, \mathbf{T}_2^4)$, we have proved the proposition under the assumption that $i > 1$.

Case 4. Here $i = 1$ and $S_{3,j-1} \in \mathrm{Shape}(\mathbf{T}_1^1)$. Since $\bar{\mathbf{T}}_1^1$ is special we then have that $S_{4,j-1} \in \mathrm{Shape}(\mathbf{T}_1^1)$. Let $r = \kappa_{j-1}(\mathbf{T}_1^1)$, so $r \geqslant 4$. Set $P_1 = \{S_{r-1,j-1}, S_{r,j-1}\}$, and set $P_2 = \{S_{r-3,j-1}, S_{r-2,j-1}\}$. The rest of the argument is as in cases 1 and 2, except that we take $\Sigma_3 = (\alpha_n, \alpha_{n-1})$ and $(\mathbf{T}_1^3, \mathbf{T}_2^3) = T_{\alpha_n\alpha_{n-1}}^L((\mathbf{T}_1^2, \mathbf{T}_2^2))$.

Case 5. Here $i = 1$, $S_{3,j-1} \notin \mathrm{Shape}(\mathbf{T}_1^1)$, and $S_{3,j-2} \in \mathrm{Shape}(\mathbf{T}_1^1)$. Set $P_2 = \{S_{1,j-1}, S_{2,j-1}\}$. Set $r = \kappa_{j-2}(\mathbf{T}_1^1)$. If $r = 3$ then $\bar{\mathbf{T}}_1^1$ special implies that $j > 3$ and $\kappa_{j-3}(\mathbf{T}_1^1) = 3$ so let $P_1 = \{S_{3,j-3}, S_{3,j-2}\}$. If $r > 3$ set $P_1 = \{S_{r-1,j-2}, S_{r,j-2}\}$ (again, $\bar{\mathbf{T}}_1^1$ special says that P_1 is boxed). The rest of the argument is as in case 4.

Case 6. Here $i = 1, j > 2$, and $S_{3,j-2} \notin \mathrm{Shape}(\mathbf{T}_1^1)$. Set $P_1 = \{S_{2,j-2}, S_{2,j-1}\}$ and $P_2 = \{S_{1,j-2}, S_{1,j-1}\}$. The rest of the argument is analogous to that of case 3. (Here we have $P(n, \mathbf{T}_1^3) = P_1$, and then $P(n, \mathbf{T}_1^4) = P'(n, \mathbf{T}_1^3) = \{S_{2,j-1}, S_{2,j}\}$.)

Case 7. Here $i = 1, j = 2$, and $S_{3,j-1} \notin \mathrm{Shape}(\mathbf{T}_1^1)$. Then $n = 2$, $(\mathbf{T}_1^1, \mathbf{T}_2^1) = (\mathbf{T}_1, \mathbf{T}_2)$, and $\mathbf{T}_1 = (\tilde{F}_1, \phi_C)$. Let $(\mathbf{T}'_1, \mathbf{T}'_2)$ be such that $T_{\alpha_2,\alpha_1}^L((\mathbf{T}_1, \mathbf{T}_2)) = \{(\mathbf{T}'_1, \mathbf{T}'_2)\}$. Then $\mathbf{T}'_1 = (F_2, \phi_C)$ is special. Setting $\Sigma = (\alpha_2, \alpha_1)$ we see that we have proved the proposition in this case. This completes the proof of Proposition 3.2.4. □

REMARK. For $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_B(M_1, M_2)$ we have to modify the above proof as follows.

We note that $\phi_B(S_{1,1}) = W$, so here i is even. In the first three cases, the assumption $i > 1$ is replaced with the assumption $i > 2$. We then proceed as before. The next three assumptions are modified accordingly, that is "$i = 1$" is replaced with "$i = 2$", etc. Finally, assume $i = 2$ and $j \leqslant 2$. Then in fact $j = 1$. Assume first $\rho_1(\mathbf{T}_1^1) > 3$. Then $\bar{\mathbf{T}}_1^1$ special implies that $\rho_1(\mathbf{T}_1^1) \geqslant 5$. Set $r = \rho_1(\mathbf{T}_1^1)$. Setting $P_1 = \{S_{1,r-1}, S_{1,r}\}$ and $P_2 = \{S_{1,r-3}, S_{1,r-2}\}$, we proceed as in cases 1 and 2. Finally, assume $i = 2$ and $j = 1$ and $\rho(\mathbf{T}_1^1) = 3$. Then we have $\mathbf{T}_1^1 = (\tilde{F}_2, \phi_B)$. This case now proceeds as in the last case of the proof of the proposition.

We now proceed to prove Theorem 3.2.2. The key ingredient in that proof is the following lemma.

3.2.9. LEMMA. *Let $(\mathbf{T}_1, \mathbf{T}_2)$ be as in Theorem 3.2.2 and suppose $\mathbf{T}_2$ is special. Suppose P' is an extremal position in $\mathbf{T}_1$. Then there is a sequence Σ for Π and a $(\mathbf{T}_1', \mathbf{T}_2') \in T_\Sigma^L((\mathbf{T}_1, \mathbf{T}_2))$ such that $P(n, \mathbf{T}_1') = P'$ and $\mathbf{T}_2' = \mathbf{T}_2$.*

Proof. As in the proof of Proposition 3.2.4 we will assume by induction on n that Theorem 3.2.2 is true when $M_1 = \{1, \ldots, n-1\}$. Let $P = P(n, \mathbf{T}_1)$. We may assume $P' \neq P$. We first prove the lemma under the additional assumption that both P and P' are boxed. As in the proof of Proposition 3.2.4 we will assume that P is horizontal (here we note that for the C grid, P is boxed if and only if ${}^tP$ is boxed, and similarly P'), so let $P = \{S_{ij}, S_{i,j+1}\}$. There are a number of cases. In the first four cases we assume that P' is also horizontal, and write $P' = \{S_{k,l}, S_{k,l+1}\}$.

Case A. Here $k = i - 1$. Then since P' is an extremal position in $\mathbf{T}_1$ we have $l \geqslant j + 2$. Set $P_1 = \{S_{i-1,l-2}, S_{i-1,l-1}\}$.

Case B. Here $k < i - 1$. Let $r = \rho_{i-1}(\mathbf{T}_1)$. Then $l \geqslant r + 1$. Set $P_1 = \{S_{i-1,r-1}, S_{i-1,r}\}$.

Case C. Here $k = i + 1$. Then $l \leqslant j - 2$. Set $P_1 = \{S_{i,j-2}, S_{i,j-1}\}$.

Case D. Here $k > i + 1$. Let $r = \rho_{k-1}(\mathbf{T}_1)$ and set $P_1 = \{S_{k-1,r-1}, S_{k-1,r}\}$.

Let $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2) = (\mathbf{T}_1, \mathbf{T}_2) - L$. In all the above cases, by induction on Theorem 3.2.2, and using Lemma 3.2.8, we can find a sequence Σ_1 for $\Pi \backslash \{\alpha_n\}$ and a $(\bar{\mathbf{T}}_1^1, \bar{\mathbf{T}}_2^1) \in T_{\Sigma_1}^L((\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2))$ such that $\bar{\mathbf{T}}_2^1 = \bar{\mathbf{T}}_2$, $P(n-1, \bar{\mathbf{T}}_1^1) = P'$ and $P(n-2, \bar{\mathbf{T}}_1^1) = P_1$. Using Lemma 3.2.6 let $(\mathbf{T}_1^1, \mathbf{T}_2^1) \in T_{\Sigma_1}^L((\mathbf{T}_1, \mathbf{T}_2))$ be such that $(\mathbf{T}_1^1, \mathbf{T}_2^1) - L = (\bar{\mathbf{T}}_1^1, \bar{\mathbf{T}}_2^1)$. By Lemma 3.2.7–3) we have $P(n, \mathbf{T}_1^1) = P(n, \mathbf{T}_1)$, in particular $\mathbf{T}_2^1 = \mathbf{T}_2$. In cases A and B set $\Sigma_2 = (\alpha_{n-1}, \alpha_n)$; in cases C and D set $\Sigma_2 = (\alpha_n, \alpha_{n-1})$. Then $\Sigma = \Sigma_1 \Sigma_2$ and $(\mathbf{T}_1', \mathbf{T}_2') = T_{\Sigma_2}^L((\mathbf{T}_1^1, \mathbf{T}_2^1))$ verify the lemma in these cases.

Now we assume that P' is vertical and write $P' = \{S_{k,l}, S_{k+1,l}\}$. Again there are several cases. The first four are easier so we treat them first.

Case E. Here $k + 1 = i - 1$. Let P_1 be as in case A.

Case F. Here $k + 1 < i - 1$. Let P_1 be as in case B.

Case G. Here $k = i + 1$ and $l < j - 1$. Let P_1 be as in case C.

Case H. Here $k > i + 1$. Suppose first that $\rho_{i+1}(\mathbf{T}_1) \leqslant j - 3$. Then let $P_1\{S_{i,j-2}, S_{i,j-1}\}$. Suppose next that $\rho_{i+1}(\mathbf{T}_1) = j - 1$. Then let $r = \kappa_{j-1}(\text{Shape}(\mathbf{T}_1)\backslash P')$ and set $P_1 = \{S_{r-1,j-1}, S_{r,j-1}\}$. Finally, suppose $\rho_{i+1}(\mathbf{T}_1) = j - 2$. Then $\mathbf{T}_1$ special and P boxed implies that $\phi_C(S_{ij}) = Z$, and thus that $\kappa_{j-2}(\text{Shape}(\mathbf{T}_1)\backslash P') \geqslant i + 2$ (since P' is vertical). Let $r = \kappa_{j-2}(\text{Shape}(\mathbf{T}_1)\backslash P')$ and set $P_1 = \{S_{r-1,j-2}, S_{r,j-2}\}$.

In cases E and F we prove the lemma as in cases A and B; in cases G and H we prove the lemma as in cases C and D.

Case I. Here $k = i - 1$, that is $P \cap P' = \{S_{i,j+1}\}$. Then since both P and P' are boxed we have $\phi_C(S_{ij}) = Z$. In particular since $\phi_C(S_{1,1}) = X$ we have that i is even. There are several subcases of this case.

Subcase (a). Here $i > 2$. Then we have $i \geqslant 4$. Set $r = \rho_{i-2}(\mathbf{T}_1)s = \rho_{i-3}(\mathbf{T}_1)$. Assume first $r = j + 1$. Set

$$P_1 = \{S_{i-2,j+1}, S_{i-1,j+1}\}$$

and

$$P_2 = \{S_{i-2,j}, S_{i-1,j}\}.$$

Then as above there is a sequence Σ_1 for $\Pi\backslash\{\alpha_n\}$ and a $(\mathbf{T}_1^1, \mathbf{T}_2^1) \in T_{\Sigma_1}^L((\mathbf{T}_1, \mathbf{T}_2))$ with $\mathbf{T}_2^1 = \mathbf{T}_2$, $P(n - 1, \mathbf{T}_1^1) = P_1$, and $P(n - 2, \mathbf{T}_1^1) = P_2$. Setting $\Sigma_2 = (\alpha_{n-1}, \alpha_n)$, $\Sigma = \Sigma_1\Sigma_2$, and $(\mathbf{T}_1', \mathbf{T}_2') = T_{\alpha_{n-1}\alpha_n}^L((\mathbf{T}_1^1, \mathbf{T}_2^1))$, we see that we have verified the lemma under this assumption.

Assume next that $s = r > j + 1$ and set $P'' = \{S_{i-3,r}, S_{i-2,r}\}$. Then by case F there is a sequence Σ_1 for Π and a $(\mathbf{T}_1^1, \mathbf{T}_2^1) \in T_{\Sigma_1}^L((\mathbf{T}_1, \mathbf{T}_2))$ such that $\mathbf{T}_2^1 = \mathbf{T}_2$ and $P(n, \mathbf{T}_1^1) = P''$. By case A or B (transposed) there is a sequence Σ_2 for Π and a $(\mathbf{T}_1^2, \mathbf{T}_2^2) \in T_{\Sigma_2}^L((\mathbf{T}_1^1, \mathbf{T}_2^1))$ such that $\mathbf{T}_2^2 = \mathbf{T}_2^1$ and $P(n, \mathbf{T}_1^2) = P'$. Setting $\Sigma = \Sigma_1\Sigma_2$ and $(\mathbf{T}_1', \mathbf{T}_2') = (\mathbf{T}_1^2, \mathbf{T}_2^2)$ verifies the lemma under this assumption.

Finally, assume $s > r$. Then $\mathbf{T}_1$ special implies that $\phi_C(S_{i-3,s}) = Y$ and $s \geqslant r + 2$. Then we set $P'' = \{S_{i-3,s-1}, S_{i-3,s}\}$ and argue as in the previous paragraph, using case B in place of case F and case H in place of case A or B. This completes the proof of subcase (a).

Subcase (b). Here $j > 1$. This subcase is parallel to subcase (a). We omit the argument.

Subcase (c). Here we have $i = 2$ and $j = 1$. Then we must have $n = 2$ and $\mathbf{T}_1 = (F_2, \phi_C)$. Then, setting $\Sigma = (\alpha_1, \alpha_2)$, $\mathbf{T}_1' = (F_1, \phi_C)$, and $\mathbf{T}_2' = \mathbf{T}_2$, we see that $(\mathbf{T}_1', \mathbf{T}_2') \in T_\Sigma^L((\mathbf{T}_1, \mathbf{T}_2))$, as was to be shown. This completes the proof of case I.

Case J. Here $k = i + 1$ and $l = j - 1$. Then again the hypothesis that P and P' are boxed implies that $\phi_C(S_{ij}) = Z$. Again, i is even, so $i \geqslant 2$. Let $r = \rho_{i-1}(\mathbf{T}_1)$. If $r = j + 1$ let $P'' = \{S_{i-1,j+1}, S_{i,j+1}\}$. Using case I we find a sequence Σ_1 for Π and a $(\mathbf{T}_1^1, \mathbf{T}_2^1) \in T_{\Sigma_1}^L((\mathbf{T}_1, \mathbf{T}_2))$ such that $\mathbf{T}_2^2 = \mathbf{T}_2$ and $P(n, \mathbf{T}_1^1) = P''$. Then, using case B (transposed) we can find a sequence Σ_2 for Π and a $(\mathbf{T}_1', T_2') \in T_{\Sigma_2}^L((\mathbf{T}_1^1, \mathbf{T}_2^1))$ such that $\mathbf{T}_2' = \mathbf{T}_2^1$ and $P(n, \mathbf{T}_2') = P'$. Setting as usual $\Sigma = \Sigma_1\Sigma_2$ verifies the lemma under this assumption. If instead $r > j + 1$ then $\mathbf{T}_1$ special implies that $r \geqslant j + 3$, so set $P'' = \{S_{i-1,r-1}, S_{i-1,r}\}$. We now proceed, as under the assumption $r = j + 1$, to move the n first to P'' and then to P', using the previous cases. This completes the proof of the lemma in the case where both P and P' are boxed.

We now assume that P' is boxed and P is unboxed. Again we assume that P is horizontal and set $P = \{S_{ij}, S_{i,j+1}\}$. Since $\mathbf{T}_1$ is special we have $\phi_C(S_{ij}) = W$ and $\rho_{i-1}(\mathbf{T}_1) = j + 1$. Set $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2) = (\mathbf{T}1, \mathbf{T}_2) - L$. By Proposition 3.2.4 there is a sequence Σ_1 for $\Pi\backslash\{\alpha_n\}$ and a $(\bar{\mathbf{T}}_1^1, \bar{\mathbf{T}}_2^1) \in T_{\Sigma_1}^L((\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2))$ such that $\bar{\mathbf{T}}_1^1$ is special. Let $(\mathbf{T}_1^1, \mathbf{T}_2^1) \in T_{\Sigma_1}^L((\mathbf{T}_1, \mathbf{T}_2))$ be such that $(\bar{\mathbf{T}}_1^1, \bar{\mathbf{T}}_2^1) = (\mathbf{T}_1^1, \mathbf{T}_2^1) - L$. Then $\text{Shape}(\bar{\mathbf{T}}_1^1) = (\text{Shape}(\bar{\mathbf{T}}_1)\backslash\{S_{i-1,j+1}\}) \cup \{S_{ij}\}$, and so by Lemma 3.2.7–1) we have

$$P(n, \mathbf{T}_1^1) = P'(n, \mathbf{T}_1) = \{S_{i-1,j+1}, S_{i,j+1}\},$$

which is boxed. Now we have already proved this lemma in the case where the two positions are boxed, that is, we have shown that there is a sequence Σ_2 for Π and a $(\mathbf{T}_1', \mathbf{T}_2') \in T_{\Sigma_2}^L((\mathbf{T}_1^1, \mathbf{T}_2^1))$ such that $P(n, \mathbf{T}_1') = P'$. Setting $\Sigma = \Sigma_1\Sigma_2$ proves the lemma under this second set of assumptions for P and P'.

Finally we treat the case where P' is unboxed. Here we may assume that P' is horizontal, so we set $P' = \{S_{ij}, S_{i,j+1}\}$. Then as in the previous case we have $\phi_C(S_{ij}) = W$ and $\rho_{i-1}(\mathbf{T}_1) = j + 1$. Set $P'' = \{S_{i-1,j+1}, S_{i,j+1}\}$. Then P'' is an extremal position in $\mathbf{T}_1$ and P'' is boxed. Thus, using the cases of the lemma which we have already proved, we have a sequence Σ_1 for Π and a $(\mathbf{T}_1^1, \mathbf{T}_2^1) \in T_{\Sigma_1}^L((\mathbf{T}_1, \mathbf{T}_2))$ such that $\mathbf{T}_2^1 = \mathbf{T}_2$ and $P(n, \mathbf{T}_1^1) = P''$. Now let $(\bar{\mathbf{T}}_1^1, \bar{\mathbf{T}}_2^1) = (\mathbf{T}_1^1, \mathbf{T}_2^1) - L$. By Theorem 2.2.3, there is a cycle $c \in OC(\bar{\mathbf{T}}_1^1)$ with $S_b(c) = S_{ij}$ and $S_f(c) = S_{i-1,j+1}$. By Lemma 3.2.8 there is a tableau $\bar{\mathbf{T}} \in \mathscr{T}_C(n-1)$ with $\text{Shape}(\bar{\mathbf{T}}) = \text{Shape}(\bar{\mathbf{T}}_1^1)$ and $P(n-1, \bar{\mathbf{T}}) = \{S_{i-1,j}, S_{ij}\}$. Then $\{n-1\}$ is an extended cycle in $\bar{\mathbf{T}}$ relative to $\bar{\mathbf{T}}_2^1$. Set $(\bar{\mathbf{T}}_1^2, \bar{\mathbf{T}}_2^2) = \mathbf{E}((\bar{\mathbf{T}}, \bar{\mathbf{T}}_2^1), \{n-1\}, L)$, so

$$\text{Shape}(\bar{\mathbf{T}}_1^2) = (\text{Shape}(\bar{\mathbf{T}})\backslash\{S_{ij}\}) \cup \{S_{i-1,j+1}\}.$$

Now $\mathbf{S}(\bar{\mathbf{T}}_2^2) = \bar{\mathbf{T}}_2^1$, so by induction on Theorem 3.2.2 there is a sequence Σ_2 for $\Pi\backslash\{\alpha_n\}$ such that $(\bar{\mathbf{T}}_1^2, \bar{\mathbf{T}}_2^2) \in T_{\Sigma_2}^L((\bar{\mathbf{T}}_1^1, \bar{\mathbf{T}}_2^1))$. Let $(\mathbf{T}_1', \mathbf{T}_2') \in T_{\Sigma_2}^L((\mathbf{T}_1^1, \mathbf{T}_2^1))$ be such that $(\mathbf{T}_1', \mathbf{T}_2') - L = (\bar{\mathbf{T}}_1^2, \bar{\mathbf{T}}_2^2)$. Then $P(n, \mathbf{T}_1') = P'(n, \mathbf{T}_1^1) = P'$, so, setting $\Sigma = \Sigma_1\Sigma_2$, we have prove the lemma in this case. □

REMARK. If $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_B(M_1, M_2)$ we must modify the proof of Lemma 3.2.9

as follows. The argument in subcase (a) of case I breaks down if $i = 3$ and $r > j + 1$ (since now i is odd, it is not now true that $i > 2$ implies $i \geqslant 4$). If $r > j + 3$ we can set $P'' = \{S_{1,r-1}, S_{1,r}\}$ and proceed as before. Similarly the argument in subcase (b) of case I breaks down if $l = 3$ and $\kappa_1(\mathbf{T}_1) > k + 1$. Again, if $\kappa_1(\mathbf{T}_1) > k + 3$ we can proceed as usual. We are left (in case I) with the situation where $i = l = 3$, $\rho_1(\mathbf{T}_1) = \kappa_1(\mathbf{T}_1) = 5$, $P(n, \mathbf{T}_1) = \{S_{3,2}, S_{3,3}\}$, and $P' = \{S_{2,3}, S_{3,3}\}$. In this case set $P'' = \{S_{1,4}, S_{1,5}\}$ and $P^* = \{S_{4,1}, S_{5,1}\}$. Then, using cases B, H, and the transposed case D, we can proceed as in case J to move the n first to P'', from there to P^*, and finally from P^* to P'.

We must also modify the proof of case J. In case J we need not have $i > 1$. If $i > 1$, we proceed as before. If $i = 1$ but $j > 2$ we can proceed in an analogous fashion, that is we set $r = \kappa_{j-2}(\mathbf{T}_1)$. If $r = i + 2$ we set $P'' = \{S_{i+2,j-2}, S_{i+2,j-1}\}$; if $r > i + 2$ we set $P'' = \{S_{r-1,j-2}, S_{r,j-2}\}$. In either case we proceed as before. We are left with the situation where $i = 1$ and $j = 2$. But then we must have $\mathbf{T}_1 = (F_1, \phi_B)$, so we may take $\Sigma = (\alpha_2, \alpha_1)$ and $(\mathbf{T}'_1, \mathbf{T}'_2) = ((F_2, \phi_B), \mathbf{T}_2)$.

3.2.10. COROLLARY. *Theorem* 3.2.2 *holds when* $\mathbf{T}_2$ *and* $\mathbf{T}'_2$ *are special.*

Proof. As usual we assume by induction on n that Theorem 3.2.2 is true when $M_1 = \{1, \ldots, n-1\}$. Let $P' = P(n, \mathbf{T}'_1)$. Applying Lemma 3.2.9, let Σ_1 be a sequence for Π and $(\mathbf{T}^1_1, \mathbf{T}^1_2) \in T^L_{\Sigma_1}((\mathbf{T}_1, \mathbf{T}_2))$ be such that $P(n, \mathbf{T}^1_1) = P'$ and $\mathbf{T}^1_2 = \mathbf{T}_2$. Since by hypothesis $\mathbf{T}'_2 = \mathbf{T}_2$, we have $\mathbf{T}^1_2 = \mathbf{T}'_2$. Now let $(\bar{\mathbf{T}}^1_1, \bar{\mathbf{T}}^1_2) = (\mathbf{T}^1_1, \mathbf{T}^1_2) - L$ and $(\bar{\mathbf{T}}'_1, \bar{\mathbf{T}}'_2) = (\mathbf{T}'_1, \mathbf{T}'_2) - L$. Since $P(n, \mathbf{T}^1_1) = P(n, \mathbf{T}'_1)$ and $\mathbf{T}^1_2 = \mathbf{T}'_2$ we have $\bar{\mathbf{T}}^1_2 = \bar{\mathbf{T}}'_2$. Thus by induction on Theorem 3.2.2 there is a sequence Σ_2 for $\Pi \backslash \{\alpha_n\}$ such that $(\bar{\mathbf{T}}'_1, \bar{\mathbf{T}}'_2) \in T^L_{\Sigma_2}((\bar{\mathbf{T}}^1_1, \bar{\mathbf{T}}^1_2))$. Let (by Lemma 3.2.6) $(\mathbf{T}'_3, \mathbf{T}'_4) \in T^L_{\Sigma_2}((\mathbf{T}^1_1, \mathbf{T}^1_2))$ be such that $(\bar{\mathbf{T}}'_1, \bar{\mathbf{T}}'_2) = (\mathbf{T}'_3, \mathbf{T}'_4) - L$. Setting $\Sigma = \Sigma_1 \Sigma_2$, we will have proved the corollary once we show that $(\mathbf{T}'_3, \mathbf{T}'_4) = (\mathbf{T}'_1, \mathbf{T}'_2)$. Now $\text{Shape}(\bar{\mathbf{T}}'_1) = \text{Shape}(\bar{\mathbf{T}}^1_1)$, and thus by Lemma 3.2.7–3) we have $P(n, \mathbf{T}'_3) = P(n, \mathbf{T}^1_1)$. Since $P(n, \mathbf{T}^1_1) = P(n, \mathbf{T}'_1)$ and $\mathbf{T}'_3 - n = \bar{\mathbf{T}}'_1 = \mathbf{T}'_1 - n$ we have $\mathbf{T}'_3 = \mathbf{T}'_1$. It follows that $\text{Shape}(\mathbf{T}'_4) = \text{Shape}(\mathbf{T}'_2)$, and since $\mathbf{T}^1_2 = \mathbf{T}'_2$ we have $\mathbf{T}'_4 = \mathbf{T}^1_2 = \mathbf{T}'_2$. □

We can now complete the proof of Theorem 3.2.2.

PROOF OF THEOREM 3.2.2. By Proposition 3.2.4 we can find sequences Σ_1 and Σ_2 for Π and pairs of tableaux $(\mathbf{T}^1_1, \mathbf{T}^1_2) \in T^L_{\Sigma_1}((\mathbf{T}_1, \mathbf{T}_2))$ and $(\mathbf{T}^2_1, \mathbf{T}^2_2) \in T^L_{\Sigma_2}((\mathbf{T}'_1, \mathbf{T}'_2)$ such that $\mathbf{T}^1_2$ and $\mathbf{T}^2_2$ are special, that is, $\mathbf{T}^1_2 = \mathbf{S}(\mathbf{T}_2)$ and $\mathbf{T}^2_2 = \mathbf{S}(\mathbf{T}'_2)$. By hypothesis then $\mathbf{T}^1_2 = \mathbf{T}^2_2$. By Corollary 3.2.10 there is a sequence Σ_3 for Π such that $(\mathbf{T}^2_1, \mathbf{T}^2_2) \in T^L_{\Sigma_3}((\mathbf{T}^1_1, \mathbf{T}^1_2))$. Setting $\Sigma = \Sigma_1 \Sigma_3 \Sigma_2^{-1}$, we have $(\mathbf{T}'_1, \mathbf{T}'_2) \in T^L_{\Sigma}((\mathbf{T}_1, \mathbf{T}_2))$ (this uses Remark 3.2.1), as was to be shown. □

Section 3

3.3.1. NOTATION. (1) If $S \in \mathscr{F}$ and $S = S_{ij}$ we write $\rho(S) = i$ and $\kappa(S) = j$.

(2) Let $\mathbf{T} \in \mathscr{T}_K(M)$ with $K = B$, C, or D. We write

$$n_h(\mathbf{T}) = |\{k \mid k \in M \quad \text{and } P(k, \mathbf{T}) \text{ is horizontal}\}| \quad \text{and}$$

$$n_v(\mathbf{T}) = |\{k \mid k \in M \quad \text{and} \quad P(k, \mathbf{T}) \text{ is vertical}\}|.$$

3.3.2. PROPOSITION. *Let $\mathbf{T} \in \mathscr{T}_K(M)$ with $K = B, C$, or D.*

(1) *Let $c, c' \in \mathrm{OC}^*(\mathbf{T})$. Then*

(a) $\rho(S_b(c)) < \rho(S_b(c')) < \rho(S_f(c))$ *if and only if* $\rho(S_b(c)) < \rho(S_f(c')) < \rho(S_f(c))$,

(b) $\rho(S_f(c)) < \rho(S_b(c')) < \rho(S_b(c))$ *if and only if* $\rho(S_f(c)) < \rho(S_f(c')) < \rho(S_b(c))$.

(2) *If c and c' satisfy either side of either equivalence of part* (1), *we have* $\inf c < \inf c'$.

(3) *Suppose $K = B$ or C, $c \in \mathrm{OC}(\mathbf{T}) \backslash \mathrm{OC}^*(\mathbf{T})$, and $c' \in \mathrm{OC}^*(\mathbf{T})$. Then* $\rho(S(c)) < \rho(S_b(c'))$ *if and only if* $\rho(S(c)) < \rho(S_f(c'))$.

(4) *Statements* (1), (2), *and* (3) *also hold with κ in place of ρ.*

3.3.3. DEFINITION. If c and c' are related by one of the pairs of inequalities of Proposition 3.3.2–1) then we say that c' is nested in c. By Lemma 3.3.5 this is equivalent to a similar relation involving κ's.

We will use the following lemmas in the proof of Proposition 3.3.2.

3.3.4. LEMMA. *Let $\mathbf{T} \in \mathscr{T}_K(M)$ with $K = B, C$, or D, and let $c \in \mathrm{OC}^*(\mathbf{T})$. Then we have*

(1) $\rho(S_b(c)) \neq \rho(S_f(c))$.

(2) *If $c' \in \mathrm{OC}^*(\mathbf{T})$ and $c \neq c'$ then* $\{\rho(S_b(c')), \rho(S_f(c'))\} \cap \{\rho(S_b(c)), \rho(S_f(c))\} = \varnothing$.

(3) *If $c' \in \mathrm{OC}(\mathbf{T}) \backslash \mathrm{OC}^*(\mathbf{T})$ then* $\rho(S(c')) \notin \{\rho(S_b(c)), \rho(S_f(c))\}$.

(4) *Statements* (1), (2), *and* (3) *also hold with κ in place of ρ.*

Proof. These statements assert that any row has at most one corner or hole in it, and that a corner and a hole cannot occur in the same row (and similarly with column in place of row), which facts are obvious from the definitions. □

3.3.5. LEMMA. *Let $\mathbf{T} \in \mathscr{T}_K(M)$ with $K = B, C$, or D, and let $c, c' \in \mathrm{OC}^*(\mathbf{T})$. Then we have*

(1) $\rho(S_b(c)) < \rho(S_b(c'))$ *if and only if* $\kappa(S_b(c)) > \kappa(S_b(c'))$,

(2) $\rho(S_b(c)) < \rho(S_f(c'))$ *if and only if* $\kappa(S_b(c)) > \kappa(S_f(c'))$,

(3) $\rho(S_f(c)) < \rho(S_b(c'))$ *if and only if* $\kappa(S_f(c)) > \kappa(S_b(c'))$,

(4) $\rho(S_f(c)) < \rho(S_f(c'))$ *if and only if* $\kappa(S_f(c)) > \kappa(S_f(c'))$.

Proof. Assume first $c \neq c'$. Then it suffices to prove (1), since then to prove (2) (respectively (3) or (4)), we can, by Proposition 1.5.33–2), replace $\mathbf{T}$ with $\mathbf{E}(\mathbf{T}, c')$ (repectively $\mathbf{E}(\mathbf{T}, c)$ or $\mathbf{E}(\mathbf{T}, c, c')$). Using the symmetry between rows and columns, it suffices to prove the foward implication of (1). So assume $\rho(S_b(c)) < \rho(S_b(c'))$. Let $S_{ij} = S_b(c)$ and $S_{kl} = S_b(c')$. Then $\rho_i(\mathbf{T}) = j$, $\rho_{i+1}(\mathbf{T}) < \rho_i(\mathbf{T})$, and $\rho_k(\mathbf{T}) = l$. Then we have $i < k$, so $i + 1 \leqslant k$ so $\rho_k(\mathbf{T}) \leqslant \rho_{i+1}(\mathbf{T})$. Since $\rho_{i+1}(\mathbf{T}) < \rho_i(\mathbf{T})$, we have $\rho_k(\mathbf{T}) < \rho_i(\mathbf{T})$, that is $l < j$, that is $\kappa(S_b(c')) < \kappa(S_b(c))$, as was to be shown.

Now assume $c = c'$. Then (1) and (4) are vacuously true, and again by symmetry it suffices to prove the foward implications of (2) and (3). Set $S_{ij} = S_b(c)$ and $S_{kl} = S_f(c)$. We have $\rho_i(\mathbf{T}) = j$, $\rho_{i+1}(\mathbf{T}) < \rho_i(\mathbf{T})$, $\rho_k(\mathbf{T}) = l - 1$ and $\rho_{k-1}(\mathbf{T}) > \rho_k(\mathbf{T})$. Assume first $\rho(S_b(c)) < \rho(S_f(c))$. Then as is the first part of the proof we have $\rho_k(\mathbf{T}) < \rho_i(\mathbf{T})$, so $l - 1 < j$ so $l \leqslant j$. But by Lemma 3.3.4–4) $l \neq j$, so we have $\kappa(S_f(c)) < \kappa(S_b(c))$, which proves the foward implication of (2). For (3), assume $\rho(S_f(c)) < \rho(S_b(c'))$, that is, $k < i$. Then $\rho_k(\mathbf{T}) \geqslant \rho_i(\mathbf{T})$, that is $l - 1 \geqslant j$. But then $l > j$, so $\kappa(S_f(c)) > \kappa(S_b(c))$, as was to be shown. □

PROOF OF PROPOSITION 3.3.2. By Lemma 3.3.5 it suffices to prove statements (1), (2), and (3). The proof uses induction on $|M|$, the proposition being vacuously true when $|M| = 0$. Let $e = \sup M$. We prove first parts (1) and (2). If $e \notin c \cup c'$ then the proposition is true by induction (using the appropriate c.s.p.b. of Proposition 2.2.4) so assume $e \in c \cup c'$. Set $\mathbf{T}_0 = \mathbf{T} - e$. We will also assume that $P(e, \mathbf{T})$ is horizontal. (If not, we can look at ${}^t\mathbf{T}$. A comparison of row-indices in $\mathbf{T}$ becomes a comparison of column-indices in ${}^t\mathbf{T}$, which by Lemma 3.3.5 is equivalent to a comparison of row-indices in ${}^t\mathbf{T}$.) If $P'(e, \mathbf{T})$ is also horizontal then again by induction the proposition is true, that is, we consider the cycles $c_0 = c \cap (M \setminus \{e\})$ and $c'_0 = c' \cap (M \setminus \{e\})$ in $\mathbf{T}_0$. By Proposition 2.2.4 we have $\rho(S_b(c_0)) = \rho(S_b(c))$, and similarly for $\rho(S_f(c_0))$, $\rho(S_b(c'_0))$, and $\rho(S_f(c'_0))$. For part (2) we note that $\inf c_0 = \inf c$ and $\inf c'_0 = \inf c'$.

We are left with three cases. They are (1) $c' = \{e\}$, (2) $e \in c', c' \neq \{e\}$, and $P'(e, \mathbf{T})$ is vertical, and (3) $e \in c, c \neq \{e\}$, and $P'(e, \mathbf{T})$ is vertical. Assume first $c' = \{e\}$. Then (since $P(e, \mathbf{T})$ is horizontal) $\rho(S_f(c')) = \rho(S_b(c')) + 1$. So Lemma 3.3.5 proves part (1) of the proposition, and part (2) is obviously true.

Now assume that $e \in c'$, $c' \neq \{e\}$, and $P'(e, \mathbf{T})$ is vertical. Let $P(e, \mathbf{T}) = \{S_{i,j-1}, S_{ij}\}$. Then $P'(e, \mathbf{T}) = \{S_{i-1,j}, S_{ij}\}$. By Proposition 2.2.4 c is a cycle in $\mathbf{T}_0$ (with the same S_b and S_f), and we can write $c' \setminus \{e\} = c'_1 \cup c'_2$ where c'_1 and c'_2 are cycles in $\mathbf{T}_0$ such that $S_b(c'_1) = S_b(c')$, $S_f(c'_1) = S_{i,j-1}$, $S_b(c'_2) = S_{i-1,j}$, and $S_f(c'_2) = S_f(c')$. We prove first the foward implication of part (1) (a) of the proposition. Assume then $\rho(S_b(c)) < \rho(S_b(c')) < \rho(S_f(c))$. By induction we have

$$\rho(S_b(c)) < \rho(S_f(c'_1)) < \rho(S_f(c)),$$

then by Lemma 3.3.4 we have

$\rho(S_b(c)) < \rho(S_b(c'_2)) < \rho(S_f(c))$ (since $\rho(S_b(c'_2)) = \rho(S_f(c'_1)) - 1$). Again by induction we have

$$\rho(S_b(c)) < \rho(S_f(c'_2)) < \rho(S_f(c));$$

and thus $\rho(S_b(c)) < \rho(S_f(c')) < \rho(S_f(c))$, as desired. This argument also shows that the conclusion of part (2) holds under the stated hypotheses, since $\inf c' = \inf\{\inf c_1', \inf c_2'\}$, and we have shown that both c_1' and c_2' satisfy the two sides of the implication of part (1) (a). The rest of parts (1) and (2) of the proposition are proved in an analogous fashion.

Finally, assume tht $e \in c$, $c \neq \{e\}$, and $P'(e, \mathbf{T})$ is vertical. Let $P(e, \mathbf{T}) = \{S_{i,j-1}, S_{ij}\}$. Then $P'(e, \mathbf{T}) = \{S_{i-1,j}, S_{ij}\}$. We can write $c \backslash \{e\} = c_1 \cup c_2$ where c_1 and c_2 are cycles in $\mathbf{T}_0$ such that $S_b(c_1) = S_b(c)$, $S_f(c_1) = S_{i,j-1}$, $S_b(c_2) = S_{i-1,j}$, and $S_f(c_2) = S_f(c)$. Replacing if necessary $\mathbf{T}$ with $\mathbf{E}(\mathbf{T}, c')$ it suffices to prove one implication of each of (1) (a) and (1) (b).

We will prove first the foward implication of part (1) (a). Assume $\rho(S_b(c)) < \rho(S_b(c')) < \rho(S_f(c))$. Then (using Lemma 3.3.4) we must have either $\rho(S_b(c)) < i - 1$ or $\rho(S_f(c)) > i$. We will assume that $\rho(S_b(c)) < i - 1$; the proof under the other assumption is similar. Then $\rho(S_b(c_1)) < \rho(S_b(c_2)) < \rho(S_f(c_1))$, so by induction we have

$$\rho(S_b(c_1)) < \rho(S_f(c_2)) < \rho(S_f(c_1)).$$

Now, since $\rho(S_b(c')) < \rho(S_f(c)) = \rho(S_f(c_2))$ and since $S_b(c_1) = S_b(c)$, we have

$$\rho(S_b(c_1)) < \rho(S_b(c')) < \rho(S_f(c_1)),$$

so again by induction

$$\rho(S_b(c_1)) < \rho(S_f(c')) < \rho(S_f(c_1)).$$

So we have $\rho(S_b(c)) < \rho(S_f(c'))$. It remains to show that $\rho(S_f(c')) < \rho(S_f(c))$. To do this we will assume the contrary and derive a contradiction. So assume $\rho(S_f(c')) > \rho(S_f(c))$. Now $S_f(c_2) = S_f(c)$, and we have already $\rho(S_f(c')) < \rho(S^f(c_1)) = i$, so (again using Lemma 3.3.4) we have

$$\rho(S_f(c_2)) < \rho(S_f(c')) < i - 1 = \rho(S_b(c_2));$$

so by induction we conclude that $\rho(S_f(c_2)) < \rho(S_b(c')) < \rho(S_b(c_2))$. But $S_f(c_2) = S_f(c)$, so this contradicts $\rho(S_b(c')) < \rho(S_f(c))$. So this proves the foward implication of part (1) (a) under these hypotheses. The conclusion of part (2) in this situation can now also be easily seen. By induction and by what we have shown above, we have $\inf c_1 < \inf c'$ and also $\inf c_1 < \inf c_2$. Since $\inf c = \inf\{\inf c_1, \inf c_2\}$, we have $\inf c < \inf c'$, as desired.

We now prove the foward implication of part (1) (b). Assume $\rho(S_f(c)) < \rho(S_b(c')) < \rho(S_b(c))$. We first show that

$$\rho(S_b(c_1)) > i \qquad \text{and} \qquad \rho(S_f(c_2)) < i - 1. \tag{3.3.6}$$

To do this, assume not. So suppose $\rho(S_b(c_1)) < i - 1$. (As usual, $\rho(S_b(c_1)) \in \{i - 1, i\}$ is ruled out by Lemma 3.3.4.) Then $\rho(S_b(c_1)) < \rho(S_b(c_2)) < \rho(S_f(c_1))$, so by induction we have $\rho(S_b(c_1)) < \rho(S_f(c_2))$. But $S_b(c_1) = S_b(c)$ and $S_f(c_2) = S_f(c)$, so this contradicts $\rho(S_f(c)) < \rho(S_b(c))$. Similarly, $\rho(S_f(c_2)) > i$ is impossible, so we have established 3.3.6. Given this, we have either $\rho(S_f(c_1)) < \rho(S_b(c')) < \rho(S_b(c_1))$ or $\rho(S_f(c_2)) < \rho(S_b(c')) < \rho(S_b(c_2))$. Then by induction we have either $\rho(S_f(c_1)) < \rho(S_f(c')) < \rho(S_b(c_1))$ or $\rho(S_f(c_2)) < \rho(S_f(c')) < \rho(S_b(c_2))$. In the first case, since $S_b(c_1) = S_b(c)$ and since $\rho(S_f(c_1)) = i > \rho(S_f(c_2)) = \rho(S_f(c))$, we have $\rho(S_f(c)) < \rho(S_f(c')) < \rho(S_b(c))$ (as desired); in the second case an analogous argument gives the same conclusion. Also, the conclusion of part (2) of the proposition under these hypotheses follows from what we have shown; by induction we have either $\inf c_1 < \inf c'$ or $\inf c_2 < \inf c'$, and $\inf c = \inf\{\inf c_1, \inf c_2\}$. This completes the proof of parts (1) and (2) of the proposition.

Part (3) of the proposition is proved in a similar fashion; we omit the details.

□

3.3.7. PROPOSItION. *Let* $\mathbf{T} \in \mathscr{T}_K(M)$ *with* $K = B, C,$ *or* D, *and let* $c \in \mathrm{OC}^*(\mathbf{T})$. *The following are equivalent:*

(1) $\rho(S_f(c)) < \rho(S_b(c))$
(2) $\kappa(S_b(c)) < \kappa(S_f(c))$
(3) $n_h(\mathbf{E}(\mathbf{T}, c)) = n_h(\mathbf{T}) + 1$
(4) $n_v(\mathbf{E}(\mathbf{T}, c)) = n_v(\mathbf{T}) - 1$
(5) $P(\inf c, \mathbf{T})$ *is vertical.*

3.3.8. PROPOSITION. *Let* $\mathbf{T} \in \mathscr{T}_K(M)$ *with* $K = B, C,$ *or* D, *and let* $c \in \mathrm{OC}^*(\mathbf{T})$. *The following are equivalent*:

(1) $\rho(S_b(c)) < \rho(S_f(c))$
(2) $\kappa(S_f(c)) < \kappa(S_b(c))$
(3) $n_h(\mathbf{E}(\mathbf{T}, c)) = n_h(\mathbf{T}) - 1$
(4) $n_v(\mathbf{E}(\mathbf{T}, c)) = n_v(\mathbf{T}) + 1$
(5) $P(\inf c, \mathbf{T})$ *is horizontal.*

3.3.9. DEFINITION. A cycle $c \in \mathrm{OC}^*(\mathbf{T})$ which satisfies the equivalent conditions of Proposition 3.3.7 is called an up cycle (in $\mathbf{T}$). A cycle $c \in \mathrm{OC}^*(\mathbf{T})$ which satisfies the equivalent conditions of Proposition 3.3.8 is called a down cycle (in $\mathbf{T}$).

REMARK. Of course (e.g. by the conditions (5)) every $c \in \mathrm{OC}^*(\mathbf{T})$ is either an up cycle or a down cycle. This implies the non-obvious fact that $n_h(\mathbf{E}(\mathbf{T}, c)) = n_h(\mathbf{T}) \pm 1$.

PROOF OF PROPOSITIONS 3.3.7 and 3.3.8. We note first that the equivalence of conditions (3) and (4) (of either proposition) is obvious, and that the equivalence of conditions (1) and (2) is part of Lemma 3.3.5. Thus it suffices to prove that given a cycle $c \in \mathrm{OC}^*(\mathbf{T})$, either c satisfies conditions (1), (3), and (5) of Proposition 3.3.7 or c satisfies conditions (1), (3), and (5) of Proposition 3.3.8. The proof uses induction on $|M|$, the propositions being vacuously true when $|M| = 0$. Let $e = \sup M$. If $e \notin c$ the propositions are true by induction, so assume $e \in c$. If $c = \{e\}$ the propositions are obvious, so assume $c \neq \{e\}$.

We will assume $P(e, \mathbf{T})$ is horizontal (when $P(e, \mathbf{T})$ is vertical we have an analogous argument, interchanging rows and columns). Let

$$P(e, \mathbf{T}) = \{S_{i,j-1}, S_{ij}\}.$$

Let $\mathbf{T}_0 = \mathbf{T} - e$. Suppose first that $P'(e, \mathbf{T})$ is also horizontal. Let $c_0 = c\backslash\{e\}$. Then $c_0 \in \mathrm{OC}^*(\mathbf{T}_0)$. We have $\rho(S_f(c_0)) = \rho(S_f(c))$, $\rho(S_b(c_0)) = \rho(S_b(c))$, $n_h(\mathbf{T}_0) = n_h(\mathbf{T})$, $n_h(\mathbf{E}(\mathbf{T}_0, c_0)) = n_h(\mathbf{E}(\mathbf{T}, c))$, and $\inf c_0 = \inf c$. By induction either c_0 satisfies conditions (1), (3), and (5) of Proposition 3.3.7 or c_0 satisfies conditions (1), (3), and (5) of Proposition 3.3.8, and thus the same is true for c.

Henceforth assume that $P'(e, \mathbf{T})$ is vertical. Then $P'(e, \mathbf{T}) = \{S_{i-1,j}, S_{ij}\}$. We can write $c\backslash\{e\} = c_1 \cup c_2$ where c_1 and c_2 are cycles in $\mathbf{T}_0$ such that $S_b(c_1) = S_b(c)$, $S_f(c_1) = S_{i,j-1}$, $S_b(c_2) = S_{i-1,j}$, and $S_f(c_2) = S_f(c)$. By induction we have that each of c_1 and c_2 satisfy the equivalent conditions of either Proposition 3.3.7 or Proposition 3.3.8.

Suppose first that both c_1 and c_2 are up cycles. We will show that c satisfies conditions (1), (3), and (5) of Proposition 3.3.7. Note first that

$$\rho(S_f(c)) = \rho(S_f(c_2)) < \rho(S_b(c_2)) < \rho(S_f(c_1)) < \rho(S_b(c_1)) = \rho(S_b(c)),$$

so statement (1) of Proposition 3.3.7 holds. Next we have (since $P(e, \mathbf{E}(\mathbf{T}, c)) = P'(e, \mathbf{T})$ is vertical and since $\mathbf{E}(\mathbf{T}_0, c_1, c_2) = \mathbf{E}(\mathbf{T}, c) - e$) that

$$n_h(\mathbf{E}(\mathbf{T}, c)) = n_h(\mathbf{E}(\mathbf{T}_0, c_1, c_2)) = n_h(\mathbf{T}_0) + 2 = n_h(\mathbf{T}) + 1$$

so statement (3) of Proposition 3.3.7 holds. Finally, if $a = \inf c$ then either $a = \inf c_1$ or $a = \inf c_2$ so statement (5) of Proposition 3.3.7 holds.

Suppose next that c_1 is a down cycle. We will show that c satisfies conditions (1), (3), and (5) of Proposition 3.3.8. We first establish that c_2 is an up cycle. To see this, note first that $\rho(S_b(c_1)) < \rho(S_b(c_2)) < \rho(S_f(c_1))$ (this uses Lemma 3.3.4). Then by Proposition 3.3.2 we have $\rho(S_b(c_1)) < \rho(S_f(c_2)) < \rho(S_f(c_1))$. Thus $\rho(S_f(c_2)) < \rho(S_b(c_2))$ (again using Lemma 3.3.4), so c_2 is an up cycle. Since $S_b(c) = S_b(c_1)$ and $S_f(c) = S_f(c_2)$, we have also established that

$\rho(S_b(c)) < \rho(S_f(c))$, that is, statement (1) of Proposition 3.3.8 holds for c. Now

$$n_h(\mathbf{E}(\mathbf{T}, c)) = n_h(\mathbf{E}(\mathbf{T}_0, c_1, c_2)) = n_h(\mathbf{T}_0) = n_h(\mathbf{T}) - 1$$

so statement (3) of Proposition 3.3.8 holds. Finally, by Proposition 3.3.2 we have $\inf c_1 < \inf c_2$ so $\inf c = \inf c_1$. Thus since statement (5) of Proposition 3.3.8 holds for c_1 it holds for c.

Finally, suppose that c_2 is a down cycle. Then, in the same way as in the previous paragraph, we show first that c_1 is an up cycle, and then that c sataisfies conditions (1), (3), and (5) of Proposition 3.3.8. This completes the proof of Propositions 3.3.7 and 3.3.8. □

3.3.10. PROPOSITION. *Suppose* $\mathbf{T} \in \mathcal{T}_K(M)$ *with* $K = B, C,$ *or* D. *Let* c *be a closed cycle in* $\mathbf{T}$. *The following are equivalent:*

(1) $n_h(\mathbf{E}(\mathbf{T}, c)) = n_h(\mathbf{T}) + 2$
(2) $n_v(\mathbf{E}(\mathbf{T}, c)) = n_v(\mathbf{T}) - 2$
(3) $P(\inf c, \mathbf{T})$ *is vertical.*
(4) $P(\sup c, \mathbf{T})$ *is vertical.*

3.3.11. PROPOSITION. *Suppose* $\mathbf{T} \in \mathcal{T}_K(M)$ *with* $K = B$, C, *or* D. *Let* c *be a closed cycle in* $\mathbf{T}$. *The following are equivalent:*

(1) $n_h(\mathbf{E}(\mathbf{T}, c)) = n_h(\mathbf{T}) - 2$
(2) $n_v(\mathbf{E}(\mathbf{T}, c)) = n_v(\mathbf{T}) + 2$
(3) $P(\inf c, \mathbf{T})$ *is horizontal.*
(4) $P(\sup c, \mathbf{T})$ *is horizontal.*

3.3.12. DEFINITION. Suppose $\mathbf{T} \in \mathcal{T}_K(M)$ with $K = B, C,$ or D. Let c be a closed cycle in $\mathbf{T}$. Then c is called an up cycle if it satisfies the equivalent conditions of Proposition 3.3.10, otherwise c is called a down cycle.

PROOF OF PROPOSITIONS 3.3.10 and 3.3.11. The proof uses induction on $|M|$, the propositions being vacuously true when $|M| = 0$. Let $e = \sup M$. If $e \notin c$ the propositions are true by induction, so assume $e \in c$. Since the two conditions (4) are mutually exclusive, it suffices to prove, for each proposition, that (4) implies (1), (2), and (3). We will prove this for Proposition 3.3.11 (an analogous argument works for Proposition 3.3.10). So assume $P(e, \mathbf{T})$ is horizontal, say $P(e, \mathbf{T}) = \{S_{i,j-1}, S_{ij}\}$. Since c is closed we have $P'(e, \mathbf{T}) = \{S_{i-1,j}, S_{ij}\}$. Let $\mathbf{T}_0 = \mathbf{T} - e$ and let $c_0 = c \backslash \{e\}$. Then $c_0 \in \mathrm{OC}^*(\mathbf{T}_0)$, $S_b(c_0) = S_{i-1,j}$, and $S_f(c_0) = S_{i,j-1}$. Thus c_0 satisfies condition (1) of Proposition 3.3.8, and so c_0 satisfies conditions (5) and (3) of Proposition 3.3.8. So we have that $P(\inf c_0, \mathbf{T}_0)$ is horizontal. Since $\inf c = \inf c_0$ we have $P(\inf c, \mathbf{T})$ is horizontal, that is, condition (3) of Proposition 3.3.11 holds. Also, we have $n_h(\mathbf{E}(\mathbf{T}_0, c_0)) = n_h(\mathbf{T}_0) - 1$. But then

$$n_h(\mathbf{E}(\mathbf{T}, c)) = n_h(\mathbf{E}(\mathbf{T}_0, c_0)) = n_h(\mathbf{T}_0) - 1 = n_h(\mathbf{T}) - 2,$$

that is, condition (1) of Proposition 3.3.11 holds. Since clearly condition (1) implies condition (2), we are done. □

3.3.13. REMARK. Let c be an up cycle in $\mathbf{T}$. Then c is a down cycle in ${}^t\mathbf{T}$. (This follows from condition (5) of Propositions 3.3.7 and 3.3.8 when c is open, and from condition (3) of Propositions 3.3.10 and 3.3.11 when c is closed.) Also, c is a down cycle in $\mathbf{E}(\mathbf{T}, c)$. (This follows from condition (3) of Propositions 3.3.7 and 3.3.8 when c is open, and from condition (1) of Propositions 3.3.10 and 3.3.11 when c is closed.)

3.3.14. DEFINITION. Let $\mathbf{T} \in \mathscr{T}_K(M)$ and let c be a non-empty subset of M. We set

$$V_i(c) = \{k \mid k \in c \quad \text{and} \quad \rho(S) = i \quad \text{for some } S \in P(k, \mathbf{T})\}.$$

We let $\rho_{\text{inf}}(c) = \inf\{i \mid V_i(c) \neq \varnothing\}$, and similarly $\rho_{\text{sup}}(c)$. If $i = \rho_{\text{inf}}(c)$ we write $V(c)$ for $V_i(c)$. If $V_i(c) \neq \varnothing$ we let $r_i(c) = \inf V_i(c)$. We write $r(c) = \inf V(c)$ and $s(c) = \sup V(c)$.

3.3.15. LEMMA. *Let* $\mathbf{T} \in \mathscr{T}_K(M)$ *and let* $c \in \mathrm{OC}^*(\mathbf{T})$. Then

(1) $\rho_{\text{inf}}(c) \leqslant \rho(S_f(c))$ *and* $\rho_{\text{sup}}(c) \geqslant \rho(S_f(c)) - 1$.

(2) *Let* $e = \sup M$ *and suppose* $e \in c$ *and* $c \neq \{e\}$. *Then* $\rho_{\text{inf}}(c \backslash \{e\}) \leqslant \rho^1(P(e, \mathbf{T}))$, that is, $\rho_{\text{inf}}(c \backslash \{e\}) = \rho_{\text{inf}}(c)$.

Proof. To prove statement (1), let $S_f(c) = S_{ij}$. Since $S_f(c)$ is empty in $\mathbf{T}$ and $S_f(c) \in P'(k, \mathbf{T})$ for some $k \in c$, then (from the definition of $P'(k, \mathbf{T})$) we must have either $S_{i-1,j} \in P(k, \mathbf{T})$ or $S_{i,j-1} \in P(k, \mathbf{T})$. Statement (2) is verified by an examination of the cases (and their transposes) of Proposition 2.2.4, using in some cases statement (1) of this lemma. □

3.3.16. LEMMA. *Let* $\mathbf{T} \in \mathscr{T}_K(M)$. *If* $c, c' \in \mathrm{OC}^*(\mathbf{T})$ *and if* c' *is nested in* c *then* $\rho_{\text{inf}}(c) < \rho_{\text{inf}}(c')$ *and* $\rho_{\text{sup}}(c) \geqslant \rho_{\text{sup}}(c')$.

Proof. The proof is by induction on $|M|$. Let $e = \sup M$. If $e \notin c \cup c'$ then the lemma is true by induction, so assume $e \in c \cup c'$. Suppose $c' = \{e\}$. Then $\{\rho_{\text{inf}}(c'), \rho_{\text{sup}}(c')\} = \{\rho(S_b(c')), \rho(S_f(c'))\}$, so this lemma is a consequence of Lemma 3.3.15–1. So assume $c' \neq \{e\}$. Let $\mathbf{T}_0 = \mathbf{T} - e$. Assume first that either $S_b(c) \in P(e, \mathbf{T})$ or $S_f(c) \in P'(e, \mathbf{T})$. Let $c_0 = c \backslash \{e\}$. Then c_0 and c' are open cycles in $\mathbf{T}_0$ and c' is nested in c_0 (since by Proposition 2.2.4, if $P(e, \mathbf{T})$ is horizontal we have $\rho(S_b(c_0)) = \rho(S_b(c))$ and $\rho(S_f(c_0)) = \rho(S_f(c))$, and if instead $P(e, \mathbf{T})$ is vertical we have the corresponding statement with κ in place of ρ). Now clearly $\rho_{\text{inf}}(c) \leqslant \rho_{\text{inf}}(c_0)$ and $\rho_{\text{sup}}(c) \geqslant \rho_{\text{sup}}(c_0)$. Thus by induction the lemma is true in this case. Assume next that $S_b(c') \in P(e, \mathbf{T})$ or $S_f(c') \in P'(e, \mathbf{T})$. Let $c'_0 = c' \backslash \{e\}$; then, as cycles in $\mathbf{T}_0, c'_0$ is nested in c. Now Lemma 3.3.15–2 says that $\rho_{\text{inf}}(c') = \rho_{\text{inf}}(c'_0)$, so induction proves the first statement of this lemma. Now if $P(e, \mathbf{T})$ is horizontal then $\rho_{\text{sup}}(c') = \rho_{\text{sup}}(c'_0)$, if $P(e, \mathbf{T})$ is vertical and $S_b(c) \in P(e, \mathbf{T})$ then $\rho_{\text{sup}}(c') \in \{\rho_{\text{sup}}(c'_0), \rho(S_b(c'))\}$, if $P(e, \mathbf{T})$ is vertical and $S_f(c) \in P'(e, \mathbf{T})$ then $\rho_{\text{sup}}(c') \in \{\rho_{\text{sup}}(c'_0), \rho(S_f(c')) - 1\}$. So (using also Lemma 3.3.15–1 and the inequalities of Proposition 3.3.2–1 when $\rho_{\text{sup}}(c') \in \{\rho(S_b(c')), \rho(S_f(c')) - 1\}$) the second statement of the lemma is proved in this case by induction. Next assume that $e \in c'$ and $c' \backslash \{e\} = c'_1 \cup c'_2$ where c'_1 and c'_2 are open cycles in $\mathbf{T}_0$. Then the argument given in the proof of Proposition 3.3.2 shows that, considered as cycles in $\mathbf{T}_0$, both c'_1 and c'_2 are nested in c. Now by Lemma 3.3.15–2 we have $\rho_{\text{inf}}(c') = \inf\{\rho_{\text{inf}}(c'_1), \rho_{\text{inf}}(c'_2)\}$. Also, if $P(e, \mathbf{T})$ is vertical then

$$\rho_{\text{sup}}(c') = \sup\{\rho_{\text{sup}}(c'_1), \rho_{\text{sup}}(c'_2)\};$$

if $P(e, \mathbf{T})$ is horizontal then either

$$\rho_{\text{sup}}(c') = \sup\{\rho_{\text{sup}}(c'_1), \rho_{\text{sup}}(c'_2)\}$$

or

$$\rho_{\text{sup}}(c') = \rho^1(P(e, \mathbf{T})) \in \{\rho(S_f(c'_1)), \rho(S_f(c'_2))\}.$$

So again the lemma is true by induction (and using Lemma 3.3.15–1 and the inequalities of Proposition 3.3.2–1 in the last-mentioned situation). Finally assume that $e \in c$ and $c \backslash \{e\} = c_1 \cup c_2$ where c_1 and c_2 are open cycles in $\mathbf{T}_0$. Then the argument given in the proof of Proposition 3.3.2 shows that either c' is nested in c_1 or c' is nested in c_2 (considered as cycles in $\mathbf{T}_0$). Again, the lemma now follows by induction. □

3.3.17. LEMMA. *Let* $\mathbf{T} \in \mathscr{T}_K(M)$ *and let* $c \in \mathrm{OC}^*(\mathbf{T})$. *If* $\rho_{\text{inf}}(c) \leqslant i \leqslant \rho_{\text{sup}}(c)$ *then* $V_i(c) \neq \varnothing$.

Proof. The proof is by induction on $|M|$. Let $e = \sup M$. It $e \notin c$ the lemma is true by induction, so assume $e \in c$. The lemma is clear when $c = \{e\}$, so assume $c \neq \{e\}$. The lemma is also easy to verify, using induction and Lemma 3.3.15–1, when $S_b(c) \in P(e, \mathbf{T})$ or $S_f(c) \in P'(e, \mathbf{T})$. Now suppose $c \backslash \{e\} = c_1 \cup c_2$ where $c_1, c_2 \in \mathrm{OC}^*(\mathbf{T} - e)$. If c is an up cycle then the argument in the proof of Propositions 3.3.7 and 3.3.8 shows that either $P(e, \mathbf{T})$ is horizontal and c_1 and c_2 are up cycles or $P(e, \mathbf{T})$ is vertical and (say) c_1 is an up cycle and c_2 is a down cycle which is nested in c_1. The latter situation splits into two cases, depending on whether $S_f(c_1) \in P(e, \mathbf{T})$ or $S_f(c_2) \in P(e, \mathbf{T})$. We have an analogous description of three cases when c is a down cycle. Then an examination of these six cases, using Lemmas 3.3.15–1 and 3.3.16 and induction, verifies this lemma. □

3.3.18. LEMMA. *Let* c *and* c' *be as in Lemma* 3.3.16. *Then for all* $\rho_{\inf}(c') \leqslant i \leqslant \rho_{\sup}(c')$ *we have* $r_i(c) < r_i(c')$.

Proof. Given the previous lemma, this is easy to prove, using induction and the same breakdown into cases which was used to prove Lemma 3.3.16. We omit the details. □

3.3.19. PROPOSITION. *Let* $\mathbf{T} \in \mathscr{T}_K(M)$ *and let* c *be an open up cycle in* $\mathbf{T}$. *Then*

(1) $P(r_i(c), \mathbf{T})$ *is vertical for all* $\rho_{\inf}(c) \leqslant i \leqslant \rho_{\sup}(c)$.
(2) *If* $S \in P(r_i(c), \mathbf{T})$ *and* $\rho(S) = \rho^1(r_i(c), \mathbf{T})$ *then* S *is* ϕ_K*-fixed.*
(3) *If* $\rho_{\inf}(c) \neq \rho(S_f(c))$ *then* $P(s(c), \mathbf{T})$ *is vertical.*

Proof. The proof is by induction on $|M|$. Let $e = \sup M$. If $e \notin c$ then the proposition is true by induction, so assume $e \in c$. If $c = \{e\}$ then the proposition is obviously true, so assume $c \neq \{e\}$. Let $\mathbf{T}_0 = \mathbf{T} - e$. There are several cases. In the first four cases $c\backslash\{e\} \in \mathrm{OC}^*(\mathbf{T}_0)$. We set $c_0 = c\backslash\{e\}$. By Lemma 3.3.15–2 we have $\rho_{\inf}(c_0) = \rho_{\inf}(c)$.

Case 1. Here $P(e, \mathbf{T})$ is horizontal and $S_b(c) \in P(e, \mathbf{T})$. Then $S_f(c_0) = S_f(c)$. Write $P(e, \mathbf{T}) = \{S_{j,k}, S_{j,k+1}\}$. Then $S_b(c_0) = S_{j,k-1}$, in particular $N_{\mathbf{T}}(S_{j,k-1}) \in c$. Then $\rho_{\sup}(c_0) = \rho_{\sup}(c)$. Since c is an up cycle, Lemma 3.3.15–1 says that $\rho_{\inf}(c) < j$. Thus $s(c) = s(c_0)$ and for all $\rho_{\inf}(c) \leqslant i \leqslant \rho_{\sup}(c)$ we have $r_i(c) = r_i(c_0)$, so the proposition is true by induction.

Case 2. Here $P(e, \mathbf{T})$ is horizontal and $S_f(c) \in P'(e, \mathbf{T})$. Then $S_b(c_0) = S_b(c)$. Write $P(e, \mathbf{T}) = \{S_{j,k-1}, S_{j,k}\}$, so that $S_f(c) = S_{j,k+1}$. By Lemma 3.3.15–1 we have $\rho_{\inf}(c_0) \leqslant j$, and since c is an up cycle we have $j < \rho(S_b(c_0)) \leqslant \rho_{\sup}(c_0)$. Thus $\rho_{\sup}(c) = \rho_{\sup}(c_0)$, and, using Lemma 3.3.17, we have for all $\rho_{\inf}(c) \leqslant i \leqslant \rho_{\sup}(c)$, $r_i(c) = r_i(c_0)$. Now, either $\rho_{\inf}(c) < j$ or $\rho_{\inf}(c) = \rho(S_f(c))$, so again the proposition is true by induction.

Case 3. Here $P(e, \mathbf{T})$ is vertical and $S_b(c) \in P(e, \mathbf{T})$. Then $S_f(c) = S_f(c_0)$. Let $P(c, \mathbf{T}) = \{S_{j-1,k}, S_{j,k}\}$. Then $S_b(c_0) = S_{j-1,k}$. Here the proposition is clearly true by induction, and inspection if $\rho_{\sup}(c_0) = j - 1$.

Case 4. Here $P(e, \mathbf{T})$ is vertical and $S_f(c) \in P'(e, \mathbf{T})$. Let $P(e, \mathbf{T}) = \{S_{j-1,k}, S_{j,k}\}$, so $S_f(c_0) = S_{j-1,k}$ and $S_f(c) = S_{j+1,k}$. Then $\rho_{\inf}(c_0) \leqslant j - 1$ and $\rho_{\sup}(c_0) \geqslant \rho(S_b(c_0)) > j + 1$. Thus, as in case 2, $\rho_{\sup}(c) = \rho_{\sup}(c_0)$ and $r_i(c) = r_i(c_0)$ for all $\rho_{\inf}(c) \leqslant i \leqslant \rho_{\sup}(c)$, so statements (1) and (2) of the proposition are true by induction. For statement (3), we note that either $\rho_{\inf}(c) < j - 1$, in which case (3) is true by induction, or $\rho_{\inf}(c) = j - 1$, in which case $s(c) = e$, so again (3) is true.

In the remaining cases $c\backslash\{e\} = c_1 \cup c_2$, where $c_1, c_2 \in \mathrm{OC}^*(\mathbf{T}_0)$. Again by Lemma 3.3.15–1 we have $\rho_{\inf}(c) = \inf\{\rho_{\inf}(c_1), \rho_{\inf}(c_2)\}$. As in the proof of Propositions 3.3.7 and 3.3.8, there are three possibilities (up to interchange of c_1 and c_2).

Case 5. Here c_1 and c_2 are up cycles and (for some j and k) we have $P(e, \mathbf{T}) = \{S_{j,k-1}, S_{j,k}\}$, $P'(e, \mathbf{T}) = \{S_{j-1,k}, S_{j,k}\}$, $S_b(c_1) = S_b(c)$, $S_f(c_1) = S_{j,k-1}$, $S_b(c_2) = S_{j-1,k}$, and $S_f(c_2) = S_f(c)$. Then

$$\rho_{\sup}(c_1) \geqslant \rho(S_b(c_1)) > \rho(S_f(c_1)) = j.$$

In particular by Lemma 3.3.17 we have $e \neq r_j(c)$ and $\rho_{\sup}(c) = \sup\{\rho_{\sup}(c_1), \rho_{\sup}(c_2)\}$. So for all $\rho_{\inf}(c) \leqslant i \leqslant \rho_{\sup}(c)$ we have either $r_i(c) = r_i(c_1)$ or $r_i(c) = r_i(c_2)$. This proves statements (1) and (2) of the proposition by induction. Now

$$\rho_{\inf}(c_2) \leqslant \rho(S_f(c_2)) < \rho(S_b(c_2)) = j - 1,$$

so $e \neq s(c)$, and thus also statement (3) of the proposition is true by induction.

Case 6. Here c_1 is an up cycle, c_2 is a down cycle, $P(e, \mathbf{T}) = \{S_{j-1,k}, S_{j,k}\}$, $P'(e, \mathbf{T}) = \{S_{j,k-1}, S_{j,k}\}$, $S_b(c_1) = S_b(c)$, $S_f(c_1) = S_{j-1,k}$, $S_b(c_2) = S_{j,k-1}$, and $S_f(c_2) = S_f(c)$. Thus

$$\rho_{\sup}(c_1) \geqslant \rho(S_b(c_1)) > \rho(S_f(c_2)) > \rho(S_b(c_2)) = j$$

and

$$\rho_{\inf}(c_1) \leqslant \rho(S_f(c_1)) = j - 1.$$

In particular by Lemma 3.3.16 we have $\rho_{\inf}(c) = \rho_{\inf}(c)$ and $\rho_{\sup}(c) = \rho_{\sup}(c_1)$. By Lemma 3.3.17 (applied to c_1) we have $e \neq r_{j-1}(c)$ and $e \neq r_j(c)$. Then by Lemma 3.3.18 we have that for all

$$\rho_{\inf}(c) \leqslant i \leqslant \rho_{\sup}(c), \; r_i(c) = r_i(c_1).$$

So as usual we are done by induction (and inspection if $e = s(c)$.

Case 7. Here c_1 is an up cycle, c_2 is a down cycle, $P(e, \mathbf{T}) = \{S_{j-1,k}, S_{j,k}\}$, $P'(e, \mathbf{T}) = \{S_{j,k-1}, S_{j,k}\}$, $S_f(c_1) = S_f(c)$, $S_b(c_1) = S_{j,k-1}$, $S_f(c_2) = S_{j-1,k}$, and $S_b(c_2) = S_b(c)$. The argument in this case is like that of the previous case, so we omit it. □

3.3.20. PROPOSITION. *Let* $\mathbf{T} \in \mathcal{T}_K(M)$ *and let* c *be a closed up cycle in* $\mathbf{T}$. *Then*

(1) $P(r_i(c), \mathbf{T})$ *is vertical for all* $\rho_{\inf}(c) \leqslant i \leqslant \rho_{\sup}(c)$.
(2) *If* $S \in P(r_i(c), \mathbf{T})$ *and* $\rho(S) = \rho^1(r_i(c), \mathbf{T})$ *then* S *is* ϕ_K*-fixed.*
(3) $P(s(c), \mathbf{T})$ *is vertical.*
(4) *If* $S \in P(s(c), \mathbf{T})$ *and* $\rho(S) = \rho^1(s(c), \mathbf{T})$ *then* S *is* ϕ_K*-variable.*

Proof. The proof is by induction on $|M|$. Let $e = \sup M$. If $e \notin c$ then the lemma is true by induction, so assume $e \in c$. Then by Proposition 3.3.10–4, $P(e, \mathbf{T})$ is vertical. Let $\mathbf{T}_0 = \mathbf{T} - e$ and let $c_0 = c \backslash \{e\}$. Then c_0 is an open up cycle in $\mathbf{T}_0$ (by Proposition 3.3.7–5 and Proposition 3.3.10–3) and by Lemma 3.3.15–2, we have $\rho_{\text{inf}}(c_0) = \rho_{\text{inf}}(c)$. Write $P(e, \mathbf{T}) = \{S_{j-1,k}, S_{j,k}\}$. Then $P'(e, \mathbf{T}) = \{S_{j,k-1}, S_{j,k}\}$, $S_f(c_0) = S_{j-1,k}$, and $S_b(c_0) = S_{j,k-1}$. In particular $\rho_{\text{sup}}(c) = \rho_{\text{sup}}(c_0)$, and by Lemma 3.3.15, $\rho_{\text{inf}}(c_0) \leqslant j - 1$. Then by Lemma 3.3.17 we have that $r_i(c) = r_i(c_0)$ for all $\rho_{\text{inf}}(c) \leqslant i \leqslant \rho_{\text{sup}}(c)$. Thus statements (1) and (2) of this proposition follow from Proposition 3.3.19, as does statement (3), since $P(e, \mathbf{T})$ is vertical. Now statement (4) is an obvious consequence of statement (3). To see this, note that if $l \in M$ and $P(l, \mathbf{T}) = \{S_{t,u}, S_{t,u+1}\}$ and $S_{t,u}$ is ϕ_K-fixed then either $S_{t-1,u} \in P'(l, \mathbf{T})$ or $S_{t,u+1} \in P'(l, \mathbf{T})$. So such an l cannot be $s(c)$. □

Section 4

3.4.1. DEFINITION. Let X be either W, $\mathscr{S}(n, n)$, $\mathscr{T}_C(n, n)$ or $\mathscr{T}_B(n, n)$. We define an equivalence relation, the left generalized τ-invariant, on X, as follows. Let $x, y \in X$. We say that $x \sim_0 y$ if $\tau^L(x) = \tau^L(y)$. Inductively, for $m \geqslant 1$ we say that $x \sim_m y$ if $x \sim_{m-1} y$, and if for every sequence Σ for Π with $|\Sigma| = 1$ we have

(1) for every $z \in T_\Sigma^L(x)$ there is a $w \in T_\Sigma^L(y)$ such that $z \sim_{m-1} w$, and
(2) for every $w \in T_\Sigma^L(y)$ there is a $z \in T_\Sigma^L(x)$ such that $z \sim_{m-1} w$.

We say that $x \sim_{GTL} y$, or that x and y have the same left generalized τ-invariant, if $x \sim_m y$ for every non-negative integer m.

We define similarly the right generalized τ-invariant. For the convenience of our proofs by induction we extend (in the obvious way) the definition of $\sim_{GTL}$ to $\mathscr{T}_C(M_1, M_2)$ when $M_1 = \{1, \ldots, n\}$.

3.4.2. REMARK. (1) Since $|T_\Sigma^L(x)| \leqslant 2$ when $|\Sigma| = 1$ we see that if $x \sim_{GTL} y$ then (1) and (2) above hold, for any Σ, with $z \sim_{GTL} w$ in place of $z \sim_{m-1} w$.

(2) Let Σ be a sequence for $\Pi \backslash \{\alpha_1\}$, let $x, y \in X$, and suppose that $T_\Sigma^L(x) \neq \varnothing$. Then $x \sim_{GTL} y$ if and only if $T_\Sigma^L(x) \sim_{GTL} T_\Sigma^L(y)$.

The purpose of this section is to prove that for $(\mathbf{T}_1, \mathbf{T}_2), (\mathbf{T}'_1, \mathbf{T}'_2) \in \mathscr{T}_C(n, n)$ we have $(\mathbf{T}_1, \mathbf{T}_2) \sim_{GTL} (\mathbf{T}'_1, \mathbf{T}'_2)$ if and only if $\mathbf{S}(\mathbf{T}_1) = \mathbf{S}(\mathbf{T}'_1)$. We start by proving the easy implication, that is, Proposition 3.4.7.

3.4.3. PROPOSITION. *Let* $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_C(M_1, M_2)$ *where* $M_1 = \{1, \ldots, n\}$. *Let* U *be a set of extended cycles in* $\mathbf{T}_1$ *relative to* $\mathbf{T}_2$ *such that for every cycle* c *in* $\mathbf{T}_1$ *such that* $c \subseteq \bar{c}$ *for some* $\bar{c} \in U$ *we have* $c \in \mathrm{OC}^*(\mathbf{T}_1)$. *Let* $(\mathbf{T}'_1, \mathbf{T}'_2) = \mathbf{E}((\mathbf{T}_1, \mathbf{T}_2); U, L)$. *Then* $(\mathbf{T}'_1, \mathbf{T}'_2) \sim_{GTL} (\mathbf{T}_1, \mathbf{T}_2)$.

Proof. We prove by induction on m that $(\mathbf{T}'_1, \mathbf{T}'_2) \sim_m (\mathbf{T}_1, \mathbf{T}_2)$. When $m = 0$

this is part of Proposition 3.1.4. Assume then that $m > 1$ and that the proposition is true with $\sim_{m-1}$ in place of $\sim_{GTL}$, and let Σ be a sequence for Π with $|\Sigma| = 1$. Suppose first that $\Sigma = (\alpha, \beta)$ where $\{\alpha, \beta\} = \{\alpha_i, \alpha_{i+1}\}$ with $i \geqslant 2$. Then Proposition 3.1.5–1 and induction give the desired conclusion, since, by that proposition, $T^L_{\alpha\beta}((\mathbf{T}'_1, \mathbf{T}'_2)) = \mathbf{E}(T^L_{\alpha\beta}((\mathbf{T}_1, \mathbf{T}_2)); U', L)$ where U' is the set of extended cycles corresponding to U via the map μ. Now assume $\Sigma = (\alpha, \beta)$ with $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$. For simplicity we will assume $\alpha = \alpha_2$ (an analogous argument holds when $\alpha = \alpha_1$). Suppose first either that $\tilde{F}_1 \subseteq \mathbf{T}_1$ or that $F_1 \subseteq \mathbf{T}_1$ and $1 \notin ec(2, \mathbf{T}_1; \mathbf{T}_2)$. Then clearly one of these is also true with $\mathbf{T}'_1$ and $\mathbf{T}'_2$ in place of $\mathbf{T}_1$ and $\mathbf{T}_2$. Now $T^L_\Sigma((\mathbf{T}_1, \mathbf{T}_2))$ consists of a single element, say $T^L_\Sigma((\mathbf{T}_1, \mathbf{T}_2)) = \{Z\}$. Then clearly we have $T^L_\Sigma((\mathbf{T}'_1, \mathbf{T}'_2)) = \{\mathbf{E}(Z; U', L)\}$, where $U' = U$ if $ec(2, \mathbf{T}_1; \mathbf{T}_2) \notin U$, otherwise $U' = U \backslash \{ec(2, \mathbf{T}_1; \mathbf{T}_2)\}$. Thus by induction we have the desired result. Suppose next that $F_1 \subseteq \mathbf{T}_1$ and $1 \in ec(2, \mathbf{T}_1; \mathbf{T}_2)$. Again, the same is true with $\mathbf{T}'_1$ and $\mathbf{T}'_2$ in place of $\mathbf{T}_1$ and $\mathbf{T}_2$. Then $T^L_\Sigma((T_1, T_2)) = \{Z_1, Z_2\}$ where $Z_1 \neq Z_2$, and, say, $Z_1 = (\mathbf{T}^1_1, \mathbf{T}_2)$ with $F_2 \subseteq \mathbf{T}^1_1$. Then $T^L_\Sigma((\mathbf{T}'_1, \mathbf{T}'_2)) = \{Z'_1, Z'_2\}$ where $Z'_j = \mathbf{E}(Z_j; U, L)$ for $j = 1, 2$. So again by induction we are done. □

3.4.4. COROLLARY. *Let* $(\mathbf{T}_1, \mathbf{T}_2) \in D^L_{\alpha\beta}(\mathscr{T}_C(M_1, M_2))$ with $M_1 = \{1, \ldots, n\}$ *and* $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$. *Suppose that* $T^L_{\alpha\beta}((\mathbf{T}_1, \mathbf{T}_2)) = \{Z_1, Z_2\}$ with $Z_1 \neq Z_2$, and suppose $c(2, \mathbf{T}_1) \in \mathrm{OC}^*(\mathbf{T}_1)$. *Then* $Z_1 \sim_{GTL} Z_2$.

3.4.5. DEFINITION. Let $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_K(M_1, M_2)$ (for $K = B, C$, or D). We define

$$\mathbf{S}((\mathbf{T}_1, \mathbf{T}_2)) = (\mathbf{S}(\mathbf{T}_1), \mathbf{S}(\mathbf{T}_2)).$$

Since in general Shape(S(T)) is determined by Shape(T) we have that $\mathbf{S}((\mathbf{T}_1, \mathbf{T}_2)) \in \mathscr{T}_K(M_1, M_2)$.

As a consequence of Proposition 3.4.3 we have

3.4.6. PROPOSITION. *Suppose* $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_C(M_1, M_2)$ *with* $M_1 = \{1, \ldots, n\}$. Then $\mathbf{S}((\mathbf{T}_1, \mathbf{T}_2)) \sim_{GTL} (\mathbf{T}_1, \mathbf{T}_2)$.

3.4.7. PROPOSITION. *Let* $(\mathbf{T}_1, \mathbf{T}_2)$, $(\mathbf{T}'_1, \mathbf{T}'_2) \in \mathscr{T}_C(M_1, M_2)$ *where* $M_1 = \{1, \ldots, n\}$. *Suppose* $\mathbf{S}(\mathbf{T}_1) = \mathbf{S}(\mathbf{T}'_1)$. Then $(\mathbf{T}_1, \mathbf{T}_2) \sim_{GTL} (\mathbf{T}'_1, \mathbf{T}'_2)$.

Proof. By Proposition 3.4.6 we may assume $\mathbf{T}_1$ is special and $\mathbf{T}_1 = \mathbf{T}'_1$. We will prove by induction on m that $(\mathbf{T}_1, \mathbf{T}_2) \sim_m (\mathbf{T}'_1, \mathbf{T}'_2)$. When $m = 0$ this is obvious, so assume that $m > 1$ and that $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2) \sim_{m-1} (\bar{\mathbf{T}}'_1, \bar{\mathbf{T}}'_2)$ whenever $\bar{\mathbf{T}}_1 = \bar{\mathbf{T}}'_1$ is special. Let $\Sigma = (\alpha, \beta)$ be a sequence for Π. If $\{\alpha, \beta\} = \{\alpha_i, \alpha_{i+1}\}$ with $i \geqslant 2$ then since $\mathbf{T}_1 = \mathbf{T}'_1$ we have $T_{\alpha\beta}(\mathbf{T}_1) = T_{\alpha\beta}(\mathbf{T}'_1)$ and thus are done by induction. Suppose now $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$. There are three possibilities. First is that $c(2, \mathbf{T}_1)$ is closed. In this case $T^L_\Sigma((\mathbf{T}_1, \mathbf{T}_2)) = \{(\mathbf{T}_3, \mathbf{T}_2)\}$ and

$T_\Sigma^L((\mathbf{T}'_1, \mathbf{T}'_2)) = \{(\mathbf{T}_3, \mathbf{T}'_2)\}$ for some $\mathbf{T}_3 \in \mathscr{T}_C(n)$. The second possibility is that

$$T_\Sigma^L((\mathbf{T}_1, \mathbf{T}_2)) = \{(\mathbf{T}_1^1, \mathbf{T}_2), (\mathbf{T}_1^2, \mathbf{T}_2)\}$$

with $\mathbf{T}_1^1 \neq \mathbf{T}_1^2$. In this case

$$T_\Sigma^L((\mathbf{T}'_1, \mathbf{T}'_2)) = \{(\mathbf{T}_1^1, \mathbf{T}'_2), (\mathbf{T}_1^2, \mathbf{T}'_2)\}.$$

The third possibility is that

$$T_\Sigma^L((\mathbf{T}_1, \mathbf{T}_2)) = \{Z_1, Z_2\}$$

(where possibly $Z_1 = Z_2$) with $Z_1 = \mathbf{S}(Z_1) = \mathbf{S}(Z_2)$. In this case we have, setting $Z_1 = (\mathbf{T}_1^1, \mathbf{T}_2)$, that $T_\Sigma^L((\mathbf{T}'_1, \mathbf{T}'_2)) = \{Z'_1, Z'_2\}$, with $Z'_1 = \mathbf{S}(Z'_1) = \mathbf{S}(Z'_2)$, and $Z'_1 = (\mathbf{T}_1^1, \mathbf{T}'_2)$. Then by Proposition 3.4.6 (or Corollary 3.4.4) we have $Z_1 \sim_{GTL} Z_2$ and $Z'_1 \sim_{GTL} Z'_2$. So in all three cases we are done by induction. □

We now prove that for $(\mathbf{T}_1, \mathbf{T}_2), (\mathbf{T}'_1, \mathbf{T}'_2) \in \mathscr{T}_C(n, n)$ we have $(\mathbf{T}_1, \mathbf{T}_2) \sim_{GTL} (\mathbf{T}'_1, \mathbf{T}'_2)$ implies $\mathbf{S}(\mathbf{T}_1) = \mathbf{S}(\mathbf{T}'_1)$. A key step in the proof is the special case contained in Lemma 3.4.15. The following results will be used in its proof.

3.4.8. LEMMA. *Let* $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_C(M_1, M_2)$ *with* $M_1 = \{1, \ldots, n\}$. *Let* $1 < l < n$ *and let* $\bar{\mathbf{T}}_1$ *be the tableau obtained from* $\mathbf{T}_1$ *by removing the numbers* $l+1, \ldots, n$. *Let* $\bar{P}$ *be an extremal position in* $\bar{\mathbf{T}}_1$. *Then there is a sequence* Σ *for* $\{\alpha_1, \ldots, \alpha_l\}$ *and a* $(\mathbf{T}_3, \mathbf{T}_2) \in T_\Sigma^L((\mathbf{T}_1, \mathbf{T}_2))$ *such that* $P(l, \mathbf{T}_3) = \bar{P}$ *and for* $l+1 \leqslant r \leqslant n$, $P(r, \mathbf{T}_3) = P(r, \mathbf{T}_1)$.

Proof. Let $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2)$ be the pair of tableux obtained from $(\mathbf{T}_1, \mathbf{T}_2)$ by applying the procedure "$-L$", $n-l$-times. By Lemma 3.2.6 there is a $\bar{\mathbf{T}}_3 \in \mathscr{T}_C(\{1, \ldots, l\})$ such that $\text{Shape}(\bar{\mathbf{T}}_3) = \text{Shape}(\bar{\mathbf{T}}_1)$ and $P(l, \bar{\mathbf{T}}_3) = \bar{P}$. By Theorem 3.2.2 there is a sequence Σ for $\{\alpha_1, \ldots, \alpha_l\}$ such that $(\bar{\mathbf{T}}_3, \bar{\mathbf{T}}_2) \in T_\Sigma^L((\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2))$. Using repeatedly Lemma 3.2.5, parts (1) and (3), we find that there is a $(\mathbf{T}_3, \mathbf{T}_4) \in T_\Sigma^L((\mathbf{T}_1, \mathbf{T}_2))$ such that applying the procedure "$-L$" $n-l$-times to $(\mathbf{T}_3, \mathbf{T}_4)$ results in $(\bar{\mathbf{T}}_3, \bar{\mathbf{T}}_2)$, and such that $P(r, \mathbf{T}_3) = P(r, \mathbf{T}_1)$ for all $l+1 \leqslant r \leqslant n$. Then we have $\text{Shape}(\mathbf{T}_3) = \text{Shape}(\mathbf{T}_1)$ and thus $\mathbf{T}_4 = \mathbf{T}_2$. □

3.4.9. LEMMA. *Let* $(\mathbf{T}_1, \mathbf{T}_2), (\mathbf{T}'_1, \mathbf{T}'_2) \in \mathscr{T}_C(M_1, M_2)$ *with* $M_1 = \{1, \ldots, n\}$. *Suppose* $(\mathbf{T}_1, \mathbf{T}_2) \sim_{GTL} (\mathbf{T}'_1, \mathbf{T}'_2)$. *Then* $(\mathbf{T}_1, \mathbf{T}_2) - L \sim_{GTL} (\mathbf{T}'_1, \mathbf{T}'_2) - L$.

Proof. We want to show that $(\mathbf{T}_1, \mathbf{T}_2) \sim_m (\mathbf{T}'_1, \mathbf{T}'_2)$ implies $(\mathbf{T}_1, \mathbf{T}_2) - L \sim_m (\mathbf{T}'_1, \mathbf{T}'_2) - L$. We prove this by induction on m, the case $m = 0$ being obvious. Now the desired result is a consequence of Lemma 3.2.5–1. □

3.4.10. PROPOSITION. *Let* $\mathbf{T} \in \mathscr{T}_C(n)$ *and suppose* a *and* $k \geqslant 2$ *are such that*

$\alpha_{a+1} \in \tau(\mathbf{T})$, $\alpha_{a+2}, \ldots, \alpha_{a+k} \notin \tau(\mathbf{T})$, *and* $\kappa^2(a, \mathbf{T}) > \kappa^2(a+k, \mathbf{T})$. *Let*

$$\Sigma = (\alpha_{a+2}, \alpha_{a+1}), (\alpha_{a+3}, \alpha_{a+2}), \ldots, (\alpha_{a+k}, \alpha_{a+k-1}).$$

Then $T_\Sigma(\mathbf{T}) \neq \varnothing$, *and setting* $\mathbf{T}' = T_\Sigma(\mathbf{T})$, *we have*

$$P(a+k, \mathbf{T}') = P(a+k, \mathbf{T}),$$
$$P(a+k-1, \mathbf{T}') = P(a, \mathbf{T}),$$

and

$$P(a+j, \mathbf{T}') = P(a+j+1, \mathbf{T})$$

for $0 \leqslant j \leqslant k-2$.

Proof. The proof is by induction on k. When $k = 2$ we have $\Sigma = (\alpha_{a+2}, \alpha_{a+1})$; then our hypotheses ensure that $T_\Sigma(\mathbf{T}) = \mathrm{In}(a, a+1; \mathbf{T})$, as desired. Suppose now that $k > 2$. Since $\alpha_{a+k} \notin \tau(\mathbf{T})$ we have $\kappa^2(a+k, \mathbf{T}) > \kappa^2(a+k-1, \mathbf{T})$, and so the hypotheses of this proposition are satisfied with $k-1$ in place of k. So, setting

$$\Sigma' = (\alpha_{a+2}, \alpha_{a+1}), \ldots, (\alpha_{a+k-1}, \alpha_{a+k-2})$$

and $\bar{\mathbf{T}} = T_{\Sigma'}(\mathbf{T})$, we have $P(a+k, \bar{\mathbf{T}}) = P(a+k, \mathbf{T})$, and by induction

$$P(a+k-1, \bar{\mathbf{T}}) = P(a+k-1, \mathbf{T}),\ P(a+k-2, \bar{\mathbf{T}}) = P(a, \mathbf{T}),$$

and

$$P(a+j, \bar{\mathbf{T}}) = P(a+j+1, \mathbf{T}) \quad \text{for } 0 \leqslant j \leqslant k-3.$$

Now the hypotheses of this proposition are satisfied with $\bar{\mathbf{T}}$ in place of $\mathbf{T}$, $a+k-2$ in place of a, and 2 in place of k. So

$$T_{\alpha_{a+k}, \alpha_{a+k-1}}(\bar{\mathbf{T}}) = \mathrm{In}(a+k-2, a+k-1; \bar{\mathbf{T}}),$$

as desired. □

3.4.11. PROPOSITION. *Let* $\mathbf{T} \in \mathscr{T}_C(n)$ *and suppose that for some* $0 \leqslant b \leqslant n-2$ *and some* $l \geqslant 1$ *we have*

$$\alpha_{b+2}, \ldots, \alpha_{b+l} \notin \tau(\mathbf{T}),\ \alpha_{b+l+1} \in \tau(\mathbf{T}),\ \alpha_{b+l+2}, \ldots, \alpha_{b+2l} \notin \tau(\mathbf{T}),$$

and suppose we have

$$\kappa^2(b+j,\mathbf{T}) > \kappa^2(b+l+j,\mathbf{T}) \quad \textit{for } 1 \leqslant j \leqslant l.$$

Then there is a sequence Σ *for* $\{\alpha_{b+2}, \alpha_{b+3}, \ldots, \alpha_{b+2l}\}$ *such that* $T_\Sigma(\mathbf{T}) = \mathbf{T}'$ *with*

$$P(b+2k-1,\mathbf{T}') = P(b+k,\mathbf{T})$$

and

$$P(b+2k,\mathbf{T}') = P(b+l+k,\mathbf{T}) \quad \textit{for } 1 \leqslant k \leqslant l.$$

Proof. The proof is by induction on l, the case $l = 1$ being trivial (i.e. $\mathbf{T}' = \mathbf{T}$). So assume $l > 1$. Now the hypotheses of Proposition 3.4.10 are satisfied with $b + l$ in place of a and l in place of k, so let Σ_1 be the sequence given by that proposition, and let $\mathbf{T}_1 = T_{\Sigma_1}(\mathbf{T})$. Then

$$P(b+2l,\mathbf{T}_1) = P(b+2l,\mathbf{T}),\ P(b+2l-1,\mathbf{T}_1) = P(b+l,\mathbf{T}),$$

and for $0 \leqslant j \leqslant l-2$ we have

$$P(b+l+j,\mathbf{T}_1) = P(b+l+j+1,\mathbf{T}).$$

Now the hypotheses of Proposition 3.4.11 are satisfied with $\mathbf{T}_1$ in place of $\mathbf{T}$ and $l-1$ in place of l (we have to check that $\alpha_{b+l} \in \tau(\mathbf{T}_1)$; now by hypothesis

$$\kappa^2(b+l-1,\mathbf{T}) > \kappa^2(b+2l-1,\mathbf{T}),$$

and our conditions on $\tau(\mathbf{T})$ show that

$$\kappa^2(b+2l-1,\mathbf{T}) > \kappa^2(b+l+1),$$

so

$$\kappa^2(b+l-1,\mathbf{T}_1) > \kappa^2(b+l,\mathbf{T}_1),$$

as desired), so let Σ_2 be a sequence given by the conclusion thereof. Setting $\Sigma = \Sigma_1\Sigma_2$ we are done. □

3.4.12. PROPOSITION. *Let* $\mathbf{T} \in \mathscr{T}_C(n)$ *and suppose* a *and* $k \geqslant 2$ *are such that*

$$\alpha_{a+k} \in \tau(\mathbf{T}),\ \alpha_{a+1}, \ldots, \alpha_{a+k-1} \notin \tau(\mathbf{T}),$$

and

$$\rho^1(a+k-2, \mathbf{T}) \geqslant \rho^1(a+k, \mathbf{T}).$$

Let $\Sigma = (\alpha_{a+k-1}, \alpha_{a+k}), \ldots, (\alpha_{a+1}, \alpha_{a+2})$. *Then* $T_\Sigma(\mathbf{T}) \neq \varnothing$ *and, setting* $\mathbf{T}' = T_\Sigma(\mathbf{T})$, *we have*

$$P(a+k, \mathbf{T}') = P(a+k, \mathbf{T}),\ P(a+j, \mathbf{T}') = P(a+j-1, \mathbf{T}) \quad \text{for } 1 \leqslant j \leqslant k-1,$$

and

$$P(a, \mathbf{T}') = P(a+k-1, \mathbf{T}).$$

Proof. The proof is similar to that of Proposition 3.4.10; we omit the details. □

3.4.13. PROPOSITION. *Let* $\mathbf{T} \in \mathscr{T}_C(n)$ *and suppose that for some* $0 \leqslant b \leqslant n-3$ *and some* $l \geqslant 1$ *we have* $\alpha_{b+2k} \in \tau(\mathbf{T})$ *for* $1 \leqslant k \leqslant l$ *and* $\alpha_{b+2k+1} \notin \tau(\mathbf{T})$ for $1 \leqslant k \leqslant l$, *and suppose in addition that*

$$\rho^1(b+2k+2, \mathbf{T}) \leqslant \rho^1(b+2k, \mathbf{T}) \quad \text{for } 1 \leqslant k \leqslant l-1$$

and

$$\rho^2(b+2k+1, \mathbf{T}) \leqslant \rho^2(b+2k-1, \mathbf{T}) \quad \text{for } 1 \leqslant k \leqslant l.$$

Then there is a sequence Σ *for* $\{\alpha_{b+2}, \alpha_{b+3}, \ldots, \alpha_{b+2l+1}\}$ *such that*

$$T_\Sigma(\mathbf{T}) = \mathbf{T}' \quad \text{with } P(b+k, \mathbf{T}') = P(b+2k-1, \mathbf{T}) \quad \text{for } 1 \leqslant k \leqslant l+1$$

and

$$P(b+l+1+k, \mathbf{T}') = P(b+2k, \mathbf{T}) \quad \text{for } 1 \leqslant k \leqslant l.$$

Proof. The proof is by induction on l. When $l = 1$ we take $\Sigma = (\alpha_{b+3}, \alpha_{b+2})$; then our hypotheses ensure that $T_\Sigma(\mathbf{T}) = \mathrm{In}(b+2, b+3; \mathbf{T})$, as desired. In particular this applies with $b+2l-2$ in place of b and 1 in place of l; so let $\Sigma_1 = (\alpha_{b+2l+1}, \alpha_{b+2l})$ and $\mathbf{T}_1 = T_{\Sigma_1}(\mathbf{T})$. Then $\mathbf{T}_1 = \mathrm{In}(b+2l, b+2l+1; \mathbf{T})$. Now assume $l > 1$. The hypotheses of Proposition 3.4.13 are satisfied with $\mathbf{T}_1$ in place of $\mathbf{T}$ and $l-1$ in place of l, so let Σ_2 be a sequence for $\{\alpha_{b+2}, \ldots, \alpha_{b+2l-1}\}$ such that $\mathbf{T}_2 = T_{\Sigma_2}(\mathbf{T}_1)$ satisfies $P(b+k, \mathbf{T}_2) = P(b+2k-1, \mathbf{T}_1)$ for $1 \leqslant k \leqslant l$ and

$$P(b+l+k, \mathbf{T}_2) = P(b+2k, \mathbf{T}_1) \quad \text{for } 1 \leqslant k \leqslant l-1.$$

Then the hypotheses of Proposition 3.4.12 are satisfies with $\mathbf{T}_2$ in place of $\mathbf{T}, b + l + 1$ in place of a, and l in place of k, so let Σ_3 be the sequence given by that proposition. Setting $\Sigma = \Sigma_1\Sigma_2\Sigma_3$ we are done. □

3.4.14. PROPOSITION. *Let* $\mathbf{T} \in \mathcal{T}_C(n)$ *and suppose* $\mathbf{T}$ *satisfies the hypotheses of Proposition* 3.4.11 *for some* $0 \leqslant b \leqslant n-3$ *and some* l *with* $b + 2l + 1 \leqslant n$, *and suppose in addition that* $\rho^2(b + 2l + 1, \mathbf{T}) \leqslant \rho^2(b + l, \mathbf{T})$. *Then there is a sequence* Σ *for* $\{\alpha_{b+2}, \ldots, \alpha_{b+2l+1}\}$ *such that* $T_\Sigma(\mathbf{T}) = \mathbf{T}'$ *with*

$$P(b + k, \mathbf{T}') = P(b + k, \mathbf{T}) \quad \text{for } 1 \leqslant k \leqslant l, P(b + l + 1, \mathbf{T}') = P(b + 2l + 1, \mathbf{T}),$$

and

$$P(b + l + k, \mathbf{T}') = P(b + l + k - 1, \mathbf{T}) \quad \text{for } 2 \leqslant k \leqslant l + 1.$$

Proof. Let Σ_1 be any sequence given by Proposition 3.4.11 and let $\mathbf{T}_1 = T_{\Sigma_1}(\mathbf{T})$. Then $\mathbf{T}_1$ satisfies the hypotheses of Proposition 3.4.13, so let Σ_2 be any sequence given by that proposition. Setting $\Sigma = \Sigma_1\Sigma_2$ we are done. □

3.4.15. LEMMA. *Let* $(\mathbf{T}_1, \mathbf{T}) \in \mathcal{T}_C(M_1, M_2)$ *where* $M_1 = \{1, \ldots, n\}$ *and suppose* $\mathbf{T}_1$ *is special. Let* c *be a closed cycle in* $\mathbf{T}_1$ *and let* $\mathbf{T}_1' = \mathbf{E}(\mathbf{T}_1, c)$. *Then* $(\mathbf{T}_1, \mathbf{T}) \not\sim_{GTL} (T_1', \mathbf{T})$.

Proof. Interchanging if necessary $\mathbf{T}_1$ and $\mathbf{T}_1'$ (cf. Remark 3.3.13) we may assume that c is an up cycle in $\mathbf{T}_1$. The proof of this lemma is, in the first instance, by induction on n. Since the lemma is vacuously true when $n = 1$, we will assume the lemma is true when $M_1 = \{1, \ldots, n-1\}$. Assume first $n \notin c$. Then induction and Lemma 3.4.9 verify this lemma. So henceforth we will assume that $n \in c$. Now we will proceed to prove the lemma by a downward induction on $\inf c$. Set $a = \inf c$. When $a = n-1$ we have $\alpha_n \notin \tau(\mathbf{T}_1)$ and $\alpha_n \in \tau(\mathbf{T}_1')$, so $(\mathbf{T}_1, \mathbf{T}) \not\sim_{GTL} (\mathbf{T}_1', \mathbf{T})$. So henceforth we will assume that $a < n - 1$ and that the lemma is true for all $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}})$, etc., with $\inf \bar{c} > a$. We now let

$$W(c) = \{k \mid k > \inf c \quad \text{and} \quad \rho^1(k, \mathbf{T}_1) < \rho_{\inf}(c)\}.$$

If $W(c) \neq \varnothing$ we set $k(c) = \inf W(c)$. There are two cases:

Case A. Here we have $W(c) \neq \varnothing$. We first prove the lemma under the additional hypothesis that for all i with $a + 1 \leqslant i \leqslant k(c) - 1$ we have $\alpha_i \notin \tau(\mathbf{T}_1)$. (Note that by the definition of $k(c)$ we have $\alpha_{k(c)} \notin \tau(\mathbf{T}_1)$.) Since the argument is lengthy we state this as a lemma here and postpone the proof thereof until after completing the proof of Lemma 3.4.15.

3.4.16. LEMMA. *Suppose* $(\mathbf{T}_1, \mathbf{T})$, *etc., are as in Lemma* 3.4.15 *with* c *an up cycle, and suppose that* $n \in c$, *that* $W(c) \neq \varnothing$, *and that for all* $\inf c < i \leqslant k(c)$ *we have* $\alpha_i \notin \tau(\mathbf{T}_1)$. *Suppose further that for every* $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}}) \in \mathcal{T}_C(\bar{M}_1, \bar{M}_2)$ *and every closed* *cycle* $\bar{c}$ *of* $\bar{\mathbf{T}}_1$ *such that either* $\bar{M}_1 = \{1, \ldots, n-1\}$ *or* $\bar{M}_1 = \{1, \ldots, n\}$ *and* $\inf \bar{c} > \inf c$ *we have* $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}}) \not\approx_{GTL} (\mathbf{E}(\bar{\mathbf{T}}_1, \bar{c}), \bar{\mathbf{T}})$. *Then* $(\mathbf{T}_1, \mathbf{T}) \not\approx_{GTL} (\mathbf{T}'_1, \mathbf{T})$.

We now proceed to prove Case A of Lemma 3.4.15 by induction on $k(c)$. Set $k = k(c)$. If $k = a + 1$ then necessarily $(\mathbf{T}_1, \mathbf{T})$, etc., satisfy the hypotheses of Lemma 3.4.16, and so we are done. So assume that $k > a + 1$ and that Lemma 3.4.15 is true for all $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}})$, etc., with $\inf \bar{c} = a$, $W(\bar{c}) \neq \varnothing$, and $k(\bar{c}) < k$. If $(\mathbf{T}_1, \mathbf{T})$, etc., satisfy the hypotheses of Lemma 3.4.16 then again we are done, so assume that there is an $a + 1 \leqslant i \leqslant k - 1$ such that $\alpha_i \in \tau(\mathbf{T}_1)$, and assume further that i is maximal given this criterion. We will complete the argument by a downward induction on i. Assume first that $i = k - 1$. Since $\rho^1(k, \mathbf{T}_1) < \rho^1(k-2, \mathbf{T}_1)$ we have $T_{\alpha_k \alpha_{k-1}}(\mathbf{T}_1) = \mathrm{In}(k-1, k; \mathbf{T}_1)$. Set $\mathbf{T}_2 = T_{\alpha_k \alpha_{k-1}}(\mathbf{T}_1)$ and $\mathbf{T}'_2 = T_{\alpha_k \alpha_{k-1}}(\mathbf{T}'_1)$. Let $c_2 = c(n, \mathbf{T}_2)$. Then by Proposition 3.1.5–2 we have $\mathbf{T}'_2 = \mathbf{E}(\mathbf{T}_2, c_2)$. Since $k(c_2) = k - 1$ (we observe that $\mathbf{T}_1$ does not fall under any of the exceptional situations described in Proposition 3.1.2–1), we have by induction $(\mathbf{T}_2, \mathbf{T}) \not\approx_{GTL} (\mathbf{T}'_2, \mathbf{T})$, and so by Remark 3.4.2–2 we have $(\mathbf{T}_1, \mathbf{T}) \not\approx_{GTL} (\mathbf{T}'_1, \mathbf{T})$.

Finally, assume that $i < k - 1$. Let $\mathbf{T}_2 = T_{\alpha_{i+1}, \alpha_i}(\mathbf{T}_1)$ and $\mathbf{T}'_2 = T_{\alpha_{i+1}, \alpha_i}(\mathbf{T}'_1)$. Let $c_2 = c(n, \mathbf{T}_2)$. Then by Proposition 3.1.5–2 we have $\mathbf{T}'_2 = \mathbf{E}(\mathbf{T}_2, c_2)$, and by induction we have $(\mathbf{T}'_2, \mathbf{T}) \not\approx_{GTL} (\mathbf{T}'_1, \mathbf{T})$. (If $i = a + 1$ then possibly $\inf c_2 > a$. In this case the conclusion $(\mathbf{T}'_2, \mathbf{T}) \not\approx_{GTL} (\mathbf{T}'_1, \mathbf{T})$ follows from an appeal to our earlier induction hypothesis. If $\inf c_2 = a$ then Proposition 3.1.2–1 shows that $W(c_2) \neq \varnothing$ and $k(c_2) = k$.) So again by Remark 3.4.2–2 we are done. This completes the proof of Case A.

Case B. Here we have $W(c) = \varnothing$. Since we have already proved Case A, we will assume that the lemma is true for every $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}})$, etc., with $\inf \bar{c} = \inf c$ and $W(\bar{c}) \neq \varnothing$. Set $i = \rho_{\inf}(c)$. The proof of Case B is by induction on $s(c)$. That is, we must first prove Case B under the further assumption that $s(c) = a + 2$, so assume this. (Proposition 3.3.20 rules out $s(c) = a$, and $s(c) = a + 1$ would, by that same proposition, imply that $c = \{a, a+1\}$, contradicting our hypothesis that $\inf c < \sup c - 1$.) Then by Proposition 3.3.20 we have, for some j, $P(a, \mathbf{T}_1) = \{S_{i,j-1}, S_{i+1,j-1}\}$ and $P(a+2, \mathbf{T}_1) = \{S_{ij}, S_{i+1,j}\}$, and since $W(c) = \varnothing$ we have $\rho^1(a+1, \mathbf{T}_1) \geqslant i + 2$. (Note that it is not possible that $a + 1$ occupy the position which we have allotted to a in the previous sentence, since then we would have $c = \{a+1, a+2\}$.) Assume first that $a + 2 = n$. Then since c is closed we have $P'(n, \mathbf{T}_1) = \{S_{i+1,j-1}, S_{i+1,j}\}$ and thus $c = \{n-2, n\}$. That is, we have $P(n-2, \mathbf{T}'_1) = \{S_{i,j-1}, S_{ij}\}$ and $P(n, \mathbf{T}'_1) = \{S_{i+1,j-1}, S_{i+1,j}\}$. Then (transposing and interchanging the roles of $(\mathbf{T}_1, \mathbf{T})$ and $(\mathbf{T}'_1, \mathbf{T})$) we see that this situation corresponds to a case of Lemma 3.4.16 (with $k(c) = n - 1$), so we proceed accordingly. We note first that $P(n-1, \mathbf{T}_1) = P(n-1, \mathbf{T}'_1)$, and thus since $\rho^1(n-1, \mathbf{T}_1) \geqslant i + 2$ we have $\kappa^2(n-1, \mathbf{T}_1) < j - 1$. We let l be maximal given that $S_{l,j-2} \in \mathrm{Shape}(\mathbf{T}_1) \backslash P(n-1, \mathbf{T}_1)$. Using Lemma 3.4.8 we find a sequence Σ for $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-3}\}$ and a $(\mathbf{T}_2, \mathbf{T}) \in T^L_\Sigma((\mathbf{T}_1, \mathbf{T}))$ with

$P(n-3, \mathbf{T}_2) = \{S_{l-1,j-2}, S_{l,j-2}\}$ and $P(r, \mathbf{T}_2) = P(r, \mathbf{T}_1)$ for $n-2 \leqslant r \leqslant n$. Let $\mathbf{T}'_2 = \mathbf{E}(\mathbf{T}_2, c)$. Then (using Claim 1 of the proof of Lemma 3.4.16) we have that $(\mathbf{T}_1, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_1, \mathbf{T})$ implies $(\mathbf{T}_2, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_2, \mathbf{T})$. Now let $(\mathbf{T}_3, \mathbf{T}) = T^L_{\alpha_{n-2}\alpha_{n-1}}((\mathbf{T}_2, \mathbf{T}))$ and $(\mathbf{T}'_3, \mathbf{T}) = T^L_{\alpha_{n-2}\alpha_{n-1}}((\mathbf{T}'_2, \mathbf{T}))$. Then by Remark 3.4.2–2 we have $(\mathbf{T}_2, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_2, \mathbf{T})$ if and only if $(\mathbf{T}_3, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_3, \mathbf{T})$. Now we observe that $\mathbf{T}_3 = \mathrm{In}(n-2, n-1; \mathbf{T}_2)$ and $\mathbf{T}'_3 = \mathrm{In}(n-2, n-1; \mathbf{T}'_2)$, so $\alpha_n \notin \tau(\mathbf{T}_3)$ and $\alpha_n \in \tau(\mathbf{T}'_3)$, and so we conclude that $(\mathbf{T}_1, \mathbf{T}) \not\sim_{GTL} (\mathbf{T}'_1, \mathbf{T})$, as desired. Now assume $a+2 < n$. Then $c \neq \{a, a+2\}$, so

$$P'(a+2, \mathbf{T}_1) = \{S_{i+1,j}, S_{i+2,j}\}.$$

It follows that $S_{i+2,j-1} \in P(a+1, \mathbf{T}_1)$, and since by Proposition 3.3.20 we have that $S_{i+2,j-1}$ is ϕ_C-fixed, we have $a+1 \in c$ and $\tilde{F}_2(a; i, j-1) \subseteq \mathbf{T}'_1$. Let

$$(\mathbf{T}_2, \mathbf{T}) = T^L_{\alpha_{a+2}\alpha_{a+1}}((\mathbf{T}_1, \mathbf{T}))$$

and

$$(\mathbf{T}'_2, \mathbf{T}) = T^L_{\alpha_{a+2}\alpha_{a+1}}((\mathbf{T}'_1, \mathbf{T})),$$

and let $c_2 = c(n, \mathbf{T}_2)$. Then (by Propositions 3.1.2–1 and 3.1.5–2) we have $c_2 = c \backslash \{a, a+1\}$ (in particular $\inf c_2 > \inf c$) and $T'_2 = \mathbf{E}(\mathbf{T}_2, c_2)$. So by induction and Remark 3.4.2–2 we have proved the lemma in this case.

Henceforth assume that $s(c) > a+2$ and that the lemma is true for all $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}})$, etc., with $n \in \bar{c}$, $\inf(\bar{c}) = a$, $W(\bar{c}) \neq \varnothing$, and $s(\bar{c}) < s(c)$. Set $s = s(c)$. By Proposition 3.3.20 we have that $P(s, \mathbf{T}_1)$ is vertical, so let $P(s, \mathbf{T}_1) = \{S_{ij}, S_{i+1,j}\}$. Let $b = N_{\mathbf{T}_1}(S_{i,j-1})$. Since $i = \inf c$ and $S_{i-1,j}$ is ϕ_C-fixed (by Proposition 3.3.20) we have $P'(b, \mathbf{T}_1) = \{S_{i,j-1}, S_{ij}\}$. Now $b \leqslant s-2$, since $b = s-1$ and $P(b, \mathbf{T}_1)$ vertical implies that $c = \{s-1, s\}$, contradicting $s > a+2$, whereas $b = s-1$ and $P(b, \mathbf{T}_1)$ horizontal contradicts condition (4) of Definition 1.1.8. There are four cases:

Case 1. Here $b = r(c) = s-2$ (so $b > a$). Then by Proposition 3.3.20 we have $P(s-2, \mathbf{T}_1) = \{S_{i,j-1}S_{i+1,j-1}\}$. Then as in the previous situation (since $c \neq \{s-2, s\}$ and $W(c) = \varnothing$) we have $S_{i+2,j-1} \in P(s-1, \mathbf{T}_1)$ and $\tilde{F}_2(s-2; i, j-1) \subseteq \mathbf{T}'_1$. Let $(\mathbf{T}_2, \mathbf{T}) = T^L_{\alpha_s\alpha_{s-1}}((\mathbf{T}_1, \mathbf{T}))$ and $(\mathbf{T}'_2, \mathbf{T}) = T^L_{\alpha_s\alpha_{s-1}}((\mathbf{T}'_1, \mathbf{T}))$, and let $c_2 = c(a, \mathbf{T}_2)$. Then (by Propositions 3.1.2–1 and 3.1.5–2) we have $c_2 = c \backslash \{s-2, s-1\}$, $\mathbf{T}'_2 = \mathbf{E}(\mathbf{T}_2, c_2)$, and $\rho_{\inf}(c_2) = i+2$. Now $n \in c_2$, and $\inf c_2 = a$, but on the other hand $W(c_2) \neq \varnothing$, since

$$P(s-2, \mathbf{T}_2) = \{S_{i,j-1}, S_{i+1,j-1}\}.$$

So, as far as $(\mathbf{T}_2, \mathbf{T})$ and c_2 are concerned, we are in Case A of this lemma, that is, we have already proved that $(\mathbf{T}_2, \mathbf{T}) \not\approx_{GTL} (\mathbf{T}'_2, \mathbf{T})$. Then by Remark 3.4.2–2 we have $(\mathbf{T}_1, \mathbf{T}) \not\approx_{GTL} (\mathbf{T}'_1, \mathbf{T})$.

Case 2. Here we assume that $b = s - 2 \neq r(c)$ and that

$$P(s-2, \mathbf{T}_1) = \{S_{i,j-1}, S_{i+1,j-1}\}.$$

Then as in Case 1 we have

$$S_{i+2,j-1} \in P(s-1, \mathbf{T}_1) \quad \text{and} \quad \tilde{F}_2(s-2; i, j-1) \subseteq \mathbf{T}'_1.$$

Let $\mathbf{T}_2, \mathbf{T}'_2$, and c_2 be as in Case 1. Then $c_2 = c \backslash \{s-2, s-1\}$, $\mathbf{T}'_2 = \mathbf{E}(\mathbf{T}_2, c_2)$, and, since $s - 2 \neq r(c)$, we have $\rho_{\inf}(c_2) = \rho_{\inf}(c)$ and $s(c_2) < s(c)$. So by induction $(\mathbf{T}_2, \mathbf{T}) \not\approx_{GTL} (\mathbf{T}'_2, \mathbf{T})$, and thus $(\mathbf{T}_1, \mathbf{T}) \not\approx_{GTL} (\mathbf{T}'_1, \mathbf{T})$.

Case 3. Here we assume that $b = s - 2 \neq r(c)$ and that

$$P(s-2, \mathbf{T}_1) = \{S_{i,j-2}, S_{i,j-1}\}.$$

Then by condition (4) of Definition 1.1.8 we must have

$$P(s-1, \mathbf{T}_1) = \{S_{i+1,j-2}, S_{i+1,j-1}\},$$

that is, $\tilde{F}_1(s-2; i, j-2) \subseteq \mathbf{T}_1$. Let $\mathbf{T}_2, \mathbf{T}'_2$, and c_2 be as in Case 2. Then $c_2 = c \backslash \{s-1, s\}$ and $s(c_2) = s - 2$. If $s \neq n$ then the desired conclusion follows as in Case 2; if $s = n$ then we have $n \notin c_2$ in which case the fact that $(\mathbf{T}_2, \mathbf{T}) \not\approx_{GTL} (\mathbf{T}'_2, \mathbf{T})$ is a consequence of our first induction hypothesis.

Case 4. Here we assume that $b < s - 2$. Now since $W(c) = \varnothing$ we must have $\alpha_{b+1} \in \tau(\mathbf{T}_1)$ and $\alpha_s \notin \tau(\mathbf{T}_1)$. Let $m = m(c)$ be maximal given that $b + 1 \leqslant m \leqslant s - 1$ and $\alpha_m \in \tau(\mathbf{T}_1)$. We will prove this case by a downward induction on m. Assume first that $m = s - 1$. Let

$$(\mathbf{T}_2, \mathbf{T}) = T^L_{\alpha_s \alpha_{s-1}}((\mathbf{T}_1, \mathbf{T})), \ (\mathbf{T}'_2, \mathbf{T}) = T^L_{\alpha_s \alpha_{s-1}}((\mathbf{T}'_1, \mathbf{T})),$$

and let $c_2 = c(a, \mathbf{T}_2)$. Then by Proposition 3.1.5–2, we have $\mathbf{T}'_2 = \mathbf{E}(\mathbf{T}_2, c_2)$. Since $\rho^1(s, \mathbf{T}_1) < \rho^1(s-2, \mathbf{T}_1)$ we have $\mathbf{T}_2 = \mathrm{In}(s-1, s; \mathbf{T}_1)$, and by Proposition 3.1.2–1 we have $\inf c_2 = \inf c$, $\rho_{\inf}(c_2) = \rho_{\inf}(c)$, $W(c_2) = \varnothing$, and $s(c_2) = s - 1$. If $s \neq n$ or if $s = n$ and $n \in c_2$ then the desired conclusion follows by induction on s; if $n \notin c_2$ then instead we use induction on n.

Finally, if $m < s - 1$, we let $(\mathbf{T}_2, \mathbf{T}) = T^L_{\alpha_{m+1}\alpha_m}((\mathbf{T}_1, \mathbf{T}))$, $(\mathbf{T}'_2, \mathbf{T}) = T^L_{\alpha_{m+1}\alpha_m}((\mathbf{T}'_1, \mathbf{T}))$, and $c_2 = c(n, \mathbf{T}_2)$. As usual we have $\mathbf{T}'_2 = \mathbf{E}(\mathbf{T}_2, c_2)$. If $m = b + 1 = a + 1$ it is possible that $\inf c_2 > \inf c$, in which case we are done by the induction hypothesis involving a; if $\inf c_2 = \inf c$ then we have $\rho_{\inf}(c_2) = \rho_{\inf}(c)$, $W(c_2) = \varnothing$, $s(c_2) = s$, and $m(c_2) = m + 1$, and so we are done by induction on m. This completes the proof of Case B and of Lemma 3.4.15. $\square$

PROOF OF LEMMA 3.4.16. Let $a = \inf c$. We will assume $(\mathbf{T}_1, \mathbf{T}) \not\sim_{GTL} (\mathbf{T}'_1, \mathbf{T})$ and derive a contradiction. We will need the following:

Claim 1. Suppose $a > 2$. Let Σ be a sequence for $\{\alpha_1, \alpha_2, \ldots, \alpha_{a-1}\}$, and suppose that $(\mathbf{T}_2, \mathbf{T}) \in T^L_\Sigma((\mathbf{T}_1, \mathbf{T}))$. Then c is a closed cycle in $\mathbf{T}_2$. Let $\mathbf{T}'_2 = \mathbf{E}(\mathbf{T}_2, c)$. Suppose $(\mathbf{T}_1, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_1, \mathbf{T})$. Then $(\mathbf{T}_2, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_2, \mathbf{T})$.

The proof of this claim is by induction on $|\Sigma|$, the case $|\Sigma| = 0$ being trivial. So assume $|\Sigma| \geqslant 1$ and write $\Sigma = \Sigma'(\alpha, \beta)$. Assume first that $\alpha_1 \notin \{\alpha, \beta\}$. Let $\mathbf{T}_3 = T_{\beta\alpha}(\mathbf{T}_2)$. Then $(\mathbf{T}_3, \mathbf{T}) \in T^L_{\Sigma'}((\mathbf{T}_1, \mathbf{T}))$, so by induction c is a closed cycle in $\mathbf{T}_3$. Let $\mathbf{T}'_3 = \mathbf{E}(\mathbf{T}_3, c)$. Now by Proposition 3.1.2–1 we have that c is a closed cycle in $\mathbf{T}_2$, and by Proposition 3.1.5–2 (since $\mathbf{T}_2 = T_{\alpha\beta}(\mathbf{T}_3)$) we have that $\mathbf{T}'_2 = T_{\alpha\beta}(\mathbf{T}'_3)$. Now by Remark 3.4.2–2 we have $(\mathbf{T}_3, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_3, \mathbf{T})$ if and only if $(\mathbf{T}_2, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_2, \mathbf{T})$. So by induction we are done in this case.

Assume now that $\alpha_1 \in \{\alpha, \beta\}$. Note first that $T^L_{\beta\alpha}((\mathbf{T}_2, \mathbf{T})) \cap T^L_{\Sigma'}((\mathbf{T}_1, \mathbf{T})) \neq \varnothing$. There are three cases:

Case 1. Here $T^L_{\beta\alpha}((\mathbf{T}_2, \mathbf{T}))$ is a two-element set. Let

$$(\mathbf{T}_3, \mathbf{T}) \in T^L_{\beta\alpha}((\mathbf{T}_2, \mathbf{T})) \cap T^L_{\Sigma'}((\mathbf{T}_1, \mathbf{T})).$$

By induction c is a closed cycle in $\mathbf{T}_3$ so let $\mathbf{T}'_3 = \mathbf{E}(\mathbf{T}_3, c)$. Then by induction we have that $(\mathbf{T}_3, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_3, \mathbf{T})$. Now $T^L_{\alpha\beta}((\mathbf{T}_3, \mathbf{T})) = \{(\mathbf{T}_2, \mathbf{T})\}$, so by Proposition 3.1.6–1 we have $T^L_{\alpha\beta}((\mathbf{T}'_3, \mathbf{T})) = \{(\mathbf{T}'_2, \mathbf{T})\}$. Then by Remark 3.4.2–1 we have $(\mathbf{T}_2, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_2, \mathbf{T})$.

Case 2. Here $T^L_{\beta\alpha}((\mathbf{T}_2, \mathbf{T})) = \{(\mathbf{T}_3, \mathbf{T})\}$ and $T^L_{\alpha\beta}((\mathbf{T}_3, \mathbf{T})) = \{(\mathbf{T}_2, \mathbf{T}), (\mathbf{T}_4, \bar{\mathbf{T}})\}$ with $\bar{\mathbf{T}} \neq \mathbf{T}$. Then by Corollary 3.4.4 we have $(\mathbf{T}_2, \mathbf{T}) \sim_{GTL} (\mathbf{T}_4, \bar{\mathbf{T}})$. Now c is a closed cycle in $\mathbf{T}_3$ (by induction) and in $\mathbf{T}_2$ and $\mathbf{T}_4$ (by Proposition 3.1.3–1). Set $\mathbf{T}'_k = \mathbf{E}(\mathbf{T}_k, c)$ for $k = 3, 4$. Then by Proposition 3.1.6–1 we have $T^L_{\alpha\beta}((\mathbf{T}'_3, \mathbf{T})) = \{(\mathbf{T}'_2, \mathbf{T}), (\mathbf{T}'_4, \bar{\mathbf{T}})\}$ and so by Corollary 3.4.4 we have $(\mathbf{T}'_2, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_4, \bar{\mathbf{T}})$. Thus by Remark 3.4.2–1 we have $(\mathbf{T}_2, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_2, \mathbf{T})$ if and only if $(\mathbf{T}_3, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_3, \mathbf{T})$. So by induction we are done in this case.

Case 3. Here $T^L_{\beta\alpha}((\mathbf{T}_2, \mathbf{T})) = \{(\mathbf{T}_3, \mathbf{T})\}$ and $T^L_{\alpha\beta}((\mathbf{T}_3, \mathbf{T})) = \{(\mathbf{T}_2, \mathbf{T}), (\mathbf{T}_4, \mathbf{T})\}$ (so $c(2, \mathbf{T}_2)$ is closed and $\mathbf{T}_4 = \mathbf{E}(\mathbf{T}_2, c(2, \mathbf{T}_2))$). Set $c' = c(2, \mathbf{T}_2)$. (By hypothesis $\inf c > 2$ so $c \neq c'$; in particular $n \notin c'$.) Let $\mathbf{T}'_k = \mathbf{E}(\mathbf{T}_k, c)$ for $k = 3, 4$. Then as in the previous case we have that c is closed cycle in $\mathbf{T}_2, \mathbf{T}_3$, and $\mathbf{T}_4$, and $T^L_{\alpha\beta}((\mathbf{T}'_3, \mathbf{T})) = \{(\mathbf{T}'_2, \mathbf{T}), (\mathbf{T}'_4, \mathbf{T})\}$. Now $(\mathbf{T}_3, \mathbf{T}) \in T^L_{\Sigma'}((\mathbf{T}_1, \mathbf{T}))$, so by induction we have $(\mathbf{T}_3, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_3, \mathbf{T})$. Thus by Remark 3.4.2–1 we have either $(\mathbf{T}_2, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_2, \mathbf{T})$ or $(\mathbf{T}_2, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_4, \mathbf{T})$. We will show that $(\mathbf{T}_2, \mathbf{T}) \not\sim_{GTL} (\mathbf{T}'_4, \mathbf{T})$ and thus complete the proof of this case. Let $(\bar{\mathbf{T}}_k, \bar{\mathbf{T}}) = (\mathbf{T}_k, \mathbf{T}) - L$ for $k = 2$ and $k = 4$ (since $P(n, \mathbf{T}_2) = P(n, \mathbf{T}_4)$ the right tableaux of these two pairs are equal) and let $(\bar{\mathbf{T}}'_4, \bar{\mathbf{T}}') = (\mathbf{T}'_4, \mathbf{T}) - L$. Now $c \backslash \{n\}$ is an open cycle in $\bar{\mathbf{T}}_4$, and by Proposition 2.3.3c) we have $(\bar{\mathbf{T}}'_4, \bar{\mathbf{T}}') = \mathbf{E}((\bar{\mathbf{T}}_4, \bar{\mathbf{T}}), c\backslash\{n\}, L)$, so by Proposition 3.4.3 we have $(\bar{\mathbf{T}}'_4, \bar{\mathbf{T}}') \sim_{GTL} (\bar{\mathbf{T}}_4, \bar{\mathbf{T}})$. On the other hand, since $\mathbf{T}_4 = \mathbf{E}(\mathbf{T}_2, c')$ we have

$\bar{\mathbf{T}}_4 = \mathbf{E}(\bar{\mathbf{T}}_2, c')$. Now $(\bar{\mathbf{T}}_2, \bar{\mathbf{T}})$ is an element of some $\mathscr{T}_C(\bar{M}_1, \bar{M}_2)$ with $\bar{M}_1 = \{1, \ldots, n-1\}$, so by the hypothesis of Lemma 3.4.11 (and using again Proposition 3.4.3 if $\bar{\mathbf{T}}_2$ is not special) we have that $(\bar{\mathbf{T}}_2, \bar{\mathbf{T}}) \not\sim_{GTL} (\bar{\mathbf{T}}_4, \bar{\mathbf{T}})$. Thus $(\bar{\mathbf{T}}_2, \bar{\mathbf{T}}) \not\sim_{GTL} (\bar{\mathbf{T}}'_4, \bar{\mathbf{T}}')$ and so by Lemma 3.4.9 we have $(\mathbf{T}_2, \mathbf{T}) \not\sim_{GTL} (\mathbf{T}'_4, \mathbf{T})$. This completes the proof of Claim 1.

Now let $k = k(c)$. By hypothesis, for all $a \leqslant l < k$ we have

$$l \in c,\ \kappa^2(l, \mathbf{T}_1) < \kappa^1(l+1, \mathbf{T}_1),$$

and

$$\rho^1(l, \mathbf{T}_1) \geqslant \rho^1(l+1, \mathbf{T}_1).$$

Let $a = j_0 < j_1 < \cdots < j_r = k$ be such that for $0 \leqslant i \leqslant r-1$ we have $\rho^1(j_{i+1}, \mathbf{T}_1) < \rho^1(j_i, \mathbf{T}_1)$ and for $j_i < l < j_{i+1}$ we have $\rho^1(l, \mathbf{T}_1) = \rho^1(j_i, \mathbf{T}_1)$. Set $t_i = \rho^1(j_i, \mathbf{T}_1)$. We claim that

Claim 2. (1) For $0 \leqslant i \leqslant r-1$ we have that $P(j_i, \mathbf{T}_1)$ is vertical and, if $S \in P(j_i, \mathbf{T}_1)$ is such that $\rho(S) = \rho^1(j_i, \mathbf{T}_1)$ then S is a fixed square.

(2) For $0 \leqslant i \leqslant r-1$ and $j_i < l < j_{i+1}$ we have that $P(l, \mathbf{T}_1)$ is horizontal, and, if $S \in P(l, \mathbf{T}_1)$ is such that $\kappa(S) = \kappa^2(l, \mathbf{T}_1)$, then S is a fixed square.

(3) For $a \leqslant l < k$, if $S_{t,u} \in P(l, \mathbf{T}_1)$ is such that $t = \rho^1(l, \mathbf{T}_1)$ and $u = \kappa^2(l, \mathbf{T}_1)$ then $S_{t,u+1} \in P'(l, \mathbf{T}_1)$. Furthermore, $N_{\mathbf{T}_1}(S_{t-1,u+1}) < a$.

(4) We have $\rho^2(k, \mathbf{T}_1) < t_{r-1}$.

Part 1 of this claim follows from Proposition 3.3.20 and the observation that $j_i = r_{t_i}(c)$. We have also that $P'(j_i, \mathbf{T}_1)$ is horizontal, since else let $S \in P'(j_i, \mathbf{T}_1)$, $S \notin P(j_i, \mathbf{T}_1)$, and let $d = N_{\mathbf{T}_1}(S)$. Then $d \in c$, $d < j_i$, and $\kappa^2(d, \mathbf{T}_1) \geqslant \kappa^1(j_i, \mathbf{T}_1)$, contradicting the fact that for $a \leqslant l < k$ we have $\kappa^2(l, \mathbf{T}_1) < \kappa^1(l+1, \mathbf{T}_1)$. Thus the first statement of part 3 holds for $l = j_i$. We prove part 2 by induction on $l - j_i$. Suppose first that $l = j_i + 1$. If $P(j_i, \mathbf{T}_1)$ were vertical then by the above we would have $c = \{j_i, j_i + 1\}$, contradicting our hypothesis that $W(c) \neq \varnothing$. Thus $P(l, \mathbf{T}_1)$ is horizontal, and the statement about $S \in P(l, \mathbf{T}_1)$ is an obvious consequence of the corresponding statement in part 1. Now suppose that $j_i + 1 < l < j_{i+1}$ and that 2 holds for all m with $j_i + 1 \leqslant m \leqslant l - 1$. Suppose that $P(l, \mathbf{T}_1)$ is vertical. Let $u = \kappa^1(l, \mathbf{T}_1)$ and let $d = N_{\mathbf{T}_1}(S_{t_i+1,u-1})$. Then by condition (4) of Definition 1.1.8 we have $j_i < d < l$, contradicting our induction hypothesis. Thus $P(l, \mathbf{T}_1)$ is horizontal, and the statement about $S \in P(l, \mathbf{T}_1)$ is an obvious consequence of the corresponding statement about $l - 1$. This completes the proof of statement 2. Now the first statement of part 3 is proved for the rest of the l by the same argument as that used for $l = j_i$. The second statement of 3 is now clear, since we have that $N_{\mathbf{T}_1}(S_{t-1,u+1}) < l$, so $N_{\mathbf{T}_1}(S_{t-1,u+1}) \geqslant a$ would contradict the fact that for $a \leqslant r < k$ we have $\kappa^2(r, \mathbf{T}_1) < \kappa^1(r+1, \mathbf{T}_1)$. Now part 4 follows from 3, since by condition (4) of Definition 1.1.8, if $\rho^2(k, \mathbf{T}_1) = t_{r-1}$ then $\kappa^1(k, \mathbf{T}_1) =$

$\kappa^2(k-1, \mathbf{T}_1) + 1$, so by 3 we have $P(k, \mathbf{T}_1) \cap P'(k-1, \mathbf{T}_1) \neq \varnothing$, contradicting $k - 1 \in c$, $k \notin c$. This completes the proof of the claim.

Now for $0 \leqslant i \leqslant r - 1$ set $u_i = t_i - 1$ and let

$$v_i = \sup\{\kappa(S) \mid \rho(S) = u_i \quad \text{and} \quad N_{\mathbf{T}_1}(S) < a\}.$$

By part 3 of Claim 2 we have $v_i \geqslant \kappa^2(j_{i+1} - 1, \mathbf{T}_1) + 1$. For $0 \leqslant i \leqslant r$ set $m_i = j_i - a$ and for $0 \leqslant i \leqslant r - 1$ let

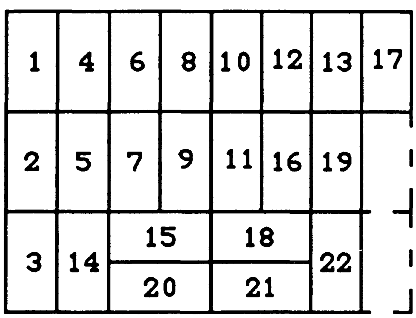
$$w_i = v_i - 2(m_{i+1} - m_i) = v_i - 2(j_{i+1} - j_i).$$

Then by part 2 of Claim 2 we have $w_i \geqslant \kappa^2(j_i, \mathbf{T}_1) + 1$. For $0 \leqslant i \leqslant r - 1$ and $m_i < l \leqslant m_{i+1}$ set

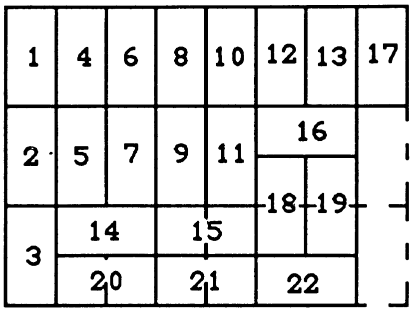
$$P_l = \{S_{u_i, w_i + 2(l - m_i) - 1}, S_{u_i, w_i + 2(l - m_i)}\}.$$

Let $b = 2a - k - 1$. By Lemma 3.4.8 (applied repeatedly) there is a sequence Σ for $\{\alpha_1, \alpha_2, \ldots, \alpha_{a-1}\}$ and a $(\mathbf{T}_2, \mathbf{T}) \in T_\Sigma^L((\mathbf{T}_1, \mathbf{T}))$ such that $P(b + l, \mathbf{T}_2) = P_l$ for $1 \leqslant l \leqslant k - a$ and $P(r, \mathbf{T}_2) = P(r, \mathbf{T}_1)$ for $a \leqslant r \leqslant n$. (That is, in $\mathbf{T}_2$ we have placed the numbers between $a - (k - a)$ and $a - 1$ (inclusive) as horizontal dominos so that the numbers between $j_i - (k - a)$ and $j_{i+1} - 1 - (k - a)$ (inclusive) are in the row above the numbers between j_i and $j_{i+1} - 1$. See the example, below.) Set $\mathbf{T}'_2 = \mathbf{E}(\mathbf{T}_2, c)$. Then Claim 1 says that $(\mathbf{T}_2, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_2, \mathbf{T})$. Now Claim 2 shows that $\mathbf{T}_2$ satisfies the hypotheses of Proposition 3.4.14 with $k - a$ in place of l, so let Σ_2 be the sequence given by that proposition, let $(\mathbf{T}_3, \mathbf{T}) = T_{\Sigma_2}^L((\mathbf{T}_2, \mathbf{T}))$, let $(\mathbf{T}'_3, \mathbf{T}) = T_{\Sigma_2}^L((\mathbf{T}'_2, \mathbf{T}))$, and let $c_3 = c(n, \mathbf{T}_3)$. Then by Proposition 3.1.5–2 (applied repeatedly) we have $\mathbf{T}'_3 = \mathbf{E}(\mathbf{T}_3, c_3)$. On the other hand we have clearly (i.e. using Claim 2) that $c_3 = (c \backslash \{a\}) \cup \{k\}$, in particular $\inf c_3 = a + 1$. By Remark 3.4.2–2 we have $(\mathbf{T}_3, \mathbf{T}) \sim_{GTL} (\mathbf{T}'_3, \mathbf{T})$, on the other hand by hypothesis, since $\inf c_3 > a$, we have $(\mathbf{T}_3, \mathbf{T}) \not\sim_{GTL} (\mathbf{T}'_3, \mathbf{T})$. This contradiction proves Lemma 3.4.16. □

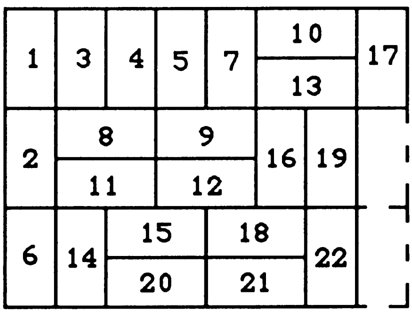
EXAMPLE. In this example $\mathbf{T}_1$ is the tableau pictured below, and c is the cycle $\{14, 15, 16, 18, 19, 20, 21, 22\}$.



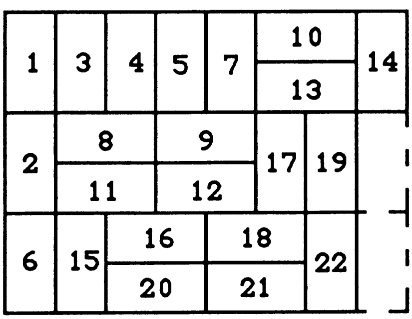
Then $\mathbf{T}'_1$ is the tableau picture here



We have $\rho_{\inf}(c) = 3$, $a = j_0 = 14$, $k = j_2 = 17$, and $j_1 = 16$. We may take for $\mathbf{T}_2$ this tableau.



Then $\mathbf{T}_3$ is the following tableau.



We have $c_3 = \{15, 16, 17, 18, 19, 20, 21, 22\}$ and in fact $W(c_3) = \varnothing$, so that for this tableau we are in Case B of the proof of Lemma 3.4.15.

3.4.17. THEOREM. *Let* $(\mathbf{T}_1, \mathbf{T}_2), (\mathbf{T}'_1, \mathbf{T}'_2) \in \mathscr{T}_C(M_1, M_2)$ *where* $M_1 = \{1, \ldots, n\}$. *Then* $(\mathbf{T}_1, \mathbf{T}_2) \sim_{GTL} (\mathbf{T}'_1, \mathbf{T}'_2)$ *if and only if* $(\mathbf{T}_1) = \mathbf{S}(\mathbf{T}'_1)$.

Proof. We have already proved one implication as Proposition 3.4.7. We will now prove the other implication. The proof is by induction on n, the theorem being vacuously true when $n = 0$. By Proposition 3.4.6 it suffices to prove the theorem when $\mathbf{T}_1$ and $\mathbf{T}'_1$ are special, so assume this. That is, we have $(\mathbf{T}_1, \mathbf{T}_2)$, $(\mathbf{T}'_1, \mathbf{T}'_2) \in \mathscr{T}_C(n, n)$ with $\mathbf{T}_1$ and $\mathbf{T}'_1$ special, and such that $(\mathbf{T}_1, \mathbf{T}_2) \sim_{GTL} (\mathbf{T}'_1, \mathbf{T}'_2)$, and we want to prove that $\mathbf{T}_1 = \mathbf{T}'_1$. Set $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2) = (\mathbf{T}_1, \mathbf{T}_2) - L$ and $(\bar{\mathbf{T}}'_1, \bar{\mathbf{T}}'_2) = (\mathbf{T}'_1, \mathbf{T}'_2) - L$. By Lemma 3.4.9 we have $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2) \sim_{GTL} (\bar{\mathbf{T}}'_1, \bar{\mathbf{T}}'_2)$. Then by induction we have $\mathbf{S}(\bar{\mathbf{T}}_1) = \mathbf{S}(\bar{\mathbf{T}}'_1)$. We will now assume that $\mathbf{T}_1 \neq \mathbf{T}'_1$ and derive a contradiction. There are several cases. In general the point of the proof is to choose carefully a sequence Σ for $\Pi \backslash \{\alpha_n\}$ and a $(\mathbf{T}_3, \mathbf{T}_4) \in T^L_\Sigma((\mathbf{T}_1, \mathbf{T}_2))$. Then by Remark 3.4.2–1 there is a $(\mathbf{T}'_3, \mathbf{T}'_4) \in T^L_\Sigma((\mathbf{T}'_1, \mathbf{T}'_2))$ with $(\mathbf{T}_3, \mathbf{T}_4) \sim_{GTL} (\mathbf{T}'_3, \mathbf{T}'_4)$. Setting $(\bar{\mathbf{T}}_3, \bar{\mathbf{T}}_4) = (\mathbf{T}_3, \mathbf{T}_4) - L$ and $(\bar{\mathbf{T}}'_3, \bar{\mathbf{T}}'_4) = (\mathbf{T}'_3, \mathbf{T}'_4) - L$ we have, by Lemma 3.4.9, that $(\bar{\mathbf{T}}_3, \bar{\mathbf{T}}_4) \sim_{GTL} (\bar{\mathbf{T}}'_3, \bar{\mathbf{T}}'_4)$. Then by induction we have $\mathbf{S}(\bar{\mathbf{T}}_3) = \mathbf{S}(\bar{\mathbf{T}}'_3)$. On the other hand, Lemma 3.2.7 gives information about information about the relation between $P(n, \mathbf{T}_3)$ and $P(n, \mathbf{T}_1)$, and similarly between $P(n, \mathbf{T}'_3)$ and $P(n, \mathbf{T}'_1)$. Now in many cases we will have been able to choose Σ so that at this point we have the contradiction that $\alpha_n \in \tau(\mathbf{T}_3)$ and $\alpha_n \notin \tau(\mathbf{T}'_3)$, or vice versa. Then there are two cases in which we make a more elaborate argument. Finally, there is one case in which neither of these arguments work. This case was handled separately as Lemma 3.4.15.

We will assume that $P(n, \mathbf{T}_1)$ is horizontal. (Our arguments can then be applied to the case where $P(n, \mathbf{T}_1)$ is vertical by interchanging rows and columns.) Let $P(n, \mathbf{T}_1) = \{S_{ij}, S_{i,j+1}\}$. Since $\mathbf{T}_1$ is special we have either (i) $\phi_C(S_{ij}) = X$ or (ii) $\phi_C(S_{ij}) = Z$ or (iii) $\phi_C(S_{ij}) = W$ and $i > 1$ and $S_{i-1,j+2} \notin \mathrm{Shape}(\mathbf{T}_1)$.

Case A. Here we have $\rho^2(n, \mathbf{T}'_1) < i$. Let $r = \rho_{i-1}(\mathbf{T}_1)$ and let $P_1 = \{S_{i-1,r-1}, S_{i-1,r}\}$. Then by Lemma 3.2.8 and Theorem 3.2.2 there is a sequence Σ for $\Pi \backslash \{\alpha_n\}$ and a $(\bar{\mathbf{T}}_3, \bar{\mathbf{T}}_4) \in T^L_\Sigma((\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2))$ such that $\mathrm{Shape}(\bar{\mathbf{T}}_3) = \mathrm{Shape}(\bar{\mathbf{T}}_1)$ and $P(n-1, \bar{\mathbf{T}}_3) = P_1$. Using Lemma 3.2.6, let $(\mathbf{T}_3, \mathbf{T}_4) \in T^L_\Sigma((\mathbf{T}_1, \mathbf{T}_2))$ be such that $(\bar{\mathbf{T}}_3, \bar{\mathbf{T}}_4) = (\mathbf{T}_3, \mathbf{T}_4) - L$. By Lemma 3.2.7–3 we have $P(n, \mathbf{T}_3) = P(n, \mathbf{T}_1)$. Then $\alpha_n \in \tau(\mathbf{T}_3)$. Let $(\mathbf{T}'_3, \mathbf{T}'_4)$ and $(\bar{\mathbf{T}}'_3, \bar{\mathbf{T}}'_4)$ be as described above. Since $\mathbf{S}(\bar{\mathbf{T}}_3) = \mathbf{S}(\bar{\mathbf{T}}'_3)$ we have $P(n-1, \mathbf{T}'_3) \in \{P(n-1, \mathbf{T}_1), P'(n-1, \mathbf{T}_1)\}$, and thus $\rho^2(n-1, \mathbf{T}'_3) \geqslant i-1$. By Lemma 3.2.7–1 we have $P(n, \mathbf{T}'_3) \in \{P(n, \mathbf{T}'_1), P'(n, \mathbf{T}'_1)\}$, and thus $\rho^1(n, \mathbf{T}'_3) \leqslant i-1$. Thus $\alpha_n \notin \tau(\mathbf{T}'_3)$.

Henceforth we assume that $\rho^2(n, \mathbf{T}'_1) \geqslant i$. Note that in general any square $S \in \mathrm{Shape}(\bar{\mathbf{T}}_1)$ which is ϕ_C-fixed is also in $\mathrm{Shape}(\bar{\mathbf{T}}'_1)$. We use this implicitly in what follows to rule out some possibilities for $P(n, \mathbf{T}'_1)$. If $P(n, \mathbf{T}'_1)$ is horizontal then $\rho^2(n, \mathbf{T}'_1) > i$ (since if $\rho^2(n, \mathbf{T}'_1) = i$ then $\mathbf{T}_1$ and $\mathbf{T}'_1$ special implies that $\mathbf{T}_1 = \mathbf{T}'_1$, contradicting our hypothesis), so, interchanging the roles of $(\mathbf{T}_1, \mathbf{T}_2)$ and $(\mathbf{T}'_1, \mathbf{T}'_2)$, we are reduced to case A. So we also assume that $P(n, \mathbf{T}'_1)$ is vertical.

Case B. Here we have that $\rho^1(n, \mathbf{T}'_1) > i$, that $S_{i+1,j-1} \in \mathrm{Shape}(\mathbf{T}_1)$, and that $P(n, \mathbf{T}'_1) \neq \{S_{i+1,j}, S_{i+2,j}\}$. Then $\kappa^2(n, \mathbf{T}'_1) < j$. Let $r = \kappa_{j-1}(\mathbf{T}_1)$, and let

$P_1 = \{S_{r-1,j-1}, S_{r,j-1}\}$. We now proceed as in case A and find that $\alpha_n \notin \tau(\mathbf{T}_3)$ and $\alpha_n \in \tau(\mathbf{T}'_3)$ (for this last fact we transpose the argument given in case A.

Case C. Here we assume that $\rho^1(n, \mathbf{T}'_1) > i$, that $S_{i+1,j-1} \notin \text{Shape}(\mathbf{T}_1)$, that $S_{i+1,j-2} \in \text{Shape}(\mathbf{T}_1)$. and that $P(n, \mathbf{T}'_1) \neq \{S_{i+1,j-1}, S_{i+2,j-1}\}$. Then $\mathbf{T}_1$ special implies that $\phi_C(S_{ij}) \neq X$. If $\phi_C(S_{ij}) = Z$ then $\mathbf{T}_1$ special implies that $S_{i+2,j-2} \in \text{Shape}(\mathbf{T}_1)$. Here our hypotheses show that $\kappa^1(n, \mathbf{T}'_1) < j - 1$. Let $r = \kappa_{j-2}(\mathbf{T}_1)$ and let $P_1 = \{S_{r-1,j-2}, S_{r,j-2}\}$. If $\phi_C(S_{ij}) = W$ and $\kappa_{j-2}(\mathbf{T}_1) > i + 1$ then again we set $r = \kappa_{j-2}(\mathbf{T}_1)$ and $P_1 = \{S_{r-1,j-2}, S_{r,j-2}\}$. Here again we have $\kappa^1(n, \mathbf{T}'_1) < j - 1$ (note that $P(n, \mathbf{T}'_1) = \{S_{i+2,j-1}, S_{i+3,j-1}\}$ is impossible since else $S_{i+3,j-1}$ is a filled corner in $\mathbf{T}'_1$). If $\phi_C(S_{ij}) = W$ and $\kappa_{j-2}(\mathbf{T}_1) = i + 1$ then $\mathbf{T}_1$ special implies that $j \geqslant 4$ and $S_{i+2,j-3} \notin \text{Shape}(\mathbf{T}_1)$, so we set $P_1 = \{S_{i+1,j-3}, S_{i+1,j-2}\}$. Since $P(n, \mathbf{T}'_1)$ is vertical we have $\kappa^1(n, \mathbf{T}'_1) < j - 2$. Now we proceed as in the previous cases and find that $\alpha_n \notin \tau(\mathbf{T}_3)$ and $\alpha_n \in \tau(\mathbf{T}'_3)$.

Case D. Here we assume that $\rho^1(n, \mathbf{T}'_1) > i$ and $S_{i+1,j-2} \notin \text{Shape}(\mathbf{T}_1)$. Since $P(n, \mathbf{T}'_1)$ is vertical we have $\kappa^2(n, \mathbf{T}'_1) \leqslant j - 2$. Let $P_1 = \{S_{i,j-2}, S_{i,j-1}\}$. We proceed as in the previous cases, and find that $\alpha_n \notin \tau(\mathbf{T}_3)$ and $\alpha_n \in \tau(\mathbf{T}'_3)$.

Case E. Here we assume that $P(n, \mathbf{T}'_1) = \{S_{i+1,j}, S_{i+2,j}\}$. Then S_{ij} is ϕ_C-variable, so $\phi_C(S_{ij}) \neq Z$. We have also $\phi_C(S_{ij}) \neq X$, since else $S_{i+2,j}$ is a filled corner in $\mathbf{T}'_1$. So $\phi_C(S_{ij}) = W$. Set $P_1 = \{S_{i-1,j}, S_{ij}\}$ and let $(\bar{\mathbf{T}}_1^1, \bar{\mathbf{T}}_2^1) = \mathbf{S}((\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2))$. Then

$$\text{Shape}(\bar{\mathbf{T}}_1^1) = (\text{Shape}(\bar{\mathbf{T}}_1) \backslash \{S_{i-1,j+1}\}) \cup \{S_{ij}\},$$

so we can apply Lemma 3.2.8 to find a tableau $\bar{\mathbf{T}}_3 \in \mathscr{T}_C(\{1, \ldots, n-1\})$ with $\text{Shape}(\bar{\mathbf{T}}_3) = \text{Shape}(\bar{\mathbf{T}}_1^1)$ and $P(n-1, \bar{\mathbf{T}}_3) = P_1$. Then we set $\bar{\mathbf{T}}_4 = \bar{\mathbf{T}}_2^1 = \mathbf{S}(\bar{\mathbf{T}}_2)$ and apply Theorem 3.2.2 to find a sequence Σ for $\Pi \backslash \{\alpha_n\}$ such that $(\bar{\mathbf{T}}_3, \bar{\mathbf{T}}_4) \in T_\Sigma^L((\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2))$. Let $(\mathbf{T}_3, \mathbf{T}_4)$, etc., be as in previous cases. Then

$$P(n, \mathbf{T}_3) = P'(n, \mathbf{T}_1) = \{S_{i-1,j+1}, S_{i,j+1}\},$$

so $\alpha_n \notin \tau(\mathbf{T}_3)$. On the other hand, since $\mathbf{S}(\bar{\mathbf{T}}'_3) = \mathbf{S}(\bar{\mathbf{T}}_3)$ and $P(n, \mathbf{T}'_3) \in \{P'(n, \mathbf{T}'_1)\}$, $P'(n, \mathbf{T}'_1)\}$, we see that $\alpha_n \in \tau(\mathbf{T}'_3)$.

Case F. Here we assume that $P(n, \mathbf{T}'_1) = \{S_{i+1,j-1}, S_{i+2,j-1}\}$. The subcase where $\phi_C(S_{ij}) = X$ is the transpose of the subcase where $\phi_C(S_{ij}) = W$, so we will assume $\phi_C(S_{ij}) = W$ or $\phi_C(S_{ij}) = Z$, in particular, since $\phi_C(S_{1,1}) = X$, we have $i > 1$. If $\phi_C(S_{ij}) = W$ then, since $\mathbf{T}_1$ is special we have $\rho_{i-1}(\mathbf{T}_1) = j + 1$ and $S_{i+1,j-1} \notin \text{Shape}(\mathbf{T}_1)$, so let $P_1 = \{S_{i-1,j}, S_{i-1,j+1}\}$ and $P_2 = \{S_{i-1,j-1}, S_{i,j-1}\}$. If $\phi_C(S_{ij}) = Z$ and $\rho_{i-1}(\mathbf{T}_1) = j + 1$ then let P_1 and P_2 be as above. If $\rho_{i-1}(\mathbf{T}_1) > j + 1$ then since $\mathbf{T}_1$ is special we have $\rho_{i-1}(\mathbf{T}_1) \geqslant j + 3$, so let $r = \rho_{i-1}(\mathbf{T}_1)$ and let $P_1 = \{S_{i-1,r-1}, S_{i-1,r}\}$ and $P_2 = \{S_{i-1,r-3}, S_{i-1,r-2}\}$. By Lemma 3.2.8 and Theorem 3.2.2 there is a sequence Σ for $\Pi \backslash \{\alpha_n\}$ and a

$(\bar{\mathbf{T}}_3, \bar{\mathbf{T}}_4) \in T_\Sigma^L((\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2))$ such that $\text{Shape}(\bar{\mathbf{T}}_3) = \text{Shape}(\bar{\mathbf{T}}_1)$, $P(n-1, \bar{T}_3) = P_1$, and $P(n-2, \bar{\mathbf{T}}_3) = P_2$. Let $(\mathbf{T}_3, \mathbf{T}_4) \in T_\Sigma^L((\mathbf{T}_1, \mathbf{T}_2))$ be such that $(\bar{\mathbf{T}}_3, \bar{\mathbf{T}}_4) = (\mathbf{T}_3, \mathbf{T}_4) - L$. Let $(\mathbf{T}_3^1, \mathbf{T}_4^1) \in T_\Sigma^L((\mathbf{T}'_1, \mathbf{T}'_2))$ be such that $(\mathbf{T}_3^1, \mathbf{T}_4^1) \sim_{GTL} (\mathbf{T}_3, \mathbf{T}_4)$. By Lemma 3.4.2–2 we have

$$P(n, \mathbf{T}_3^1) \in \{P(n, \mathbf{T}'_1), P'(n, \mathbf{T}'_1)\}.$$

Let $(\mathbf{T}'_3, \mathbf{T}'_4) = S((\mathbf{T}_3^1, \mathbf{T}_4^1))$. Then $P(n, \mathbf{T}'_3) \in \{P(n, \mathbf{T}'_1), P'(n, \mathbf{T}'_1)\}$, and by Proposition 3.4.6 we have $(\mathbf{T}'_3, \mathbf{T}'_4) \sim_{GTL} (\mathbf{T}_3, \mathbf{T}_4)$. Let $(\bar{\mathbf{T}}'_3, \bar{\mathbf{T}}'_4) = (\mathbf{T}'_3, \mathbf{T}'_4) - L$. By induction $\mathbf{S}(\bar{\mathbf{T}}'_3) = \mathbf{S}(\bar{\mathbf{T}}_3)$. In the case where $\phi_C(S_{ij}) = W$ we then have

$$P(n-2, \mathbf{T}'_3) = P_2 \quad \text{or} \quad P(n-2, \mathbf{T}'_3) = \{S_{i,j-1}, S_{i+1,j-1}\},$$
$$P(n-1, \mathbf{T}'_3) = \{S_{i-1,j}, S_{ij}\}, \text{ and } P(n, \mathbf{T}'_3) = P(n, \mathbf{T}'_1).$$

In the other cases we have

$$P(n-2, \mathbf{T}'_3) = P_2,\ P(n-1, \mathbf{T}'_3) = P_1,$$

and

$$P(n, \mathbf{T}'_3) = P(n, \mathbf{T}'_1).$$

Now let $(\mathbf{T}_5, \mathbf{T}_6) = T^L_{\alpha_{n-1},\alpha_n}((\mathbf{T}_3, \mathbf{T}_4))$ and $(\mathbf{T}'_5, \mathbf{T}'_6) = T^L_{\alpha_{n-1},\alpha_n}((\mathbf{T}'_3, \mathbf{T}'_4))$. Let $(\bar{\mathbf{T}}_5, \bar{\mathbf{T}}_6) = (\mathbf{T}_5, \mathbf{T}_6) - L$ and $(\bar{\mathbf{T}}'_5, \bar{\mathbf{T}}'_6) = (\mathbf{T}'_5, \mathbf{T}'_6) - L$. Then by induction we have $\mathbf{S}(\bar{\mathbf{T}}_5) = \mathbf{S}(\bar{\mathbf{T}}'_5)$. On the other hand, we observe directly that $\mathbf{S}(\bar{\mathbf{T}}_5) \neq \mathbf{S}(\bar{\mathbf{T}}'_5)$.

Case G. Here $P(n, \mathbf{T}'_1) = \{S_{ij}, S_{i+1,j}\}$. Then we must have $\phi_C(S_{ij}) = X$. (To see this, note that since $\mathbf{T}'_1$ is special we have that $\phi_C(S_{ij}) \neq Z$. Now assume $\phi_C(S_{ij}) = W$. Then since $\mathbf{T}_1$ is special and $S_{i-1,j+1}$ is a filled corner in $\bar{\mathbf{T}}_1$ and S_{ij} is an empty hole in $\bar{\mathbf{T}}_1$, it follows that there is a cycle $c \in \text{OC}^*(\bar{\mathbf{T}}_1)$ with $S_b(c) = S_{i-1,j+1}$ and $S_f(c) = S_{ij}$. Similarly, there is a cycle $c' \in \text{OC}^*(\bar{\mathbf{T}}'_1)$ with $S_b(c') = S_{i+1,j-1}$ and $S_f(c') = S_{ij}$. Since $\mathbf{S}(\bar{\mathbf{T}}_1) = \mathbf{S}(\bar{\mathbf{T}}'_1)$ this is impossible.) If $i = 1$ and $j = 1$ then $n = 1$, $\alpha_1 \in \tau(\mathbf{T}'_1)$, and $\alpha_1 \notin \tau(\mathbf{T}_1)$. Henceforth assume that $i > 1$. (The situation where $i = 1$ and $j > 1$ is the transpose of the situation where $j = 1$ and $i > 1$.) Since $\phi_C(S_{1,1}) = X$ we have $i \geqslant 3$. There are several cases. If $S_{i-1,j+3} \in \text{Shape}(\mathbf{T}_1)$ let $r = \rho_{i-1}(\mathbf{T}_1)$, $P_1 = \{S_{i-1,r-1}, S_{i-1,r}\}$, and $P_2 = \{S_{i-1,r-3}, S_{i-1,r-2}\}$. If $S_{i-1,j+3} \notin \text{Shape}(\mathbf{T}_1)$ and $S_{i-1,j+2} \in \text{Shape}(\mathbf{T}_1)$ then since $\mathbf{T}_1$ is special we have $S_{i-2,j+3} \notin \text{Shape}(\mathbf{T}_1)$ so let $P_1 = \{S_{i-2,j+2}, S_{i-1,j+2}\}$ and $P_2 = \{S_{i-1,j}, S_{i-1,j+1}\}$. If $S_{i-1,j+2} \notin \text{Shape}(\mathbf{T}_1)$ and $S_{i-2,j+2} \in \text{Shape}(\mathbf{T}_1)$ then since $\mathbf{T}_1$ is special we have $S_{i-2,j+3} \in \text{Shape}(\mathbf{T}_1)$ so let $r = \rho_{i-2}(\mathbf{T}_1)$, $P_1 = \{S_{i-2,r-1}, S_{i-2,r}\}$ and $P_2 = \{S_{i-1,j}, S_{i-1,j+1}\}$. If $S_{i-2,j+2} \notin \text{Shape}(\mathbf{T}_1)$ then let $P_1 = \{S_{i-2,j+1}, S_{i-1,j+1}\}$ and $P_2 = \{S_{i-2,j}, S_{i-1,j}\}$. In all four cases we proceed as in case F and arrive at the contradiction that $\mathbf{S}(\bar{\mathbf{T}}_5) \neq \mathbf{S}(\bar{\mathbf{T}}'_5)$.

Case H. Here we assume that $P(n, \mathbf{T}'_1) = \{S_{i,j-1}, S_{i+1,j-1}\}$. This case is the transpose of case E.

Case I. Here we assume that $P(n, \mathbf{T}'_1) = \{S_{i-1,j+1}, S_{i,j+1}\}$. Then $\phi_C(S_{ij}) = X$ or $\phi_C(S_{ij}) = W$, and we have $\mathbf{T}'_1 = \mathbf{E}(\mathbf{T}_1, c(n, \mathbf{T}_1))$. Here the arguments used in the previous cases will not work. We could set $P_1 = \{S_{i-1,j}, S_{i-1,j+1}\}$ and find $(\bar{\mathbf{T}}_3, \bar{\mathbf{T}}_4)$, etc., as in case A, but it is then possible that $\mathbf{T}_3 = \mathbf{T}'_3$. So instead we have proved this case by a different method, in Lemma 3.4.15.

This completes the proof of Theorem 3.4.17. □

REMARK. To prove this theorem for $(\mathbf{T}_1, \mathbf{T}_2), (\mathbf{T}'_1, \mathbf{T}'_2) \in \mathscr{T}_B(n, n)$ we need the following variations on cases F and G.

Case F. We can no longer assume that $i > 1$. So assume that $i = 1$. Then $\mathbf{T}_1$ special implies that $\phi_B(S_{ij}) = Z$. If $j > 2$ then transposing takes us to the situation where $i > 1$, so we are reduced to the case where $i = 1$ and $j = 2$. But then $n = 1$, $\alpha_1 \notin \tau(\mathbf{T}_1)$, and $\alpha_1 \in \tau(\mathbf{T}'_1)$, so we are done.

Case G. Here we have to consider the possibility that $i = 2$ and $j = 2$ and $\rho_1(\mathbf{T}_1) = 3$ and $\kappa_1(\mathbf{T}_1) = 3$. We will assume that $\bar{\mathbf{T}}_1 = \bar{\mathbf{T}}'_1 = (F_2, \phi_B)$ (the case where $\bar{\mathbf{T}}_1 = \bar{\mathbf{T}}'_1 = (F_1, \phi_B)$ is entirely similar). Then $T^L_{\alpha_1,\alpha_2}((\mathbf{T}'_1, \mathbf{T}'_2))$ consists of one element, say $(\mathbf{T}'_3, \mathbf{T}'_4)$, and we have $\alpha_3 \in \tau(\mathbf{T}'_3)$. On the other hand, $T^L_{\alpha_1,\alpha_2}((\mathbf{T}_1, \mathbf{T}_2))$ consists of two elements, and α_3 is not in the left τ-invariant of one of them.

Section 5

We now recall the equivalence relation of cells defined by Joseph in [8].

3.5.1. DEFINITION. Let X be either W, $\mathscr{S}(n, n)$, $\mathscr{T}_C(n, n)$ or $\mathscr{T}_B(n, n)$. We define an equivalence relation, the left cell relation of Joseph, on X. If $x, y \in X$ then $x \sim_{JL} y$ if and only if there is a sequence Σ for Π such that $y \in T^L_\Sigma(x)$. (That this is an equivalence relation follows from the definition of T^L_Σ and Remark 3.2.1.) We define analogously the right cell relation of Joseph, $\sim_{JR}$.

We recall here the main result of Section 2, in the form in which we will use it in this section.

3.5.2. THEOREM. *Let* $(\mathbf{T}_1, \mathbf{T}_2), (\mathbf{T}'_1, \mathbf{T}'_2) \in \mathscr{T}_C(n, n)$. Then $(\mathbf{T}_1, \mathbf{T}_2) \sim_{JR} (\mathbf{T}'_1, \mathbf{T}'_2)$ *if and only if* $\mathbf{S}(\mathbf{T}_1) = \mathbf{S}(\mathbf{T}'_1)$.

Proof. This combines Theorem 3.2.2 and Proposition 3.2.3. □

3.5.3. NOTATION. For this section we will consider the map A defined in part I of this series of papers as a map from $\mathscr{S}(n, n)$ to $\mathscr{T}_C(n, n)$ in the obvious way, that is, the image of a $\gamma \in \mathscr{S}(n, n)$ is a pair of tableaux with the C grid. We will write $A(\gamma) = (\mathbf{L}(\gamma), \mathbf{R}(\gamma))$.

3.5.4. PROPOSITION. *Let* $w_1, w_2 \in W$. *Then*

(1) $w_1 \sim_{JL} w_2$ *if and only if* $A(\delta(w_1)) \sim_{JR} A(\delta(w_2))$.
(2) $w_1 \sim_{GTR} w_2$ *if and only if* $A(\delta(w_1)) \sim_{GTL} A(\delta(w_2))$.

Proof. These statements are an obvious consequence of Remark 2.1.7–1), Proposition 2.1.18, and Theorems 2.1.19 and 2.3.8. □

3.5.5. PROPOSITION. *Let* $w_1, w_2 \in W$. *Then*

(1) $A(\delta(w_1)) \sim_{JR} A(\delta(w_2))$ *if and only if* $\mathbf{S}(\mathbf{L}(\delta(w_1))) = \mathbf{S}(\mathbf{L}(\delta(w_2)))$.
(2) $A(\delta(w_1)) \sim_{GTL} A(\delta(w_2))$ *if and only if* $\mathbf{S}(\mathbf{L}(\delta(w_1))) = \mathbf{S}(\mathbf{L}(\delta(w_2)))$.

Proof. Statement (1) is simply Theorem 3.5.2 applied to $A(\delta(w_1))$ and $A(\delta(w_2))$. Similarly, statement (2) is Theorem 3.4.17 applied to $A(\delta(w_1))$ and $A(\delta(w_2))$. □

3.5.6. COROLLARY. *Let* $w_1, w_2 \in W$. *Then*

(1) $w_1 \sim_{GTR} w_2$ *if and only if* $\mathbf{S}(\mathbf{L}(\delta(w_1))) = \mathbf{S}(\mathbf{L}(\delta(w_2)))$.
(2) $w_1 \sim_{JL} w_2$ *if and only if* $w_1 \sim_{GTR} w_2$.

Proof. This combines Propositions 3.5.4 and 3.5.5. □

Now let $\mathfrak{g}$ be a complex semisimple Lie algebra. Let $\mathrm{U}(\mathfrak{g})$ be its universal enveloping algebra, and let Prim $\mathrm{U}(\mathfrak{g})$ be the set of primitive ideals in $\mathrm{U}(\mathfrak{g})$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, and fix Δ^+ a choice of positive roots for Δ. Let $\mathfrak{n} = \Sigma_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$ and let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$. Let $\rho = \frac{1}{2} \Sigma_{\alpha \in \Delta^+} \alpha$.

If $\lambda \in \mathfrak{h}^*$ we write Δ_λ for the integral roots with respect to λ, that is,

$$\Delta_\lambda = \{\alpha \in \Delta \mid s_\alpha(\lambda) - \lambda = n_\alpha \alpha \quad \text{with } n_\alpha \in \mathbb{Z}\}.$$

We set $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$ and we let Π_λ be the simple roots of Δ_λ^+. We write W_λ for the Weyl group of Δ_λ. We define τ^L, $D_{\alpha\beta}^L$, and $T_{\alpha\beta}^L$ on W_λ exactly as they were defined on W in section 1 of [4], and similarly for their counterparts on the right, and for the equivalence relations $\sim_{JL}$, $\sim_{JR}$, $\sim_{GTL}$, and $\sim_{GTR}$. We write $\mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g}))$ for the set of primitive ideals in $\mathrm{U}(\mathfrak{g})$ with infinitesimal character λ.

Now suppose $\lambda \in \mathfrak{h}^*$ is anti-dominant and regular. Let $w \in W_\lambda$. We write $I_\lambda(w)$ for the annihilator in $\mathrm{U}(\mathfrak{g})$ of the irreducible highest weight module $L(w\lambda)$, where $L(w\lambda)$ is the unique irreducible quotient of the Verma module $M_{w\lambda} = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$.

Duflo [2] has shown that the map $W_\lambda \to \mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g}))$ given by $w \mapsto I_\lambda(w)$ is surjective. One can define τ, $D_{\alpha\beta}$, and $T_{\alpha\beta}$ for $\mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g}))$ by means of the Duflo map:

3.5.7. DEFINITION. We define $\tau(I_\lambda(w)) = \tau^R(w)$ and

$$D_{\alpha\beta}(\mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g}))) = \{I \in \mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g})) \mid \beta \in \tau(I) \quad \text{and } \alpha \notin \tau(I)\}.$$

Suppose $I_\lambda(w) \in D_{\alpha\beta}(\text{Prim}_\lambda(\text{U}(\mathfrak{g})))$. We define $T_{\alpha\beta}(I_\lambda(w)) = I_\lambda(T^R_{\alpha\beta}(w))$ when α and β have the same length, otherwise $T_{\alpha\beta}(I_\lambda(w)) = \{I_\lambda(w') \mid w' \in T^R_{\alpha\beta}(w)\}$.

Duflo [2], Borho-Jantzen [1], and Jantzen [7] have shown that the above are well-defined. Vogan [12] then defines the generalized τ-invariant as an equivalence relation on $\text{Prim}_\lambda(\text{U}(\mathfrak{g}))$, written $\sim_{GT}$, as in Definition 3.4.1. We have clearly

$$I_\lambda(w_1) \sim_{GT} I_\lambda(w_2) \quad \text{if and only if } w_1 \sim_{GTR} w_2. \tag{3.5.8}$$

3.5.9. THEOREM. Suppose all the simple factors of Δ_λ are of type A_n, B_n, or C_n. Let $I_1, I_2 \in \text{Prim}_\lambda(\text{U}(\mathfrak{g}))$. Then $I_1 \sim_{GT} I_2$ if and only if $I_1 = I_2$.

This is Conjecture 3.11 of [12]. In that paper it was proved when the simple factors of Δ_λ are of type A_n. As we will show in part IV of this series of papers, it is false as stated for type D_n, but true when $\sim_{GT}$ is replaced by a stronger equivalence relation, which will be described in that paper.

PROOF OF THEOREM 3.5.9. Let $w_1, w_2 \in W_\lambda$. Joseph has shown [8] that $w_1 \sim_{JL} w_2$ implies that $I_\lambda(w_1) = I_\lambda(w_2)$. Clearly if $I_\lambda(w_1) = I_\lambda(w_2)$ then $I_\lambda(w_1) \sim_{GT} I_\lambda(w_2)$, and the observation 3.5.8 says that $I_\lambda(w_1) \sim_{GT} I_\lambda(w_2)$ implies that $w_1 \sim_{GTR} w_2$. So it suffices to prove that $w_1 \sim_{GTR} w_2$ implies that $w_1 \sim_{JL} w_2$, and clearly it suffices to prove this when Δ_λ is simple. For type A_n this was done by Jantzen in [7], and also by Vogan in [12], where the above argument was used to complete the proof of Theorem 3.5.9 for type A_n. For type C_n we have proved that $w_1 \sim_{GTR} w_2$ implies $w_1 \sim_{JL} w_2$ as Corollary 3.5.6–2). Since the standard identification of the Weyl groups of types B_n and C_n commutes with the definitions of τ^L, etc., we have also proved this for type B_n. □

3.5.10. DEFINITION. Recall that $\mathscr{T}^S_C(n)$ was defined by $\mathscr{T}^S_C(n) = \{\mathbf{T} \in \mathscr{T}_C(n) \mid \mathbf{T}$ is special$\}$, and similarly $\mathscr{T}^S_B(n)$. We define τ on $\mathscr{T}^S_C(n)$ as on $\mathscr{T}_C(n)$, and then define $D_{\alpha\beta}$ as usual. When $\{\alpha, \beta\} = \{\alpha_i, \alpha_{i+1}\}$ with $i \geqslant 2$ we define $T_{\alpha\beta}$ on $D_{\alpha\beta}(\mathscr{T}^S_C(n))$ as on $D_{\alpha\beta}(\mathscr{T}_C(n))$, that is, using Definition 2.1.10. When $\{\alpha, \beta\} = \{\alpha_i, \alpha_{i+1}\}$ with $i = 1$ and $\mathbf{T} \in D_{\alpha\beta}(\mathscr{T}^S_C(n))$ we define $T_{\alpha\beta}(\mathbf{T})$ as a one or two element subset of $D_{\beta\alpha}(\mathscr{T}^S_C(n))$ as follows.

(1) If $\tilde{F}_1 \subseteq \mathbf{T}$ (respectively $\tilde{F}_2 \subseteq \mathbf{T}$) then, since $\mathbf{T}$ is special, we have that $c(2, \mathbf{T})$ is closed. Let $\mathbf{T}' = \mathbf{E}(\mathbf{T}, c(2, \mathbf{T}))$ and define $T_{\alpha\beta}(\mathbf{T}) = \{(\mathbf{T}' \backslash F_1) \cup F_2\}$ (respectively $T_{\alpha\beta}(\mathbf{T}) = \{(\mathbf{T}' \backslash F_2) \cup F_1\}$).

(2) If $F_1 \subseteq \mathbf{T}$ (respectively $F_2 \subseteq \mathbf{T}$) let $\mathbf{T}' = (\mathbf{T} \backslash F_1) \cup F_2$ (respectively $\mathbf{T}' = (\mathbf{T} \backslash F_2) \cup F_1$). If $c(2, \mathbf{T}')$ is open define $T_{\alpha\beta}(\mathbf{T}) = \{\mathbf{T}'\}$. If $c(2, \mathbf{T}')$ is closed, define $T_{\alpha\beta}(\mathbf{T}) = \{\mathbf{T}', \mathbf{E}(\mathbf{T}', c(2, \mathbf{T}'))\}$.

We now have the classification theorem.

3.5.11. THEOREM. *Suppose* $\mathfrak{g}$ *is of type* C_n *or type* B_n *and* $\lambda \in \mathfrak{h}$ *is integral, regular, and anti-dominant. Then the map*

$$cl\colon \mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g})) \to \mathscr{T}_C^S(n)$$

given by $cl(I_\lambda(w)) = \mathbf{S}(\mathbf{L}(\delta(w)))$ is a bijection. The map cl has the following properties:

(1) $\tau(cl(I)) = \tau(I)$ *for* $I \in \mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g}))$,
(2) $cl(T_{\alpha\beta}(I)) = T_{\alpha\beta}(cl(I))$ *for* $I \in D_{\alpha\beta}(\mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g})))$.

The map cl is the unique map from $\mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g}))$ to $\mathscr{T}_C^S(n)$ having properties (1) and (2).

Proof. The surjectivity of cl follows from the surjectivity of the map A. The injectivity of cl is a combination of Corollary 3.5.6–1 and Theorem 3.5.9. Properties (1) and (2) follow from Definitions 3.5.7 and 3.5.10 and the results cited in the proof of Proposition 3.5.4. To see the uniqueness, let ψ: $\mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g})) \to \mathscr{T}_C^S(n)$ be another map with properties (1) and (2). Then $cl^{-1} \circ \psi$ commutes with τ and the $T_{\alpha\beta}$'s, and thus for every $I \in \mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g}))$ we have $(cl^{-1} \circ \psi)(I) \sim_{GT} I$. Then by Theorem 3.5.9 we have $(cl^{-1} \circ \psi)(I) = I$, as desired. □

3.5.12. REMARK. For $\mathfrak{g}$ of type B_n and λ as in Theorem 3.5.11 it will sometimes be necessary to have a classifying map $cl\colon \mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g})) \to \mathscr{T}_B^S(n)$. This map is defined as in Theorem 3.5.11 except that $\mathbf{L}(\delta(w))$ is defined with reference to the map A^0 of Remark 1.2.14. Theorem 3.5.11 still holds for this map cl, as every input to this theorem is either true as stated for this situation and has the same proof, or we have, in the various remarks of these papers, shown how to modify the statements and proofs for this situation.

3.5.13. REMARK. Recall that for $\mathfrak{g}$ of type A_n and λ integral, regular, and anti-dominant Joseph has proved the analogue of Theorem 3.5.11 for a classifying map $\mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g})) \to \mathscr{T}_A(n)$, where $\mathscr{T}_A(n)$ is the set of standard Young tableaux of size n. It follows from this and our results that, for $\mathfrak{g}$ and λ as in Theorem 3.5.9, we can, by choosing an identification of Δ_λ with a product of root systems of type A_n, B_n, and C_n, define a classification map which takes $\mathrm{Prim}_\lambda(\mathrm{U}(\mathfrak{g}))$ to a product of $\mathscr{T}_A(n)$'s, $\mathscr{T}_B^S(n)$'s, and $\mathscr{T}_C^S(n)$'s. This map will also have the properties stated in Theorem 3.5.11. (Cf. [5], Chapter 5 and section 6.3. In that paper Joseph's classification map for type A_n is called cl^{ad} and there is another classification map cl^d. For types B_n and C_n, since the long element of the Weyl group is equal to -1, we do not need to distinguish two classification maps: they are equal.)

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Bibliography

1. Borho, W. and Jantzen, J.: Über primitive ideale in der Einhüllenden einer halbeinfacher Lie-Algebra. *Invent. Math.* 39 (1977) 1–53.
2. Duflo, M.: Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'unde algèbre de Lie semi-simple. *Ann. of Math.* 105 (1977) 107–120.
3. Garfinkle, D.: On the classification of primitive ideals for complex classical Lie algebras, I. *Comp. Math.* 75 (1990) 135–169.
4. Garfinkle, D.: On the classification of primitive ideals for complex classical Lie algebras, II. preprint.
5. Garfinkle, D.: The annihilators of irreducible Harish-Chandra modules for $SU(p, q)$ and other type A_{n-1} groups. preprint.
6. Garfinkle, D. and Vogan, D.: On the structure of Kazhdan-L usztig cells for branched Dynkin diagrams, *J. of Alg.* (to appear).
7. Jantzen, J.: Moduln mit einem höchsten Gewicht, L ecture Notes in Mathematics, vol. *750*. Berlin: Springer 1979.
8. Joseph, A.: A characteristic variety for the primitive spectrum of a semisimple Lie algebra, preprint. Short version in: (ed.) J. Carmona, and M. Vergne, Non-commutative harmonic analysis. *Lecture Notes in Mathematics vol.* 587, *pp.* 102–118. Berlin: Springer 1977.

9. Joseph, A.: Towards the Jantzen conjecture, II. *Comp. Math.* 40 (1980) 69–78.
10. Kazhdan, D., and Lusztig, G.: Representations of Coxeter groups and Hecke algebras. *Inv. Math.* 53 (1979) 165–184.
11. Lusztig, G.: A class of irreducible representations of a Weyl group. *Proc. Kon. Nederl. Akad.*, Series A 82 (1979) 323–335.
12. Vogan, D.: A generalized τ-invariant for the primitive spectrum of a semisimple Lie algebra. *Math. Ann.* 242 (1979) 209–224.