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An integral for the product of two Selberg-Jack symmetric polynomials

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1. Introduction and summary

Aomoto [Ao1] has recently given a simple and elegant proof of an extension of Selberg's integral [Se1]. Kadell [Ka4] has given two proofs of the following generalization [Ka1, Conjecture 2] of Aomoto's theorem. There exists a family $\{s_\lambda^k(t)\}$ of homogeneous symmetric functions such that if the integrand of Selberg's integral is multiplied by $s_\lambda^k(t_1, \ldots, t_n)$, then the integral has a certain closed form. It follows by comparison with the recent results of Stanley [St1] and Macdonald [Ma2, Chap. VI] that $s_\lambda^k(t)$ is the renormalized Jack symmetric function $J_\lambda(t;\alpha)$, introduced by Jack [Ja1], with $\alpha = 1/k$. We call $s_\lambda^k(t)$ the Selberg-Jack symmetric function.

Our main result is that if the integrand of Selberg's integral is multiplied by the product $s_\lambda^k(t_1, \ldots, t_n)s_\mu^k(t_1, \ldots, t_n)$ of two Selberg-Jack symmetric polynomials, then the integral has a certain closed form provided that $y = k$. Hua [Hu1] proved the $k = 1$ case of this theorem by using the Cauchy identity for the Schur symmetric functions. Our proof requires a certain homogeneous rational function identity which follows by Hua's theorem and which also follows by the Cauchy identity. We formulate our main result as an equivalent constant term identity and, using the homogeneity, we obtain a constant term orthogonality relation for the Selberg-Jack symmetric functions which was conjectured by Kadell [Ka3, Conjecture 2] and recently proved by Macdonald [Ma2, Chap. VI]. We give an expansion which is also equivalent to our main result. The constant term orthogonality mediates between this expansion and the constant term formulation of our main result.

We use throughout this paper notation and terminology related to partitions and symmetric functions from Macdonald [Ma1]. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$, be a partition. The number $l(\lambda) = \mathrm{card}(\{i \mid \lambda_i > 0\})$ of parts of λ, called the length of λ, is finite as is the norm of λ, denoted by $|\lambda| = \Sigma_{i\geqslant 1} \lambda_i$. We write $\lambda = (\lambda_1, \ldots, \lambda_n)$ to indicate that $\lambda_{n+1} = \cdots = 0$.

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Let $t=(t_1,t_2,\ldots)$ be an infinite set of indeterminates. We write $t=(t_1,\ldots,t_n)$ to indicate that $t_{n+1}=\cdots=0$. Let $m_\lambda(t)$ denote the monomial symmetric function.

Let $n\geqslant 1$ be a positive integer and let

$$\Delta_n(t_1,\ldots,t_n)=\prod_{1\leqslant i<j\leqslant n}(t_i-t_j) \tag{1.1}$$

denote the Vandermonde determinant. Let $k\geqslant 0$ be a nonnegative integer and let x and y be real numbers with $\mathrm{Re}(x)>0$, $\mathrm{Re}(y)>0$. The integrand of Selberg's integral [Se1] is

$$W_n^k(x,\, y;\, t_1,\ldots,t_n)=\prod_{i=1}^{n} t_i^{(x-1)}(1-t_i)^{(y-1)}\Delta_n^{2k}(t_1,\ldots,t_n). \tag{1.2}$$

Kadell's generalization [Ka4] of Aomoto's extension [Ao1] of Selberg's integral [Se1] is given by the following theorem.

THEOREM 1 (Kadell [Ka4]). *Let* $n\geqslant 1$, $\mathrm{Re}(x)>0$, $\mathrm{Re}(y)>0$. *For each* $k\geqslant 0$ *there exists a basis* $\{s_\lambda^k(t)\}$ *for the vector space of homogeneous symmetric functions in* t *such that the coefficient of* $m_\lambda(t)$ *in* $s_\lambda^k(t)$ *equals* 1 *and whenever* $l(\lambda)\leqslant n$ *we have*

$$\begin{aligned} I_n^k(x,y;\lambda) &= \int_0^1\cdots\int_0^1 s_\lambda^k(t_1,\ldots,t_n)W_n^k(x,\, y;\, t_1,\ldots,t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n \\ &= n!f_n^k(\lambda)\prod_{i=1}^{n}\frac{\Gamma(x+(n-i)k+\lambda_i)\Gamma(y+(n-i)k)}{\Gamma(x+y+(2n-i-1)k+\lambda_i)}, \end{aligned} \tag{I}$$

where, taking $(x)_k=x(x+1)\cdots(x+k-1)$,

$$f_n^k(\lambda)=\prod_{1\leqslant i<j\leqslant n}((j-i)k+\lambda_i-\lambda_j)_k. \tag{1.3}$$

We have (see [St1, Ka4 or Ma2, Chap. VI])

$$\prod_{i=1}^{n} t_i^A s_\lambda^k(t_1,\ldots,t_n)=s_{(\lambda_1+A,\ldots,\lambda_n+A)}^k(t_1,\ldots,t_n),\quad l(\lambda)\leqslant n, \tag{1.4}$$

where A is an integer for which both Selberg-Jack polynomials are defined. Observe that when the parameter x is a positive integer, it is subsumed by λ.

The integration formula (**I**) generalizes Selberg's integral [Se1] which is the case $\lambda=(0)$. See Andrews [An2] for a readily accessible version of Selberg's proof. Theorem 1 was also known for $k=1$ by Kadell [Ka1], for $\lambda_1=1$ by Aomoto [Ao1], and for $n=2$ by Richards [Ri1] and Kadell [Ka3].

Kadell [Ka1] showed for $k=1$ that

$$s_\lambda^1(t_1,\ldots,t_n)=\frac{\det|t_j^{n-i+\lambda_i}|_{n\times n}}{\det|t_j^{n-i}|_{n\times n}}$$
$$=s_\lambda(t_1,\ldots,t_n),\quad l(\lambda)\leqslant n, \tag{RA}$$

is the Schur function. This is the ratio of alternants formula for the Schur functions.

Aomoto [Ao1] showed for $\lambda_1=1$ that

$$s_{(1^m)}^k(t_1,\ldots,t_n)=\sum_{1\leqslant i_1<\cdots<i_m\leqslant n} t_{i_1}\cdots t_{i_m}$$
$$=e_m(t_1,\ldots,t_n),\quad m\leqslant n, \tag{1.5}$$

is the mth elementary symmetric function. Here (1^m) denotes the partition $\lambda_1=\cdots=\lambda_m=1$, $\lambda_{m+1}=\cdots=0$.

Richards [Ri1] and Kadell [Ka3] independently showed that

$$s_{(r)}^k(s,t)=\sum_{i=0}^{r}\frac{(-r)_i(k)_i}{(1-r-k)_i i!}s^i t^{r-i},$$
$$=t^r p_r^{(1-r-k,2k)}\left(\frac{s}{t}\right),\quad r\geqslant 0, \tag{1.6}$$

where

$$p_n^{(x,y)}(t)=\sum_{i=0}^{n}\frac{(-n)_i(n+x+y-1)_i}{(x)_i i!}t^i \tag{1.7}$$

is the Jacobi polynomial (see Szegö [Sz1, Chap. IV]) of degree n. Thus we may view $s_{(r)}^k(s,t)$ as a formal Jacobi polynomial.

It follows by comparison with the recent results of Stanley [St1] and Macdonald [Ma2, Chap. VI] that $s_\lambda^k(t)$ is the renormalized Jack symmetric function $J_\lambda(t;\alpha)$ with $\alpha=1/k$. We have

$$s_\lambda^k(t)=cJ_\lambda\left(t;\frac{1}{k}\right), \tag{1.8}$$

where the constant c is given by Stanley [Sta1]. We call $s_\lambda^k(t)$ the Selberg-Jack symmetric function.

We show that if the integrand of Selberg's integral is multiplied by the product $s_\lambda^k(t_1,\ldots,t_n)s_\mu^k(t_1,\ldots,t_n)$ of two Selberg-Jack symmetric polynomials, then the integral has a certain closed form provided that $y=k$. This is given by the following theorem, which is our main result.

THEOREM 2. *Let $n \geqslant 1$, $k \geqslant 0$, $\mathrm{Re}(x) > 0$, $l(\lambda) \leqslant n$ and $l(\mu) \leqslant n$. We have*

$$J_n^k(x;\lambda,\mu) = \int_0^1 \cdots \int_0^1 s_\lambda^k(t_1,\ldots,t_n) s_\mu^k(t_1,\ldots,t_n) W_n^k(x,k;t_1,\ldots,t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n$$

$$= n!\, f_n^k(\lambda) f_n^k(\mu)(\Gamma(k))^n \prod_{i=1}^n \prod_{j=1}^n \frac{1}{(x+(2n-i-j)k+\lambda_i+\mu_j)_k}. \tag{J}$$

Observe by (1.4) that when the parameter x is a positive integer, it is subsumed by λ and also by μ. The parameter $y=k$ introduces the factor $\Pi_{i=1}^n (1-t_i)^{(k-1)}$ into the integrand.

Hua [Hu1] proved that the Schur functions $s_\lambda(t)$ satisfy the product integration formula (**J**) when $k=1$. This is given by the following theorem.

THEOREM 3 (Hua[Hu1]). *Let $n \geqslant 1$, $\mathrm{Re}(x) > 0$, $l(\lambda) \leqslant n$ and $l(\mu) \leqslant n$. We have*

$$\int_0^1 \cdots \int_0^1 s_\lambda(t_1,\ldots,t_n) s_\mu(t_1,\ldots,t_n) \prod_{i=1}^n t_i^{(x-1)} \Delta_n^2(t_1,\ldots,t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n$$

$$= n! \prod_{1 \leqslant i < j \leqslant n} (j-i+\lambda_i-\lambda_j)(j-i+\mu_i-\mu_j)$$

$$\cdot \prod_{i=1}^n \prod_{j=1}^n \frac{1}{(x+2n-i-j+\lambda_i+\mu_j)}. \tag{1.9}$$

Hua's proof used the ratio of alternants (**RA**) and the Cauchy identity for the Schur symmetric functions in the form

$$\det \left| \frac{1}{(x_i+y_j)} \right|_{n\times n} = \Delta_n(x_1,\ldots,x_n)\Delta_n(y_1,\ldots,y_n) \prod_{i=1}^n \prod_{j=1}^n \frac{1}{(x_i+y_j)}. \tag{C}$$

Our proof of Theorem 2 requires a certain homogeneous rational function identity which follows by Hua's Theorem 3 (1.9) and which also follows by the Cauchy identity (**C**).

Let $[\omega]f$ denote the coefficient of the monomial ω in the Laurent expansion of f and let $\chi(A)$ be 1 or 0 according to whether A is true or false, respectively. We let $a \geqslant 0$ and $b \geqslant 0$ be nonnegative integers.

Habsieger [Hab1] and Kadell [Ka2, Ka4] (see also Section 4 of this paper) have independently shown that Selberg-type integrals can be formulated as equivalent constant term identities. We formulate Theorems 1 (**I**) and 2 (**J**) as constant term identities as follows.

The integration formula (**I**) is equivalent to the constant term identity of the following theorem.

THEOREM 4 (Kadell [Ka4]). *Let $n \geqslant 1$, $k \geqslant 0$, $a \geqslant 0$, $b \geqslant 0$, $l(\lambda) \leqslant n$ and $l(\mu) \leqslant n$. We have*

$$CS_n^k(a,b;\lambda) = [1] s_\lambda^k(t_1,\ldots,t_n) \prod_{i=1}^n (1-t_i)^a \left(1-\frac{1}{t_i}\right)^b \prod_{1 \leqslant i<j \leqslant n} \left(1-\frac{t_i}{t_j}\right)^k \left(1-\frac{t_j}{t_i}\right)^k$$
$$= n! f_n^k(\lambda)(-1)^{|\lambda|} \prod_{i=1}^n \frac{(a+b+(n-i)k)!}{(a+(n-i)k+\lambda_i)!(b+(i-1)k-\lambda_i)!}. \qquad (\mathbf{A})$$

We have

$$(1-t_i)^a \left(1-\frac{1}{t_i}\right)^b = (-t_i)^a \left(1-\frac{1}{t_i}\right)^{a+b}. \qquad (1.10)$$

Hence (1.4) gives

$$s_\lambda^k(t_1,\ldots,t_n) \prod_{i=1}^n (1-t_i)^a \left(1-\frac{1}{t_i}\right)^b$$
$$= (-1)^{na} s^k_{(\lambda_1+a,\ldots,\lambda_n+a)}(t_1,\ldots,t_n) \prod_{i=1}^n \left(1-\frac{1}{t_i}\right)^{a+b}. \qquad (1.11)$$

Multiplying (1.11) by $\Pi_{1 \leqslant i<j \leqslant n}(1-t_i/t_j)^k(1-t_j/t_i)^k$ and extracting the constant term, we have

$$CS_n^k(a,\ b;\ \lambda) = (-1)^{na} CS_n^k(0,\ a+b;\ (\lambda_1+a,\ldots,\lambda_n+a)). \qquad (1.12)$$

Thus the parameters a and b in (**A**) can be replaced by the single parameter $a+b$.

For $\lambda = (0)$, we see that (**A**) reduces to Morris' theorem [Mo1, (4.12)]. Morris obtained his theorem from a Selberg-type integral which used the Cauchy form of the beta integral. The a and b parameters of Morris' theorem are related to the x and y of Selberg's integral (see [Hab1, Ka2 or Ka4]) by

$$x = -b-(n-1)k, \qquad y = a+b+1. \qquad (1.13)$$

The product integration formula (**J**) is equivalent to the constant term identity of the following theorem.

THEOREM 5. *Let* $n \geqslant 1$, $k \geqslant 0$, $a \geqslant 0$, $l(\lambda) \leqslant n$ *and* $l(\mu) \leqslant n$. *We have*

$$\mathscr{K}_n^k(a;\lambda,\mu) = [1] s_\lambda^k(t_1,\ldots,t_n) s_\mu^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right) \prod_{i=1}^{n} (1-t_i)^a \left(1-\frac{1}{t_i}\right)^{(k-a-1)}$$
$$\cdot \prod_{1\leqslant i<j\leqslant n} \left(1-\frac{t_i}{t_j}\right)^k \left(1-\frac{t_j}{t_i}\right)^k$$
$$1 = n!\, f_n^k(\lambda) f_n^k(\mu) (-1)^{|\lambda|-|\mu|} \prod_{i=1}^{n} \frac{(k-1)!}{(k-a-1+\mu_i-\lambda_i)!(a+\lambda_i-\mu_i)!}$$
$$\cdot \prod_{1\leqslant i<j\leqslant n} \frac{1}{(1+a-k+(j-i)k+\lambda_i-\mu_j)_k(-a+(j-i)k+\mu_i-\lambda_j)_k}. \qquad \textbf{(K)}$$

Observe that setting $y = k$ in (1.13) gives $b = k-a-1$, which is the value of the parameter b that is used in **(K)**. Setting $b = k-a-1$ in (1.11), multiplying the result by $\Pi_{1\leqslant i<j\leqslant n}(1-t_i/t_j)^k(1-t_j/t_i)^k$ and extracting the constant term, we have

$$\begin{aligned}\mathscr{K}_n^k(a;\lambda,\mu) &= (-1)^{na}\mathscr{K}_n^k(0;(\lambda_1+a,\ldots,\lambda_n+a),\mu)\\ &= (-1)^{na}\mathscr{K}_n^k(0;\lambda,(\mu_1-a,\ldots,\mu_n-a)).\end{aligned} \qquad (1.14)$$

The last expression is obtained by replacing λ by $(\mu_1-a,\ldots,\mu_n-a)$ and $(t_1,\ldots,t_n)$ by $(1/t_1,\ldots,1/t_n)$ in (1.4). Thus the parameter a is subsumed by λ and also by μ.

Using the homogeneity of $s_\lambda^k(t)$, Theorem 5 implies a constant term orthogonality relation which was conjectured by Kadell [Ka3, Conjecture 2] and recently proved by Macdonald [Ma2, Chap. VI]. This orthogonality combines orthogonality relations for the Schur functions and the Jacobi polynomials. It is given by the following theorem.

THEOREM 6 (Macdonald [Ma2, Chap. VI]). *Let* $n \geqslant 1$, $k \geqslant 0$, $l(\lambda) \leqslant n$ *and* $l(\mu) \leqslant n$. *We have*

$$[1] s_\lambda^k(t_1,\ldots,t_n) s_\mu^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right) \prod_{1\leqslant i<j\leqslant n} \left(1-\frac{t_i}{t_j}\right)^k \left(1-\frac{t_j}{t_i}\right)^k$$
$$= n!\,\chi(\lambda=\mu) g_n^k(\lambda), \qquad \textbf{(O}'\textbf{)}$$

where

$$g_n^k(\lambda) = \frac{f_n^k(\lambda)}{\bar{f}_n^k(\lambda)}, \qquad (1.15)$$

$$\bar{f}_n^k(\lambda) = \prod_{1 \leqslant i < j \leqslant n} (1-k+(j-i)k+\lambda_i-\lambda_j)_k. \tag{1.16}$$

Kadell [Ka4] obtained the expansion **(H)**, given by the following theorem, which is also equivalent to **(A)**. The orthogonality **(O′)** mediates between **(H)** and **(A)**.

THEOREM 7 (Kadell [Ka4]). *Let* $k \geqslant 0$, $a \geqslant 0$. *We have*

$$\prod_{i \geqslant 1} (1-t_i)^a = \sum_{\lambda} H^k(a;\lambda) s_\lambda^k(t), \tag{H}$$

where

$$H^k(a;\lambda) = \bar{f}_n^k(\lambda) \prod_{i=1}^{n} \frac{(-a-(i-1)k)_{\lambda_i}}{((n-i)k+\lambda_i)!}, \quad l(\lambda) \leqslant n. \tag{1.17}$$

Observe that the coefficient $H^k(a;\lambda)$ is independent of n. We identify λ with its Ferrers diagram $\{(i,j) \mid 1 \leqslant i \leqslant l(\lambda),\ 1 \leqslant j \leqslant \lambda_i\}$, by letting $(i,j) \in \lambda$ be the cell in row i and column j of λ. The conjugate partition λ' of λ is given by $(i,j) \in \lambda'$ iff $(j,i) \in \lambda$. Thus $\lambda'_j = \mathrm{card}(\{i \mid \lambda_i \geqslant j\})$, $j \geqslant 1$. We define the k-content number and the k-hook number of (i,j) by

$$c_{i,j}^k(\lambda) = (j-1)-(i-1)k \tag{1.18}$$

and

$$h_{i,j}^{k,a}(\lambda) = a + A_{(i,j)}(\lambda) + kL_{(i,j)}(\lambda), \tag{1.19}$$

respectively, where

$$A_{(i,j)}(\lambda) = \lambda_i - j \quad \text{and} \quad L_{(i,j)}(\lambda) = \lambda'_j - i \tag{1.20}$$

are the arm and leg, respectively, of the cell (i,j). We have (see [Ka4])

$$H^k(a;\lambda) = \prod_{(i,j)\in\lambda} \frac{(-a+c_{i,j}^k(\lambda))}{h_{i,j}^{k,1}(\lambda)}, \tag{1.21}$$

which is independent of n.

Observe that **(H)** is a multivariable extension of the binomial theorem. It follows by using Stanley's [St1] extensions of the Cauchy identity and the specialization formula for the Schur functions. Kadell [Ka4] used the argument in the first proof [Ka4, §§3–5] of the integration formula **(I)** to give an induction on a.

Alternatively, we may use the orthogonality (**O**′) to compute the coefficient $H^k(a,\lambda)$ as follows. Use n variables in (**H**) where $l(\lambda) \leqslant n$ and multiply by $s_\lambda^k(1/t_1, \ldots, 1/t_n)$ times the weight function $\Pi_{1\leqslant i<j\leqslant n}(1-t_i/t_j)^k(1-t_j/t_i)^k$ of (**O**′). Since the constant term is unchanged by the substitution $t_i \to 1/t_i$, $1 \leqslant i \leqslant n$, we may use (**A**) to extract the required constant term. We obtain

$$\begin{aligned} H^k(a;\lambda) &= \frac{1}{n!g_n^k(\lambda)}[1]s_\lambda^k(t_1,\ldots,t_n)\prod_{i=1}^n\left(1-\frac{1}{t_i}\right)^a\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k \\ &= \frac{1}{n!g_n^k(\lambda)}\,CS_n^k(0,\,a;\,\lambda), \quad l(\lambda)\leqslant n. \end{aligned} \tag{1.22}$$

Observe that the formula (1.17) for $H^k(a;\lambda)$ follows by substituting (**A**) and the normalization (1.15) of the orthogonality (**O**′) into (1.22).

By (1.12) and (1.22), we see that if we know two of the three quantities $CS_n^k(a,b;\lambda)$, $H^k(a;\lambda)$ and $g_n^k(\lambda)$, then we can evaluate the third. Macdonald [Ma2, Chap. VI] proves Theorem 6 (**O**′), giving an elegant evaluation of the normalization factor $g_n^k(\lambda)$. By (**H**), we obtain the value of $CS_n^k(a,b;\lambda)$, thus proving the extension (**A**) of Morris' theorem [Mo1, (4.12)]. Thus (**O**′) mediates between (**H**) and (**A**). Since (**A**) and (**I**) are equivalent, this gives the second proof [Ka4, §§9–10] of the integration formula (**I**).

Macdonald [Ma2, Chap. VI] proves that the constant term orthogonality (**O**′) may be used to define the Selberg-Jack symmetric functions $\{s_\lambda^k(t)\}$. Let us now assume that we use (**O**′) to define $\{s_\lambda^k(t)\}$ and that we want to evaluate the normalization factor $g_n^k(\lambda)$. We have (see [Hab1, Ka2, Ka4] or Section 4 of this paper) that the constant term identity (**A**) is equivalent to the integration formula (**I**). Alternatively, we may prove (**A**) by giving the first proof [Ka4, §§3–5] of (**I**) in the constant term setting. One may then obtain both the coefficient $H^k(a;\lambda)$ and the normalization factor $g_n^k(\lambda)$ by the following argument.

Since $H^k(a;\lambda)$ is independent of n, we see that (1.22) gives

$$\frac{1}{n!g_n^k(\lambda)}\,CS_n^k(0,\,a;\,\lambda) = \frac{1}{(n+1)!g_{n+1}^k(\lambda)}\,CS_{n+1}^k(0,\,a;\,\lambda), \quad l(\lambda)\leqslant n. \tag{1.23}$$

We want to use the known values of $CS_n^k(0,a;\lambda)$ and $CS_{n+1}^k(0,a;\lambda)$ to evaluate $g_n^k(\lambda)$ by induction on n. Rearranging (1.23) gives

$$g_{n+1}^k(\lambda) = \frac{1}{(n+1)}\,\frac{CS_{n+1}^k(0,\,a;\,\lambda)}{CS_n^k(0,\,a;\,\lambda)}\,g_n^k(\lambda), \quad l(\lambda)\leqslant n. \tag{1.24}$$

This only applies if $l(\lambda) \leqslant n$. We may circumvent this difficulty by using (1.4).

Replace A by $-A$ in (1.4) and use the substitution $t_i \to 1/t_i$, $1 \leqslant i \leqslant n$.

Multiplying our results gives

$$s_\lambda^k(t_1, \ldots, t_n) s_\lambda^k\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right)$$
$$= s^k_{(\lambda_1 - A, \ldots, \lambda_n - A)}(t_1, \ldots, t_n) s^k_{(\lambda_1 - A, \ldots, \lambda_n - A)}\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right), \quad l(\lambda) \leqslant n. \tag{1.25}$$

Multiplying (1.25) by $\Pi_{1 \leqslant i < j \leqslant n}(1 - t_i/t_j)^k (1 - t_j/t_i)^k$ and extracting the constant term, we have

$$g_n^k(\lambda) = g_n^k(\lambda_1 - A, \ldots, \lambda_n - A), \quad l(\lambda) \leqslant n. \tag{1.26}$$

Observe that (1.26) follows from the explicit formulas (1.15) and (1.16). Replacing n by $n+1$ and setting $A = -\lambda_{n+1}$ in (1.26) gives

$$g^k_{n+1}(\lambda) = g^k_{n+1}(\lambda_1 - \lambda_{n+1}, \ldots, \lambda_n - \lambda_{n+1}), \quad l(\lambda) \leqslant n+1. \tag{1.27}$$

If $l(\lambda) \leqslant n+1$ then $l(\lambda_1 - \lambda_{n+1}, \ldots, \lambda_n - \lambda_{n+1}) \leqslant n$. Using this in (1.24) gives

$$g^k_{n+1}(\lambda_1 - \lambda_{n+1}, \ldots, \lambda_n - \lambda_{n+1})$$
$$= \frac{1}{(n+1)} \frac{CS^k_{n+1}(0, a; (\lambda_1 - \lambda_{n+1}, \ldots, \lambda_n - \lambda_{n+1}))}{CS^k_n(0, a; (\lambda_1 - \lambda_{n+1}, \ldots, \lambda_n - \lambda_{n+1}))} g_n^k(\lambda), \quad l(\lambda) \leqslant n+1. \tag{1.28}$$

Substituting (1.28) into (1.27), we have

$$g^k_{n+1}(\lambda) = \frac{1}{(n+1)} \frac{CS^k_{n+1}(0, a; (\lambda_1 - \lambda_{n+1}, \ldots, \lambda_n - \lambda_{n+1}))}{CS^k_n(0, a; (\lambda_1 - \lambda_{n+1}, \ldots, \lambda_n - \lambda_{n+1}))}$$
$$\cdot g_n^k(\lambda_1 - \lambda_{n+1}, \ldots, \lambda_n - \lambda_{n+1}), \quad l(\lambda) \leqslant n+1. \tag{1.29}$$

We may evaluate $g_n^k(\lambda)$ by (1.29) by induction on n and then evaluate $H^k(a; \lambda)$ directly by (1.22).

There is an expansion (**G**), given by the following theorem, which is also equivalent to (**K**). The orthogonality (**O**′) mediates between (**G**) and (**K**) just as it mediates between (**H**) and (**A**).

THEOREM 8 (Kadell [Ka4]). *Let $k \geqslant 0$. We have*

$$s_\mu^k(t) \prod_{i \geqslant 1} (1 - t_i)^{(k-1)} = \sum_\lambda G^k(\lambda, \mu) s_\lambda^k(t), \tag{G}$$

where

$$G^k(\lambda,\mu)=(-1)^{|\lambda|-|\mu|}\prod_{i=1}^{n}\frac{(k-1)!}{(k-1+\mu_i-\lambda_i)!(\lambda_i-\mu_i)!}$$
$$\cdot\prod_{1\leqslant i<j\leqslant n}\frac{(1-k+(j-i)k+\lambda_i-\lambda_j)_k((j-i)k+\mu_i-\mu_j)_k}{(1-k+(j-i)k+\lambda_i-\mu_j)_k((j-i)k+\mu_i-\lambda_j)_k},$$
$$l(\lambda)\leqslant n,\ l(\mu)\leqslant n. \qquad (1.30)$$

In Section 2, we reprise the first proof [Ka4, §§3–5] of the integration formula Theorem 1 (**I**). We use (**I**) to guess the coefficients $U^k(\nu/\lambda)$ and $V^k(\lambda/\mu)$ occurring in the extensions [Ka4, (**U**) and (**V**)] of the Pieri formula and the combinatorial representation, respectively, of the Schur functions. The (**U**) property overdetermines the Selberg-Jack symmetric functions $\{s_\lambda^k(t)\}$ by a double induction on the number λ_1 of columns of λ and on the number $\text{cl}(\lambda)=\lambda'_{\lambda_1}$ of cells (i,λ_1) in the last column of λ.

In Section 3, we prove the product integration formula Theorem 2 (**J**) by using the double induction given in Section 2. The proof requires a certain homogeneous rational function identity which follows by Hua's Theorem 3 (1.9) and which also follows by the Cauchy identity (**C**).

In Section 4, we formulate the product integration formula Theorem 2 (**J**) as the constant term identity Theorem 5 (**K**). Using the homogeneity of $s_\lambda^k(t)$, we prove the constant term orthogonality Theorem 6 (**O′**), including the evaluation of the normalization factor $g_n^k(\lambda)$.

In Section 5, we use the constant term orthogonality (**O′**) and the constant term identity (**K**) to obtain the expansion Theorem 8 (**G**). The orthogonality (**O′**) mediates between (**G**) and (**K**) just as it mediates between (**H**) and (**A**).

In Section 6, we outline alternative proofs of the constant term identity Theorem 5 (**K**) and the expansion Theorem 8 (**G**). We show that the argument used to prove the product integration formula Theorem 2 (**J**) in Section 3 can also be used to prove (**K**) and (**G**) and that the identities which arise for (**K**) and (**G**) are both equivalent to the identity which arises for (**J**), which may be formulated as a homogeneous rational function identity.

2. A reprise of the first proof [Ka4, §§3–5] of the integration formula Theorem 1 (I)

In this section, we reprise the first proof [Ka4, §§3–5] of the integration formula Theorem 1 (**I**). We use (**I**) to guess the coefficients $U^k(\nu/\lambda)$ and $V^k(\lambda/\mu)$ occurring in the extensions [Ka4, (**U**) and (**V**)] of the Pieri formula and the combinatorial representation, respectively, of the Schur functions. The (**U**) property overdetermines the Selberg-Jack symmetric functions $\{s_\lambda^k(t)\}$ by a double induction on the number λ_1 of columns of λ and on the number $\mathrm{cl}(\lambda) = \lambda'_{\lambda_1}$ of cells (i, λ_1) in the last column of λ.

Let the skew diagram $\nu/\lambda = \{(i,j) \mid 1 \leqslant i \leqslant l(\nu), \lambda_i < j \leqslant \nu_i\}$ consist of those cells of ν which are not in λ. We call ν/λ a vertical m-strip if $\lambda_i \leqslant \nu_i \leqslant \lambda_i + 1$, for all i, $i \geqslant 1$, and $|\nu| = |\lambda| + m$. We call λ/μ a horizontal r-strip if λ'/μ' is a vertical r-strip.

The first proof [Ka4, §§3–5] of Theorem 1 uses the integration formula (**I**) to find the coefficients $U^k(\nu/\lambda)$ and $V^k(\lambda/\mu)$ occurring in the following two properties of $s_\lambda^k(t)$.

$$e_m(t)s_\lambda^k(t) = \sum_{\substack{\nu \\ \nu/\lambda \text{ is a vertical} \\ m\text{-strip}}} U^k(\nu/\lambda)s_\nu^k(t) \tag{U}$$

and

$$s_\lambda^k(t_1, \ldots, t_n) = \sum_{r \geqslant 0} t_n^r \sum_{\substack{\mu \\ \lambda/\mu \text{ is a horizontal} \\ r\text{-strip}}} V^k(\lambda/\mu)s_\mu^k(t_1, \ldots, t_{n-1}). \tag{V}$$

The (**U**) property reduces for $k = 1$ to a special case of the Littlewood-Richardson rule which is dual to the Pieri formula. The (**V**) property extends the combinatorial representation of the Schur functions. Each of these properties determines the Selberg-Jack functions.

Let $0 \leqslant m \leqslant n$ and $l(\nu) \leqslant n$. Observe that if ν/λ is a vertical m-strip, then no two cells of ν/λ are in the same row. Thus $\nu = \lambda + \chi_M$ where $M = \{i \mid \nu_i = \lambda_i + 1\} \subseteq [1, n]$, where $[1, n] = \{i \mid 1 \leqslant i \leqslant n\}$ denotes the interval from 1 to n. We have

$$I_n^k(x, y+1; \lambda) = \sum_{M \subseteq [1,n]} (-1)^{|M|} U^k(\lambda + \chi_M/\lambda) I_n^k(x, y; \lambda + \chi_M). \tag{2.1}$$

For $l(\lambda) \leqslant n - 1$, we have

$$I_n^k(x, 1; \lambda) = \frac{n}{\left(\lambda + 2k\binom{n}{2} + nx\right)} \sum_{\substack{\mu \\ \lambda/\mu \text{ is a horizontal} \\ \text{strip}}} V^k(\lambda/\mu) I_{n-1}^k(x, 2k+1; \mu). \tag{2.2}$$

All of the coefficients are identically equal to 1 when $k = 1$, in which case Theorem 1 is known [Ka1]. Thus (2.1) and (2.2) hold for $k = 1$ and we may convert them into polynomial identities. Observe that if $U^k(\nu/\lambda)$ and $V^k(\lambda/\mu)$ are rational functions, then (2.1) and (2.2) may be converted into polynomial identities. We boldly define $U^k(\nu/\lambda)$ so that when (2.1) is written as a polynomial identity, it is homogeneous in x, y, k and λ_i, $1 \leqslant i \leqslant n$. Using $n \to nk$,

$x \to x+1-k$, we define $V^k(\lambda/\mu)$ so that when (2.2) is written as a polynomial identity, it is homogeneous in $1/k$ and λ'_j, $1 \leqslant j \leqslant \lambda_1$. This gives

$$U^k(\nu/\lambda) = \prod_{\substack{1 \leqslant i < j \leqslant n \\ \nu_i = \lambda_i \\ \nu_j = \lambda_j + 1}} \frac{(-k+(j-i)k+\lambda_i-\lambda_j)(k-1+(j-i)k+\lambda_i-\lambda_j)}{(-1+(j-i)k+\lambda_i-\lambda_j)((j-i)k+\lambda_i-\lambda_j)}$$

$$= \prod_{\nu_i=\lambda_i} \frac{h^{k,k}_{i,t}(\nu) h^{k,1}_{i,t}(\lambda)}{h^{k,k}_{i,t}(\lambda) h^{k,1}_{i,t}(\nu)}$$

and

$$V^k(\lambda/\mu) = \prod_{1 \leqslant i < j \leqslant n} \frac{(1-k+(j-i)k+\mu_i-\lambda_j)_{k-1}(1-k+(j-i)k+\lambda_i-\mu_{j-1})_{k-1}}{(1-k+(j-i)k+\lambda_i-\lambda_j)_{k-1}(1-k+(j-i)k+\mu_i-\mu_{j-1})_{k-1}}$$

$$= \prod_{\lambda'_i=\mu'_i} \frac{h^{k,1}_{i,t}(\lambda) h^{k,k}_{i,t}(\mu)}{h^{k,1}_{i,t}(\mu) h^{k,k}_{i,t}(\lambda)}. \tag{2.4}$$

Observe that

$$h^{k,k}_{i,j}(\lambda) = k^{h_{i,j}(\lambda)} h^{1/k,1}_{j,i}(\lambda'),$$
$$h^{k,1}_{i,j}(\lambda) = k^{h_{i,j}(\lambda)} h^{1/k,1/k}_{j,i}(\lambda'). \tag{2.5}$$

Using (2.5) to compare (2.3) and (2.4), we have the elegant identity

$$V^k(\lambda/\mu) = U^{1/k}(\lambda'/\mu'), \tag{U-V}$$

which is related to Macdonald's extension (see [Ma2, Chap. VI]) of the duality property of the Schur functions.

There now arises an important quandry. While the (V) property determines $\{s^k_\lambda(t)\}$ by induction on the number of variables, it is not clear that the resulting functions are symmetric. However, the (U) property overdetermines $\{s^k_\lambda(t)\}$, although it is clear that if there exist functions satisfying (U), then they must be symmetric. This quandry is resolved and Theorem 1 (I) is proved by a double induction on n and y by showing that the functions defined by (V) satisfy (U). This requires induction on the number of variables and the identity

$$\sum_{\substack{M \subseteq [1,n] \\ |M|=m}} V^k(\lambda+\chi_M/\varphi) U^k(\lambda+\chi_M/\lambda)$$
$$= \sum_{\substack{M \subseteq [1,n-1] \\ |M|=m}} V^k(\lambda/\varphi-\chi_M) U^k(\varphi/\varphi-\chi_M)$$
$$+ \sum_{\substack{M \subseteq [1,n-1] \\ |M|=m-1}} V^k(\lambda/\varphi-\chi_M) U^k(\varphi/\varphi-\chi_M), \tag{UV}$$

which is homogeneous in $k-1$ and $\lambda_i+(n-i)k$, $1 \leqslant i \leqslant n$, and $\varphi_i+(n-1-i)k$, $1 \leqslant i \leqslant n-1$.

We require some of the calculations from [Ka4] for the homogeneous polynomial identity obtained from (2.1). Divide both sides of (2.1) by

$$n! \prod_{i=1}^{n} \frac{\Gamma(x+(n-i)k+\lambda_i)\Gamma(y+(n-i)k)}{\Gamma(x+y+1+(2n-i-1)k+\lambda_i)}. \tag{2.6}$$

We obtain

$$\begin{aligned} & f_n^k(\lambda) \prod_{i=1}^{n} (y+(n-i)k) \\ &= \sum_{M \subseteq [1,n]} (-1)^{|M|} U^k(\lambda+\chi_M/\lambda) f_n^k(\lambda+\chi_M) \\ &\quad \cdot \prod_{\substack{i=1 \\ i \in M}}^{n} (x+(n-i)k+\lambda_i) \prod_{\substack{i=1 \\ i \notin M}}^{n} (x+y+(2n-i-1)k+\lambda_i). \end{aligned}$$

The coefficients $U^k(\lambda+\chi_M/\lambda)$ are chosen so that (2.7) becomes the homogeneous polynomial identity

$$\begin{aligned} & \prod_{1 \leqslant i < j \leqslant n} ((j-i)k+\lambda_i-\lambda_j) \prod_{i=1}^{n} (y+(n-i)k) \\ &= \sum_{M \subseteq [1,n]} (-1)^{|M|} \prod_{1 \leqslant i < j \leqslant n} ((j-i)k+\lambda_i+k\chi(i \in M)-\lambda_j-k\chi(j \in M)) \\ &\quad \cdot \prod_{\substack{i=1 \\ i \in M}}^{n} (x+(n-i)k+\lambda_i) \prod_{\substack{i=1 \\ i \notin M}}^{n} (x+y+(2n-i-1)k+\lambda_i). \end{aligned} \tag{2.8}$$

This may be done by dividing (2.7) by

$$\Omega_n^k(\lambda) = \prod_{1 \leqslant i < j \leqslant n} (1+(j-i)k+\lambda_i-\lambda_j)_{k-1}. \tag{2.9}$$

Thus we have

$$\frac{f_n^k(\lambda)}{\Omega_n^k(\lambda)} = \prod_{1 \leqslant i < j \leqslant n} ((j-i)k+\lambda_i-\lambda_j), \tag{2.10}$$

which is the $M = \varnothing$ case of

$$\begin{aligned} & \frac{U^k(\lambda+\chi_M/\lambda) f_n^k(\lambda+\chi_M)}{\Omega_n^k(\lambda)} \\ &= \prod_{1 \leqslant i < j \leqslant n} ((j-i)k+\lambda_i+k\chi(i \in M)-\lambda_j-k\chi(j \in M)), \quad M \subseteq [1,n]. \end{aligned} \tag{2.11}$$

We now recall the proof [Ka4, §2] of the fact that the (U) property overdetermines the Selberg-Jack symmetric functions $s_\lambda^k(t)$. We proceed by a double induction on the number λ_1 of columns of λ and on the number $\mathrm{cl}(\lambda) = \lambda'_{\lambda_1}$ of cells (i, λ_1) in the last column of λ. We begin the induction by using the initial condition

$$s_{(0)}^k(t) = 1. \tag{2.12}$$

We may assume that $\lambda_1 > 0$. We set $m = \mathrm{cl}(\lambda) > 0$ and let $\theta(\lambda)$ be the partition obtained by deleting the last column of λ. Thus

$$\begin{aligned} \theta(\lambda)_i &= \lambda_i - 1, \quad 1 \leqslant i \leqslant \mathrm{cl}(\lambda), \\ \theta(\lambda)_i &= \lambda_i, \qquad i > \mathrm{cl}(\lambda). \end{aligned} \tag{2.13}$$

We have

$$\lambda = \theta(\lambda) + \chi_{[1,m]} \tag{2.14}$$

and, since the coefficient of $m_\lambda(t)$ in $s_\lambda^k(t)$ equals 1,

$$U^k(\lambda/\theta(\lambda)) = 1. \tag{2.15}$$

By the (U) property, we have

$$e_m(t) s_{\theta(\lambda)}^k(t) = \sum_{\substack{M \subseteq \{1,2,\ldots\} \\ |M| = m}} U^k(\theta(\lambda) + \chi_M/\theta(\lambda)) s_{\theta(\lambda)+\chi_M}^k(t). \tag{2.16}$$

By (2.14) and (2.15), we see that the term $s_\lambda^k(t)$ occurs on the right side of (2.16) when $M = [1, m]$. Solving for this term, we obtain

$$s_\lambda^k(t) = e_m(t) s_{\theta(\lambda)}^k(t) - \sum_{\substack{M \subseteq \{1,2,\ldots\} \\ |M| = m \\ M \neq [1,m]}} U^k(\theta(\lambda) + \chi_M/\theta(\lambda)) s_{\theta(\lambda)+\chi_M}^k(t). \tag{2.17}$$

Since all of the partitions which occur in the terms on the right side of (2.17) have fewer cells in column λ_1, our induction is complete and the (U) property determines the Selberg-Jack symmetric functions $\{s_\lambda^k(t)\}$.

Observe that the coefficient $U^k(\nu/\lambda)$ is constrained so that (2.16) holds for all $m \geqslant 1$. Thus the (U) property overdetermines the Selberg-Jack symmetric functions $\{s_\lambda^k(t)\}$.

3. A proof of the product integration formula Theorem 2 (J)

In this section, we prove the product integration formula Theorem 2 (**J**) by using the double induction given in Section 2. The proof requires a certain homogeneous rational function identity which follows by Hua's Theorem 3 (1.9) and which also follows by the Cauchy identity (**C**).

We proceed by a double induction on the number λ_1 of columns of λ and on the number $\mathrm{cl}(\lambda) = \lambda'_{\lambda_1}$ of cells (i, λ_1) in the last column of λ.

We begin the induction by setting $\lambda = (0)$ and letting μ be any partition with $l(\mu) \leqslant n$. Using the integration formula (**I**), we have

$$\begin{aligned} J_n^k(x;(0),\mu) &= \int_0^1 \cdots \int_0^1 s_\mu^k(t_1,\ldots,t_n) W_n^k(x,k;t_1,\ldots,t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n \\ &= I_n^k(x,k;\mu) \\ &= n!f_n^k(\mu)\prod_{i=1}^n \frac{\Gamma(x+(n-i)k+\mu_i)\Gamma(k+(n-i)k)}{\Gamma(x+k+(2n-i-1)k+\mu_i)} \\ &= n!f_n^k(\mu)\prod_{i=1}^n \frac{\Gamma(k+(n-i)k)}{(x+(n-i)k+\mu_i)_{nk}}. \end{aligned} \tag{3.1}$$

We easily have

$$\begin{aligned} \prod_{i=1}^n \Gamma(k+(n-i)k) &= (\Gamma(k))^n \prod_{i=1}^n (k)_{(n-i)k} \\ &= (\Gamma(k))^n \prod_{i=1}^{n-1}\prod_{j=i+1}^n ((j-i)k)_k \\ &= (k))^n \prod_{1\leqslant i<j\leqslant n} ((j-i)k)_k \\ &= (\Gamma(k))^n f_n^k(0) \end{aligned} \tag{3.2}$$

and

$$\prod_{i=1}^n \frac{1}{(x+(n-i)k+\mu_i)_{nk}} = \prod_{i=1}^n\prod_{j=1}^n \frac{1}{(x+(2n-i-j)k+\mu_i)_k}. \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1) yields

$$J_n^k(x;(0),\mu) = n!f_n^k(0)f_n^k(\mu)(\Gamma(k))^n \prod_{i=1}^n\prod_{j=1}^n \frac{1}{(x+(2n-i-j)k+\mu_i)_k}, \tag{3.4}$$

as required.

We let $l(\lambda) \leqslant n$ and we may assume that $\lambda_1 > 0$. We set $m = \mathrm{cl}(\lambda) > 0$ and let $\theta(\lambda)$ be the partition obtained by deleting the last column of λ.

Using n variables in the (U) property, we may expand $e_m(t_1, \ldots, t_n)$ times either $s^k_{\theta(\lambda)}(t_1, \ldots, t_n)$ or $s^k_\mu(t_1, \ldots, t_n)$. This gives

$$\begin{aligned} e_m(t_1, \ldots, t_n) & s^k_{\theta(\lambda)}(t_1, \ldots, t_n) s^k_\mu(t_1, \ldots, t_n) \\ &= \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\theta(\lambda) + \chi_M / \theta(\lambda)) s^k_{\theta(\lambda) + \chi_M}(t_1, \ldots, t_n) s^k_\mu(t_1, \ldots, t_n) \\ &= \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\mu + \chi_M / \mu) s^k_{\theta(\lambda)}(t_1, \ldots, t_n) s^k_{\mu + \chi_M}(t_1, \ldots, t_n). \end{aligned} \tag{3.5}$$

Multiplying (3.5) by $W^k_n(x, k; t_1, \ldots, t_n)$ and then integrating using (J), we obtain

$$\begin{aligned} \int_0^1 \cdots \int_0^1 & e_m(t_1, \ldots, t_n) s^k_{\theta(\lambda)}(t_1, \ldots, t_n) s^k_\mu(t_1, \ldots, t_n) W^k_n(x, k;\, t_1, \ldots, t_n)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n \\ &= \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\theta(\lambda) + \chi_M / \theta(\lambda)) J^k_n(x;\, \theta(\lambda) + \chi_M,\, \mu) \\ &= \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\mu + \chi_M / \mu) J^k_n(x;\, \theta(\lambda),\, \mu + \chi_M). \end{aligned} \tag{3.6}$$

We now show that (3.6) holds when we use the product integration formula Theorem 2 (J) for $J^k_n(x; \lambda, \mu)$. Substituting (J) into (3.6) and dividing by $n!\,(\Gamma(k))^n$ gives

$$\begin{aligned} \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} & U^k(\theta(\lambda) + \chi_M / \theta(\lambda)) f^k_n(\theta(\lambda) + \chi_M) f^k_n(\mu) \\ & \cdot \prod_{i=1}^n \prod_{j=1}^n \frac{1}{(x + (2n - i - j)k + \theta(\lambda)_i + \mu_j + \chi(i \in M))_k} \\ = \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} & U^k(\mu + \chi_M / \mu) f^k_n(\theta(\lambda)) f^k_n(\mu + \chi_M) \\ & \cdot \prod_{i=1}^n \prod_{j=1}^n \frac{1}{(x + (2n - i - j)k + \theta(\lambda)_i + \mu_j + \chi(j \in M))_k}. \end{aligned} \tag{3.7}$$

Multiply both sides of (3.7) by

$$\prod_{i=1}^n \prod_{j=1}^n (x + (2n - i - j)k + \theta(\lambda)_i + \mu_j + 1)_{k-1}. \tag{3.8}$$

We obtain

$$\sum_{\substack{M\subseteq[1,n]\\|M|=m}} U^k(\theta(\lambda)+\chi_M/\theta(\lambda))f_n^k(\theta(\lambda)+\chi_M)f_n^k(\mu)$$

$$\cdot\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{(x+(2n-i-j)k+\theta(\lambda)_i+\mu_j+k\chi(i\in M))}$$

$$=\sum_{\substack{M\subseteq[1,n]\\|M|=m}} U^k(\mu+\chi_M/\mu)f_n^k(\theta(\lambda))f_n^k(\mu+\chi_M)$$

$$\cdot\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{(x+(2n-i-j)k+\theta(\lambda)_i+\mu_j+k\chi(j\in M))}. \tag{3.9}$$

Using (2.10) and (2.11), we divide (3.9) by $\Omega_n^k(\theta(\lambda))\Omega_n^k(\mu)$. This gives

$$\sum_{\substack{M\subseteq[1,n]\\|M|=m}}\prod_{1\leqslant i<j\leqslant n}((j-i)k+\theta(\lambda)_i+k\chi(i\in M)-\theta(\lambda)_j$$

$$-k\chi(j\in M))((j-i)k+\mu_i-\mu_j)$$

$$\cdot\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{(x+(2n-i-j)k+\theta(\lambda)_i+\mu_j+k\chi(i\in M))}$$

$$=\sum_{\substack{M\subseteq[1,n]\\|M|=m}}\prod_{1\leqslant i<j\leqslant n}((j-i)k+\theta(\lambda)_i-\theta(\lambda)_j)((j-i)k+\mu_i$$

$$+k\chi(i\in M)-\mu_j-k\chi(j\in M))$$

$$\cdot\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{(x+(2n-i-j)k+\theta(\lambda)_i+\mu_j+k\chi(j\in M))}. \tag{3.10}$$

Observe that (3.10) and hence (3.6) is homogeneous in k and x as well as $\theta(\lambda)_i$, $1\leqslant i\leqslant n$, and μ_j, $1\leqslant j\leqslant n$.

Let us set

$$\theta(\lambda)_i^k=(n-i)k+\theta(\lambda)_i,\quad 1\leqslant i\leqslant n,$$

$$\mu_i^k=(n-i)k+\mu_i,\qquad 1\leqslant i\leqslant n. \tag{3.11}$$

Recognizing the Vandermonde determinants, we see that (3.10) becomes

$$\sum_{\substack{M\subseteq[1,n]\\|M|=m}} \Delta_n(\theta(\lambda)_1^k+k\chi(1\in M),\dots,\theta(\lambda)_n^k+k\chi(n\in M))\Delta_n(\mu_1^k,\dots,\mu_n^k)$$

$$\cdot\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{(x+\theta(\lambda)_i^k+\mu_j^k+k\chi(i\in M))}$$

$$=\sum_{\substack{M\subseteq[1,n]\\|M|=m}} \Delta_n(\theta(\lambda)_1^k,\dots,\theta(\lambda)_n^k)\Delta_n(\mu_1^k+k\chi(1\in M),\dots,\mu_n^k+k\chi(n\in M))$$

$$\cdot\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{(x+\theta(\lambda)_i^k+\mu_j^k+k\chi(j\in M))}. \tag{3.12}$$

Let us set

$$x_i = x+\theta(\lambda)_i^k, \qquad y_i=\mu_i^k, \quad 1\leqslant i\leqslant n. \tag{3.13}$$

Observe that since the Vandermonde determinant is unchanged if all of its arguments are shifted by the same amount, we may just as easily incorporate the parameter x into the y_i, $1\leqslant i\leqslant n$. Using the substitutions $x_i\to x_i+k\chi(i\in M)$, $1\leqslant i\leqslant n$, and $y_j\to y_j+k\chi(j\in M)$, $1\leqslant j\leqslant n$, for the left and right sides of (3.12), respectively, we have

$$\sum_{\substack{M\subseteq[1,n]\\|M|=m}} \Delta_n(x_1+k\chi(1\in M),\dots,x_n+k\chi(n\in M))\Delta_n(y_1,\dots,y_n)$$

$$\cdot\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{(x_i+y_j+k\chi(i\in M))}$$

$$=\sum_{\substack{M\subseteq[1,n]\\|M|=m}} \Delta_n(x_1,\dots,x_n)\Delta_n(y_1+k\chi(1\in M),\dots,y_n+k\chi(n\in M))$$

$$\cdot\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{(x_i+y_j+k\chi(j\in M))}.$$

Observe that (3.14) is homogeneous in k as well as x_i, $1\leqslant i\leqslant n$, and y_j, $1\leqslant j\leqslant n$.

The $k=1$ case of the product integration formula Theorem 2 (**J**) was proved by Hua [Hu1]. Thus (3.6) and hence (3.14) holds for $k=1$. Since it is homogeneous in k as well as x_i, $1\leqslant i\leqslant n$, and y_j, $1\leqslant j\leqslant n$, we see that (3.14) holds for all $k\geqslant 0$. Hence (3.6) holds for all $k\geqslant 0$ when we use the formula (**J**) for $J_n^k(x;\lambda,\mu)$.

We pause to show that (3.14) and hence (3.6) also follows by the Cauchy identity (**C**). Using (**C**), we see that (3.14) becomes

$$\sum_{\substack{M\subseteq[1,n]\\|M|=m}} \det\left|\frac{1}{(x_i+y_j+k\chi(i\in M))}\right|_{n\times n}$$
$$=\sum_{\substack{M\subseteq[1,n]\\|M|=m}} \det\left|\frac{1}{(x_i+y_j+k\chi(j\in M))}\right|_{n\times n}. \tag{3.15}$$

Expanding the determinants in (3.15), we obtain

$$\sum_{\substack{M\subseteq[1,n]\\|M|=m}} \sum_{\pi\in S_n} \operatorname{sgn}(\pi)\prod_{i=1}^{n}\frac{1}{(x_i+y_{\pi(i)}+k\chi(i\in M))}$$
$$=\sum_{\substack{\mathscr{M}\subseteq[1,n]\\|\mathscr{M}|=m}} \sum_{\pi\in S_n} \operatorname{sgn}(\pi)\prod_{i=1}^{n}\frac{1}{(x_i+y_{\pi(i)}+k\chi(\pi(i)\in \mathscr{M}))}. \tag{3.16}$$

This follows directly using the bijection

$$\mathscr{M}=\pi(M)=\{\pi(i)\mid i\in M\} \tag{3.17}$$

and, since it is equivalent to (3.14), we obtain (3.6).

By (2.14) and (2.15), we see that the term $J_n^k(x;\lambda,\mu)$ occurs on the left side of (3.6) when $M=[1,m]$. Solving for this term, we obtain

$$J_n^k(x;\lambda,\mu)=\sum_{\substack{M\subseteq[1,n]\\|M|=m}} U^k(\mu+\chi_M/\mu)J_n^k(x;\theta(\lambda),\mu+\chi_M)$$
$$-\sum_{\substack{M\subseteq[1,n]\\|M|=m\\M\neq[1,m]}} U^k(\theta(\lambda)+\chi_M/\theta(\lambda))J_n^k(x;\theta(\lambda)+\chi_M,\mu). \tag{3.18}$$

Since all of the partitions which occur as a first argument of $J_n^k(x;\cdot,\cdot)$ in the terms on the right side of (3.18) have fewer cells in column λ_1, our induction is complete and the product integration formula Theorem 2 (**J**) is established.

4. Proofs of the constant term identity Theorem 5 (K) and the constant term orthogonality Theorem 6 (O′)

In this section, we formulate the product integration formula Theorem 2 (**J**) as the constant term identity Theorem 5 (**K**). Using the homogeneity of $s_\lambda^k(t)$, we prove the constant term orthogonality Theorem 6 (**O′**), including the evaluation of the normalization factor $g_n^k(\lambda)$.

By (1.14), the parameter a is subsumed in the constant term identity (**K**) by λ and also by μ. Thus in order to prove Theorem 5 (**K**), we need only prove the $a=0$ case of (**K**). This is given by

$$\mathcal{K}_n^k(0;\lambda,\mu) = [1]s_\lambda^k(t_1,\ldots,t_n)s_\mu^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)\prod_{i=1}^n\left(1-\frac{1}{t_i}\right)^{(k-1)}$$
$$\cdot\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k$$
$$= n!f_n^k(\lambda)f_n^k(\mu)(-1)^{|\lambda|-|\mu|}\prod_{i=1}^n\frac{(k-1)!}{(k-1+\mu_i-\lambda_i)!(\lambda_i-\mu_i)!}$$
$$\cdot\prod_{1\leqslant i<j\leqslant n}\frac{1}{(1-k+(j-i)k+\lambda_i-\mu_j)_k((j-i)k+\mu_i-\lambda_j)_k},$$
$$l(\lambda)\leqslant n,\ l(\mu)\leqslant n. \tag{4.1}$$

We assume throughout this section that $l(\lambda)\leqslant n$, $l(\mu)\leqslant n$. Let us set

$$s_\lambda^k(t_1,\ldots,t_n)s_\mu^k(t_1,\ldots,t_n)\Delta_n^{2k}(t_1,\ldots,t_n) = \sum_{\alpha_1\geqslant 0,\ldots,\alpha_n\geqslant 0} c_n^k(\lambda,\mu;\alpha)\prod_{i=1}^n t_i^{\alpha_i}, \tag{4.2}$$

where by homogeneity the sum on the right side of (4.2) is also restricted to

$$\sum_{i=1}^n \alpha_i = |\lambda|+|\mu|+2k\binom{n}{2}. \tag{4.3}$$

Substituting (4.2) into the product integration formula Theorem 2 (**J**), we can carry out each integral separately. We obtain

$$J_n^k(x;\lambda,\mu) = \int_0^1\cdots\int_0^1 s_\lambda^k(t_1,\ldots,t_n)s_\mu^k(t_1,\ldots,t_n)W_n^k(x,k;t_1,\ldots,t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n$$
$$= \sum_{\alpha_1\geqslant 0,\ldots,\alpha_n\geqslant 0} c_n^k(\lambda,\mu;\alpha)\prod_{i=1}^n\frac{\Gamma(x+\alpha_i)\Gamma(k)}{\Gamma(x+k+\alpha_i)}$$
$$= (\Gamma(k))^n\sum_{\alpha_1\geqslant 0,\ldots,\alpha_n\geqslant 0} c_n^k(\lambda,\mu;\alpha)\prod_{i=1}^n\frac{1}{(x+\alpha_i)_k}. \tag{4.4}$$

Equating this with (**J**) and dividing by $(\Gamma(k))^n$, we have

$$\mathcal{S}_n^k(x;\lambda,\mu) = \sum_{\alpha_1\geqslant 0,\ldots,\alpha_n\geqslant 0} c_n^k(\lambda,\mu;\alpha)\prod_{i=1}^n\frac{1}{(x+\alpha_i)_k}$$
$$= n;f_n^k(\lambda)f_n^k(\mu)\prod_{i=1}^n\prod_{j=1}^n\frac{1}{(x+(2n-i-j)k+\lambda_i+\mu_j)_k}. \tag{4.5}$$

Observe by (4.3) that the sum in (4.5) is a finite sum. Thus (4.5) is valid for all x since both sides are rational functions in x. We can convert (4.5) into a polynomial identity if we wish.

We are going to require the closely related function

$$\mathscr{R}_n^k(x;\lambda,\mu)=\sum_{\alpha_1\geq 0,\ldots,\alpha_n\geq 0} c_n^k(\lambda,\mu;\alpha)\prod_{i=1}^{n}\frac{1}{(x-\alpha_i)_k}. \tag{4.6}$$

The identity for reversing a finite product is

$$\begin{aligned}(a)_N&=a\cdots(a+N-1)\\ &=(-1)^N(-a)\cdots(-a-N+1)\\ &=(-1)^N(1-N-a)_N,\quad N\geq 0.\end{aligned} \tag{4.7}$$

By (4.7), we have

$$\frac{1}{(x-\alpha_i)_k}=(-1)^k\frac{1}{(1-k-x+\alpha_i)_k},\quad 1\leq i\leq n. \tag{4.8}$$

Hence (4.5) gives

$$\begin{aligned}\mathscr{R}_n^k(x;\lambda,\mu)&=(-1)^{nk}\sum_{\alpha_1\geq 0,\ldots,\alpha_n\geq 0} c_n^k(\lambda,\mu;\alpha)\prod_{i=1}^{n}\frac{1}{(1-k-x+\alpha_i)_k}\\ &=(-1)^{nk}\mathscr{S}_n^k(1-k-x;\lambda,\mu)\\ &=(-1)^{nk}n!f_n^k(\lambda)f_n^k(\mu)\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{(1-k-x+(2n-i-j)k+\lambda_i+\mu_j)_k}.\end{aligned} \tag{4.9}$$

By (4.7), we have

$$\frac{1}{(1-k-x+(2n-i-j)k+\lambda_i+\mu_j)_k}=(-1)^k\frac{1}{(x-(2n-i-j)k-\lambda_i-\mu_j)_k}. \tag{4.10}$$

Observe that

$$(-1)^{nk}(-1)^{n^2k}=(-1)^{(n^2+n)k}=(-1)^{n(n+1)k}=1, \tag{4.11}$$

since either n or $n+1$ must be even. Thus we see that (4.9) becomes

$$\mathscr{R}_n^k(x;\lambda,\mu) = n!f_n^k(\lambda)f_n^k(\mu)\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{(x-(2n-i-j)k-\lambda_i-\mu_j)_k}. \tag{4.12}$$

Set

$$\hat{\lambda}_i = -\lambda_{n+1-i}, \quad 1 \leqslant i \leqslant n. \tag{4.13}$$

By (1.4), we see that we may define $s_\lambda^k(t_1,\ldots,t_n)$ even when some of the parts of λ are negative provided that the parts are in nonincreasing order. Kadell [Ka4] has shown that

$$s_\lambda^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right) = s_{\hat{\lambda}}^k(t_1,\ldots,t_n). \tag{4.14}$$

Observe that

$$\begin{aligned}\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k &= \prod_{1\leqslant i<j\leqslant n}\left(\frac{-1}{t_it_j}\right)^k(t_i-t_j)^{2k}\\ &= (-1)^{k\binom{n}{2}}\prod_{i=1}^{n}t_i^{-(n-1)k}\Delta_n^{2k}(t_1,\ldots,t_n).\end{aligned} \tag{4.15}$$

By (4.2), (4.14) and (4.15), we have

$$\begin{aligned}&s_\lambda^k(t_1,\ldots,t_n)s_\mu^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)\prod_{i=1}^{n}\left(1-\frac{1}{t_i}\right)^{(k-1)}\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k\\ &\quad= s_\lambda^k(t_1,\ldots,t_n)s_{\hat{\mu}}^k(t_1,\ldots,t_n)\prod_{i=1}^{n}\left(1-\frac{1}{t_i}\right)^{(k-1)}(-1)^{k\binom{n}{2}}\\ &\qquad\cdot\prod_{i=1}^{n}t_i^{-(n-1)k}\Delta_n^{2k}(t_1,\ldots,t_n)\\ &\quad= (-1)^{k\binom{n}{2}}\sum_{\alpha_1\geqslant 0,\ldots,\alpha_n\geqslant 0}c_n^k(\lambda,\hat{\mu};\alpha)\prod_{i=1}^{n}t_i^{\alpha_i-(n-1)k}\left(1-\frac{1}{t_i}\right)^{(k-1)}.\end{aligned} \tag{4.16}$$

Substituting (4.16) into (4.1) and extracting the constant term, we have

$$\mathscr{K}_n^k(0;\lambda,\mu) = (-1)^{k\binom{n}{2}}\sum_{\alpha_1\geqslant 0,\ldots,\alpha_n\geqslant 0}c_n^k(\lambda,\hat{\mu};\alpha)\prod_{i=1}^{n}[1]t_i^{\alpha_i-(n-1)k}\left(1-\frac{1}{t_i}\right)^{(k-1)}. \tag{4.17}$$

Let A be an integer. We have

$$[1]s^A\left(1-\frac{1}{s}\right)^{k-1}=(-1)^A\binom{k-1}{A}$$
$$=(-1)^A\frac{(k-1)!}{A!(k-1-A)!}. \tag{4.18}$$

Observe by the standard convention

$$\frac{1}{(-n)!}=0 \quad \text{if } n>0 \tag{4.19}$$

that the constant term (4.18) is 0 if $A<0$ or $A>k-1$. We assume that $0\leqslant A\leqslant k-1$. This gives

$$(-A)_A=(-1)^A(1)_A=(-1)^A A!. \tag{4.20}$$

Observe that

$$\lim_{\varepsilon\to 0}\varepsilon\frac{(k-1)!}{(\varepsilon-A)_k}=\lim_{\varepsilon\to 0}\varepsilon\frac{(k-1)!}{(\varepsilon-A)_A\varepsilon(1+\varepsilon)_{k-1-A}}$$
$$=\frac{(k-1)!}{(-A)_A(k-1-A)!}$$
$$=(-1)^A\frac{(k-1)!}{A!(k-1-A)!}. \tag{4.21}$$

Comparing (4.18) and (4.21), we have

$$[1]s^A\left(1-\frac{1}{s}\right)^{k-1}=\lim_{\varepsilon\to 0}\varepsilon\frac{(k-1)!}{(\varepsilon-A)_k}. \tag{4.22}$$

Since the right side of (4.22) is 0 if $A<0$ or $A>k-1$, we see that (4.22) holds for all integers A.

Using (4.22) to extract the constant term in (4.17), we have

$$\mathscr{K}_n^k(0;\lambda,\mu)=(-1)^{k\binom{n}{2}}\sum_{\alpha_1\geqslant 0,\ldots,\alpha_n\geqslant 0}c_n^k(\lambda,\hat{\mu};\alpha)\prod_{i=1}^{n}\lim_{\varepsilon\to 0}\varepsilon\frac{(k-1)!}{(\varepsilon+(n-1)k-\alpha_i)_k}$$
$$=(-1)^{k\binom{n}{2}}((k-1)!)^n\lim_{\varepsilon\to 0}\varepsilon^n\mathscr{R}_n^k(\varepsilon+(n-1)k;\lambda,\hat{\mu}). \tag{4.23}$$

Using the rational function identity (4.12), we obtain

$$\mathscr{K}_n^k(0;\lambda,\mu) = n!f_n^k(\lambda)f_n^k(\hat{\mu})(-1)^{k\binom{n}{2}}((k-1)!)^n \cdot \lim_{\varepsilon\to 0} \varepsilon^n \prod_{i=1}^n \prod_{j=1}^n \frac{1}{(\varepsilon+(n-1)k-(2n-i-j)k-\lambda_i-\hat{\mu}_j)_k}. \tag{4.24}$$

We now reconcile (4.24) with the formula (4.1) for $\mathscr{K}_n^k(0;\lambda,\mu)$. Using the substitution $i\to n+1-j$, $j\to n+1-i$, we have

$$\begin{aligned} f_n^k(\hat{\mu}) &= \prod_{1\leqslant i<j\leqslant n} ((j-i)k+\hat{\mu}_i-\hat{\mu}_j)_k \\ &= \prod_{1\leqslant i<j\leqslant n} (((n+1-i)-(n+1-j))k+\hat{\mu}_{n+1-j}-\hat{\mu}_{n+1-i})_k \\ &= \prod_{1\leqslant i<j\leqslant n} ((j-i)k+\mu_i-\mu_j)_k \\ &= f_n^k(\mu). \end{aligned} \tag{4.25}$$

Observe that under the substitution $j\to n+1-j$, we have

$$\begin{aligned} &\varepsilon+(n-1)k-(2n-i-j)k-\lambda_i-\hat{\mu}_j \\ &\quad\to \varepsilon+(n-1)k-(2n-i-(n+1-j))k-\lambda_i-\hat{\mu}_{n+1-j} \\ &\quad= \varepsilon-(j-i)k-\lambda_i+\mu_j, \quad 1\leqslant i\leqslant n,\ 1\leqslant j\leqslant n. \end{aligned} \tag{4.26}$$

Thus (4.24) becomes

$$\mathscr{K}_n^k(0;\lambda,\mu) = n!f_n^k(\lambda)f_n^k(\mu)(-1)^{k\binom{n}{2}}((k-1)!)^n \cdot \lim_{\varepsilon\to 0} \varepsilon^n \prod_{i=1}^n \prod_{j=1}^n \frac{1}{(\varepsilon-(j-i)k-\lambda_i+\mu_j)_k}. \tag{4.27}$$

Let us investigate when $\mathscr{K}_n^k(0;\lambda,\mu)$ is 0. Observe that

$$\begin{aligned} &(\varepsilon-((j+1)-i)k-\lambda_i+\mu_{j+1})+k \\ &\quad\leqslant \varepsilon-(j-i)k-\lambda_i+\mu_j, \quad 1\leqslant i\leqslant n,\ 1\leqslant j\leqslant n-1. \end{aligned} \tag{4.28}$$

Thus for a fixed i, $1\leqslant i\leqslant n$, when we consider

$$\prod_{j=1}^n \frac{1}{(\varepsilon-(j-i)k-\lambda_i+\mu_j)_k} \tag{4.29}$$

as a function of ε, it has only simple poles. If it has a pole at $\varepsilon = 0$, then there exists a unique $j(i)$, $1 \leqslant j(i) \leqslant n$, such that

$$\frac{1}{(\varepsilon-(j(i)-i)k-\lambda_i+\mu_{j(i)})_k} \tag{4.30}$$

has a pole at $\varepsilon = 0$. This requires that

$$1-k \leqslant -(j(i)-i)k-\lambda_i+\mu_{j(i)} \leqslant 0. \tag{4.31}$$

Multiplying by -1 and reading the inequality from right to left gives

$$0 \leqslant (j(i)-i)k+\lambda_i-\mu_{j(i)} \leqslant k-1. \tag{4.32}$$

We observe from the factor ε^n on the right side of (4.27) that $\mathscr{K}_n^k(0; \lambda, \mu)$ is 0 unless for each i, $1 \leqslant i \leqslant n$, the function (4.29) has a simple pole at $\varepsilon = 0$. Thus for each i, $1 \leqslant i \leqslant n$, there exists a unique $j(i)$, $1 \leqslant j(i) \leqslant n$, which satisfies (4.32).

Replacing i by $i+1$ in (4.31) gives

$$1-k \leqslant -(j(i+1)-(i+1))k-\lambda_{i+1}+\mu_{j(i+1)} \leqslant 0, \quad 0 \leqslant i \leqslant n-1. \tag{4.33}$$

Adding (4.32) and (4.33) together gives

$$1-k \leqslant (j(i)-j(i+1)+1)k+\lambda_i-\lambda_{i+1}+\mu_{j(i+1)}-\mu_{j(i)} \leqslant k-1, \quad 1 \leqslant i \leqslant n-1. \tag{4.34}$$

Since λ is a partition, we have $\lambda_i-\lambda_{i+1} \geqslant 0$, $1 \leqslant i \leqslant n-1$. Hence

$$(j(i)-j(i+1)+1)k+\mu_{j(i+1)}-\mu_{j(i)} \leqslant k-1, \quad 1 \leqslant i \leqslant n-1. \tag{4.35}$$

Observe that

$$j(i) \geqslant j(i+1) \text{ implies } (j(i)-j(i+1)+1)k+\mu_{j(i+1)}-\mu_{j(i)} \geqslant k. \tag{4.36}$$

Since this contradicts (4.35), we have $j(i) < j(i+1)$, $1 \leqslant i \leqslant n-1$. Thus $j(i) = i$, $1 \leqslant i \leqslant n$, and (4.32) becomes

$$0 \leqslant \lambda_i-\mu_i \leqslant k-1, \quad 1 \leqslant i \leqslant n. \tag{4.37}$$

We have that $\mathscr{K}_n^k(0; \lambda, \mu)$ is 0 unless (4.37) holds. Assume that this is the case.

Then we have

$$(-1)^{k\binom{n}{2}}((k-1)!)^n \lim_{\varepsilon\to 0} \varepsilon^n \prod_{i=1}^{n}\prod_{j=1}^{n} \frac{1}{(\varepsilon-(j-i)k-\lambda_i+\mu_j)_k}$$

$$= \prod_{i=1}^{n} \lim_{\varepsilon\to 0} \varepsilon \frac{(k-1)!}{(\varepsilon-\lambda_i+\mu_i)_k} (-1)^{k\binom{n}{2}} \prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{(-(j-i)k-\lambda_i+\mu_j)_k}. \tag{4.38}$$

Putting $A=\lambda_i-\mu_i$ in (4.21), we have

$$\lim_{\varepsilon\to 0} \varepsilon \frac{(k-1)!}{(\varepsilon-\lambda_i+\mu_i)_k} = (-1)^{\lambda_i-\mu_i} \frac{(k-1)!}{(\lambda_i-\mu_i)!(k-1+\mu_i-\lambda_i)!}, \quad 1\leqslant i\leqslant n. \tag{4.39}$$

Hence

$$\prod_{i=1}^{n} \lim_{\varepsilon\to 0} \varepsilon \frac{(k-1)!}{(\varepsilon-\lambda_i+\mu_i)_k} = (-1)^{|\lambda|-|\mu|} \prod_{i=1}^{n} \frac{(k-1)!}{(\lambda_i-\mu_i)!(k-1+\mu_i-\lambda_i)!}. \tag{4.40}$$

Using the substitution $i\leftrightarrow j$ and the identity (4.7) for reversing a finite product, we have

$$(-1)^{k\binom{n}{2}} \prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{(-(j-i)k-\lambda_i+\mu_j)_k}$$

$$= \prod_{1\leqslant i<j\leqslant n} \frac{(-1)^k}{(-(j-i)k-\lambda_i+\mu_j)_k(-(i-j)k-\lambda_j+\mu_i)_k}$$

$$= \prod_{1\leqslant i<j\leqslant n} \frac{1}{(1-k+(j-i)k+\lambda_i-\mu_j)_k((j-i)k+\mu_i-\lambda_j)_k}. \tag{4.41}$$

Substituting (4.40) and (4.41) into (4.38), we have

$$(-1)^{k\binom{n}{2}}((k-1)!)^n \lim_{\varepsilon\to 0} \varepsilon^n \prod_{i=1}^{n}\prod_{j=1}^{n} \frac{1}{(\varepsilon-(j-i)k-\lambda_i+\mu_j)_k}$$

$$= (-1)^{|\lambda|-|\mu|} \prod_{i=1}^{n} \frac{(k-1)!}{(\lambda_i-\mu_i)!(k-1+\mu_i-\lambda_i)!}$$

$$\cdot \prod_{1\leqslant i<j\leqslant n} \frac{1}{(1-k+(j-i)k+\lambda_i-\mu_j)_k((j-i)k+\mu_i-\lambda_j)_k}. \tag{4.42}$$

Since the first factor on the right side of (4.42) is 0 if (4.37) does not hold, we see that (4.42) holds for all partitions λ and μ. Substituting (4.42) into (4.27) gives the formula (4.1) for $\mathcal{K}_n^k(0;\lambda,\mu)$. This completes the proof of the constant term identity Theorem 5 (**K**).

We have that $s_\lambda^k(t_1,\ldots,t_n)$ is homogeneous of total degree $|\lambda|$. Similarly, $s_\mu^k(1/t_1,\ldots,1/t_n)$ is homogeneous of total degree $-|\mu|$. Since $\prod_{1\leqslant i<j\leqslant n}(1-t_i/t_j)^k(1-t_j/t_i)^k$ is homogeneous of total degree 0, we have

$$[1]s_\lambda^k(t_1,\ldots,t_n)s_\mu^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k=0 \quad \text{if } |\lambda|\neq|\mu|. \tag{4.43}$$

Let us assume that $|\lambda|=|\mu|$. Then we have

$$\begin{aligned}
&[1]s_\lambda^k(t_1,\ldots,t_n)s_\mu^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k\\
&=[1]s_\lambda^k(t_1,\ldots,t_n)s_\mu^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)\prod_{i=1}^n\left(1-\frac{1}{t_i}\right)^{k-1}\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k\\
&=\mathcal{K}_n^k(0;\lambda,\mu).
\end{aligned}\tag{4.44}$$

We have already seen that this is 0 unless (4.37) holds. We require only the weaker condition

$$0\leqslant\lambda_i-\mu_i,\quad 1\leqslant i\leqslant n. \tag{4.45}$$

Since $|\lambda|=|\mu|$, this gives $\lambda=\mu$. Thus we have

$$[1]s_\lambda^k(t_1,\ldots,t_n)s_\mu^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k=0 \quad \text{if } \lambda\neq\mu. \tag{4.46}$$

This gives the constant term orthogonality (**O**′). We may evaluate the normalization factor $g_n^k(\lambda)$ by setting $\mu=\lambda$ in (4.44). We obtain

$$\begin{aligned}
n!g_n^k(\lambda)&=[1]s_\lambda^k(t_1,\ldots,t_n)s_\lambda^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k\\
&=\mathcal{K}_n^k(0;\lambda,\lambda)\\
&=n!(f_n^k(\lambda))^2\prod_{1\leqslant i<j\leqslant n}\frac{1}{(1-k+(j-i)k+\lambda_i-\lambda_j)_k((j-i)k+\lambda_i-\lambda_j)_k}\\
&=n!\,\frac{f_n^k(\lambda)}{\bar{f}_n^k(\lambda)},
\end{aligned}\tag{4.47}$$

in agreement with the formula (1.15) for $g_n^k(\lambda)$. This completes the proof of the constant term orthogonality Theorem 6 (**O**′).

5. A proof of the expansion Theorem 8 (G)

In this section, we use the constant term orthogonality (**O**′) and the constant term identity (**K**) to obtain the expansion Theorem 8 (**G**). The orthogonality (**O**′) mediates between (**G**) and (**K**) just as it mediates between (**H**) and (**A**).

Since it is a symmetric function, we may expand $s_\mu^k(t)\Pi_{i\geqslant 1}(1-t_i)^{k-1}$ in terms of the Selberg-Jack symmetric functions $\{s_\lambda^k(t)\}$. Let $G^k(\lambda,\mu)$ denote the coefficient.

We may use the constant term orthogonality (**O**′) to compute the coefficient $G^k(\lambda,\mu)$. Using n variables in the expansion (**G**) where $l(\mu)\leqslant n$, we have

$$s_\mu^k(t_1,\ldots,t_n)\prod_{i=1}^n(1-t_i)^{k-1}=\sum_{\substack{\lambda\\ l(\lambda)\leqslant n}}G^k(\lambda,\mu)s_\lambda^k(t_1,\ldots,t_n),\quad l(\mu)\leqslant n. \tag{5.1}$$

Multiply (5.1) by $s_\lambda^k(1/t_1,\ldots,1/t_n)$ times the weight function

$$\Pi_{1\leqslant i<j\leqslant n}(1-t_i/t_j)^k(1-t_j/t_i)^k$$

of the orthogonality (**O**′). By (**O**′), we may compute the coefficient $G^k(\lambda,\mu)$ by extracting the constant term. Thus we have

$$G^k(\lambda,\mu)=\frac{1}{n!g_n^k(\lambda)}[1]s_\mu^k(t_1,\ldots,t_n)s_\lambda^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)\prod_{i=1}^n(1-t_i)^{k-1}$$
$$\cdot\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k,\quad l(\lambda)\leqslant n,\ l(\mu)\leqslant n. \tag{5.2}$$

Since the constant term is unchanged by the substitution $t_i\to 1/t_i$, $1\leqslant i\leqslant n$, we may use the constant term identity (**K**) to extract the required constant term. We obtain

$$G^k(\lambda,\mu)=\frac{1}{n!g_n^k(\lambda)}[1]s_\lambda^k(t_1,\ldots,t_n)s_\mu^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)\prod_{i=1}^n\left(1-\frac{1}{t_i}\right)^{k-1}$$
$$\cdot\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k$$
$$=\frac{1}{n!g_n^k(\lambda)}\mathscr{K}_n^k(0;\lambda,\mu),\quad l(\lambda)\leqslant n,\ l(\mu)\leqslant n. \tag{5.3}$$

By (1.3), (1.15) and (1.16), we have

$$\frac{1}{n!g_n^k(\lambda)} = \frac{1}{n!} \prod_{1 \leqslant i<j \leqslant n} \frac{(1-k+(j-i)k+\lambda_i-\lambda_j)_k}{((j-i)k+\lambda_i-\lambda_j)_k}, \quad l(\lambda) \leqslant n. \tag{5.4}$$

Substituting (1.3) into (4.1) gives

$$\mathscr{K}_n^k(0; \lambda, \mu) = n!(-1)^{|\lambda|-|\mu|} \prod_{i=1}^{n} \frac{(k-1)!}{(k-1+\mu_i-\lambda_i)!(\lambda_i-\mu_i)!} \cdot \prod_{1 \leqslant i<j \leqslant n} \frac{((j-i)k+\lambda_i-\lambda_j)_k((j-i)k+\mu_i-\mu_j)_k}{(1-k+(j-i)k+\lambda_i-\mu_j)_k((j-i)k+\mu_i-\lambda_j)_k}, \quad l(\lambda) \leqslant n,\ l(\mu) \leqslant n. \tag{5.5}$$

Substituting (5.4) and (5.5) into (5.3) gives the formula Theorem 8 (1.30) for $G^k(\lambda, \mu)$. This completes the proof of the expansion Theorem 8 (**G**).

By (1.14) and (5.3), we see that if we know two of the three quantities $\mathscr{K}_n^k(a; \lambda, \mu)$, $G^k(\lambda, \mu)$ and $g_n^k(\lambda)$, then we can evaluate the third. Thus (**O′**) mediates between (**G**) and (**K**) just as it mediates between (**H**) and (**A**).

6. Alternative proofs of the constant term identity Theorem 5 (K) and the expansion Theorem 8 (G)

In this section, we outline alternative proofs of the constant term identity Theorem 5 (**K**) and the expansion Theorem 8 (**G**). We show that the argument used to prove the product integration formula Theorem 2 (**J**) in Section 3 can also be used to prove (**K**) and (**G**) and that the identities which arise for (**K**) and (**G**) are both equivalent to the identity which arises for (**J**), which may be formulated as a homogeneous rational function identity.

Observe by Aomoto's theorem [Ao1] (1.5) that the $\lambda = (1^m)$ case of (4.14) is

$$e_m\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right) = \prod_{i=1}^{n} \frac{1}{t_i}\, e_{n-m}(t_1, \ldots, t_n). \tag{6.1}$$

This plays a key role in Macdonald's elegant evaluation [Ma2, Chap. VI] of the normalization factor $g_n^k(\lambda)$ of the constant term orthogonality (**O′**). It is important to note that (4.14) and (1.4) are implicit in the orthogonality (**O′**). The following result is also implicit in (4.14).

$$U^k(\hat{\lambda} + \chi_{[1,n]-\bar{M}}/\hat{\lambda}) = U^k(\lambda + \chi_M/\lambda), \tag{6.2}$$

where

$$\bar{M} = \{n+1-i \mid i \in M\}, \quad M \subseteq [1, n]. \tag{6.3}$$

In [Ka4], we verified (6.2) and used it to obtain (4.14).

For the constant term identity Theorem 5 (**K**), we may assume that $a = 0$ since by (1.14) the parameter a is subsumed by λ and also by μ. By (6.1), we may proceed by induction on λ or on μ. We proceed by a double induction on the number μ_1 of columns of μ and on the number $\mathrm{cl}(\mu) = \mu'_{\mu_1}$ of cells (i, μ_1) in the last column of μ. We begin the induction by observing that for $\mu = (0)$, the expansion (**K**) reduces to the $b = k-1$ case of (**A**). Assume that $l(\lambda) \leqslant n$. We have

$$\mathscr{K}_n^k(0; \lambda, (0)) = CS_n^k(0, k-1; \lambda). \tag{6.4}$$

We let $l(\mu) \leqslant n$ and we may assume that $\mu_1 > 0$. We set $m = \mathrm{cl}(\mu) > 0$ and let $\theta(\mu)$ be the partition obtained by deleting the last column of μ. Replacing λ by μ in (2.14) and (2.15), we have

$$\mu = \theta(\mu) + \chi_{[1,m]} \tag{6.5}$$

and

$$U^k(\mu/\theta(\mu)) = 1. \tag{6.6}$$

Using the n variables $(1/t_1, \ldots, 1/t_n)$ in the (**U**) property, we have

$$e_m\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right) s_{\theta(\mu)}^k\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right)$$
$$= \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\theta(\mu) + \chi_M/\theta(\mu)) s_{\theta(\mu)+\chi_M}^k\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right). \tag{6.7}$$

Multiplying (6.7) by $s_\lambda^k(t_1, \ldots, t_n)\Pi_{i=1}^n(1-1/t_i)^{k-1}$ times the weight function $\Pi_{1 \leqslant i < j \leqslant n}(1-t_i/t_j)^k(1-t_j/t_i)^k$ of the orthogonality (**O′**) and using (**K**) to extract the constant term, we obtain

$$[1]e_m\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right) s_\lambda^k(t_1, \ldots, t_n) s_{\theta(\mu)}^k\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right) \prod_{i=1}^n \left(1 - \frac{1}{t_i}\right)^{(k-1)}$$

$$\cdot \prod_{1 \leqslant i < j \leqslant n} \left(1 - \frac{t_i}{t_j}\right)^k \left(1 - \frac{t_j}{t_i}\right)^k$$

$$= \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\theta(\mu) + \chi_M / \theta(\mu))[1] s_\lambda^k(t_1, \ldots, t_n) s^k_{\theta(\mu) + \chi_M}\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right)$$

$$\cdot \prod_{i=1}^{n} \left(1 - \frac{1}{t_i}\right)^{(k-1)} \prod_{1 \leqslant i < j \leqslant n} \left(1 - \frac{t_i}{t_j}\right)^k \left(1 - \frac{t_j}{t_i}\right)^k$$

$$= \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\theta(\mu) + \chi_M / \theta(\mu)) \mathscr{K}_n^k(0;\ \lambda,\ \theta(\mu) + \chi_M). \tag{6.8}$$

Use n variables in the (**U**) property and replace the index M of summation by $\mathscr{M}$. By (6.1), we have

$$e_m\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right) s_\lambda^k(t_1, \ldots, t_n)$$

$$= \prod_{i=1}^{n} \frac{1}{t_i}\, e_{n-m}(t_1, \ldots, t_n) s_\lambda^k(t_1, \ldots, t_n)$$

$$= \prod_{i=1}^{n} \frac{1}{t_i} \sum_{\substack{\mathscr{M} \subseteq [1,n] \\ |\mathscr{M}| = n - m}} U^k(\lambda + \chi_{\mathscr{M}} / \lambda) s^k_{\lambda + \chi_{\mathscr{M}}}(t_1, \ldots, t_n).$$

Setting $M = [1, n] - \mathscr{M}$, we have $|M| = m$ and $\mathscr{M} = [1, n] - M$. By (1.4), we have

$$\prod_{i=1}^{n} \frac{1}{t_i}\, s^k_{\lambda + \chi_{\mathscr{M}}}(t_1, \ldots, t_n) = s^k_{\lambda - \chi_{\mathscr{M}}}(t_1, \ldots, t_n). \tag{6.10}$$

Hence (6.9) becomes

$$e_m\left(\frac{1}{t_1}, \ldots, \frac{1}{t_n}\right) s_\lambda^k(t_1, \ldots, t_n)$$

$$= \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\lambda + \chi_{[1,n]-M} / \lambda) s^k_{\lambda - \chi_M}(t_1, \ldots, t_n). \tag{6.11}$$

Multiplying (6.11) by $s^k_{\theta(\mu)}(1/t_1, \ldots, 1/t_n) \prod_{i=1}^n (1 - 1/t_i)^{k-1}$ times the weight function $\prod_{1 \leqslant i < j \leqslant n} (1 - t_i/t_j)^k (1 - t_j/t_i)^k$ of the orthogonality (**O′**) and using (**K**) to extract the constant term, we obtain

$$[1]e_m\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)s_\lambda^k(t_1,\ldots,t_n)s_{\theta(\mu)}^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)\prod_{i=1}^n\left(1-\frac{1}{t_i}\right)^{(k-1)}$$
$$\cdot\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k$$
$$=\sum_{\substack{M\subseteq[1,n]\\|M|=m}}U^k(\lambda+\chi_{[1,n]-M}/\lambda)[1]s_{\lambda-\chi_M}^k(t_1,\ldots,t_n)s_{\theta(\mu)}^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)$$
$$\cdot\prod_{i=1}^n\left(1-\frac{1}{t_i}\right)^{(k-1)}\cdot\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{t_i}{t_j}\right)^k\left(1-\frac{t_j}{t_i}\right)^k$$
$$=\sum_{\substack{M\subseteq[1,n]\\|M|=m}}U^k(\lambda+\chi_{[1,n]-M}/\lambda)\mathscr{K}_n^k(0;\,\lambda-\chi_M,\,\theta(\mu)).\tag{6.12}$$

Equating (6.8) and (6.12) gives

$$\sum_{\substack{M\subseteq[1,n]\\|M|=m}}U^k(\theta(\mu)+\chi_M/\theta(\mu))\mathscr{K}_n^k(0;\,\lambda,\,\theta(\mu)+\chi_M)$$
$$=\sum_{\substack{M\subseteq[1,n]\\|M|=m}}U^k(\lambda+\chi_{[1,n]-M}/\lambda)\mathscr{K}_n^k(0;\,\lambda-\chi_M,\,\theta(\mu)).\tag{6.13}$$

The homogeneous rational function identity (3.14), which is equivalent to (3.6), is central to our proof in Section 3 of the product integration formula Theorem 2 (**J**). We now show that (6.13) is equivalent to (3.6).

Substituting (4.9) into (4.23) gives

$$\mathscr{K}_n^k(0;\lambda,\mu)=(-1)^{k\binom{n}{2}}((k-1)!)^n\lim_{\varepsilon\to0}\varepsilon^n\mathscr{R}_n^k(\varepsilon+(n-1)k;\,\lambda,\,\hat{\mu})$$
$$=(-1)^{k\binom{n}{2}+nk}((k-1)!)^n\lim_{\varepsilon\to0}\varepsilon^n\mathscr{S}_n^k(1-\varepsilon-nk;\,\lambda,\,\hat{\mu}).\tag{6.14}$$

By (4.4) and (4.5), we have

$$J_n^k(x;\,\lambda,\,\mu)=(\Gamma(k))^n\mathscr{S}_n^k(x;\,\lambda,\,\mu).\tag{6.15}$$

Comparing (6.14) and (6.15), we have

$$\mathscr{K}_n^k(0;\,\lambda,\,\mu)=(-1)^{k\binom{n}{2}+nk}\lim_{\varepsilon\to0}\varepsilon^nJ_n^k(1-\varepsilon-nk;\,\lambda,\,\hat{\mu}).\tag{6.16}$$

We require identity expressing $\mathscr{K}_n^k(0;\lambda,\mu)$ in terms of $J_n^k(-\varepsilon-(n-1)k;\hat{\lambda},\mu)$. To obtain this we observe by (4.14) and (1.4) that

$$s_\lambda^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right) s_\mu^k(t_1,\ldots,t_n) \prod_{i=1}^n (1-t_i)^{k-1}$$
$$= (-1)^{n(k-1)} \prod_{i=1}^n t_i^{k-1} s_\mu^k(t_1,\ldots,t_n) s_\lambda^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right) \prod_{i=1}^n \left(1-\frac{1}{t_i}\right)^{k-1}$$
$$= (-1)^{n(k-1)} s^k_{(\mu_1+k-1,\ldots,\mu_n+k-1)}(t_1,\ldots,t_n) s_\lambda^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right) \prod_{i=1}^n \left(1-\frac{1}{t_i}\right)^{k-1},$$
$$l(\lambda) \leqslant n,\ l(\mu) \leqslant n. \qquad (6.17)$$

Multiply (6.17) by $\Pi_{1\leqslant i<j\leqslant n}(1-t_i/t_j)^k(1-t_j/t_i)^k$ and extract the constant term. Using the fact that the constant term is unchanged by the substitution $t_i \to 1/t_i$, $1 \leqslant i \leqslant n$, we have the symmetry

$$\mathscr{K}_n^k(0;\lambda,\mu) = [1] s_\lambda^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right) s_\mu^k(t_1,\ldots,t_n) \prod_{i=1}^n (1-t_i)^{k-1}$$
$$\cdot \prod_{1\leqslant i<j\leqslant n} \left(1-\frac{t_i}{t_j}\right)^k \left(1-\frac{t_j}{t_i}\right)^k$$
$$= (-1)^{n(k-1)} [1] s^k_{(\mu_1+k-1,\ldots,\mu_n+k-1)}(t_1,\ldots,t_n) s_\lambda^k\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)$$
$$\cdot \prod_{i=1}^n \left(1-\frac{1}{t_i}\right)^{k-1} \prod_{1\leqslant i<j\leqslant n} \left(1-\frac{t_i}{t_j}\right)^k \left(1-\frac{t_j}{t_i}\right)^k$$
$$= (-1)^{n(k-1)} \mathscr{K}_n^k(0; (\mu_1+k-1,\ldots,\mu_n+k-1), \lambda),$$
$$l(\lambda) \leqslant n,\ l(\mu) \leqslant n. \qquad (6.18)$$

Using (6.16), the $A = k-1$ case of (1.4), and the fact that $J_n^k(x;\lambda,\mu)$ is symmetric in λ and μ, we have

$$\mathscr{K}_n^k(0;\lambda,\mu) = (-1)^{n(k-1)} \mathscr{K}_n^k(0; (\mu_1+k-1,\ldots,\mu_n+k-1), \lambda)$$
$$= (-1)^{k\binom{n}{2}+n(2k-1)} \lim_{\varepsilon\to 0} \varepsilon^n J_n^k(1-\varepsilon-nk; (\mu_1+k-1,\ldots,\mu_n+k-1), \hat{\lambda})$$
$$= (-1)^{k\binom{n}{2}+n(2k-1)} \lim_{\varepsilon\to 0} \varepsilon^n J_n^k(-\varepsilon-(n-1)k; \hat{\lambda}, \mu),$$
$$l(\lambda) \leqslant n,\ l(\mu) \leqslant n. \qquad (6.19)$$

Substituting (6.19) into (6.13) and dividing both sides by $(-1)^{k\binom{n}{2}+n(2k-1)}$, we obtain

$$\sum_{\substack{M\subseteq[1,n]\\|M|=m}} U^k(\theta(\mu)+\chi_M/\theta(\mu)) \lim_{\varepsilon\to 0} \varepsilon^n J_n^k(-\varepsilon-(n-1)k;\, \hat{\lambda},\, \theta(\mu)+\chi_M)$$
$$= \sum_{\substack{M\subseteq[1,n]\\|M|=m}} U^k(\lambda+\chi_{[1,n]-M}/\lambda) \lim_{\varepsilon\to 0} \varepsilon^n J_n^k(-\varepsilon-(n-1)k;\, (\lambda-\chi_M)\hat{}\,,\, \theta(\mu)). \tag{6.20}$$

Observe that

$$(\lambda-\chi_M)\hat{} = \hat{\lambda}+\chi_{\bar{M}}, \quad M\subseteq[1,n]. \tag{6.21}$$

Replacing λ by $\hat{\lambda}$ and M by $\bar{M}$ in (6.2) and using the facts that $\hat{\hat{\lambda}} = \lambda$ and $\bar{\bar{M}} = M$, we have

$$U^k(\lambda+\chi_{[1,n]-M}/\lambda) = U^k(\hat{\lambda}+\chi_{\bar{M}}/\hat{\lambda}), \quad M\subseteq[1,n]. \tag{6.22}$$

Hence (6.20) becomes

$$\sum_{\substack{M\subseteq[1,n]\\|M|=m}} U^k(\theta(\mu)+\chi_M/\theta(\mu)) \lim_{\varepsilon\to 0} \varepsilon^n J_n^k(-\varepsilon-(n-1)k;\, \hat{\lambda},\, \theta(\mu)+\chi_M)$$
$$= \sum_{\substack{M\subseteq[1,n]\\|M|=m}} U^k(\hat{\lambda}+\chi_{\bar{M}}/\hat{\lambda}) \lim_{\varepsilon\to 0} \varepsilon^n J_n^k(-\varepsilon-(n-1)k;\, \hat{\lambda}+\chi_{\bar{M}},\, \theta(\mu)). \tag{6.23}$$

Since $|M| = |\bar{M}|$ and the parameter x in (3.6) is subsumed by λ and also by μ, it is easy to see that (6.23) is equivalent to (3.6).

We now return to our alternative proof of the constant term identity Theorem 5 (**K**). We require that (6.13) holds when we use the constant term identity (**K**) for $\mathscr{K}_n^k(0;\lambda,\mu)$. We may simply verify (6.19) and (6.22) formally or we may explicitly massage (6.13) into (3.14) as was done for (3.6). The preceding analysis shows that this may be done using the substitution

$$x_i = (1-i)k+\hat{\lambda}_i,\ y_i = (n-i)k+\theta(\mu)_i, \quad 1\leqslant i\leqslant n. \tag{6.24}$$

As in Section 3, we may use the Cauchy identity (**C**) to establish the homogeneous rational function identity (3.14). Thus (6.13) holds when we use the constant term identity (**K**) for $\mathscr{K}_n^k(0;\lambda,\mu)$. We may use Hua's argument [Hu1] in the constant term setting to establish the $k=1$ case of (**K**). This requires the Cauchy identity (**C**). We again obtain (6.13) since (3.14) is homogeneous in k as well as x_i, $1\leqslant i\leqslant n$, and y_j, $1\leqslant j\leqslant n$.

By (6.5) and (6.6), we see that the term $\mathscr{K}_n^k(0; \lambda, \mu)$ occurs on the left side of (6.13) when $M = [1, m]$. Solving for this term, we obtain

$$\mathscr{K}_n^k(0; \lambda, \mu) = \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\lambda + \chi_{[1,n]-M}/\lambda)\mathscr{K}_n^k(0; \lambda - \chi_M, \theta(\mu))$$
$$- \sum_{\substack{M \subseteq [1,n] \\ |M| = m \\ M \neq [1,m]}} U^k(\theta(\mu) + \chi_M/\theta(\mu))\mathscr{K}_n^k(0; \lambda, \theta(\mu) + \chi_M) \tag{6.25}$$

Since all of the partitions which occur as a second argument of $\mathscr{K}_n^k(0; \cdot, \cdot)$ in the terms on the right side of (6.25) have fewer cells in column μ_1, our induction is complete and the constant term identity Theorem 5 (**K**) is established.

For the expansion Theorem 8 (**G**), we proceed as with (**G**) by a double induction on the number μ_1 of columns of μ and on the number $\mathrm{cl}(\mu) = \mu'_{\mu_1}$ of cells (i, μ_1) in the last column of μ. We begin the induction by observing that for $\mu = (0)$, the expansion (**G**) reduces to the $a = k-1$ case of (**H**). That is

$$G^k(\lambda, (0)) = H^k(k-1; \lambda). \tag{6.26}$$

We let $l(\mu) \leqslant n$ and we may assume that $\mu_1 > 0$. We set $m = \mathrm{cl}(\mu) > 0$ and let $\theta(\mu)$ be the partition obtained by deleting the last column of μ.

Using n variables in the (**U**) property, we have

$$e_m(t_1, \ldots, t_n)s^k_{\theta(\mu)}(t_1, \ldots, t_n)$$
$$= \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\theta(\mu) + \chi_M/\theta(\mu))s^k_{\theta(\mu)+\chi_M}(t_1, \ldots, t_n). \tag{6.27}$$

Multiplying (6.27) by $\Pi_{i=1}^n (1 - t_i)^{k-1}$ and using the expansion (**G**), we have

$$e_m(t_1, \ldots, t_n)s^k_{\theta(\mu)}(t_1, \ldots, t_n) \prod_{i=1}^n (1 - t_i)^{k-1}$$
$$= \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\theta(\mu) + \chi_M/\theta(\mu))s^k_{\theta(\mu)+\chi_M}(t_1, \ldots, t_n) \prod_{i=1}^n (1 - t_i)^{k-1}$$
$$= \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\theta(\mu) + \chi_M/\theta(\mu)) \sum_{\substack{\lambda \\ l(\lambda) \leqslant n}} G^k(\lambda, \theta(\mu) + \chi_M)s^k_\lambda(t_1, \ldots, t_n). \tag{6.28}$$

Using the expansion (**G**) first gives

$$s^k_{\theta(\mu)}(t_1, \ldots, t_n) \prod_{i=1}^n (1 - t_i)^{k-1} = \sum_{\substack{\nu \\ l(\nu) \leqslant n}} G^k(\nu, \theta(\mu))s^k_\nu(t_1, \ldots, t_n). \tag{6.29}$$

Multiplying (6.29) by $e_m(t_1, \ldots, t_n$ and using n variables in the (U) property, we have

$$\begin{aligned} e_m(t_1, \ldots, t_n)s^k_{\theta(\mu)}(t_1, \ldots, t_n) \prod_{i=1}^{n} (1-t_i)^{k-1} \\ &= \sum_{\substack{\nu \\ l(\nu)\leqslant n}} G^k(\nu, \theta(\mu))e_m(t_1, \ldots, t_n)s^k_\nu(t_1, \ldots, t_n) \\ &= \sum_{\substack{\nu \\ l(\nu)\leqslant n}} G^k(\nu, \theta(\mu)) \sum_{\substack{M\subseteq[1,n] \\ |M|=m}} U^k(\nu+\chi_M/\nu)s^k_{\nu+\chi_M}(t_1, \ldots, t_n). \end{aligned}$$

We set

$$\lambda = \nu + \chi_M, \quad M \subseteq [1, n], \tag{6.31}$$

where we assume that $l(\nu) \leqslant n$. We have $l(\lambda) \leqslant n$ and we may equate coefficients in (6.28) and (6.30). We obtain

$$\begin{aligned} \sum_{\substack{M\subseteq[1,n] \\ |M|=m}} U^k(\theta(\mu)+\chi_M/\theta(\mu))G^k(\lambda, \theta(\mu)+\chi_M) \\ = \sum_{\substack{M\subseteq[1,n] \\ |M|=m}} U^k(\lambda/\lambda-\chi_M)G^k(\lambda-\chi_M, \theta(\mu)). \end{aligned} \tag{6.32}$$

The homogeneous rational function identity (3.14), which is equivalent to (3.6), is central to our proof in Section 3 of the product integration formula Theorem 2 (J). We now show that (6.32) is equivalent to (3.6).

Substituting (5.3) into (6.32) and multiplying by $n!g^k_n(\lambda)$ gives

$$\begin{aligned} \sum_{\substack{M\subseteq[1,n] \\ |M|=m}} U^k(\theta(\mu)+\chi_M/\theta(\mu))\mathscr{K}^k_n(0; \lambda, \theta(\mu)+\chi_M) \\ = \sum_{\substack{M\subseteq[1,n] \\ |M|=m}} U^k(\lambda/\lambda-\chi_M)\frac{g^k_n(\lambda)}{g^k_n(\lambda-\chi_M)}\mathscr{K}^k_n(0; \lambda-\chi_M, \theta(\mu)). \end{aligned} \tag{6.33}$$

Observe that substituting

$$U^k(\lambda/\lambda-\chi_M)\frac{g^k_n(\lambda)}{g^k_n(\lambda-\chi_M)} = U^k(\lambda+\chi_{[1,n]-M}/\lambda), \quad M \subseteq [1, n], \tag{6.34}$$

into (6.33) gives (6.13), which is equivalent to (3.6). We give an elegant argument due to Macdonald [Ma2, Chap. VI] which establishes (6.34) directly. By (6.1) and (1.4), we have

$$\begin{aligned}
&e_m(t_1,\ldots,t_n)s^k_{\lambda-\chi_M}(t_1,\ldots,t_n)s^k_\lambda\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)\\
&= t_1\cdots t_n e_{n-m}\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)s^k_{\lambda-\chi_M}(t_1,\ldots,t_n)s^k_\lambda\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)\\
&= s^k_{\lambda+\chi_{[1,n]-M}}(t_1,\ldots,t_n)e_{n-m}\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right)s^k_\lambda\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right). \qquad (6.35)
\end{aligned}$$

Expanding the extreme left and right sides of (6.35) by the (**U**) property, multiplying by the weight function $\Pi_{1\leqslant i<j\leqslant n}(1-t_i/t_j)^k(1-t_j/t_i)^k$ of the orthogonality (**O**′), and using (**O**′) to extract the constant term, we obtain

$$U^k(\lambda/\lambda-\chi_M)g^k_n(\lambda) = U^k(\lambda+\chi_{[1,n]-M}/\lambda)g^k_n(\lambda+\chi_{[1,n]-M}),\quad M\subseteq[1,n]. \qquad (6.36)$$

By (1.26), we have

$$g^k_n(\lambda+\chi_{[1,n]-M}) = g^k_n(\lambda-\chi_M),\quad M\subseteq[1,n]. \qquad (6.37)$$

Substituting (6.37) into (6.36) and rearranging gives (6.34). Thus (6.32) is equivalent to (3.6).

Solving (6.34) for $g^k_n(\lambda)$ gives

$$g^k_n(\lambda) = \frac{U^k(\lambda/\lambda-\chi_M)}{U^k(\lambda+\chi_{[1,n]-M}/\lambda)}\,g^k_n(\lambda-\chi_M),\quad M\subseteq[1,n]. \qquad (6.38)$$

Observe that

$$g^k_n(0) = CS^k_n(0,0;(0)). \qquad (6.39)$$

Macdonald [Ma4, Chap. VI] evaluates the normalization factor $g^k_n(\lambda)$ of the constant term orthogonality (**O**′). He proceeds by induction on λ starting with $\lambda=(0)$ (6.39) and uses the $|M|=1$ case of (6.38). Observe that we may evaluate $g^k_n(\lambda)$ by using any nonempty subset M and that the coefficient $U^k(\nu/\lambda)$ is constrained so that we obtain the same value for $g^k_n(\lambda)$ for all $M\neq\varnothing$. This constraint is also implicit in the fact that the (**U**) property overdetermines the Selberg-Jack symmetric functions $\{s^k_\lambda(t)\}$.

We now return to our alternative proof of the expansion Theorem 8 (**G**). We require that (6.32) holds when we use the formula Theorem 8 (1.30) for $G^k(\lambda,\mu)$. We may simply verify (5.3) and (6.34) formally or we may explicitly massage

(6.32) into (3.14) as was done for (3.6). The preceding analysis shows that this may be done using the substitution (6.24).

As in Section 3, we may use the Cauchy identity (**C**) to establish the homogeneous rational function identity (3.14). Thus (6.32) holds when we use the formula Theorem 8 (1.30) for $G^k(\lambda, \mu)$.

By (6.5) and (6.6), we see that the term $G^k(\lambda, \mu)$ occurs on the left side of (6.32) when $M = [1, m]$. Solving for this term, we obtain

$$G^k(\lambda, \mu) = \sum_{\substack{M \subseteq [1,n] \\ |M| = m}} U^k(\lambda/\lambda - \chi_M) G^k(\lambda - \chi_M, \theta(\mu)) - \sum_{\substack{M \subseteq [1,n] \\ |M| = m \\ M \neq [1,m]}} U^k(\theta(\mu) + \chi_M/\theta(\mu)) G^k(\lambda, \theta(\mu) + \chi_M). \tag{6.40}$$

Since all of the partitions which occur as a second argument of $G^k(\cdot, \cdot)$ in the terms on the right side of (6.40) have fewer cells in column μ_1, our induction is complete and the expansion Theorem 8 (**G**) is established.

Since the constant term identity Theorem 5 (**K**) and the product integration formula Theorem 2 (**J**) both reduce to the rational function identity (4.4), we see that they are equivalent. Since it follows from (**K**), let us assume that the constant term orthogonality (**O**′) is known including the value of the normalization factor $g_n^k(\lambda)$. The orthogonality (**O**′) mediates between the constant term identity (**K**) and the expansion (**G**). Thus (**J**), (**K**) and (**G**) are equivalent. It is not surprising that the argument used to prove (**J**) in Section 3 can also be used to prove (**K**) and (**G**) and that the identities (6.13) and (6.32) which arise for (**K**) and (**G**), respectively, are both equivalent to the identity (3.6) which arises for (**J**), which may be formulated (3.14) as a homogeneous rational function identity.

The product integration formula (**J**), the constant term identity (**K**), and the expansion (**G**) are equivalent formulations of the constant term orthogonality (**O**′). Since the rational function identity (3.14) is homogeneous, the orthogonality (**O**′) is in some sense independent of k. That is, the orthogonality (**O**′) is in some sense the same for all $k \geqslant 0$ as the orthogonality of the Schur functions. As is well-known (see [Ma1, Chap. I]), the Cauchy identity (**C**) is a formulation of the orthogonality of the Schur functions. Accordingly, we may use (**C**) to establish the homogeneous rational function identity (3.14) which is central to our proof of (**J**), (**K**) and (**G**).

References

[An2] G. E. Andrews: *q-Series: Their Development and Applications in Analysis, Number Theory, Combinatorics, Physics and Computer Algebra*, CBMS Regional Conference Series in Mathematics, number 66, American Mathematical Society, Providence, RI, 1986.

[Ao1] K. Aomoto: Jacobi polynomials associated with Selberg's integral, *SIAM J. Math. Analysis* 18 (1987), 545–549.

[Hab1] L. Habsieger: Une q-intégrale de Selberg-Askey, *SIAM J. Math. Analysis* 19 (1988), 1475–1489.

[Han1] P. Hanlon: Jack symmetric functions and some combinatorial properties of Young symmetrizers, *J. Combinatorial Theory* (A) 47 (1988), 37–80.

[Hu1] L. K. Hua: Harmonic analysis of functions of several complex variables in the complex domains. Translations of mathematical monographs, Vol. 6, *Amer. Math. Soc.*, Providence, RI, 1963.

[Ja1] H. Jack: A class of symmetric polynomials with a parameter, *Proc. Royal Soc. Edinburgh* (A) 69 (1969–70), 1–17.

[Ka1] K. W. J. Kadell: A proof of some q-analogues of Selberg's integral for $k = 1$, *SIAM J. Math. Analysis* 19 (1988), 944–968.

[Ka2] K. W. J. Kadell: A proof of Askey's conjectured q-analogue of Selberg's integral and a conjecture of Morris, *SIAM J. Math. Analysis* 19 (1988), 969–986.

[Ka3] K. W. J. Kadell: The q-Selberg polynomials for $n = 2$, *Trans. Amer. Math. Soc.* 310 (1988), 535–553.

[Ka4] K. W. J. Kadell: The Selberg-Jack symmetric functions, *Advances in Math.*, to appear.

[Ma1] I. G. Macdonald: *Symmetric Functions and Hall Polynomials*, Oxford University Press, Oxford, 1979.

[Ma2] I. G. Macdonald: Second edition of [Ma1], to appear.

[Ma3] I. G. Macdonald: Orthogonal polynomials associated with root systems, to appear.

[Ri1] D. St. P. Richards: *Analogs and Extensions of Selberg's integral*, Proceedings of the IMA workshop on q-series and partitions, March 1988.

[Se1] A. Selberg: Bemerkninger om et multipelt integral, *Norsk. Mat. Tidsskr.* 26 (1944), 71–78.

[St1] R. P. Stanley: Some combinatorial properties of Jack symmetric functions, *Advances in Math.* 77 (1989), 76–115.

[Sz1] G. Szegö: *Orthogonal Polynomials*, Colloquium Publications, Vol. 23, American Mathematical Society, Providence, RI, 1975.

[Ze1] D. Zeilberger: A proof of the G_2 case of Macdonald's root system-Dyson conjecture, *SIAM J. Math. Analysis* 18 (1987), 880–883.