COMPOSITIO MATHEMATICA

WINFRIED KOHNEN

Estimates for Fourier coefficients of Siegel cusp forms of degree two

Compositio Mathematica, tome 87, nº 2 (1993), p. 231-240
<http://www.numdam.org/item?id=CM_1993__87_2_231_0>

Compositio Mathematica **87**: 231–240, 1993.

Estimates for Fourier coefficients of Siegel cusp forms of degree two

WINFRIED KOHNEN
Max-Planck-Institut für Mathematik, Gottfried-Claren-Str. 26, W-5300 Bonn 3, Germany

Received 26 May 1992; accepted 9 June 1992

Introduction

An important problem in the theory of modular forms is to obtain good estimates for Fourier coefficients of cusp forms. On the one hand, this may lead to asymptotic expansions of interesting arithmetical functions, on the other hand one sometimes can derive growth estimates for Hecke eigenvalues. Needless to say that the problem is also of intrinsic interest.

Estimates for the Fourier coefficients $a(T)$ (T a positive definite symmetric half-integral (n, n)-matrix) of a Siegel cusp form F of integral weight k on the group $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z})$ have been given by several authors. The classical Hecke argument immediately shows that $a(T) \ll_F (\det T)^{k/2}$. Using Rankin's method, this was improved by Böcherer and Raghavan [2] and independently by Fomenko [4] to $a(T) \ll_{\varepsilon,F} (\det T)^{k/2-\delta_n+\varepsilon}$ $(\varepsilon > 0)$ where

$$\delta_n^{-1} := 2n + 2 + 4\left[\frac{n}{2}\right] + \frac{2}{n+1}$$

with $[x]$ = integral part of x.

In the case $n = 2$ better estimates have been known. On the one-hand side, by a variant of Rankin's method it was shown by Raghavan and Weissauer [10] that $a(T) \ll_{\varepsilon,F} (\min T)^{1/2}(\det T)^{k/2-1/4-3/38+\varepsilon}$ $(\varepsilon > 0)$, where $\min T$ denotes the least positive integer represented by T; on the other hand, using the method of Poincaré series and generalized Kloosterman sums, Kitaoka [7] for k even proved that $a(T) \ll_{\varepsilon,F} (\det T)^{k/2-1/4+\varepsilon}$ $(\varepsilon > 0)$.

In the present paper we would like to give a new method which is largely based on the theory of Jacobi forms and which may lead to considerable improvements upon estimates of the above type (at least) in the case $n = 2$. As an example, we shall prove:

THEOREM 1. *Let F be a Siegel cuspform of integral weight k on $\Gamma_2 = \mathrm{Sp}_2(\mathbb{Z})$ and* *let* $a(T)$ *(*T *a positive definite symmetric half-integral* $(2,2)$*-matrix) be its Fourier coefficients. Let* min T *be the least positive integer represented by* T. *Then*

$$a(T) \ll_{\varepsilon,F} (\min T)^{5/18+\varepsilon} (\det T)^{(k-1)/2+\varepsilon} \quad (\varepsilon > 0)$$

provided that $-4\det T$ *is a fundamental discriminant.*

Recall that a negative integer is called a fundamental discriminant if it is the discriminant of an imaginary quadratic field.

Very probably, the restriction to fundamental discriminants in Theorem 1 is not essential, and a more general statement could be proved by the same method.

Since by reduction theory $\min T \leqslant \frac{2}{\sqrt{3}} (\det T)^{1/2}$, we obtain:

COROLLARY. *One has*

$$a(T) \ll_{\varepsilon,F} (\det T)^{k/2-13/36+\varepsilon} \quad (\varepsilon > 0)$$

provided $-4\det T$ *is a fundamental discriminant.*

The proof of Theorem 1 is rather short. The main idea is a combination of appropriate estimates for both Fourier coefficients of Jacobi forms and Petersson norms of Fourier-Jacobi coefficients of Siegel modular forms (the latter again is based on Rankin's method). More precisely, in Section 1 (Proposition) we shall prove an estimate for the Fourier coefficients $c(n,r)$ ($n, r \in \mathbb{Z}$, $D := r^2 - 4mn$ a negative fundamental discriminant) of a Jacobi cusp form ϕ of weight $k > 2$ and index m on the Jacobi group $\Gamma_1^J := \Gamma_1 \ltimes \mathbb{Z}^2$ ($\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$) which is uniform in m, involves the Petersson norm $\|\phi\|$ of ϕ and when $|D| \to \infty$ essentially bounds $c(n,r)$ by $|D|^{(k-1)/2+\varepsilon}$ ($\varepsilon > 0$). The proof depends on the fact that the Fourier coefficients of Poincaré series of weight k and index m on Γ_1^J involve certain finite linear combinations of Kloosterman sums which can be evaluated rather explicitly [5, Chap. II] and as a consequence can be estimated in a simple manner. An analogous situation is known in the context of modular forms of half-integral weight where the Fourier coefficients of Poincaré series are closely connected with Salié sums (cf. e.g. [6]).

In Section 2 (Theorem 2) we shall show that the norms of the Fourier-Jacobi coefficients ϕ_m ($m \geqslant 1$) of a Siegel cusp form F of integral weight k on Γ_2 are bounded by $m^{k/2-2/9+\varepsilon}$ ($\varepsilon > 0$). The proof is based on the analytic properties of the Rankin-Dirichlet series $\Sigma_{m\geqslant 1} \|\phi_m\|^2 m^{-s}$ ($\mathrm{Re}(s) \gg 0$) which was introduced and studied by Skoruppa and the author in [8] and on a theorem of Landau (cf. [9, 11]; note that Landau's theorem was also used in [2, 10]).

The proof of Theorem 1 formally follows from the Proposition and Theorem 2. It should be noted that if instead of Theorem 2 we would only use the weaker bound $\|\phi_m\| \ll_F m^{k/2}$ which follows by applying a variant of the classical Hecke argument [8, Lemma 1], one would arrive at an estimate for $a(T)$ like Kitaoka's one mentioned above.

The bound for the linear combinations of Kloosterman sums used in the proof of the Proposition still is what in the context of modular forms of half-integral weight corresponds to the "trivial bound" for Salié sums. In fact, using some more sophisticated arguments like in Iwaniec's paper [6], it seems possible that one can improve upon the estimate given in the Proposition (for D fundamental) and hence on the bound in Theorem 1 (supposing that $-4 \det T$ is fundamental).

Also, in principle it seems possible to apply our method to arbitrary genus n. This hopefully would lead to some improvements upon the estimates for Fourier coefficients of cusp forms given in [2, 4].

NOTATION. We denote by $\mathscr{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ and $\mathscr{H}_2 = \{Z \in \mathbb{C}^{(2,2)} \mid Z$ symmetric, $\mathrm{Im}(Z) > 0\}$ the upper half-plane and the Siegel upper half-space of degree two, respectively.

We set $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_2 := \mathrm{Sp}_2(\mathbb{Z})$.

The greatest common divisor (a, b) of two non-zero integers a and b is always understood to be positive. In a sum $\Sigma_{d|a}$ where $a \in \mathbb{Z}\backslash\{0\}$, we understand that the summation is over positive divisors only. For $a \in \mathbb{Z}\backslash\{0\}$, we write $\sigma_0(a)$ for the number of positive divisors of a.

1. Estimates for Fourier coefficients of Jacobi forms

For details on Jacobi forms we refer to [3]. We denote by $\Gamma_1^J := \Gamma_1 \ltimes \mathbb{Z}^2$ the Jacobi group. It acts on $\mathscr{H} \times \mathbb{C}$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \circ (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right).$$

Recall that a Jacobi cusp form of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{N}$ on Γ_1^J is a holomorphic function $\phi\colon \mathscr{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the two transformation formulas

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \exp\left(2\pi i m \frac{cz^2}{c\tau + d}\right) \phi(\tau, z) \quad \left(\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1\right)$$

and

$$\phi(\tau, z+\lambda\tau+\mu) = \exp(-2\pi\, im(\lambda^2\tau+2\lambda z))\phi(\tau, z) \quad (\forall(\lambda, \mu)\in\mathbb{Z}^2)$$

and having a Fourier expansion

$$\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z}\\ r^2<4mn}} c(n, r)\,\mathrm{e}^{2\pi i(n\tau+rz)}$$

with $c(n, r)\in\mathbb{C}$. The coefficients $c(n, r)$ depend only on the residue class $r(\mathrm{mod}\ 2m)$ and the discriminant r^2-4mn.

We denote by $J_{k,m}^{\mathrm{cusp}}$ the complex vector space of Jacobi cusp forms of weight k and index m on Γ_1^J. It is a finite-dimensional Hilbert space under the Petersson scalar product

$$\langle\phi, \psi\rangle_{k,m} = \int_{\Gamma_1^J\backslash\mathscr{H}\times\mathbb{C}} \phi(\tau, z)\overline{\psi(\tau, z)}\exp(-4\pi my^2/v))v^k\,\mathrm{d}V_J \quad (\tau=u+iv,\ z=x+iy)$$

where $\mathrm{d}V_J=v^{-3}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}y$ is the normalized Γ_1^J-invariant measure on $\mathscr{H}\times\mathbb{C}$. We usually write $\langle\phi, \psi\rangle$ for $\langle\phi, \psi\rangle_{k,m}$.

The aim of this section is to prove

PROPOSITION. *Let ϕ be a Jacobi cusp form of weight $k>2$ and index m on Γ_1^J and let $c(n,r)$ $(n, r\in\mathbb{Z}, r^2<4mn)$ be its Fourier coefficients. Put $D:=r^2-4mn$. Then*

$$c(n, r) \ll_{\varepsilon,k} (m+|D|^{1/2+\varepsilon})^{1/2}\cdot\frac{|D|^{k/2-3/4}}{m^{(k-1)/2}}\cdot\|\phi\| \quad (\varepsilon>0)$$

provided that D is a fundamental discriminant, where the constant implied in $\ll$ depends only on ε and k (and not on m).

The rest of this section is devoted to the *proof* of the Proposition.

Let $P_{n,r}=P_{k,m;n,r}$ be the (n, r)th Poincaré series in $J_{k,m}^{\mathrm{cusp}}$ characterized by

$$\langle\psi, P_{n,r}\rangle = \lambda_{k,m,D}b_{n,r}(\psi) \quad (\forall\psi\in J_{k,m}^{\mathrm{cusp}})$$

where $b_{n,r}(\psi)$ denotes the (n, r)th Fourier coefficient of ψ and

$$\lambda_{k,m,D}:=\tfrac{1}{2}\Gamma(k-\tfrac{3}{2})\pi^{-k+3/2}m^{k-2}|D|^{-k+3/2}.$$

Then the Cauchy-Schwarz inequality gives

$$|c(n, r)|^2 = \lambda_{k,m,D}^{-2}|\langle \phi, P_{n,r}\rangle|^2 \leqslant \lambda_{k,m,D}^{-2}\|\phi\|^2\langle P_{n,r}, P_{n,r}\rangle$$
$$= \lambda_{k,m,D}^{-1} b_{n,r}(P_{n,r})\|\phi\|^2,$$

so it suffices to prove

$$b_{n,r}(P_{n,r}) \ll_{\varepsilon,k} 1 + \frac{|D|^{1/2+2\varepsilon}}{m} \tag{1}$$

for any $\varepsilon > 0$.

By the Proposition on p. 519 in [5], which gives all Fourier coefficients of $P_{n,r}$, one has

$$b_{n,r}(P_{n,r}) = 1 + (-1)^k\delta_m(r) + \frac{i^k\pi\sqrt{2}}{\sqrt{m}} \sum_{c\geqslant 1} (H_{m,c}(n, r, n, r)$$
$$+ (-1)^k H_{m,c}(n, r, n, -r)) \cdot J_{k-3/2}\left(\frac{\pi|D|}{mc}\right) \tag{2}$$

where

$$\delta_m(r) = \begin{cases} 1 & \text{if } r \equiv 0 (\operatorname{mod} m) \\ 0 & \text{otherwise,} \end{cases}$$

$$H_{m,c}(n, r, n, \pm r) = c^{-3/2} \sum_{\rho(c)^*, \lambda(c)} e_c((m\lambda^2 + r\lambda + n)\rho^{-1} + n\rho \pm r\lambda) e_{2mc}(\pm r^2) \tag{3}$$

and $J_{k-3/2}$ is the Bessel function of order $k-3/2$. In (3), ρ resp. λ run through $(\mathbb{Z}/c\mathbb{Z})^*$ resp. $\mathbb{Z}/c\mathbb{Z}$, ρ^{-1} denotes an inverse of $\rho (\operatorname{mod} c)$ and $e_a(b) := e^{2\pi i b/a}$ $(a \in \mathbb{N},\ b \in \mathbb{Z}/a\mathbb{Z})$.

LEMMA. *Suppose that D is a fundamental discriminant. Then*

$$H_{m,c}(n, r, n, \pm r) \ll_\varepsilon c^{-1/2+\varepsilon}(D, c)$$

for any $\varepsilon > 0$.

Proof. By the Lemma on p. 524 in [5] one has

$$H_{m,c}(n, r, n, \pm r) = c^{-1/2} \sum_{\substack{b(2mc) \\ b \equiv \pm r^2 (2m) \\ b^2 \equiv D^2 (4mc)}} \chi_D\left(\left[cm, b, \frac{b^2 - D^2}{4mc}\right]\right) e_{2mc}(b)$$

where $\left[cm,\ b,\ \dfrac{b^2-D^2}{4mc}\right]$ denotes the quadratic form with coefficients cm, b, $\dfrac{b^2-D^2}{4mc}$, respectively and χ_D is a certain generalized $\Gamma_0(m)$-genus character defined in [5, Chap. I, §2]. Since χ_D takes values in $\{\pm 1, 0\}$, we deduce that

$$|H_{m,c}(n,\ r,\ n,\ \pm r)| \leqslant c^{-1/2}\ \#\{b(2mc)\,|\,b \equiv D(2m),\ b^2 \equiv D^2(4mc)\}$$

(note that $D \equiv r^2(4m)$).

We claim that the number $\#$ on the right-hand side is less or equal to $\sigma_0(c)(D,c)$. Since $\sigma_0(c) \ll_\varepsilon c^\varepsilon$ for any $\varepsilon > 0$, this will prove our assertion.

Write $b = D + 2mx$ with x determined $\bmod c$. Then the congruence $b^2 \equiv D^2(4mc)$ is equivalent to $x(mx + D) \equiv 0(c)$. Since the number of solutions of the latter congruence and the functions $c \mapsto \sigma_0(c)$, $c \mapsto (D, c)$ are multiplicative, it is sufficient to prove our claim for c a prime power, $c = p^\nu$. Write $x = p^\mu y$ with $\mu \in \{0, 1, \ldots, \nu\}$, y determined $\bmod p^{\nu-\mu}$ and $p \nmid y$. The number of solutions $y(\bmod p^{\nu-\mu})$ of $m(p^\mu y) + D \equiv 0(p^{\nu-\mu})$ is $(mp^\mu, p^{\nu-\mu})$ or 0 according as $(mp^\mu, p^{\nu-\mu})$ divides D or not, hence is $\leqslant (D, p^\nu)$. This proves our claim.

From now on we shall suppose that $\varepsilon < k-2$. To simplify our notation we shall write "$\ll$" instead of "$\ll_{\varepsilon,k}$".

From (2) and the Lemma we see that

$$b_{n,r}(P_{n,r}) \ll 1 + m^{-1/2} \sum_{c \geqslant 1} c^{-1/2+\varepsilon}(D,\ c) \left| J_{k-3/2}\left(\frac{\pi|D|}{mc}\right)\right|.$$

Using the estimate

$$J_{k-3/2}(x) \ll \min\{x^{-1/2},\ x^{k-3/2}\} \quad (x > 0)$$

(cf. e.g. [1, p. 4 and p. 74]) one easily finds

$$\sum_{n \geqslant 1} n^{-1/2+\varepsilon} \left| J_{k-3/2}\left(\frac{A}{n}\right)\right| \ll A^{1/2+\varepsilon} \quad (A > 0).$$

Hence, writing t for (D, c) and n for c/t, we find

$$\sum_{c \geqslant 1} c^{-1/2+\varepsilon}(D,\ c) \left| J_{k-3/2}\left(\frac{\pi|D|}{mc}\right)\right| \leqslant \sum_{t|D} t \sum_{n \geqslant 1} (nt)^{-1/2+\varepsilon} \left| J_{k-3/2}\left(\frac{\pi|D/t|}{mn}\right)\right|$$

$$\ll \sum_{t|D} t^{1/2+\varepsilon} \left(\frac{|D/t|}{m}\right)^{1/2+\varepsilon} \ll m^{-1/2}|D|^{1/2+2\varepsilon}$$

which proves (1).

REMARK. We expect that the assertion of the Proposition is also true if $k = 2$. However, our method in this case is not applicable since the series giving the Fourier coefficients of the Poincaré series of weight 2 (i.e. the right-hand side of (2) with $k = 2$) is not absolutely convergent.

2. Estimates for Petersson norms of Fourier-Jacobi coefficients of Siegel modular forms

Let F be a Siegel cusp form of integral weight k on Γ_2. If $Z \in \mathscr{H}_2$ and we write $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$, then F has a partial Fourier expansion

$$F(Z) = \sum_{m \geqslant 1} \phi_m(\tau, z) e^{2\pi i m \tau'}.$$

The functions ϕ_m are in $J_{k,m}^{\text{cusp}}$ and are called the Fourier-Jacobi coefficients of F (for details cf. [3, Chap. II, §6]). We shall prove:

THEOREM 2. *Let F be a Siegel cusp form of weight $k \in \mathbb{Z}$ on Γ_2 with Fourier-Jacobi coefficients $\phi_m (m \geqslant 1)$. Then*

$$\|\phi_m\| \ll_{\varepsilon,F} m^{k/2-2/9+\varepsilon} \quad (\varepsilon > 0).$$

Proof. We denote by

$$D_F(s) := \zeta(2s-2k+4) \sum_{m \geqslant 1} \|\phi_m\|^2 m^{-s}$$

the formal Rankin-Dirichlet series which was introduced and studied in [8]. Recall that it was proved in [8] that $D_F(s)$ is convergent for $\mathrm{Re}(s) > k+1$, has a meromorphic continuation to $\mathbb{C}$ and is entire (of finite order) except for a possible simple pole at $s = k$.

Moreover, if one puts

$$D_F^*(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)D_F(s)$$

then the functional equation

$$D_F^*(2k-2-s) = D_F^*(s)$$

holds.

Since $D_F(s)$ has non-negative coefficients, a classical theorem of Landau says that $D_F(s)$ must have a singularity at its abscissa of convergence, hence it follows that $D_F(s)$ converges for $\mathrm{Re}(s) > k$.

In [11] a modified version of Landau's Hauptsatz proved in [9] was given. A special case of this is the following

THEOREM ([9, 11]). *Suppose $\xi(s) = \Sigma_{n\geqslant 1} c(n)n^{-s}$ is a Dirichlet series with non-negative coefficients which converges for* $\mathrm{Re}(s) > \sigma_0$, *has a meromorphic continuation to $\mathbb{C}$ with finitely many poles and satisfies a functional equation*

$$\xi^*(s) = \pm\xi^*(\delta - s),$$

where

$$\xi^*(s) := A^{-s}\cdot\prod_{j=1}^{J}\Gamma(a_j s + b_j)\cdot\xi(s) \quad (A > 0,\ a_j > 0,\ b_j\in\mathbb{R}).$$

Suppose that

$$\kappa := (2\sigma_0 - \delta)\sum_{j=1}^{J} a_j - \frac{1}{2} > 0.$$

Then

$$\sum_{n\leqslant x} c(n) = \sum_{\text{all poles}} \mathrm{Res}\left(\frac{\xi(s)}{s}x^s\right) + \mathcal{O}_\eta(x^\eta)$$

for any $\eta > \eta_0 := (\delta + \sigma_0(\kappa - 1))/(\kappa + 1)$.

(To deduce this formulation from the one given in [11], we must write the functional equation asymmetrically as

$$\xi(\delta - s) = A^{\delta - 2s}\cdot\prod_{j=1}^{J}\Gamma(a_j s + b_j)/\Gamma(a_j\delta + b_j - a_j s)\cdot\xi(s)$$

and replace each $\Gamma(x)$ in the denominator by $\dfrac{\pi}{\sin(\pi x)}\Gamma(1-x)^{-1}$.)

In our case, writing $D_F(s) = \Sigma_{n\geqslant 1} c(n)n^{-s}$ we have $\delta = 2k-2$, $\sigma_0 = k$, $J = 2$, $a_1 = a_2 = 1$. Hence $\kappa = \frac{7}{2}$, $\eta_0 = k - \frac{4}{9}$ and we deduce

$$\sum_{n\leqslant x} c(n) = Cx^k + \mathcal{O}_\varepsilon(x^{k-4/9+\varepsilon}) \quad (\forall\varepsilon > 0)$$

with

$$C = \mathrm{Res}_{s=k} D_F(s)/s.$$

Taking $x = m$ and $x = m - 1$ and subtracting we find

$$c(m) \ll_{\varepsilon,F} m^{k-4/9+\varepsilon}$$

and hence

$$\|\phi_m\|^2 = \sum_{d^2|m} \mu(d) d^{2k-4} c(m/d^2) \ll_{\varepsilon,F} m^{k-4/9+\varepsilon} \sum_{d \geqslant 1} d^{-28/9-2\varepsilon} \ll m^{k-4/9+\varepsilon}.$$

This proves the assertion of Theorem 2.

2. Proof of Theorem 1

We may suppose that $F \neq 0$. As is well-known this implies $k \geqslant 10$ (for the proof of the Theorem it actually would be sufficient to assume $k > 2$).

Fix $\varepsilon > 0$. Both sides of the inequality to be proved are invariant if T is replaced by $U'TU$ ($U \in \mathrm{GL}_2(\mathbb{Z})$), where U' denotes the transpose of U. Hence we may assume that

$$T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \quad \text{with } m = \min T.$$

We put $D := r^2 - 4mn$.

Let ϕ_m be the mth Fourier-Jacobi coefficient of F so that $A(T)$ is the (n, r)th Fourier coefficient of ϕ_m. Combining the Proposition and Theorem 2 we infer that

$$A(T) \ll_{\varepsilon,F} m^{5/18+\varepsilon}(m + |D|^{1/2+2\varepsilon})^{1/2} |D|^{k/2-3/4}.$$

By reduction theory we have $m = \min T \leqslant \frac{2}{\sqrt{3}} |D|^{1/2}$. If we apply this in the bracket above, the assertion of Theorem 1 follows.

References

[1] Bateman, H.: *Higher Transcendental Functions.* II. New York-Toronto-London, McGraw-Hill, 1953.

[2] Böcherer, S. and Raghavan, S.: On Fourier coefficients of Siegel modular forms, *J. reine angew. Math.* 384 (1988), 80–101.
[3] Eichler, M. and Zagier, D.: *The Theory of Jacobi Forms* (Progress in Maths. vol. 55), Birkhäuser, Boston, 1985.
[4] Fomenko, O. M.: Fourier coefficients of Siegel cusp forms of genus *n*, *J. Soviet. Math.* 38 (1987), 2148–2157.
[5] Gross, B., Kohnen, W. and Zagier, D.: Heegner points and derivatives of *L*-series. II. *Math. Ann.* 278 (1987), 497–562.
[6] Iwaniec, H.: Fourier coefficients of modular forms of half-integral weight. *Invent Math.* 87 (1987), 385–401.
[7] Kitaoka, Y.: Fourier coefficients of Siegel cusp forms of degree two, *Nagoya Math. J.* 93 (1984), 149–171.
[8] Kohnen, W. and Skoruppa, N.-P.: A certain Dirichlet series attached to Siegel modular forms of degree two. *Invent. Math.* 95 (1989), 541–558.
[9] Landau, E.: *Über die Anzahl der Gitterpunkte in Gewissen Bereichen.* II. Göttinger Nachr. (1915), 209–243. (Collected works, vol. 6, pp. 308–342. Essen: Thales, 1986.)
[10] Raghavan, S. and Weissauer, R.: Estimates for Fourier coefficients of cusp forms. In M. M. Dodson and J. A. G. Vickers eds., *Number Theory and Dynamical Systems*, pp. 87–102. London Math. Soc. Lect. Not. ser. vol. 134. Cambridge University Press, Cambridge, 1989.
[11] Sato, M. and Shintani, T.: On zeta functions associated with prehomogeneous vector spaces, *Ann. of Math.* 100 (1974), 131–170.