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Asymptotic estimates for rational points of bounded height on flag varieties

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Introduction

In the area of Diophantine equations one studies polynomial equations and their solutions. Typically, these are polynomials defined over the rationals $\mathbb{Q}$ or a number field K, and one is interested in solutions in the same field. Often there are infinitely many solutions, and optimally one would like a method that explicitly constructs all, or at least infinitely many, of these solutions. If no such method is known, one may take a different approach and try to find how many solutions there are of certain "size." In order to carry through with this line of thought, one needs a quantification for the "size" of a solution. This is the purpose of a *height*. If there are infinitely many solutions, one would like to know how many solutions exist with height less than or equal to a given bound.

This approach is often phrased in arithmetic-geometric terms. Suppose $\mathscr{V}$ is a projective variety defined over a number field and equipped with a height H. Usually this is done via an object called a *metrized line bundle* (see [CS] chapter VI). If $N(\mathscr{V}, B)$ denotes the number of points $\mathbf{x} \in \mathscr{V}$ with $H(\mathbf{x}) \leqslant B$, the asymptotic behavior of $N(\mathscr{V}, B)$ gives some insight into the algebraic/geometric properties of $\mathscr{V}$. The question then becomes: Can one describe the asymptotic behavior of $N(\mathscr{V}, B)$?

Explicit answers to this last question are few and far between. Two known examples are the number of points on an abelian variety with bounded height and S. Schanuel's asymptotic result in [S] for the number of points in projective space with height $\leqslant B$. Manin et al. in [FMT] gave asymptotics for the number of rational points on flag varieties of bounded height (which includes Schanuel's result as a special case) using deep results on analytic continuation of Langlands-style L-series.

Recently the author was able to give explicit estimates for the number of points on grassmannians, rational over a given number field, with height $\leqslant B$ (see [T1]). The method of proof resembled that used by Schanuel in that it involved geometry of numbers and a result on the number of lattice points in a bounded domain in $\mathbb{R}^n$. It is the purpose of this paper to generalize the work of

[T1] to include a broader notion of height and then to extend this to flag varieties.

This paper is organized as follows. In part I we give definitions for heights and flag varieties. In part II we prove asymptotics for grassmannians which generalize those given in [T1]. Finally, in part III we show how this can be used to give explicit terms in the asymptotics for general flag varieties. For example, we can show the asymptotic relation

$$N(\mathcal{V}, B) \sim c(\mathcal{V}) B \log^{l-1}(B),$$

for flag varieties $\mathcal{V}$, where $c(\mathcal{V})$ is an explicitly given constant (depending on the number field, $\mathcal{V}$, and the height chosen) and l is the rank of $\mathrm{Pic}(\mathcal{V})$ (see theorem 5 below).

PART I. HEIGHTS ON FLAG VARIETIES

1. Heights on projective space

Let K be a number field of degree $\kappa = r_1 + 2r_2$ over the field $\mathbb{Q}$. Let $M(K)$ be the set of places of K, and let

$$a \mapsto a^{(i)} \quad 1 \leqslant i \leqslant \kappa$$

denote the embeddings of K into the complex numbers, ordered so that the first r_1 are real and

$$a^{(i+r_2)} = \overline{a^{(i)}}$$

for $r_1 + 1 \leqslant i \leqslant r_1 + r_2$, where $\bar{a}$ denotes the complex conjugate of the number a.

For each non-archimedean place $v \in M(K)$ let $|\cdot|_v$ be the corresponding absolute value on K, normalized to extend the p-adic absolute value on $\mathbb{Q}$, where v lies above the rational prime p. We also have the absolute value $|\cdot|_v$ for each archimedean place $v \in M(K)$, defined by

$$|a|_v = \begin{cases} |a^{(i)}| & \text{for } 1 \leqslant i \leqslant r_1, \\ |a^{(i)} a^{(i+r_2)}|^{1/2} & \text{for } r_1 < i \leqslant r_1 + r_2, \end{cases}$$

where v corresponds to the embedding $a \mapsto a^{(i)}$ and $|\cdot|$ denotes the usual absolute value on $\mathbb{R}$.

For each place $v \in M(K)$ let n_v be the local degree. We have the *product formula*

$$\prod_{v \in M(K)} |a|_v^{n_v} = 1$$

for all $a \in K^* = K\backslash\{0\}$ (see [L], Chapter 5).

Classically, the height on $\mathbb{P}^{n-1}(K)$ is defined as follows. Given a vector $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in K^n$ and a $v \in M(K)$, put

$$\|\mathbf{a}\|_v = \begin{cases} \left(\sum_{i=1}^{n} |a_i|_v^2\right)^{n_v/2}, & \text{if } v \text{ is archimedean,} \\ \max_{1 \leqslant i \leqslant n} |a_i|_v^{n_v} & \text{otherwise.} \end{cases}$$

Note that by the product formula and the definitions

$$\prod_{v \in M(K)} \|a\mathbf{a}\|_v = \prod_{v \in M(K)} |a|_v^{n_v} \cdot \prod_{v \in M(K)} \|\mathbf{a}\|_v = \prod_{v \in M(K)} \|\mathbf{a}\|_v$$

for all $a \in K^*$. One then may define the height of a $K\mathbf{a} \in \mathbb{P}^{n-1}(K)$ to be

$$H(K\mathbf{a}) = \prod_{v \in M(K)} \|\mathbf{a}\|_v.$$

The reader should be aware that often $\|\mathbf{a}\|_v$ is defined to be $\max\{|a_i|_v^{n_v}\}$ at the archimedean places as well. This is the case for instance, in Schanuel's paper. The definition we use here has the advantage of being "rotation invariant" in a sense (see, for example, the parenthetical remark before theorem 2 below). It may also give better error terms in the type of estimates we are dealing with (cf. Schanuel's result in the case $n = 2$ with that of theorem 4 below).

In what follows, we will need more flexibility with the choice of coordinates. Specifically, we must allow *local* changes in the coordinates. With this in mind we proceed as follows.

For each place $v \in M(K)$ let K_v denote the completion of K at v. Let λ_v denote the canonical injection of K into K_v. We will also use λ_v to denote the canonical injection of K^n into K_v^n. Let $A_v \in GL_n(K_v)$ with $A_v = I_n$, the identity matrix, for all but finitely many v.

Now for $\mathbf{a} \in K^n$ and any place v we have a point $\mathbf{a}_v = \lambda_v(\mathbf{a})A_v \in K_v^n$. We get a height on $\mathbb{P}^{n-1}(K)$ given by these A_v's defined by

$$H_A(K\mathbf{a}) = \prod_{v \in M(K)} \|\mathbf{a}_v\|_v.$$

Such a height is well defined by the product formula and since the A_v's are non-trivial at only finitely many places. Note that the "classical" height above is the height obtained when all the A_v's are the identity matrix. We will write H instead of H_A whenever the A_v's are understood.

2. Heights on flag varieties via line bundles

Let V be an n-dimensional vector space over a number field K. Let G be the automorphism ring Aut(V). We fix once and for all a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of V and identify V with K^n and G with $GL_n(K)$, with matrices acting on the right by matrix multiplication. For $d = 1, 2, \ldots, n-1$ let P_d denote the maximal parabolic subgroup consisting of matrices $(p_{ij}) \in G$ with $p_{ij} = 0$ if $d < j \leqslant n$ and $1 \leqslant i \leqslant d$. Elements of P_d are nonsingular matrices of the form

$$p = \begin{pmatrix} A_p & O_d \\ B_p & C_p \end{pmatrix} \tag{1}$$

where $A_p \in GL_d(K)$, $C_p \in GL_{n-d}(K)$, B_p is an $(n-d) \times d$ matrix, and O_d is the $d \times (n-d)$ matrix all of whose entries are 0. We let P_0 denote the minimal parabolic subgroup

$$P_0 = \bigcap_{i=1}^{n-1} P_i.$$

The subgroup P_d fixes the d-dimensional subspace $V_d = \oplus_{i=1}^{d} K\mathbf{v}_i \subset V$. The homogeneous space $P_d \backslash G$ is thus $Gr_d(V)$, the *grassmannian* of d-dimensional subspaces of V. More generally, if

$$P = P_{d_1} \cap P_{d_2} \cap \cdots \cap P_{d_l} \quad 0 < d_1 < d_2 < \cdots < d_l < n \tag{2}$$

is any parabolic subgroup containing P_0, then P fixes the nested sequence of subspaces $V_{d_1} \subset V_{d_2} \subset \cdots \subset V_{d_l}$ and $P \backslash G$ is the corresponding *flag variety*.

On each P_d we have the character χ_d, where $\chi_d(p) = \det(A_p)$ for $p \in P_d$ written as in (1). Let $X'(P_d)$ be the subgroup of the character group $X(P_d)$ generated by χ_d. More generally, for P a parabolic subgroup as in (2), let $X'(P)$ be the subgroup of the character group $X(P)$ generated by $\chi_{d_1}, \ldots, \chi_{d_l}$.

Let $c(n,d)$ denote the set of ordered d-tuples of integers $(i_1, i_2, \ldots, i_d)$ satisfying $1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant n$. Order the elements of $c(n,d)$ lexicographically. For $\alpha \in c(n,d)$ and $g = (g_{ij}) \in GL_n(K)$, define

$$f_\alpha(g) = \det_{\substack{1 \leqslant i \leqslant d \\ j \in \alpha}} (g_{ij}).$$

Note that these f_α's satisfy

$$f_\alpha(pg) = \chi_d(p) f_\alpha(g)$$

for all $p \in P_d$ and $g \in G$. These f_α's are *global sections* generating the (very ample) *line bundle* L_{χ_d}. They give a morphism of $P_d \backslash G$ into $\mathbb{P}^{\binom{n}{d}-1}(K)$. For a point $P_d g$ corresponding to the d-dimensional subspace $S \subset V$, this morphism takes $P_d g$ to the *grassmann coordinates* of S.

Similarly, if $m \in \mathbb{N}$, then

$$(f_\alpha(pg))^m = \chi_d^m(p)(f_\alpha(g))^m$$

for all $p \in P_d$ and $g \in G$, and the f_α^m's generate the line bundle $L_{\chi_d^m}$. More generally, if $\chi = \chi_{d_1}^{m_1} \cdots \chi_{d_l}^{m_l} \in X'(P)$ for P as in (2) and $m_i \in \mathbb{N}$, then $\{\Pi f_{\alpha_i}^{m_i}\colon \alpha_i \in c(n, d_i)\}$ are global sections generating the line bundle L_χ on $P \backslash G$. All ample line bundles on $P \backslash G$ are of this form since the rank of $\mathrm{Pic}(P \backslash G)$ is equal to the rank of $X'(P)$ (see [FMT]).

In this manner, for any ample line bundle on the variety $P \backslash G$ we get a morphism into $\mathbb{P}^m(K)$ for some m, whence a height on $P \backslash G$. Such a height will depend on the height on $\mathbb{P}^m(K)$. If one were to choose different global sections generating the line bundle, the corresponding height would change for individual points, but not affect the asymptotics of $N(P \backslash G, B)$.

Since we will be dealing with heights on flags, we cannot allow arbitrary heights on the projective spaces $\mathbb{P}^{\binom{n}{d}-1}(K)$. These heights must be consistent, in some sense. Once we have chosen a height on $\mathbb{P}^{n-1}(K)$ (i.e., on $Gr_1(V)$) we get heights on the other grassmannians in the following way.

Let matrices $A_v \in GL_n(K)$ be as in section 1, giving a height on $\mathbb{P}^{n-1}(K)$. Recall that we think of the A_v's as representing local changes of coordinates. Suppose $S \subset V$ is a d-dimensional subspace with basis $\mathbf{s}_1 = (s_{11}, s_{12}, \ldots, s_{1n}), \ldots, \mathbf{s}_d = (s_{d1}, s_{d2}, \ldots, s_{dn})$, i.e., $\mathbf{s}_i = \Sigma_{j=1}^n s_{ij}\mathbf{v}_j$ for $i = 1, 2, \ldots, d$. Then

$$S_v = \bigoplus_{i=1}^{d} K_v \lambda_v(\mathbf{s}_i)$$

is a d-dimensional subspace of K_v^n. Identify S with the matrix (s_{ij}) and S_v with $(\lambda_v(s_{ij}))$. Write $A_v = (a_{ij}^v)$. Grassmann coordinates of $S_v A_v = (b_{ij}^v)$ are given by

$$\det_{\substack{1 \leqslant i \leqslant d \\ j \in \alpha}} (b_{ij}^v) = \sum_{\beta \in c(n,d)} \det_{\substack{1 \leqslant i \leqslant d \\ j \in \beta}} (\lambda_v(s_{ij})) \det_{\substack{i \in \beta \\ j \in \alpha}} (a_{ij}^v) \quad \alpha \in c(n, d).$$

Thus, A_v induces a linear change of the grassmann coordinates given by the matrix

$$A_v^{(d)} = \left(\det_{\substack{i \in \beta \\ j \in \alpha}} (a_{ij}^v) \right).$$

This matrix is called the *d-th compound* of A_v. One easily verifies that $I_n^{(d)} = I_{\binom{n}{d}}$.

We define a height H on $Gr_d(V)$ given by the A_v's via the line bundle $L_{\chi d}$ with the global sections described above and the height on $\mathbb{P}^{\binom{n}{d}-1}(K)$ given by the matrices $A_v^{(d)} \in GL_{\binom{n}{d}}(K_v)$. Finally, we define $H(\{\mathbf{0}\}) = 1$ and

$$H(V) = \prod_{v \in M(K)} |\det(A_v)|_v^{n_v}.$$

In [T1] the number of points on $Gr_d(K^n)$ with height $\leqslant B$ is estimated, where the height is the one given by $A_v = I_n$ at all places v. This is the height introduced in [Sch] and also used by Bombieri and Vaaler in [BV], where they prove the existence of points on $Gr_d(K^n)$ with small height. As soon as one looks at points on general flag varieties, however, one is forced to consider the more general notion of height on grassmannians we have described here, i.e., allowing local changes of coordinates. In the language of arithmetic geometry, we have a line bundle L_χ with a metrization determined by the A_v's.

With this definition of height there is a (perhaps well known) duality which will be useful to exploit later.

DUALITY THEOREM. *Let H be the height given by matrices $A_v \in GL_n(K_v)$ and H^* be the height given by the matrices $B_v = (A_v^T)^{-1} \in GL_n(K_v)$, where the superscript ^{T} denotes the transpose. Let S be a d-dimensional subspace where $0 \leqslant d \leqslant n$ and let S^* be the dual space defined by*

$$S^* = \{\mathbf{y} \in V\colon \mathbf{x} \cdot \mathbf{y} = 0 \textit{ for all } \mathbf{x} \in S\},$$

where the dot product is defined with respect to the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of V. Then

$$H^*(S^*) = \frac{H(S)}{H(V)}.$$

Proof. The theorem is obvious if $d = 0$ or n, so we will assume $0 < d < n$. For $\alpha \in c(n,d)$ let α' be its complement in $\{1, 2, \ldots, n\}$. Let ε_α be 1 if (α, α') is an even permutation of $(1, 2, \ldots, n)$ or -1 if it is an odd permutation. Let $\mathbf{s}_1, \ldots, \mathbf{s}_d$ be a basis for S as above and let $\mathbf{s}_1^*, \ldots, \mathbf{s}_{n-d}^*$ be a basis for S^*. Identify S with the matrix (s_{ij}) and similarly for S^*. For $\alpha \in c(n,d)$ and $\beta \in c(n, n-d)$ let

$$S_\alpha = \det_{\substack{1 \leqslant i \leqslant d \\ j \in \alpha}} (s_{ij}) \quad S_\beta^* = \det_{\substack{1 \leqslant i \leqslant n-d \\ j \in \beta}} (s_{ij}^*).$$

By theorem 1 in chapter VII, section 3 of [HP], there is an $a \in K$ with $\varepsilon_\alpha S_{\alpha'}^* = a S_\alpha$ for all $\alpha \in c(n,d)$. But by equation (6) on page 296 of [HP] (this is in the proof of the theorem just quoted), we then have

$$\varepsilon_\alpha(S_v^* B_v)_{\alpha'} = \frac{a}{\det(A_v)} (S_v A_v)_\alpha.$$

Our theorem follows.

3. Heights on Grassmannians via Lattices

Let V be an n-dimensional space over K as in section 2. Let matrices A_v be given, yielding a height on $Gr_d(V)$ as above. We will reformulate this height in terms of determinants of lattices in a Euclidean space. This section is a straightforward generalization of work done in [Sch] and [T1].

By a lattice Λ in a Euclidean space E we will mean a discrete subgroup of the additive group E. The dimension of the lattice Λ is the dimension of the subspace it spans. Suppose $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m$ are a basis for Λ, so that

$$\Lambda = \bigoplus_{i=1}^{m} \mathbb{Z}\mathbf{X}_i.$$

The determinant of Λ is defined to be

$$\det(\Lambda) = (\det(\mathbf{X}_i * \mathbf{X}_j))^{1/2},$$

where $\mathbf{X} * \mathbf{Y}$ denotes the inner product in E of $\mathbf{X}$ and $\mathbf{Y}$. By convention $\det(\{\mathbf{0}\}) = 1$.

For $\mathbf{X} \in \mathbb{R}^{nr_1} \oplus \mathbb{C}^{2nr_2}$ we write

$$\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\kappa),$$

where

$$\mathbf{x}_i \in \begin{cases} \mathbb{R}^n & \text{for } 1 \leqslant i \leqslant r_1, \\ \mathbb{C}^n & \text{for } r_1 < i \leqslant \kappa. \end{cases}$$

Let $\mathbb{E}^{n\kappa} \subset \mathbb{R}^{nr_1} \oplus \mathbb{C}^{2nr_2}$ be given by the set of points satisfying

$$\mathbf{x}_{i+r_2} = \overline{\mathbf{x}_i} \quad r_1 < i \leqslant r_1 + r_2.$$

For $\mathbf{X}$ and $\mathbf{Y}$ in $\mathbb{E}^{n\kappa}$ we define their inner product to be $\mathbf{X} \cdot \bar{\mathbf{Y}}$, the usual inner product in $\mathbb{C}^{n\kappa}$. Thus, for $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\kappa)$ and $\mathbf{Y} = (y_1, y_2, \ldots, y_\kappa)$ in $\mathbb{E}^{n\kappa}$,

$$\mathbf{X} \cdot \bar{\mathbf{Y}} = \sum_{i=1}^{r_1} \mathbf{x}_i \cdot \mathbf{y}_i + \sum_{i=r_1+1}^{r_1+2r_2} \mathbf{x}_i \cdot \bar{\mathbf{y}}_i.$$

One easily verifies that $\mathbb{E}^{n\kappa}$ is a Euclidean vector space of dimension $n\kappa$ with this inner product. In particular, $\mathbf{X} \cdot \bar{\mathbf{Y}}$ is real and $\mathbf{X} \cdot \bar{\mathbf{Y}} = \mathbf{Y} \cdot \bar{\mathbf{X}}$.

The archimedean places give an embedding of V into $\mathbb{E}^{n\kappa}$ as follows. For each archimedean place v and $\mathbf{v} \in V$ let $\rho_i(\mathbf{v}) = \lambda_v(\mathbf{v}) A_v$, where v corresponds to the embedding $a \mapsto a^{(i)}$, $1 \leqslant i \leqslant r_1 + r_2$, and let $\rho_{i+r_2}(\mathbf{v}) = \overline{\rho_i(\mathbf{v})}$ for $r_1 < i \leqslant r_1 + r_2$. Define $\rho\colon V \to \mathbb{E}^{n\kappa}$ by $\rho = \rho_1 \times \rho_2 \times \cdots \times \rho_\kappa$. The nonarchimedean places give a lattice via this embedding. For each finite place v, multiplication by A_v gives a change of coordinates in K_v^n from those with respect to the basis $\lambda_v(\mathbf{v}_1), \lambda_v(\mathbf{v}_2), \ldots, \lambda_v(\mathbf{v}_n)$ to a new basis $\mathbf{v}_1^v, \mathbf{v}_2^v, \ldots, \mathbf{v}_n^v$. Letting $\mathfrak{O}$ denote the ring of integers in K and $\mathfrak{O}_v$ denote the integers in K_v, we have

$$\mathfrak{M}_v = \bigoplus_{i=1}^{n} \mathfrak{O}_v \mathbf{v}_i^v$$

is an $\mathfrak{O}_v$-module in K_v^n and $\mathfrak{M}_v = \mathfrak{O}_v^n$ for all but finitely many v's. Thus,

$$\mathfrak{M}_V = \bigcap_{v \nmid \infty} (K^n \cap \mathfrak{M}_v)$$

is an $\mathfrak{O}$-module spanning V with localization $\mathfrak{M}_v$ at each finite place v (see [W] chapter V, section 2, for example). For $S \subseteq V$ a subspace we will write $I(S)$ for the $\mathfrak{O}$-module $S \cap \mathfrak{M}_V$.

THEOREM 1. *Let $0 \leqslant d \leqslant n$ and suppose S is a d-dimensional subspace of V. Then $\rho(I(S))$ is a $d\kappa$-dimensional lattice in $\mathbb{E}^{n\kappa}$ with*

$$\det(\rho(I(S))) = \Delta^d H(S),$$

where Δ is the square root of the absolute value of the discriminant of K.

Proof. This is a generalization of [Sch] theorem 1, where $V = K^n$ and the height is the "classical" height ($A_v = I_n$ at each place v). For $d = 0$ the theorem simply follows from the definitions. We will now suppose that $1 \leqslant d$.

Let $\mathbf{s}_1, \ldots, \mathbf{s}_d$ be a basis for S over K with $\mathbf{s}_i \in \mathfrak{M}_V$ for each i. For the moment, let $\mathfrak{M} = \oplus \mathfrak{O}\mathbf{s}_i$. Using the notation in the proof of the duality theorem, a minor variation of [Sch] lemma 4 shows

$$\det(\rho(\mathfrak{M})) = \Delta^d \prod_{v \mid \infty} \left(\sum_{\alpha \in c(n,d)} |(S_v A_v)_\alpha|_v^2 \right)^{1/2}.$$

At each finite place v let

$$\max_{\alpha\in c(n,d)} \{|(S_v A_v)_\alpha|_v^{n_v}\} = N(\mathfrak{P}_v)^{a_v},$$

where v corresponds to the prime ideal $\mathfrak{P}_v$ and N denotes the norm of an ideal. Note that by our hypotheses on the $\mathbf{s}_i$'s, $S_v A_v$ is a $d \times n$ matrix with entries in $\mathfrak{O}_v$, so that the rational integers a_v are all less than or equal to 0. Our proof will be complete if we show that the index of $\mathfrak{M}$ in $I(S)$ (and hence $\rho(\mathfrak{M})$ in $\rho(I(S))$) is $N(\mathfrak{A})$, where $\mathfrak{A} = \prod \mathfrak{P}_v^{-a_v}$.

So suppose $b_1\mathbf{s}_1 + b_2\mathbf{s}_2 + \cdots + b_d\mathbf{s}_d$ is an element of $I(S)$. If we let

$$B_v = \begin{pmatrix} \lambda_v(b_1) & \lambda_v(b_2) & \cdots & \lambda_v(b_d) \\ 0 & & & \\ \vdots & & I_{d-1} & \\ 0 & & & \end{pmatrix}$$

then the entries of $B_v S_v A_v$ are in $\mathfrak{O}_v$, and

$$(B_v S_v A_v)_\alpha = \lambda_v(b_1)(S_v A_v)_\alpha \in \mathfrak{O}_v$$

for all $\alpha \in c(n, d)$. Similarly, one can show

$$\lambda_v(b_i)(S_v A_v)_\alpha \in \mathfrak{O}_v$$

for all $\alpha \in c(n, d)$, $v \nmid \infty$, and $i = 1, 2, \ldots, d$. So if $a \in \mathfrak{A}$, then $c_i = b_i a \in \mathfrak{O}$ for $i = 1, 2, \ldots, d$. It suffices to show that the number of d-tuples $c_i, c_2, \ldots, c_d$ of $\mathfrak{O}$ modulo (a) with

$$\sum_{i=1}^{d} c_i \mathbf{s}_i \in aI(S)$$

equals $N(\mathfrak{A})$, and this is shown in the proof of [Sch] lemma 5.

Of course, we may reverse the order in which we have defined height and start with lattices. Specifically, suppose we are given an n-dimensional vector space V and a finitely generated $\mathfrak{O}$-module $\mathfrak{M}_V$ spanning V. For $1 \leqslant i \leqslant \kappa$ let ρ_i be a $1-1$ map

$$\rho_i\colon V \to \begin{cases} \mathbb{R}^n & \text{if } 1 \leqslant i \leqslant r_1 \\ \mathbb{C}^n & \text{if } r_1 < i \leqslant \kappa \end{cases}$$

satisfying

$$\rho_i(a\mathbf{a} + b\mathbf{b}) = a^{(i)}\rho_i(\mathbf{a}) + b^{(i)}\rho_i(\mathbf{b})$$

and

$$\rho_{i+r_2}(\mathbf{a}) = \overline{\rho_i(\mathbf{a})} \quad r_1 < i \leqslant r_1 + r_2.$$

Then $\rho = \rho_1 \times \rho_2 \times \cdots \times \rho_\kappa$ and $\mathfrak{M}_V$ will give a height on $Gr_d(V)$ by defining the height of a d-dimensional subspace S to be

$$H(S) = \Delta^{-d}\det(\rho(I(S))).$$

Theorem 1 is the key ingredient of our method. Questions regarding points on grassmannians, and ultimately points on flag varieties, are translated via this theorem into questions about lattices in a Euclidean domain. We will put this to use immediately by constructing a height on factor spaces with a certain useful property.

For S a subspace of V, $S_{v_i}A_{v_i}$ will span a subspace of $\mathbb{R}^n$ if $1 \leqslant i \leqslant r_1$, or $\mathbb{C}^n$ if $r_1 < i \leqslant \kappa$, where v_i is the archimedean place corresponding to the embedding $a \mapsto a^{(i)}$. For a subspace $W \in \mathbb{R}^n$, let $W^\perp$ be its orthogonal complement. Similarly, for W a subspace of $\mathbb{C}^n$, let $W^\perp$ be the orthogonal complement:

$$W^\perp = \{\mathbf{x} \in \mathbb{C}^n \colon \mathbf{x} \cdot \bar{\mathbf{y}} = 0 \text{ for all } \mathbf{y} \in W\}.$$

Let π^i be the orthogonal projection from $\mathbb{R}^n$ or $\mathbb{C}^n$ onto the space spanned by $(S_{v_i}A_{v_i})^\perp$ when $1 \leqslant i \leqslant r_1$ or $r_1 < i \leqslant \kappa$, respectively. Define

$$\pi = \pi^1 \times \pi^2 \times \cdots \times \pi^\kappa,$$

so that

$$\pi(\mathbf{X}) = (\pi^1(\mathbf{x}_1), \ldots, \pi^\kappa(\mathbf{x}_\kappa))$$

for all $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_\kappa) \in \mathbb{E}^{n\kappa}$. Note that $\pi \circ \rho$ is linear on V and vanishes only on S. When we write π we assume the subspace S is given.

Let S be a d_0-dimensional subspace of V. Denote the $(n - d_0)$-dimensional factor space V/S by $\tilde{S}$. We then have the $\mathfrak{O}$-module $\mathfrak{M}_{\tilde{S}} = \mathfrak{M}_V + S$ and the map $\rho_{\tilde{S}}$ on $\tilde{S}$ defined by

$$\rho_{\tilde{S}}(\tilde{\mathbf{a}}) = \pi \circ \rho(\mathbf{a})$$

for $\tilde{\mathbf{a}} = \mathbf{a} + S \in \tilde{S}$, giving a height $H_{\tilde{S}}$ on $Gr_d(\tilde{S})$. For $\tilde{T} \subset \tilde{S}$ a d-dimensional subspace,

$$H_{\tilde{S}}(\tilde{T}) = \Delta^{-d} \det(\rho_{\tilde{S}}(\tilde{I}(\tilde{T}))),$$

where $\tilde{I}(\tilde{T}) = \mathfrak{M}_{\tilde{S}} \cap \tilde{T}$. (Strictly speaking, $\rho_{\tilde{S}}$ maps $\tilde{S}$ to an $(n - d_0)$-dimensional subspace of $\mathbb{E}^{n\kappa}$. But after an appropriate unitary transformation τ, we have $\tau \circ \rho_{\tilde{S}}: \tilde{S} \to \mathbb{E}^{(n-d_0)\kappa}$, and this transformation will not affect the determinants of lattices).

THEOREM 2. *Let S, $\tilde{S}$, and $H_{\tilde{S}}$ be given as above. For $\tilde{T}$ a subspace of $\tilde{S}$, let $S + T$ be the subspace of V defined by*

$$S + T = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in S \text{ and } \mathbf{b} + S \in \tilde{T}\}.$$

Then

$$H(S + T) = H(S) H_{\tilde{S}}(\tilde{T}).$$

In particular

$$H_{\tilde{S}}(\tilde{S}) = \frac{H(V)}{H(S)}.$$

Proof. We will use theorem 1. Each point in $\rho(I(S + T))$ will either be in $\rho(I(S))$ or will be the sum of a point in $\rho(S)$ and a point in $\pi \circ \rho(I(S + T)) = \rho_{\tilde{S}}(\tilde{I}(\tilde{T}))$. But for $\mathbf{s} \in S$ and $\mathbf{v} \in V$, we have $\rho(\mathbf{s})$ and $\pi \circ \rho(\mathbf{v})$ are orthogonal. Thus,

$$\det(\rho(I(S + T))) = \det(\rho(I(S))) \cdot \det(\rho_{\tilde{S}}(\tilde{I}(\tilde{T}))),$$

which proves the theorem.

We make a final remark. Let $K\mathbf{a} \subset V$ be a one-dimensional subspace. Let H be a height on $Gr_1(V)$ given by the $\mathfrak{O}$-module $\mathfrak{M}_V$ and embedding ρ. Define

$$\mathfrak{J}(\mathbf{a}) = \{a \in K : a\mathbf{a} \in \mathfrak{M}_V\}$$

so that $I(K\mathbf{a}) = \mathfrak{J}(\mathbf{a})\mathbf{a}$. Clearly $\mathfrak{J}(\mathbf{a})$ is a fractional ideal. One easily verifies that

$$\det(\rho(\mathfrak{O}\mathbf{a})) = \Delta \prod_{i=1}^{\kappa} \|\rho_i(\mathbf{a})\|,$$

where $\|\cdot\|$ denotes the usual norm on $\mathbb{R}^n$ if $1 \leqslant i \leqslant r_1$, or on $\mathbb{C}^n$ if $r_1 < i \leqslant \kappa$. We then have

$$H(K\mathbf{a}) = N(\mathfrak{I}(\mathbf{a})) \prod_{i=1}^{\kappa} \|\rho_i(\mathbf{a})\|. \tag{3}$$

(See lemma 4 below).

PART II. ASYMPTOTICS FOR GRASSMANNIANS

1. Auxiliary results and statement of theorems

Our method for giving asymptotics on flags hinges on giving asymptotics for grassmannians. Once we have an asymptotic result for grassmannians with a reasonable error term, the generalization to flag varieties will follow from simple partial summation arguments. Before we state any theorems, however, it is convenient to introduce some ideas from the geometry of numbers.

We first map $\mathbb{E}^{n\kappa}$ into $\mathbb{R}^{n\kappa}$:

$$t: \mathbb{E}^{n\kappa} \to \mathbb{R}^{n\kappa},$$

is defined by

$$\begin{aligned} t(\mathbf{X}) &= t((\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\kappa)) \\ &= (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{r_1}, \mathbf{x}'_{r_1+1}, \mathbf{x}'_{r_1+2}, \ldots, \mathbf{x}'_{r_1+r_2}), \end{aligned}$$

where, for $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ and $i = r_1 + 1, r_1 + 2, \ldots, r_1 + r_2$, we define

$$\mathbf{x}'_i = (\mathrm{Re}(x_{i1}), \mathrm{Im}(x_{i1}), \ldots, \mathrm{Re}(x_{in}), \mathrm{Im}(x_{in})).$$

One sees that the determinant of T is $2^{-r_2 n}$. For $\mathbf{Y} \in \mathbb{R}^{n\kappa}$, we write

$$\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{r_1+r_2}),$$

where

$$\mathbf{y}_i \in \begin{cases} \mathbb{R}^n & \text{for } 1 \leqslant i \leqslant r_1, \\ \mathbb{R}^{2n} & \text{for } r_1 + 1 \leqslant i \leqslant r_1 + r_2. \end{cases}$$

Now let Λ be an l-dimensional lattice in $\mathbb{R}^k$ and let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_l$ be the successive minima of Λ with respect to the unit ball. Pick linearly independent points $\mathbf{y}_i \in \Lambda$ (the choice is not necessarily unique) satisfying

$$|\mathbf{y}_i| = \lambda_i \qquad i = 1, 2, \ldots, l.$$

Define

$$\Lambda^{-i} = \Lambda \cap \bigoplus_{j=1}^{l-i} \mathbb{R}\mathbf{y}_j \qquad i = 1, 2, \ldots, l-1$$

and

$$\Lambda^{-l} = \{\mathbf{0}\}.$$

Minkowski's second convex bodies theorem ([C] chapter VIII) asserts that

$$\frac{2^l}{l!} \det(\Lambda) \leqslant \lambda_1 \lambda_2 \cdots \lambda_l V(l) \leqslant 2^l \det(\Lambda),$$

where $V(l)$ denotes the volume of the unit ball in $\mathbb{R}^l$. Since the successive minima of Λ^{-i} are, by construction, $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{l-i}$ for $i < l$, we have $\det(\Lambda^{-i})$ is minimal among sublattices of Λ of dimension $l - i$.

Let V be an n-dimensional K-vector space. The successive minima of $t \circ \rho(\mathfrak{M}_V)$ were investigated in [T1]. They correspond to "minimal" subspaces. Let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n\kappa}$ be the successive minima of $t \circ \rho(\mathfrak{M}_V)$. We define i-dimensional subspaces $V_i \subseteq V$ and minima $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ as follows:

$$V_0 = \{\mathbf{0}\},$$

and recursively

$$\mu_{i+1} = \min\{\lambda_j\text{: there exists an } \mathbf{a}_{i+1} \in \mathfrak{M}_V \text{ with } \mathbf{a}_{i+1} \notin V_i \text{ and } \|t \circ \rho(\mathbf{a}_{i+1})\| = \lambda_j\}$$

and

$$V_{i+1} = V_i \oplus K\mathbf{a}_{i+1} \qquad 0 \leqslant i \leqslant n-1.$$

These subspaces are not necessarily uniquely defined by these conditions; at each stage one may need to make a choice. The following lemma is proven in [T1].

LEMMA 1. *Let V_i be as above and let $\lambda'_1 \leqslant \lambda'_2 \leqslant \cdots \leqslant \lambda'_{i\kappa}$ be the successive minima of $t \circ \rho(I(V_i))$. Then*

$$\lambda'_{(l-1)\kappa+j} \ll \mu_l \leqslant \lambda'_{(l-1)\kappa+j} \qquad 1 \leqslant j \leqslant \kappa,\ 1 \leqslant l \leqslant i.$$

By lemma 1 and Minkowski's theorem, the subspaces V_i are i-dimensional subspaces of minimal height. Moreover, suppose $S \subset V$ is a d-dimensional subspace and $0 \leqslant i \leqslant n-d$. By the duality theorem $((S^*)_i)^*$ is an $(n-i)$-dimensional subspace containing S of minimal height, so we define

$$S_i = ((S^*)_{n-i})^* \qquad d < i \leqslant n.$$

We will also use the following result which relates the successive minima arising from a height and its dual.

LEMMA 2. *Let V, ρ, and $\mathfrak{M}_V$ be as above. Let H^* be the dual height with corresponding embedding ρ^* and $\mathfrak{O}$-module $\mathfrak{M}_V^*$. Suppose S is a d-dimensional subspace and write $\tilde{S}$ for the factor space V/S. If $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{(n-d)\kappa}$ are the successive minima of $t \circ \rho_{\tilde{S}}(\mathfrak{M}_{\tilde{S}})$ and $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{(n-d)\kappa}$ are the successive minima of $t \circ \rho^*(I^*(S^*))$, then*

$$\lambda_i \mu_{(n-d)\kappa-i+1} \gg\ll 1 \qquad 1 \leqslant i \leqslant (n-d)\kappa.$$

Proof. Consider the subspaces $\tilde{S}_i$. These correspond to $(d+i)$-dimensional subspaces $T_i \supseteq S$ of minimal height, by theorem 2. Thus, by lemma 1

$$\prod_{j=1}^{i\kappa} \lambda_j \gg\ll \frac{H(T_i)}{H(S)} = \frac{H^*(T_i^*)}{H^*(S^*)} \gg\ll \frac{\prod_{j=1}^{(n-d-i)\kappa} \mu_j}{\prod_{j=1}^{(n-d)\kappa} \mu_j} = \prod_{j>(n-d-i)\kappa} u_j^{-1},$$

where the empty product is interpreted as 1. This shows

$$\prod_{j=(n-d-i)\kappa+1}^{(n-d)\kappa} \lambda_j \gg\ll \prod_{j=1}^{\kappa} u_j^{-1},$$

and inductively

$$\prod_{j=(i-1)\kappa+1}^{i\kappa} \lambda_j \gg\ll \prod_{j=(n-d-i)\kappa+1}^{(n-d-i+1)\kappa} \mu_j^{-1}.$$

Our lemma now follows from lemma 1.

This result is strongly reminiscent of a theorem of Mahler ([C] theorem VI, p. 219) relating the successive minima of a lattice with those of the polar lattice.

This is not coincidental. In fact, in the case $K = \mathbb{Q}$, the dual lattice *is* the polar lattice, which gives an alternate method for proving the duality theorem.

The next lemma will be used in the proof of theorem 3 below.

LEMMA 3. *Let V be an n-dimensional vector space over K and let S be a d-dimensional subspace where $n > d > 0$. Let $T = S_1$ and suppose $S \cap V_{n-d} = \{\mathbf{0}\}$. Then*

$$T_{n-d+1} \cap S = T.$$

Equivalently, if S is an $(n-d)$-dimensional subspace with $S \cap V_d = \{\mathbf{0}\}$, then

$$(S_{n-1})_{d-1} \cap S = \{\mathbf{0}\}.$$

Proof. The two statements are seen to be equivalent by taking duals and invoking lemma 2. We will prove the first statement. It is obvious if $d = 1$, so we may assume $d > 1$. Let $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_d$ be elements of $I(S)$ from the construction of the S_i's, i.e.,

$$S_i = \bigoplus_{j=1}^{i} K\mathbf{s}_j \qquad 1 \leqslant i \leqslant d,$$

and define

$$W = \bigoplus_{j=2}^{d} K\mathbf{s}_j.$$

Let

$$\phi\colon V \to V/T = \tilde{T}$$

be the canonical homomorphism. Then by lemma 1 and Minkowski's theorem

$$|\rho_{\tilde{T}}(\mathbf{w})| \gg |\rho(\mathbf{s}_1)| \qquad \mathbf{w} \in \phi(W) \cap \mathfrak{M}_{\tilde{T}}.$$

But since $S \cap V_{n-d} = \{\mathbf{0}\}$, there are $n - d$ linearly independent elements of $\rho_{\tilde{T}}(\mathfrak{M}_{\tilde{T}})$ (namely $\rho_{\tilde{T}}(\phi(\mathbf{a}_1)), \ldots, \rho_{\tilde{T}}(\phi(\mathbf{a}_{n-d}))$, where the $\mathbf{a}_i$'s are the elements of $\mathfrak{M}_V$ in the construction of the V_i's) with smaller length. We thus have

$$\tilde{T}_{n-d} \cap \phi(W) = \{\mathbf{0}\},$$

which implies the desired result.

Finally, we will need the following result which is lemma 1 of [T1].

LEMMA 4. *Let $\mathfrak{M}$ be an $\mathfrak{O}$-module in a vector space over K spanning a subspace of dimension $d \geqslant 1$ and let $\mathfrak{A}$ be an integral ideal. If we let $\mathfrak{A}\mathfrak{M}$ denote the set of finite sums*

$$\{\sum a_i m_i \colon a_i \in \mathfrak{A} \text{ and } m_i \in \mathfrak{M}\},$$

then the index of $\mathfrak{A}\mathfrak{M}$ in $\mathfrak{M}$ is $N(\mathfrak{A})^d$.

As an example of how this lemma will be used, suppose $S'' V$ is a d-dimensional subspace and let $\mathfrak{A}$ be an arbitrary ideal. Write $\mathfrak{A} = \mathfrak{P}\mathfrak{C}^{-1}$, where $\mathfrak{P}$ and $\mathfrak{C}$ are integral ideals. By lemma 4

$$\begin{aligned}\Delta^d N(\mathfrak{P})^d H(S) &= \det(\rho(\mathfrak{P} I(S)))\\ &= \det(\rho(\mathfrak{C}\mathfrak{A} I(S)))\\ &= N(\mathfrak{C})^d \det(\rho(\mathfrak{A} I(S)))\end{aligned}$$

giving

$$(\Delta N(\mathfrak{A}))^d H(S) = \det(\rho(\mathfrak{A} I(S))). \tag{4}$$

We are now ready to state our theorems for grassmannians.

THEOREM 3. *Let V be an n-dimensional space over K. Let $A_v \in GL_n(K_v)$ be matrices giving a height H. For $0 < d < n$ let $M(d, V, B)$ denote the number of d-dimensional subspaces $S \subset V$ with $S \cap V_{n-d} = \{\mathbf{0}\}$ and $H(S) \leqslant B$. Then*

$$M(d, V, B) = a(n, d)\, \frac{B^n}{H(V)^d} + O\left(\frac{B^{n-b(n,d)}}{H(V)^{d-b(n,d)} H(V_{n-d})^{b(n,d)}}\right)$$

as $B \to \infty$ and the constant implicit in the O notation depends only on n and K.

The values of the constants above are as follows.

$$b(n, d) = \max\left\{\frac{1}{\kappa d}, \frac{1}{\kappa(n-d)}\right\}.$$

Let R be the regulator, h be the class number, and w be the number of roots of unity of K. Further, let ζ_K be the Dedekind zeta function of K, and introduce the function $V_2(n) = V(2n)$. Given a function f defined for $n = 2, 3, \ldots$ and having non-zero values, introduce the generalized binomial symbol

$$(f|_d^n) = \frac{f(n)f(n-1)\cdots f(n-d+1)}{f(2)f(3)\cdots f(d)},$$

defined for $0 < d < n$, with $f(2)f(3)\cdots f(d)$ to be interpreted as 1 when $d = 1$. Note that $(f|_d^n) = (f|_{n-d}^n)$. With this notation,

$$a(n,d) = \frac{hR}{wn}\left(\frac{2^{r_2}}{\Delta}\right)^{(n-d)d+1}\binom{n}{d}^{r_1+r_2}(V|_d^n)^{r_1}(V_2|_d^n)^{r_2}(\zeta_K|_d^n)^{-1}.$$

We remark that the main term here is larger than the error term only for $B \geqslant H(V)/H(V_{n-d})$. In fact, we will assume this, since $M(d, V, B) = 0$ for B smaller than $H(V)/H(V_{n-d})$ (see proof of lemma 6 below). In the case $d = 1$ we may estimate points in projective space using Schanuel's method, but incorporating some ideas we have introduced above and substituting a stronger lattice point estimate. Since we will use this case to prove theorem 3, we state it separately.

THEOREM 4. *Let V, A_v, and H be as above. The number of one-dimensional subspaces $S \subset V$ with $H(S) \leqslant B$ is asymptotically*

$$a(n,1)\frac{B^n}{H(V)} + O\left(\frac{B^{n-b(n,1)}}{H(V)^{1-b(n,1)}H(V_{n-1})^{b(n,1)}} + \sum_{i=1}^{n-1}\frac{B^i}{H(V_i)}\right)$$

as $B \to \infty$, where the constant implicit in the O notation depends only on n and K.

COROLLARY. *Let V, A_v, and H be as above. The number of $(n-1)$-dimensional subspaces $S \subset V$ with $H(S) \leqslant B$ is asymptotically*

$$a(n,n-1)\frac{B^n}{H(V)^{n-1}} + O\left(\frac{B^{n-b(n,n-1)}}{H(V)^{n-1-b(n,n-1)}H(V_1)^{b(n,n-1)}} + \sum_{i=1}^{n-1}\frac{B^i}{H(V)^{i-1}H(V_{n-i})}\right)$$

as $B \to \infty$, where the constant implicit in the O notation depends only on n and K.

This corollary follows immediately from theorem 4 and the duality theorem. As for theorem 4, the proof follows exactly as that given in chapter 5 of [T2], except that the lattice point estimate used (the one used by Schanuel in [S]) is replaced by theorem 4 of [T3]. The exception is the case $n = 2$ and $K = \mathbb{Q}$ (i.e., $\kappa = 1$). In this case one must use an even better estimate for the number of lattice points of length $\leqslant B$ in $\mathbb{R}^2$ than that given by theorem 4 of [T3]. Earlier this century Sierpinski showed in [Si] that the number of integral points in the circle of radius B is $\pi B^2 + O(B^{2/3})$. In general, his method gives the number of lattice points $\mathbf{x} \in \Lambda$ in the circle of radius B to be

$$\frac{\pi B^2}{\det(\Lambda)} + O\left(\frac{(B\lambda_2)^{1/2}}{\lambda_1} + \frac{B^{2/3}}{(\lambda_1\lambda_2)^{1/3}}\right)$$

uniformly for $B \geqslant \lambda_2$, where $\lambda_1 \leqslant \lambda_2$ are the successive minima of Λ.* This suffices to prove theorem 4 in the case $n = 2$ and $K = \mathbb{Q}$.

2. Proof of theorem 3

We will prove theorem 3 by induction on n, using a reduction argument and an estimate for the number of lattice points in a bounded domain. By theorem 4, theorem 3 is true in the case $n = 2$, starting the induction. By the duality theorem and the corollary to theorem 4, it suffices to prove theorem 3 for $1 < n/2 \leqslant d < n - 1$. We may even make a further assumption.

Let ρ be the embedding corresponding to the height H and let c be any non-zero rational number. We then get a new height H' by replacing the embedding ρ with $\rho' = c\rho$. Then $H' = c^{\kappa d}H$ on $Gr_d(V)$. Using this new embedding has the effect of multiplying the successive minima $\lambda_1 \leqslant \cdots \leqslant \lambda_{n\kappa}$ of $t \circ \rho(\mathfrak{M}_V)$ by c. So without loss of generality we may assume

$$\lambda_{n\kappa} \gg 1,\ \lambda_i \leqslant 1 \qquad 1 \leqslant i \leqslant n\kappa. \tag{5}$$

By lemma 1 this implies

$$H(V) \gg\ll H(V_{n-1}). \tag{6}$$

We now make a reduction. Let $W \subset V$ be an $(n-1)$-dimensional subspace. Using the notation of section 3 of Part I, let $K\tilde{\mathbf{a}} = V/W = \tilde{W}$ and $\mathfrak{J}_0 = \mathfrak{J}(\tilde{\mathbf{a}})$. In the notation of lemma 4 we have $\tilde{\mathbf{a}} \in \mathfrak{J}_0^{-1}\mathfrak{M}_{\tilde{W}}$. In other words, there is a $\mathbf{w} \in W$ with $\mathbf{a} + \mathbf{w} \in \mathfrak{J}_0^{-1}\mathfrak{M}_V$. So we may assume $\mathfrak{J}_0\mathbf{a} \subset \mathfrak{M}_V$. But if $\mathfrak{A}\mathbf{a} \subset \mathfrak{M}_V$ for some ideal $\mathfrak{U}$, then we must have $\mathfrak{A}'' \mathfrak{J}_0$, by the definition of $\mathfrak{J}(\tilde{\mathbf{a}})$. So what we may assume, then, is

$$\mathfrak{J}_0 = \{a \in K : a\mathbf{a} \in \mathfrak{M}_V\}. \tag{7}$$

This only determines $\mathbf{a}$ up to multiplication by units. Let π be the projection of Part I, section 3, defined with respect to W, and define

$$a_i = \|\pi^i \circ \rho_i(\mathbf{a})\| \qquad 1 \leqslant i \leqslant \kappa.$$

*Professor H. Montgomery has shown me a short proof of this using Poisson summation.

By an application of Dirichlet's unit theorem, we may assume

$$a_i \gg\ll a_j \qquad 1 \leqslant i, j \leqslant \kappa. \tag{8}$$

As a notational convenience, we will write A for the product of the a_i's, so that

$$H(V) = AN(\mathfrak{J}_0)H(W) \tag{9}$$

by equation (3).

LEMMA 5. *Let $S \subset W$ be a $(d-1)$-dimensional subspace and let $\tilde{\mathbf{b}} = \mathbf{b} + S \in W/S$. We then get a unique d-dimensional subspace of V*

$$S^+ = S \oplus K(\mathbf{a} + \mathbf{b})$$

with height

$$H(S^+) = H(S)N(\mathfrak{J}(\tilde{\mathbf{b}}) \cap \mathfrak{J}_0) \prod_{i=1}^{\kappa} \|\pi^i \circ \rho_i(\mathbf{a} + \mathbf{b})\|,$$

where π is the projection defined with respect to S. If $(T, \tilde{\mathbf{c}})$ is an another such pair giving rise to the same subspace, then $T = S$ and $\tilde{\mathbf{b}} = \tilde{\mathbf{c}}$. All d-dimensional subspaces of V not contained in W have such a decomposition.

Proof. Only the statement about the height is not immediately obvious. By equation (3) this is true if we can show that $\mathfrak{J}(\widetilde{\mathbf{a} + \mathbf{b}}) = \mathfrak{J}(\tilde{\mathbf{b}}) \cap \mathfrak{J}_0$, where

$$\mathfrak{J}(\widetilde{\mathbf{a} + \mathbf{b}}) = \{a \in K : a\mathbf{a} + a\mathbf{b} \in S + \mathfrak{M}_V\}.$$

Certainly $\mathfrak{J}(\widetilde{\mathbf{a} + \mathbf{b}}) \supseteq \mathfrak{J}(\tilde{\mathbf{b}}) \cap \mathfrak{J}_0$ by (7). Since $\mathbf{b} \in W$, we also have $\mathfrak{J}(\widetilde{\mathbf{a} + \mathbf{b}}) \subseteq \mathfrak{J}_0$. Suppose $a \in \mathfrak{J}(\widetilde{\mathbf{a} + \mathbf{b}})$. Then both $a\mathbf{a} + a\mathbf{b}$ and $a\mathbf{a}$ are in $S + \mathfrak{M}_V$, giving $a \in \mathfrak{J}(\tilde{\mathbf{b}})$. This proves the lemma.

One may think of the product $N(\mathfrak{J}(\tilde{\mathbf{b}}) \cap \mathfrak{J}_0) \Pi \|\pi^i \circ \rho_i(\mathbf{a} + \mathbf{b})\|$ as giving an inhomogeneous height on the factor space W/S. It is this inhomogeneity which makes theorem 3 more difficult to prove than theorem 4. As for proving theorem 3, we will use $W = V_{n-1}$, so that $W_i = V_i$ for $i < n$. The statement of the lemma remains true in this case even when one restricts to the type of subspaces counted in theorem 3, i.e., if $S \cap V_{n-d} = \{\mathbf{0}\}$ then $S^+ \cap V_{n-d} = \{\mathbf{0}\}$ and vice versa.

Consider the following sets for arbitrary ideals $\mathfrak{U}$ and $(d-1)$-dimensional subspaces $S \subset W$ with $S \cap W_{n-d} = \{\mathbf{0}\}$:

$$L(\mathfrak{A}, S, B) = \left\{\tilde{\mathbf{b}} \in W/S : \mathfrak{J}(\tilde{\mathbf{b}}) \supseteq \mathfrak{A} \text{ and } \prod_{i=1}^{\kappa} \|\pi^i \circ \rho_i(\mathbf{a} + \mathbf{b})\| \leqslant B/H(S)N(\mathfrak{A})\right\}$$

$$L'(\mathfrak{A}, S, B) = \{\tilde{\mathbf{b}} \in L(\mathfrak{A}, S, B) : \mathfrak{J}(\tilde{\mathbf{b}}) \cap \mathfrak{J}_0 = \mathfrak{A}\}.$$

Denote their cardinalities by $l(\mathfrak{A}, S, B)$ and $l'(\mathfrak{A}, S, B)$, respectively. Note that $l'(\mathfrak{A}, S, W) = 0$ unless $\mathfrak{A} \subseteq \mathfrak{J}_0$ and $l(\mathfrak{A}, S, B) = l'(\mathfrak{A}, S, B) = 0$ if $H(S)N(\mathfrak{A})A > B$. We then have

$$\sum l'(\mathfrak{A}, S, B) = M(d, V, B), \tag{10}$$

where the sum is over ideals $\mathfrak{A} \subseteq \mathfrak{J}_0$ and subspaces S satisfying $H(S)N(\mathfrak{A}) \leqslant B/A$. We will estimate $l'(\mathfrak{A}, S, B)$ by estimating $l(\mathfrak{A}, S, B)$ and using an inversion.

Let μ be the Möbius function on integral ideals, defined by

$$\mu(\mathfrak{P}^a) = \begin{cases} (-1)^a & \text{if } a \leqslant 1, \\ 0 & \text{otherwise} \end{cases}$$

for $\mathfrak{P}$ prime and

$$\mu(\mathfrak{A}\mathfrak{P}) = \mu(\mathfrak{A})\mu(\mathfrak{P})$$

for $\mathfrak{A}$ and $\mathfrak{P}$ relatively prime. A routine argument (cf. [T1] lemma 3) shows

$$l'(\mathfrak{A}\mathfrak{J}_0, S, B) = \sum_{\mathfrak{C} \mid \mathfrak{A}} \mu(\mathfrak{C}) l(\mathfrak{A}\mathfrak{J}_0\mathfrak{C}^{-1}, S, BN(\mathfrak{C})^{-1}) \tag{11}$$

for integral $\mathfrak{A}$.

Now fix for the moment a subspace S and an integral $\mathfrak{A}$. Let t' denote the composition $t \circ \rho_{\tilde{S}}$. Then $l(\mathfrak{A}\mathfrak{J}_0\mathfrak{C}^{-1}, S, BN(\mathfrak{C})^{-1})$ is the number of lattice points

$$\mathbf{Y} \in t'((\mathfrak{A}\mathfrak{J}_0)^{-1}\mathfrak{C}\mathfrak{M}_{\tilde{S}})$$

in some translation of the domain

$$\left\{ \mathbf{Y} \in \mathbb{R}^{n\kappa} \colon \prod_{i=1}^{r_1+r_2} (\|\mathbf{y}_i\|^2 + a_u^2)^{e_i/2} \leqslant \frac{B}{N(\mathfrak{A}\mathfrak{J}_0)H(S)} \right\},$$

where

$$e_i = \begin{cases} 1 & \text{if } i \leqslant r_1, \\ 2 & \text{if } r_1 < i \leqslant r_1 + r_2. \end{cases}$$

For $0 < x \leqslant BH(W)/N(\mathfrak{A})H(V)$ and $0 < m \leqslant (n-d)\kappa$, let $V(\mathfrak{A}, x, B, m)$

denote the sum of the m-dimensional volumes of the projections of the domain

$$D(x) = \left\{ \mathbf{Y} \in \mathbb{R}^{(n-d)\kappa}: \prod_{i=1}^{r_1+r_2} \left(\frac{\|\mathbf{y}_i\|^2}{a_i^2} + 1 \right)^{e_i/2} \leqslant \frac{BH(W)}{xN(\mathfrak{A})H(V)} \right\}$$

on the various coordinate spaces obtained by equating $(n-d)\kappa - m$ of the coordinates to 0. Let $V(\mathfrak{A}, x, B, 0) = 1$. We define the *main term* to be

$$\sum_{\mathfrak{A}} \sum_{\mathfrak{C}|\mathfrak{A}} \sum_{S} \mu(\mathfrak{C}) \frac{V(\mathfrak{A}, H(S), B, (n-d)\kappa)}{\det(t'((\mathfrak{A}\mathfrak{I}_0)^{-1}\mathfrak{C}\mathfrak{M}_{\bar{S}}))}$$

and the *main error term* to be

$$\sum_{\mathfrak{A}} \sum_{\mathfrak{C}|\mathfrak{A}} \sum_{S} \sum_{m=0}^{(n-d)\kappa-1} \frac{V(\mathfrak{A}, H(S), B, m)}{\det[(t'((\mathfrak{A}\mathfrak{I}_0)^{-1}\mathfrak{C}\mathfrak{M}_{\bar{S}}))^{-[(n-d)\kappa-m]}]},$$

where the sums are over integral ideals $\mathfrak{A}$ with

$$N(\mathfrak{A}) \leqslant \frac{BH(V_{n-d})}{H(V)}$$

and $(d-1)$-dimensional subspaces $S \subset W$ with $S \cap V_{n-d} = \{\mathbf{0}\}$ and

$$H(S) \leqslant \frac{BH(W)}{N(\mathfrak{A})H(V)}.$$

LEMMA 6. *We have*

$$|M(d, V, B) - \textit{main term}| \ll \textit{main error term}.$$

Proof. Let $S \subset W$ be a $(d-1)$-dimensional subspace with $S \cap V_{n-d} = \{\mathbf{0}\}$. Then $W = S \oplus V_{n-d}$. By theorem 2 we see $H(S)H(V_{n-d}) \geqslant H(W)$. The lemma then follows from equations (9), (10), and (11), and the lattice point estimate [T3] theorem 5.

To estimate the main term, we first have

$$\sum_{S} \frac{V(\mathfrak{A}, H(S), B, (n-d)\kappa)}{\det(t'(\mathfrak{A}\mathfrak{I}_0)^{-1}\mathfrak{C}\mathfrak{M}_{\bar{S}}))} = a(n, d) \frac{\zeta_K(n)N(\mathfrak{A}\mathfrak{C}^{-1})^{n-d}}{\zeta_K(d)H(V)^d} \left(\frac{B}{N(\mathfrak{A})} \right)^n$$
$$+ O\left(\frac{N(\mathfrak{A}\mathfrak{C}^{-1})^{n-d}}{H(V)^{d-b(n-1,d-1)}H(V_{n-d})^{b(n-1,d-1)}} \left(\frac{B}{N(\mathfrak{A})} \right)^{n-b(n-1,d-1)} \right).$$

The argument follows that given in section 8 of [T1] almost verbatum. Also as in section 8 of [T1], we have

$$\sum_{\mathfrak{A}} \sum_{\mathfrak{C}|\mathfrak{A}} \mu(\mathfrak{C}) \frac{N(\mathfrak{A}\mathfrak{C}^{-1})^{n-d}}{N(\mathfrak{A})^n} = \frac{\zeta_K(d)}{\zeta_K(n)} + O\left(\left(\frac{BH(V_{n-d})}{H(V)}\right)^{1-d}\right),$$

and easily

$$\frac{B^{n-d+1}}{H(V)H(V_{n-d})^{d-1}} \leqslant \frac{B^{n-b(n,d)}}{H(V)^{d-b(n,d)}H(V_{n-d})^{b(n,d)}}$$

for $B \geqslant H(V)/H(V_{n-d})$ and $d > 1$. Similarly,

$$\sum_{\mathfrak{A}} \sum_{\mathfrak{C}|\mathfrak{A}} N(\mathfrak{C})^{d-n} N(\mathfrak{A})^{b(n-1,d-1)-d} \ll 1$$

except in the case $n = 4$, $d = 2$, and $\kappa = 1$ (so that $b(3,1) = 1$), and

$$\frac{B^{n-b(n-1,d-1)}}{H(V)^{d-b(n-1,d-1)}H(V_{n-d})^{b(n-1,d-1)}} \leqslant \frac{B^{n-b(n,d)}}{H(V)^{d-b(n,d)}H(V_{n-d})^{b(n,d)}}$$

for $B \geqslant H(V)/H(V_{n-d})$. In the exceptional case above, we have

$$\sum_{\mathfrak{A}} \sum_{\mathfrak{C}|\mathfrak{A}} N(\mathfrak{C})^{-2} N(\mathfrak{A})^{-1} \ll \left(\frac{BH(V_2)}{H(V)}\right)^{1/2},$$

giving an error of

$$\frac{B^3}{H(V)H(V_2)} \left(\frac{BH(V_2)}{H(V)}\right)^{1/2} = \frac{B^{4-b(4,2)}}{H(V)^{2-b(4,2)}H(V_2)^{b(4,2)}}.$$

We have shown

$$\text{main term} = a(n,d)\frac{B^n}{H(V)^d} + O\left(\frac{B^{n-b(n,d)}}{H(V)^{d-b(n,d)}H(V_{n-d})^{b(n,d)}}\right). \tag{12}$$

It remains to estimate the main error term. Again we introduce some notation from [T1]. For m a positive rational integer, define

$$\Sigma_m = \{(n_1, c_1, n_2, c_2, \ldots, n_m, c_m) \colon c_i \in \{1,2\} \text{ and } c_i \leqslant n_i \leqslant c_i(n-d) \text{ for all } i\}.$$

For $\sigma \in \Sigma_m$ and $x \in (0, 1]$, define

$$f_\sigma(x) = x \int \cdots \int_{D_\sigma(x)} \prod_{j=1}^{m} u_j^{n_j - 1}\, du_j,$$

where $D_\sigma(x)$ is given by

$$\prod_{j=1}^{m} (u_j^2 + 1)^{c_j/2} \leqslant 1/x \qquad u_j \geqslant 0.$$

We now put to use our assumptions made at the beginning of this section. By (5) and lemma 2 the successive minima of $t \circ \rho^*(I^*(S^*))$ are all $\gg 1$, so by lemma 2 the successive minima of $t'(\mathfrak{M}_{\tilde{S}})$ are all $\ll 1$. Thus, by lemmas 1 and 4 we have

$$\frac{1}{\det[t'((\mathfrak{A}\mathfrak{J}_0)^{-1}\mathfrak{C}\mathfrak{M}_{\tilde{S}})^{-[(n-d)\kappa - m]}]} \ll \frac{H(S)N(\mathfrak{A}\mathfrak{J}_0\mathfrak{C}^{-1})^{m/\kappa}}{H(W)^{1-(1\kappa)}H(S_{n-2})^{1/\kappa}}\; 0 \leqslant m \leqslant (n-d)\kappa - 1.$$

As shown in [T1] in the proof of lemma 16, we have

$$V(\mathfrak{A}, H(S), B, m) \ll \frac{BA^{m/\kappa}}{H(S)N(\mathfrak{A})}\, f_\sigma\left(\frac{H(S)N(\mathfrak{A})}{B}\right),$$

by (6) and (8), for a certain σ with $\Sigma\, n_j c_j = m$ and $n_j/c_j \leqslant n - d$. Also, by (6) and (9) $(AN(\mathfrak{J}_0))^{m/\kappa} \ll 1$. What we conclude from these facts, together with lemmas 2 and 4, is

$$\text{main error term} \ll \sum_{\mathfrak{A}} \sum_{\mathfrak{C} \mid \mathfrak{A}} \sum_{S} \frac{N(\mathfrak{A}\mathfrak{C}^{-1})^{n-d-(1/\kappa)}M}{H(W)^{1-(1/\kappa)}H(S_{n-2})^{1/\kappa}}\, f_\sigma(H(S)/M), \tag{13}$$

where $M = B/N(\mathfrak{A})$, i.e., we need only estimate the sum in the main error term when $m = (n-d)\kappa - 1$.

First consider the sum in (13) only over those S satisfying

$$H(S_{n-2}) \geqslant \left(\frac{H(S)H(V_{n-d})}{H(W)}\right)^{\frac{1}{n-d}} H(W).$$

Then estimating exactly as above for the main term, we have the sum in (13) over such S is

$$\ll \frac{B^{n-b(n,d)}}{H(V)^{d-b(n,d)}H(V_{n-d})^{b(n,d)}}$$

(recall we are assuming $d \geqslant n/2$, so that $b(n, d) = 1/\kappa(n-d)$). It remains to estimate the sum in (13) over those S satisfying

$$H(S_{n-2}) < \left(\frac{H(S)H(V_{n-d})}{H(W)}\right)^{1/n-d} H(W). \tag{14}$$

By lemma 3 we may use the induction hypothesis to theorem 3 and partial summation estimates (see [T1] lemma 12) to get

$$\begin{aligned}
\sum_S f_\sigma(H(S)/M) &\ll \frac{1}{H(S_{n-2})^{d-1}} \int_0^M x^{n-3} f_\sigma(x/M)\,dx \\
&= \frac{M^{n-2}}{H(S_{n-2})^{d-1}} \int_0^1 y^{n-3} f_\sigma(y)\,dy \\
&= \frac{M^{n-2}}{H(S_{n-2})^{d-1}} \int_0^1 y^{n-2} \int_{D_\sigma(y)} \cdots \int \prod_{j=1}^m u_j^{n_j-1}\,du_j dy \\
&= \frac{M^{n-2}}{H(S_{n-2})^{d-1}} \int_0^\infty \cdots \int_0^\infty \int_0^{\prod_{j=1}^m (u_j^2+1)^{-c_j/2}} y^{n-2}dy \prod_{j=1}^m u_j^{n_j-1}\,du_j \\
&\ll \frac{M^{n-2}}{H(S_{n-2})^{d-1}} \prod_{j=1}^m \int_0^\infty u_j^{n_j-1}(u_j^2+1)^{-(n-1)c_j/2}\,du_j \\
&\ll \frac{M^{n-2}}{H(S_{n-2})^{d-1}}
\end{aligned}$$

since $n-1 > n-d \geqslant n_j/c_j$, where the sum is over $(d-1)$-dimensional subspaces $S \subset S_{n-2}$ with $H(S) \leqslant M$. So the sum in (13) over those S satisfying (14) is

$$\begin{aligned}
&\ll \sum_{\mathfrak{A}} \sum_{\mathfrak{C}\mid\mathfrak{A}} \sum_{S_{n-2}} \frac{B^{n-1}N(\mathfrak{A}\mathfrak{C}^{-1})^{n-d-(1/\kappa)}}{H(W)^{1-(1/\kappa)}N(\mathfrak{A})^{n-1}H(S_{n-2})^{d-1+(1/\kappa)}} \\
&\ll \sum_T \frac{B^{n-1}}{H(W)^{1-(1/\kappa)}H(T)^{d-1+(1/\kappa)}},
\end{aligned}$$

where the second sum is over $(n-2)$-dimensional $T \subset W$ with

$$\frac{H(W)H(W_{n-d-1})}{H(W_{n-d})} \leqslant H(T) \leqslant \left(\frac{BH(V_{n-d})}{H(W)}\right)^{1/n-d} H(W).$$

(By theorem 2 and lemma 1, we certainly have $H(S_{n-2}) \geqslant H(W)H(W_{n-d-1})/H(W_{n-d})$ if $S \cap W_{n-d} = \{\mathbf{0}\}$).

By the corollary to theorem 4 and partial summation estimates, we have

$$\sum_{T} H(T)^{1-d-(1/\kappa)} \ll \sum_{i=1}^{n-d} \frac{1}{H(W)^{n-(i+1)}H(W_{i-1})} \int_0^{M_0} \frac{x^{n-(i+1)}}{x^{d-1+(1/\kappa)}}\,dx \tag{15}$$

where

$$M_0 = \left(\frac{BH(V_{n-d})}{H(W)}\right)^{1/n-d} H(W).$$

Now assume for the moment that $\kappa \neq 1$, so that
$n-(i+1)-(d-1+\frac{1}{\kappa}) > -1$.
By (15) we have

$$\sum_{T} \frac{B^{n-1}}{H(W)^{1-(1/\kappa)}H(T)^{d-1+(1/\kappa)}} \ll \frac{B^{n-1}}{H(W)^{1-(1/\kappa)}} \sum_{i=1}^{n-d} \frac{M_0^{n-d-i+1-(1/\kappa)}}{H(W)^{n-(i+1)}H(W_{i-1})}$$
$$= \frac{B^{n-1}}{H(W)^{d-1}} \sum_{i=1}^{n-d} \frac{B_0^{n-d-i+1-(1/\kappa)}}{H(W_{i-1})},$$

where $B_0 = M_0/H(W)$.

Note the power of B in the expression above is $\leqslant n - b(n,d)$. Hence, it suffices to show this is $\ll$ the error term of theorem 3 for B as small as possible, i.e., when

$$B = \frac{X^{n-d}H(W)}{H(V_{n-d})},\ X = \frac{H(W_{n-d-1})}{H(W_{n-d})}.$$

Note that $X \gg 1$ by (5) and lemma 1. For this value of B, the sum above is

$$\ll \frac{H(V)^{n-d}X^{(n-d)(n-1)}}{H(V_{n-d})^{n-1}} \sum_{i=1}^{n-d} \frac{X^{n-d-i+1-(1/\kappa)}}{H(W_{i-1})}$$
$$\ll \frac{H(V)^{n-d}X^{(n-d)(n-b(n,d))}}{H(V_{n-d})^{n}} \sum_{i=1}^{n-d} X^{1-i}$$
$$\ll \frac{H(V)^{n-d}X^{(n-d)(n-b(n,d))}}{H(V_{n-d})^{n}}$$

by (5), (6), and lemma 1. But for this value of B, the error term in theorem 3 is

$$\frac{H(V)^{n-d}X^{(n-d)(n-b(n,d))}}{H(V_{n-d})^{n}}.$$

In the case $\kappa = 1$ one may argue as above except for $i = n - d$. But we may replace the last summand in (15) with

$$\frac{\left(H(W)\dfrac{H(W_{n-d-1})}{H(W_{n-d})}\right)^{-1}}{H(W)^{d-1}H(W_{n-d-1})}\int_0^{M_0}\frac{x^d}{x^d}\,dx$$

which is

$$\ll \frac{B_0 H(W_{n-d})}{H(W)^{d-1}H(W_{n-d-1})^2}.$$

Then for B as above

$$\begin{aligned}\frac{B^{n-1}B_0H(W_{n-d})}{H(W)^{d-1}H(W_{n-d-1})^2} &\ll \frac{H(V)^{n-d}X^{(n-d)(n-1)}}{H(V_{n-d})^{n-1}H(W_{n-d-1})}\\ &\ll \frac{H(V)^{n-d}X^{(n-d)(n-b(n,d))}}{H(V_{n-d})^n}.\end{aligned}$$

So in all cases the main error term is $\ll$ the error term in theorem 3. This together with (12) proves theorem 3.

3. Error terms for Grassmannians

It is best to estimate the number of points on $Gr_d(V)$ with height $\leqslant B$ by focusing on the different possibilities for how one can build up a subspace from one-dimensional pieces of the various minimal subspaces. To be precise, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in c(n, d)$ let $E_\alpha(V)$ denote the set of d-dimensional subspaces $S \subset V$ with

$$\dim(S \cap V_i) = \begin{cases} 0 & \text{if } i < \alpha_1,\\ \max_{\alpha_j \leqslant i}\{j\} & \text{if } i \geqslant \alpha_1.\end{cases}$$

Let $N(\alpha, V, B)$ be the number of $S \in E_\alpha(V)$ with $H(S) \leqslant B$. We may estimate the total number of d-dimensional subspaces $S \subset V$ with $H(S) \leqslant B$ by estimating the $\binom{n}{d}$ terms $N(\alpha, V, B)$ separately. Now getting good estimates in general for $N(\alpha, V, B)$ would be a cumbersome task. Here we will give sharp estimates only when this is relatively painless. To this end we make the following definitions.

DEFINITION. *Let* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in c(n, d)$. *We say* α *is of type one if* $d = 1$ *or* $\alpha_{i+1} = \alpha_i + 1$ *for* $1 \leqslant i < d$. *We say* α *if of type two if* $d > 1$ *and* $\alpha_{i+1} > \alpha_i + 2$ *for* $1 \leqslant i < d$. *We say* α *is of type three if* $d > 1$, $\alpha_2 = \alpha_1 + 2$ *and* $\alpha_{i+1} = \alpha_i + 1$ *for* $i > 1$.

Note that, using this terminology, theorem 3 estimates $N(\alpha, V, B)$ for α of type one. To give estimates for α of type two and three we introduce the following terminology. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in c(n, d)$ let V_α be an element of $E_\alpha(V)$ of smallest height. If $d > 1$ define

$$\alpha^{-1} = (\alpha_1, \alpha_2, \ldots, \alpha_{d-1})$$

and more generally

$$\alpha^{-j} = (\alpha_1, \alpha_2, \ldots, \alpha_{d-j})$$

for $j < d$.

LEMMA 7. *Let* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in c(n, d)$ *be of type two. Then*

$$N(\alpha, V, B) \gg\ll \frac{B^{\alpha_d - d + 1}}{H(V_{\alpha_d})} \prod_{j=1}^{d-1} H(V_{\alpha_j})^{-1} H(V_{\alpha^{-(d-j)}}) a^{j+2-\alpha_{j+1}}.$$

Proof. Analogous to lemma 5 we have

$$N(\alpha, V, B) = \sum_T M(1, \tilde{T}, B/H(T)) \gg\ll \sum_T \frac{B^{\alpha_d - d + 1}}{H(V_{\alpha_d}) H(T)^{\alpha_d - d}}, \tag{16}$$

where the sum is over $T \in E_{\alpha^{-1}}(V)$ with $H(T) \leqslant BH(V_{\alpha_d - 1})/H(V_{\alpha_d})$, and $\tilde{T}$ is the $(\alpha_d + 1 - d)$-dimensional factor space V_{α_d}/T. Note that (16) is valid for α of any type. We now proceed by induction on d. For $d = 2$ partial summation and theorem 3 give

$$\sum_T \frac{1}{H(T)^{\alpha_2 - 2}} \gg\ll \frac{1}{H(V_{\alpha_1})} \int_{H(V_{\alpha^{-1}})}^{BH(V_{\alpha_2 - 1})/H(V_{\alpha_2})} \frac{x^{\alpha_1 - 1}}{x^{\alpha_2 - 2}} \, dx$$
$$\gg\ll \frac{H(V_{\alpha^{-1}})^{\alpha_1 + 2 - \alpha_2}}{H(V_{\alpha_1})}$$

since $a_2 > a_1 + 2$.

Now assume $d > 2$ and the result holds for $d - 1$. We then have

$$\sum_T \frac{1}{H(T)^{\alpha_d - d}} \gg \ll \frac{\prod_{j=1}^{d-2} H(V_{\alpha_j})^{-1} H(V_{\alpha^{-(d-j)}})^{\alpha_j + 2 - \alpha_{j+1}}}{H(V_{\alpha_{d-1}})} \times \int_{H(V_{\alpha^{-1}})}^{BH(V_{\alpha_d - 1})/H(V_{\alpha_d})} \frac{x^{\alpha_{d-1} - d + 1}}{x^{\alpha_d - d}} \, dx$$

$$\gg \ll \prod_{j=1}^{d-1} H(V_{\alpha_j})^{-1} H(V_{\alpha^{-(d-j)}})^{\alpha_j + 2 - \alpha_{j+1}},$$

so the lemma follows from (16).

An entirely analogous argument gives

LEMMA 8. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in c(n, d)$ be of type three. Then*

$$N(\alpha, V, B) \gg \ll \frac{B^{\alpha_d - 1}}{H(V_{\alpha_d})^{d-1} H(V_{\alpha_1})} \log\left(\frac{BH(V_{\alpha_2 - 1})}{H(V_{\alpha_d}) H(V_{\alpha^{-(d-1)}})}\right).$$

For $\alpha \in c(n, d)$ not of type 1, 2, or 3 a general statement is difficult to write down, although by using (16) one could easily give sharp estimates for a given α. In general, the power of B would necessarily be $< a_d - 1$, but the power of $\log B$ could be as great as $d - 1$. At the very least, we can say (using the notation in the introduction)

$$N(Gr_d(V), B) \sim a(n, d) \frac{B^n}{H(V)^d}.$$

PART III. ASYMPTOTICS FOR FLAG VARIETIES

1. The anti-canonical bundle

In the previous part we gave asymptotic estimates for the number of points on grassmannians of bounded height via the line bundle L_{χ_d}. With the particular choice for the global sections we used, this line bundle gives a morphism of the grassmannian into $\mathbb{P}^{\binom{n}{d}-1}(K)$ called the *Plücker embedding*. From the definition of height on projective space, it is immediate that the height given by $L_{\chi_d^m}$ is the m-th power of the height given by the Plücker embedding for any positive integer m. Similarly, if

$$\chi = \prod_{i=1}^{l} \chi_{d_i}^{m_i}$$

is an element of $X'(P)$ for P as in (2), then the height given by L_χ is the product of the m_i-th powers of the heights given by Plücker embeddings. For example, if $P = P_1 \cap P_3$ and $\chi = \chi_1^2\chi_3^1$, then the height of the point on the flag variety corresponding to the nested sequence $T \subset S$ given by L_χ is $H(T)^2H(S)$, where H denotes the height given by the Plücker embedding.

In the language of arithmetic geometry, the line bundle L_{χ_1} on $\mathbb{P}^{n-1}(K)$ is $\mathcal{O}(1)$. The *anticanonical bundle is* $\mathcal{O}(n)$, i.e., $L_{\chi_1^n}$. This is a particular case of the following definition.

DEFINITION. *Let*

$$P = P_{d_1} \cap \cdots \cap P_{d_l}$$

be a parabolic subgroup as in (2). *Set* $d_0 = 0$ *and* $d_{l+1} = n$. *The anti-canonical bundle of* $P\backslash G$ *is* L_χ *where*

$$\chi = \prod_{i=1}^{l} \chi_{d_i}^{m_i} \qquad m_i = d_{i+1} - d_{i-1}.$$

2. Asymptotics for flag varieties

As with grassmannians, we will not give complete estimates for the asymptotic behavior of the number of points of bounded height, but just evaluate the main term. We will also only deal with the height given by the anti-canonical bundle (the height used in [FMT]), though certainly the method would work with any ample line bundle. With this goal in mind, we proceed as follows.

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \in c(n, l)$, where $l > 1$ and $\alpha_1 < n$, let $M(\alpha, V, B)$ denote the number of flags

$$S^{\alpha_1} \subset S^{\alpha_2} \subset \cdots \subset S^{\alpha_l} \subset V$$

with height (with respect to the anti-canonical bundle) less than or equal to B satisfying

$$\dim S^{\alpha_i} = \alpha_i,$$

$$S\alpha_i \cap S^{\alpha_{i+1}}_{\alpha_{i+1}-\alpha_i} = \{\mathbf{0}\} \qquad 1 \leqslant i \leqslant l-2,$$

and

$$S^{\alpha_l} \cap V_{n-\alpha_l} = \{\mathbf{0}\}.$$

For such an α, define

$$a(\alpha) = \prod_{i=1}^{l} \frac{\alpha_{i+1} a(\alpha_{i+1}, \alpha_i)}{\alpha_{i+1} - \alpha_{i-1}},$$

where we let $\alpha_{l+1} = n$ and $\alpha_0 = 0$. Note that this agrees with the previous definition when $l = 1$.

THEOREM 5. *Let α be as above and set $d = \alpha_l$. Then*

$$M(\alpha, V, B) = \frac{a(\alpha)B}{(l-1)!\,H(V)^d} \log^{l-1}\left(\frac{B}{H(V)^d}\right) + O\left(\frac{B}{H(V)^{d - b(n,d)(n-d)/n} H(V_{n-d})^{b(n,d)}} \log^{l-2}\left(\frac{B}{H(V)^d}\right)\right),$$

where the constant implicit in the O notation depends only on n and K.

We will prove theorem 5 by induction on l. As in section 2 of part II, we multiply ρ by $H(V)^{-1/\kappa n}$ so that we get a new height H' satisfying $H' = H(V)^{-d/n} \cdot H$ on $Gr_d(V)$. In particular, $H'(V) = 1$. Note that

$$S^{\alpha_1} \subset S^{\alpha_2} \subset \cdots \subset S^d \subset V$$

is a flag counted in $M(\alpha, V, B)$ if and only if it is counted in $M'(\alpha, V, B/H(V)^d)$, and that $H'(S^{\alpha_l}) \gg 1$ by lemma 1, since $S^{\alpha_l} \cap V_{n-\alpha_l} = \{\mathbf{0}\}$. So we will prove theorem 5 under the assumption $H(V) = 1$.

We now proceed with the case $l = 2$ of theorem 5. Let $\alpha = (e, d)$. Then

$$M(\alpha, V, B) = \sum_W M(e, W, (B/H(W)^{n-e})^{1/d})$$
$$= \sum_W \frac{a(e,d)B}{H(W)^n} + O\left(\frac{B^{1 - b(d,e)/d}}{H(W)^{n - b(d,e)(n+d-e)/d} H(W_{d-e})^{b(d,e)}}\right), \qquad (17)$$

where the sum is over d-dimensional $W \subset V$ with $W \cap V_{n-d} = \{\mathbf{0}\}$ and $H(W) \leqslant B^{1/(n-e)}$. The sum over the main term in (17) is

$$a(n,d)a(d,e)B \int_1^{B^{1/(n-e)}} \frac{nx^{n-1}}{x^n}\,dx + O\left(\frac{B}{H(V_{n-d})^{b(n,d)}} \int_1^{B^{1/(n-e)}} \frac{x^{n-1-b(n,d)}}{x^n}\,dx\right)$$
$$= a(\alpha)B \log B + O\left(\frac{B}{H(V_{n-d})^{b(n,d)}}\right).$$

For the sum over the error term in (17), we pass to duals. Note that

$W^* \subset W^*_{d-e}$, $\dim W^* = n - d$, $\dim W^*_{d-e} = n - (d - e)$, and $H(W_{d-e})^{d/(d-e)} \leqslant H^*(W^*) = H(W)$ by lemma 1. Also by lemma 1 $H(W_{d-e})^{e/(d-e)} \leqslant H(S^e)$ for any e-dimensional subspace $S^e \subset W$ such that the flag $S^e \subset W$ is counted in $M(\alpha, V, B)$, so that $H(W_{d-e}) \leqslant B^{(d-e)/nd}$. The sum over the error term in (17) is

$$\ll \sum_S \frac{B^{1-b(d,e)/d}}{H(S)^{b(d,e)} H^*(S^*)^{n-d}} \int_{H(S)^{d/(d-e)}}^{\infty} \frac{x^{n-(d-e)-1}}{x^{n-b(d,e)(n+d-e)/n}} \, dx$$

$$\ll \sum_S \frac{B^{1-b(d,e)/d}}{H(S)^{n-b(d,e)n/(d-e)}}$$

$$\ll B^{1-b(d,e)/d} \int_0^{B^{(d-e)/nd}} \frac{x^{n-1}}{x^{n-b(d,e)n/(d-e)}} \, dx$$

$$\ll B,$$

where the sum is over $(d - e)$-dimensional $S \subset V$ with $H(S) \leqslant B^{(d-e)/nd}$. This proves theorem 5 for $l = 2$.

Now assume $l > 2$. Let $\alpha = (\alpha_1, \ldots, e, d)$. Then by the induction hypothesis

$$\begin{aligned} M(\alpha, V, B) &= \sum_W M(\alpha^{-1}, W, B/H(W)^{n-e}) \\ &= \sum_W \frac{a(\alpha^{-1})B}{(l-2)!\,H(W)^n} \log^{l-2}\left(\frac{B}{H(W)^n}\right) \\ &\quad + O\left(\frac{B}{(H(W)^{n-b(d,e)(d-e)/d} H(W_{d-e})^{b(d,e)}} \log^{l-3}\left(\frac{B}{H(W)^n}\right)\right), \end{aligned}$$

where the sum is over d-dimensional $W \subset V$ with $W \cap V_{n-d} = \{0\}$ and $H(W) \leqslant B^{1/(n-e)}$. Theorem 5 follows by partial summation, as above, and passing to duals for the sum over the error term.

3. Concluding remarks

Here we have given estimates for the main term of the asymptotics of $N(\mathscr{V}, B)$ for flag varieties $\mathscr{V}$. This method can yield explict error terms as well. We have not done so here since a general statement would be much too cumbersome. Of course, this is not so daunting a task if one is considering a particular flag variety. One could certainly say

$$N(P\backslash G, B) = M(\alpha, V, B) + O(B \log^{l-2} B),$$

but here the constant implicit in the O notation would depend upon V and the metrization. This is something we have tried to avoid here. Indeed, one of the advantages of this method is that this dependency can be explicitly found given a particular flag variety $\mathscr{V}$ and a metrization. These methods may also be used to give upper bounds for points of small height on the variety as in [BV].

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