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Constructions of automorphic forms and related cohomology classes for arithmetic subgroups of G_2

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1. Introduction

In the study of the cohomology of arithmetic groups and its relationship with the theory of automorphic forms it is of interest to have non-vanishing results for cohomology classes of various types. In constructing cohomology classes one uses analytic methods, e.g. liftings of automorphic forms, Eisenstein series, or geometric ones as e.g. associating cycles to closed subgroups of the underlying real Lie group. Some techniques and results are summarized in [44].

The object of this paper is to study some of these ideas in the case of an arithmetic subgroup Γ of a simple algebraic group of type G_2 defined over a totally real algebraic number field k of degree n. In particular, we construct automorphic forms of cohomological interest. Assume that G is split at the places $v_1, \ldots, v_m$, $m \leqslant n$ and compact at the remaining real places. Given any integer r with $0 \leqslant r \leqslant m$ and an n-tuple $(\Lambda_1, \ldots, \Lambda_n)$ of non-negative integers we show that there exists an irreducible automorphic representation $\pi = \otimes \pi_v$ of $G(\mathbb{A})$ occurring in the cuspidal spectrum of $L^2(G(k)\backslash G(\mathbb{A}))$ such that the representation π_{v_j} of $G(k_{v_j})$ is equivalent to a *discrete series* representation $A(\Lambda_j)$ for $1 \leqslant j \leqslant r$, to a given *non-tempered* irreducible unitary representation $B(\Lambda_j)$ for $r < j \leqslant m$ and to an irreducible finite dimensional representation $C(\Lambda_j)$ for $m < j \leqslant n$. The representations $A(\Lambda_j)$ resp. $B(\Lambda_j)$ have non-trivial relative Lie algebra cohomology with respect to a suitable coefficient system depending on Λ_j. The representations $B(\Lambda_j)$ cover one half of the list (up to equivalence) of non-tempered irreducible unitary representations of $G(k_{v_j})$ with non-trivial relative Lie algebra cohomology. As immediate consequences one obtains various results pertaining to the cohomology of arithmetic groups. We mention as one example in the case that G is split at all places v_j the following one: The cusp cohomology $H^*_{\text{cusp}}(X, \mathbb{C})$ of the arithmetic group (see Section 5 for definition) does not vanish in degree $4r + 3(n-r)$ resp. $4r + 5(n-r)$ for any integer $0 \leqslant r \leqslant n$. In terms of relative Lie algebra cohomology, these non-vanishing classes correspond to irreducible unitary representations $\pi = \otimes \pi_v$ of

$G(\mathbb{A})$ occurring in the cuspidal spectrum $L^2_0(G(k)\backslash G(\mathbb{A}))$ such that the representation π_{v_j} is equivalent to $A(0)$ for $j = 1, \dots, r$ resp. $B(0)$ for $j = r+1, \dots, n$.

The cuspidal representations π are constructed by means of the global theta lifting related to the dual reductive pair (H_Q, SL_2) where H_Q denotes a suitable orthogonal group containing G as a subgroup. Of course, this procedure is inspired by the work of Rallis and Schiffmann [38]. In doing so we need to construct specific automorphic forms on the metaplectic two-fold cover $\widetilde{SL_2(\mathbb{A})}$ of $SL_2(\mathbb{A})$.

In the following sections the theory of Eisenstein series will be used to construct by analytical means certain cohomology classes in $H^*(X, \mathbb{C})$ which are represented by a regular value of a suitable Eisenstein series or a residue of such. As one result we show for $k = \mathbb{Q}$ that the cuspidal cohomology classes, so constructed in $H^3(X, \mathbb{C})$, are shadows of residual Eisenstein cohomology classes. Here we mean that the latter ones correspond to irreducible unitary automorphic representations $\pi = \otimes \pi_v$ of $G(\mathbb{A})$ occurring in the residual spectrum $L^2_{\mathrm{res}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ such that the representation π_∞ is again equivalent to $B(0)$.

The paper is organized in the following way. In Section 2 we describe (up to equivalence) the non-tempered unitary representations with non-trivial relative Lie algebra cohomology of the split simple real Lie group G_2. This is done by making explicit the general parametrization of unitary representations with non-zero cohomology given in [52].

Via the local theta correspondence, a holomorphic representation of half integral weight m of the metaplectic two-fold cover $\widetilde{SL_2(\mathbb{R})}$ of $SL_2(\mathbb{R})$ corresponds to an irreducible unitary representation $\pi(m)$ of $O(4,3)$ resp. $SO_0(4,3)$. Viewing the real Lie group G_2 as a subgroup of $SO_0(4,3)$ it is the main purpose of Section 3 to prove that for $m \geqslant \frac{7}{2}$ the restriction of $\pi(m)$ to G_2 is equivalent to one of the non-tempered representations $A_{\mathfrak{q}}(\lambda)$ discussed in Section 2 with a suitable parameter λ and hence irreducible. Starting with an antiholomorphic representation of half integral weight m, $m \leqslant -\frac{7}{2}$, of $\widetilde{SL_2(\mathbb{R})}$ one obtains via the local theta correspondence a discrete series representation of G_2 in this way.

By use of the global theta lifting it is shown in Section 4 that for a given n-tuple Λ of parameters Λ_j, $j = 1, \dots, n$ there exists an irreducible representation $\pi = \otimes \pi_v$ of $G(\mathbb{A})$ occurring in the cuspidal spectrum of $L^2(G(k)\backslash G(\mathbb{A}))$ such that the representation π_{v_j} of $G(k_{v_j})$ is of the type as described above. The necessary construction of specific genuine cuspidal representations of $\widetilde{SL_2(\mathbb{A})}$ is dealt with as well. The consequences of this result for the cohomology of arithmetic subgroups of G alluded to above are discussed in Section 5.

In Section 6 we review the procedure for the construction of Eisenstein cohomology classes for arithmetic groups as discussed e.g. in [10], [42]. We make explicit in our case the cohomology of the strata at infinity, and we perform some calculations concerning the Eisenstein series we need. These results are used in Section 7 to describe, in the case $k = \mathbb{Q}$, the image of the restriction map

$$r_P: H^*(X, \mathbb{C}) \to H^*_{\text{cusp}}(X^P, \mathbb{C})$$

of the cohomology of X to the cusp cohomology of a stratum X^P corresponding to a maximal parabolic $\mathbb{Q}$-subgroup P of G. In particular, we exhibit regular Eisenstein cohomology classes, and it is discussed under which conditions residual Eisenstein cohomology classes exist. This condition is formulated in terms of a non-vanishing condition for certain partial L-series of GL_2. These automorphic L-functions naturally appear in the constant terms of the Eisenstein series under consideration.

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NOTATION AND CONVENTIONS. (1) Let $k/\mathbb{Q}$ be an arbitrary finite extension of the field $\mathbb{Q}$, and denote by $\mathcal{O}_k$ its ring of integers. The set of places of k will be denoted by V, and V_∞ (resp. V_f) refers to the set of archimedian (resp. non-archimedian) places of k. The completion of k at a place $v \in V$ is denoted by k_v; its ring of integers by $\mathcal{O}_v$ ($v \in V_f$). For a given place $v \in V$ the normalized absolute value $\| _v$ on k_v is defined as usual: If $v \in V_\infty$ is a real place we let $\| _v$ to be the absolute value; if $v \in V_\infty$ is a complex place we put $|x_v|_v = x_v \cdot \bar{x}_v$ and if $v \in V_f$ is a finite place we define $|x_v|_v = N_v^{-\operatorname{ord}_v(x_v)}$ where N_v denotes the cardinality of the (finite) residue field at the place v.

We denote by $\mathbb{A} = \mathbb{A}_k$ (resp. $\mathbb{I} = \mathbb{I}_k$) the ring of adeles (resp. the group of ideles) of k. There is the usual decomposition of $\mathbb{A}$ (resp. $\mathbb{I}$) into the infinite and the finite part $\mathbb{A} = \mathbb{A}_\infty \times \mathbb{A}_f$ (resp. $\mathbb{I} = \mathbb{I}_\infty \times \mathbb{I}_f$).

(2) The algebraic groups considered are linear. If H is an algebraic group defined over a field F, and F' is a commutative F-algebra, we denote by $H(F')$ the group of F'-valued points of H. When F' is a field we denote by H/F' the F'-algebraic group $H \times_F F'$ obtained from H by extending the ground field from F to F'. When F' is the field $\mathbb{R}$ of real numbers we occasionally write H_∞ instead of $H(\mathbb{R})$.

(3) With respect to the algebraic groups of type G_2 we refer the reader to [16] and [38], Section 1. There one finds a description of an explicit model for the split group of type G_2 and basic facts about roots and parabolic subgroups we need.

2. Unitary representations with non-zero cohomology

In this section we describe (up to equivalence) the non-tempered unitary representations with non-zero cohomology of the split simple real Lie group G_2. This is done by making explicit the general parametrization of unitary representations with non-zero cohomology given in [52].

2.1. Let G be the split simple real Lie group of type G_2; the group G_2 is connected. We write $\mathfrak{g}_o$ for the Lie algebra of G, and $\mathfrak{g} = (\mathfrak{g}_o)_{\mathbb{C}}$ for its complexification. Analogous notation is used for other groups. Let $K \subseteq G$ be a maximal compact subgroup, let $\theta = \theta_K$ be the Cartan involution, and let $\mathfrak{g}_o = \mathfrak{k}_o + \mathfrak{p}_o$ be the corresponding Cartan decomposition. It is well known that $\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. Fix non-zero elements ix, iy belonging to the first and second summand respectively, and let $i\mathfrak{t}_0$ be the real vector space spanned by ix, iy. Then $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ is a Cartan subalgebra. We have $\mathfrak{t}_0 \cong \mathbb{R} \oplus \mathbb{R} \cong \mathbb{R}^2$ resp. $\mathfrak{t} \cong \mathbb{C}^2$. If $\varepsilon_1, \varepsilon_2$ denotes the evaluation in the first and second coordinate respectively then the roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$ may be described by

$$\Delta(\mathfrak{g}, \mathfrak{t}) = \Delta(\mathfrak{k}, \mathfrak{t}) \cup \Delta(\mathfrak{p}, \mathfrak{t})$$

where $\Delta(\mathfrak{k}, \mathfrak{t}) = \{\pm 2\varepsilon_1, \pm 2\varepsilon_2\}$ resp. $\Delta(\mathfrak{p}, \mathfrak{t}) = \{\pm(\varepsilon_2 \pm \varepsilon_1), \ \pm(3\varepsilon_1 \pm \varepsilon_2)\}$. Taking $\alpha_1 = \varepsilon_2 - \varepsilon_1$ and $\alpha_2 = 3\varepsilon_1 - \varepsilon_2$ as simple roots we have

$$\Delta^+(\mathfrak{k}, \mathfrak{t}) = \{\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}$$

$$\Delta^+(\mathfrak{p}, \mathfrak{t}) = \{\alpha_1, \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$$

Given a fixed element $x = (a, b) \in \mathfrak{t}_o$ there is an associated θ-stable parabolic subalgebra $\mathfrak{q}$ (in the sense of [52], 2.2) of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, defined by $\mathfrak{q}$ = sum of non-negative eigenspaces of $ad(x)$, $\mathfrak{u}$ = sum of positive eigenspaces of $ad(x)$, and $\mathfrak{l}$ = sum of zero eigenspaces of $ad(x)$ = centralizer of x. Recall that the linear transformation $ad(x)$ of $\mathfrak{g}$ is diagonalizable, with real eigenvalues. Let L be the connected subgroup of G with Lie algebra $\mathfrak{L}_o = \mathfrak{l} \cap \mathfrak{g}_o$. Let λ be the differential of a unitary character of L such that $\langle \alpha, \lambda|_{\mathfrak{t}} \rangle \geqslant 0$ for each root α of $\mathfrak{u}$ with respect to $\mathfrak{t}$. Such a one-dimensional representation $\lambda: \mathfrak{l} \to \mathbb{C}$ is called admissible. Given the data $\mathfrak{q}$ and λ there is a unique irreducible unitary representation $A_{\mathfrak{q}}(\lambda)$ with non-zero cohomology (cf. [52], Thm. 5.3); its Harish–Chandra module is denoted by the same letter.

If $x = 0$, then we obtain the trivial representation. If $\mathfrak{l} \subseteq \mathfrak{k}$ (and in fact only then), $A_{\mathfrak{q}}(\lambda)$ is a discrete series representation ([52], p. 58). Observe that there are (up to equivalence) exactly three discrete series representations having the same infinitesimal character.

For $j = 1, 2$, let $x_j = (a, b) \in \mathfrak{t}_o$ be an element in $\mathfrak{t}_o$ with $\alpha_j(x_j) > 0$ and $\alpha_k(x_j) = 0$

for $k \neq j$. Such a choice determines a θ-stable parabolic subalgebra $\mathfrak{q}_j = \mathfrak{l}_j + \mathfrak{u}_j$. One sees that $L_j = U(1) \times SL_2(\mathbb{R}), j = 1, 2$. For example, $\mathfrak{u}_1$ is the sum of the root spaces corresponding to the roots $\alpha_1 + \alpha_2$, $3\alpha_1 + \alpha_2$, $3\alpha_1 + 2\alpha_2$, $2\alpha_1 + \alpha_2$, $-(2\alpha_1 + \alpha_2)$.

2.2. LEMMA. *Let $\mathfrak{q}_j = \mathfrak{l}_j + \mathfrak{u}_j$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ as constructed from a given x_j chosen as above, and let $\lambda: \mathfrak{l}_j \to \mathbb{C}$ be an admissible character.*

(1) *Then the unique irreducible unitary representation $A_{\mathfrak{q}_j}(\lambda)$ is non-tempered.*
(2) *The restriction of $A_{\mathfrak{q}_j}(\lambda)$ to $\mathfrak{k}$ contains the representation of K of highest weight $\lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ [where we use the notation $\rho(\mathfrak{u} \cap \mathfrak{p})$ as introduced in* [52], 2.3].
(3) *The center $Z(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$ acts by $\chi_{\lambda+\rho}$ on $A_{\mathfrak{q}_j}(\lambda)$ where $\chi_{\lambda+\rho}: Z(\mathfrak{g}) \to \mathbb{C}$ is given by composition of the Harish–Chandra homomorphism with evaluation at $\lambda + \rho$* ([52, 2.9).
(4) *Suppose F is a finite dimensional irreducible representation of $\mathfrak{g}$, of lowest weight $-\gamma \in \mathfrak{t}^*$. Then $H^*(\mathfrak{g}, K; A_{\mathfrak{q}_j}(\lambda) \otimes F) = 0$ unless $\gamma = \lambda|_{\mathfrak{t}}$, in which case*

$$H^i(\mathfrak{g}, K; A_{\mathfrak{q}_j}(\lambda) \otimes F) = \begin{cases} \mathbb{C} & i = 3,5 \\ 0 & i \neq 3,5 \end{cases}$$

Proof. (1) follows from the fact that $[\mathfrak{l}_i, \mathfrak{l}_i] \not\subseteq \mathfrak{k}$, $i = 1, 2$ and a criterion for temperedness (see [52], p. 58) and (2), (3) resp. (4) are Thm. 5.3 resp. 5.5 in [52].

2.3. REMARKS. (1) For the sake of completeness we note that a discrete series representations ω of G_2 has non-trivial $(\mathfrak{g}, K)$-cohomology with respect to a finite dimensional irreducible representation E if (and only if) the infinitesimal character of ω coincides with the one of the contragredient representation E^* of E. In such a case one has ([6], II, 5.3, 5.4)

$$H^i(\mathfrak{g}, K; \omega \otimes E) = \begin{cases} \mathbb{C} & q = (\tfrac{1}{2})(\dim G/K) = 4 \\ 0 & \text{otherwise} \end{cases}$$

(2) Together with the trivial representation, the three discrete series representations and the two non-tempered representations $A_{\mathfrak{q}_j}(0)$, $j = 1, 2$, exhaust all irreducible unitary representations (π, H_π) of G with non-trivial $(\mathfrak{g}, K)$-cohomology $H^*(\mathfrak{g}, K; H_\pi \otimes \mathbb{C})$ up to equivalence.

2.4. THE COMPACT REAL FORM. Let G be the compact real form of the simple real Lie group G_2. Let $i\mathfrak{t}_o$ be a Cartan subalgebra of Lie$(G) = \mathfrak{g}$. Then one can define the representations $A_{\mathfrak{q}}(\lambda)$ exactly as before. However, it turns out that the representation $A_{\mathfrak{q}}(\lambda)$ is nothing but the irreducible finite dimensional representation of G with extremal weight $\lambda|_{\mathfrak{t}}$.

3. Local lifting from $\widetilde{SL_2(\mathbb{R})}$

The split simple Lie group G_2 may be viewed as a subgroup of the connected component containing the identity $SO_o(4,3)$ of $SO(4,3)$. On the other hand, via the local theta correspondence a holomorphic representation of half integral weight k of the metaplectic two-fold cover $\widetilde{SL_2(\mathbb{R})}$ of $SL_2(\mathbb{R})$ corresponds to an irreducible unitary representation $\pi(k)$ of $O(4,3)$ resp. $SO_o(4,3)$. It is the main purpose of this section to prove that for $k \geqslant \frac{7}{2}$ the restriction $\pi(k)|_G$ to G is irreducible and equivalent to $A_{\mathfrak{q}}(\lambda)$ with λ defined by $\lambda|_{\mathfrak{t}} = (k-\frac{7}{2})(2\alpha_1+\alpha_2)$ and $\mathfrak{q}=\mathfrak{q}_1$ as in 2.2. Since the representation $A_{\mathfrak{q}}(\lambda)$ is uniquely determined by the action of the Casimir operator Ω of $\mathfrak{g}$ in its Harish–Chandra module and its lowest $\mathfrak{k}_o$-type, the proof reduces to determining these data for $\pi(k)|_G$. Starting with an antiholomorphic representation of half integral weight k, $k \leqslant -\frac{7}{2}$, of $\widetilde{SL_2(\mathbb{R})}$ one obtains via the local theta correspondence a discrete series representation of G_2 in this way.

3.1. As before G denotes the split real Lie group G of type G_2; it is connected and may be viewed as a subgroup of (the connected component of the identity of) $SO(4,3)$ (cf. 4.1). Consider $\mathfrak{g}$ as a subalgebra of the Lie algebra of $SO(4,3)$; thus its elements may be written as (7×7)-matrices. An invariant form $\langle\, ,\rangle$ on $\mathfrak{g}$ is defined by

$$\langle X,\ Y\rangle := \operatorname{trace}(XY) \quad X,\ Y \in \mathfrak{g} \tag{1}$$

If $(X_i)_i$ resp. $(Y_i)_i$ are dual bases of $\mathfrak{g}$ with respect to $\langle\, ,\rangle$ then the corresponding Casimir element is given by

$$\Omega = \sum X_i Y_i \tag{2}$$

it is a multiple of the Casimir element defined by the Killing form. The restriction of $\langle\, ,\rangle$ to $\mathfrak{t}$ is non degenerate. Thus we may identify $\mathfrak{t}^*$ with $\mathfrak{t}$ via $\langle\, ,\rangle$. Observe that we then have

$$\|\alpha_1\|^2 = \langle\alpha_1,\ \alpha_1\rangle = \tfrac{1}{3},\ \|\alpha_2\| = 1,\ \langle\alpha_1,\ \alpha_2\rangle = \tfrac{1}{2}\langle\alpha_1,\ \check{\alpha}_2\rangle = -\tfrac{1}{2} \tag{3}$$

The real Lie groups $O(4,3)$ and $SL_2(\mathbb{R})$ form a reductive dual pair inside $Sp_{14}(\mathbb{R})$. Let ω be the oscillator representation of the metaplectic two-fold cover of $Sp_{14}(\mathbb{R})$. Use the same letter for the restriction of ω to $\mathfrak{o}(4,3)$ and $\mathfrak{sl}_2(\mathbb{R})$. The Casimir element of $SL_2(\mathbb{R})$ defined by the Killing form is denoted by $\Omega_{SL_2(\mathbb{R})}$.

3.2. LEMMA. *There is the following relation between* Ω *as defined in* 3.1(2) *and* $\Omega_{SL_2(\mathbb{R})}$ *with respect to the oscillator representation* ω

$$\omega(\Omega) = \tfrac{8}{3}\omega(\Omega_{SL_2(\mathbb{R})}) - \tfrac{7}{4} \tag{1}$$

Proof. Denote by e_{ij} $(1 \leqslant i, j \leqslant 7)$ the matrix whose (i, j)-entry is equal to 1 and otherwise zero. Then the elements

$$X_{ij} := e_{ij} - e_{ji} \; (1 \leqslant i < j \leqslant 4 \text{ or } 5 \leqslant i < j \leqslant 7)$$
$$Z_{ij} := e_{ij} + e_{ji} \; (1 \leqslant i \leqslant 4,\ 5 \leqslant j \leqslant 7) \tag{2}$$

form a basis for the Lie algebra of $SO(4, 3)$. Every element of $\mathfrak{g}$ is a linear combination of these elements, and thus Ω may be written in terms of these. A slightly tedious but completely straightforward computation shows

$$\begin{aligned} \Omega = {} & \tfrac{10}{3}\Omega_{O(4,3)} + \tfrac{1}{3}(X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23}) \\ & - \tfrac{1}{3}\{X_{67}(X_{12} + X_{34}) + X_{57}(X_{24} - X_{13}) + X_{56}(X_{14} + X_{23})\} \\ & + \tfrac{1}{3}\{Z_{46} \cdot Z_{37} + Z_{45} \cdot Z_{27} - Z_{25}Z_{47} + Z_{35}Z_{26} \\ & - Z_{15}(Z_{46} - Z_{37}) + Z_{16}(Z_{45} - Z_{27}) \\ & - Z_{17}(Z_{35} - Z_{26}) - Z_{36}(Z_{25} + Z_{47})\} \end{aligned} \tag{3}$$

where $\Omega_{O(4,3)}$ denotes the Casimir element of $O(4, 3)$ defined by the Killing form. Now we use the realization of the oscillator representation ω in the Schrödinger model (cf. [39]), i.e. ω acts on the space of Schwartz–Bruhat functions on the matrix space $\mathbb{R}^7$. Let x_k $(k = 1, \ldots, 7)$ be the coordinates in the canonical basis of $\mathbb{R}^7$. Then the action of the basis elements in (2) is given as follows:

$$\omega(X_{ij}) = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \tag{4}$$

$$\omega(Z_{ij}) = x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i} \tag{5}$$

These formulas imply the identities

$$\omega(X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23}) = 0 \tag{6}$$

$$\omega(X_{ij}X_{kl} + Z_{ki}Z_{lj} - Z_{li}Z_{kj}) = 0 \; (5 \leqslant i < j \leqslant 7,\ 1 \leqslant k < l \leqslant 4) \tag{7}$$

Thus formula (3) may be written

$$\omega(\Omega) = \tfrac{10}{3}\omega(\Omega_{O(4,3)}) \tag{8}$$

But from Lemma 3.10 in [1] (cf. also [14]) one knows

$$\omega(\Omega_{O(4,3)}) = \tfrac{4}{5}\omega(\Omega_{SL_2(\mathbb{R})}) - \tfrac{21}{40}, \tag{9}$$

hence our assertion holds.

3.3. Let σ be a (genuine) holomorphic discrete series representation of $\widetilde{SL_2(\mathbb{R})}$ of half-integral weight k (see e.g. [8]). The Casimir operator $\Omega_{SL_2(\mathbb{R})}$ (defined with respect to the Killing form) acts on the space of σ by $\frac{1}{8}k(k-2)$. Via the local theta correspondence σ corresponds to an irreducible unitary representation $\pi(k)$ of $O(4,3)$ (cf. [36] resp. [34, 35]). It is easy to see that the restriction of $\pi(k)$ to $SO_o(4,3)$ is again irreducible (cf. [25]). We have the following

3.4. PROPOSITION. *Let $\pi(k)$ denote the irreducible unitary representation of $O(4,3)$ corresponding via the local theta correspondence to a genuine holomorphic discrete series representation of $\widetilde{SL_2(\mathbb{R})}$ of half integral weight $k \geqslant \frac{7}{2}$. Then the restriction $\pi(k)|_G$ of $\pi(k)$ to G is irreducible and equivalent to $A_{\mathfrak{q}}(\lambda_k)$ with λ_k defined by*

$$\lambda_{k|\mathfrak{t}} = (k-\tfrac{7}{2})(2\alpha_1+\alpha_2).$$

Proof. Using 3.2 we see that $\omega(\Omega)$ acts on the space of $\pi(k)|_G$ by the scalar

$$\tfrac{1}{3}k(k-2)-\tfrac{7}{4} \tag{1}$$

On the other hand we have $\lambda|_{\mathfrak{t}} = \Lambda(2\alpha_1+\alpha_2)$, $\Lambda = k-\frac{7}{4}$ and $\rho = 5\alpha_1+3\alpha_2$, hence Ω acts on the Harish–Chandra module of $A_{\mathfrak{q}}(\lambda)$ by

$$\langle \lambda|_{\mathfrak{t}}+\rho,\ \lambda|_{\mathfrak{t}}+\rho\rangle - \langle \rho,\rho\rangle = \|(2\Lambda+5)\alpha_1+(\Lambda+3)\alpha_2\|^2 - \|\rho\|^2 \tag{2}$$

Using formulas 3.1(3) we obtain

$$\tfrac{1}{3}\Lambda(\Lambda+5) = \tfrac{1}{3}(k-\tfrac{7}{2})(k+\tfrac{3}{2}) = \tfrac{1}{3}k(k-2)-\tfrac{7}{4}$$

which agrees with the scalar given in (1).

Now we determine the $\mathfrak{k}_o$-types occurring in $\pi(k)|_G$. The maximal compact subgroup of $SO_o(4,3)$ is $SO(4)\times SO(3)$. We fix a maximal torus $i\mathfrak{t}'_o$ in $\mathfrak{so}(4)\times\mathfrak{so}(3)$. In the usual coordinates the compact roots of $\mathfrak{t}'$ in $\mathfrak{so}(4,3)$ are $\pm e_1 \pm e_2$, $\pm e_3$. A representation of $\mathfrak{so}(4)\times\mathfrak{so}(3)$ is therefore parametrized by a weight

$$\delta = \sum_{i=1}^{3} \delta_i e_i \ (i = 1, 2, 3) \quad \text{with } \delta_1 \geqslant |\delta_2|,\ \delta_3 > 0.$$

Let $m(\delta)$ be the multiplicity with which the representation corresponding to δ occurs in $\pi(k)_{|\mathfrak{so}(4)\times\mathfrak{so}(3)}$. By Theorem 5 in [36] we know that

$$m(\delta) = \begin{cases} 1, & \text{if } \delta_2 = 0,\ \tfrac{1}{2}(\delta_1-\delta_3-\Lambda-3)\in\mathbb{Z}^+ \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

with $\Lambda = k - \frac{7}{2}$.

Now we recall that $\mathfrak{k}_o \cong \mathfrak{sp}(1) \times \mathfrak{sp}(1)$. The representations of $\mathfrak{k}_o$ are therefore parametrized by weights $\mu\varepsilon_1 + \nu\varepsilon_2$ with non-negative integers μ, ν. The multiplicity of the corresponding representation in $\pi(k)|_{\mathfrak{k}_o}$ is denoted by $m(\mu, \nu)$. Since $\mathfrak{k}_o$ is naturally embedded into $\mathfrak{so}(4) \times \mathfrak{so}(3)$ we have

$$m(\mu, \nu) = \sum_{\delta} m(\delta) m(\mu, \nu; \delta) \tag{4}$$

where the sum ranges over all representations π_δ of $\mathfrak{so}(4) \times \mathfrak{so}(3)$ occurring in $\pi(k)|_{\mathfrak{so}(4) \times \mathfrak{so}(3)}$, and $m(\mu, \nu; \delta)$ denotes the multiplicity of the representation corresponding to $\mu\varepsilon_1 + \nu\varepsilon_2$ in $\pi_{\delta|\mathfrak{k}_o}$. Thus we have to determine $m(\mu, \nu; \delta)$.

There are standard isomorphisms $r: \mathfrak{sp}(1) \times \mathfrak{sp}(1) \xrightarrow{\sim} \mathfrak{so}(4)$ and $s: \mathfrak{sp}(1) \xrightarrow{\sim} \mathfrak{so}(3)$. Putting these together we obtain

$$\tau = r \times s : \mathfrak{sp}(1) \times \mathfrak{sp}(1) \times \mathfrak{sp}(1) \xrightarrow{\sim} \mathfrak{so}(4) \times \mathfrak{so}(3)$$

The map $(x, y) \mapsto \tau(x, y, x)$ gives the embedding of $\mathfrak{k}_o = \mathfrak{sp}(1) \times \mathfrak{sp}(1)$ into $\mathfrak{so}(4) \times \mathfrak{so}(3)$. The pull back of a given representation π_δ of $\mathfrak{so}(4) \times \mathfrak{so}(3)$ with $\delta = \delta_1 e_1 + \delta_3 e_3$ to $\mathfrak{sp}(1) \times \mathfrak{sp}(1) \times \mathfrak{sp}(1)$ has highest weight $(\delta_1, \delta_1, 2\delta_3)$. Thus one has $m(\mu, \nu; \delta) = 0$ unless $\nu = \delta_1$. In this case $m(\mu, \nu; \delta)$ is equal to the multiplicity with which the representation corresponding to μ occurs in the inner tensor product of the representations corresponding to ν resp. $2\delta_3$. But this is easily computed using Steinberg's formula (cf. e.g. [15], 24.4). We find $m(\mu, \nu; \delta) = 0$ unless $\nu = \delta_1$ and $\mu + \nu$ is even in which case we have

$$m(\mu, \nu; \delta) = \begin{cases} 1, & \text{if } \frac{1}{2}|\mu - \nu| \leqslant \delta_3 \leqslant \frac{1}{2}|\mu + \nu| \text{ and } \nu - \delta_3 - \Lambda - 3 \in 2\mathbb{Z}^+ \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Combining this with (3) and (4) we obtain ($\mu + \nu$ even)

$$m(\mu, \nu) = \#\{\delta_3 | \tfrac{1}{2}|\mu - \nu| \leqslant \delta_3 \leqslant \tfrac{1}{2}(\mu + \nu) \text{ and } \nu - \delta_3 - \Lambda - 3 \in 2\mathbb{Z}^+\} \tag{6}$$

In particular we find that the representation of $\mathfrak{k}_o$ with highest weight $(\Lambda + 3)(\varepsilon_1 + \varepsilon_2)$ occurs with multiplicity one. Thus we have that $A_{\mathfrak{q}}(\lambda)$ is contained in $\pi(k)|_G$. On the other hand, Theorem 6.17 in [52] gives by use of Vogan's generalized Blattner formula the multiplicity with which a representation of $\mathfrak{k}_o$ of highest weight $\mu\varepsilon_1 + \nu\varepsilon_2$ occurs in $A_{\mathfrak{q}}(\lambda)$. It is a straightforward computation to show that the expression given there coincides with the one of (6) for $\mu + \nu$ even and is zero otherwise. This implies that $\pi(k)|_G$ is equivalent to $A_{\mathfrak{q}}(\lambda)$; in particular the former one is irreducible.

3.5. Write $W = W(\mathfrak{g}, \mathfrak{t})$ for the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Recall the

Harish–Chandra isomorphism $\chi\colon Z(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{t})^W$. If $\mu \in t^*$, then composition of χ with evaluation at μ gives $\chi_\mu\colon Z(\mathfrak{g}) \to \mathbb{C}$ with $\chi_\mu = \chi_\lambda$ if and only if $\mu \in W \cdot \lambda$. We have seen that a holomorphic representation of $\widetilde{SL_2(\mathbb{R})}$ of lowest weight k, $k \geqslant \frac{7}{2}$ which has infinitesimal character $\lambda' = k-1$, corresponds to the representation $A_{\mathfrak{q}}(\lambda)$ which has infinitesimal character χ_γ with $\gamma = \lambda|_{\mathfrak{t}} + \rho = 2\lambda'\alpha_1 + (\lambda' + \frac{1}{2})\alpha_2$. More generally, the following holds

PROPOSITION. *Let σ be an irreducible admissible genuine representation of $\widetilde{SL_2(\mathbb{R})}$. Then σ corresponds (via the theta correspondence to $SO(4,3)$ followed by restriction) to a representation of G with infinitesimal character χ_γ with $\gamma = 2\lambda'\alpha_1 + (\lambda' + \frac{1}{2})\alpha_2$ where λ' denotes the infinitesimal character of σ.*

Proof. Let $\mathfrak{g}'$ be the complexified Lie algebra of $\widetilde{SL_2(\mathbb{R})}$. There exists a homomorphism $\psi\colon Z(\mathfrak{g}) \to Z(\mathfrak{g}')$ such that $\omega(\psi(z)) = \omega(z)$ for $z \in Z(\mathfrak{g})$ (see Theorem 3, Section 4 in [38]). This gives rise to the correspondence of infinitesimal characters $\psi^*\colon (\mathfrak{t}')^*/W' \to \mathfrak{t}^*/W$ where $\mathfrak{t}'$ is a Cartan subalgebra of $\mathfrak{g}'$ and $W' = W(\mathfrak{g}', \mathfrak{t}')$ denotes the Weyl group of $\mathfrak{g}'$ with respect to $\mathfrak{t}'$. In the above we have already seen the effect of this correspondence on the set of infinitesimal characters of discrete series of weight $k \geqslant \frac{7}{2}$ which is a Zariski dense subset of $(\mathfrak{t}')^*/W'$. Thus, ψ^* is as given in the claim.

3.6. Now suppose $0 < k < \frac{7}{2}$ for the lowest weight k of a given irreducible admissible genuine representation of $\widetilde{SL_2(\mathbb{R})}$. If we replace $A_{\mathfrak{q}}(\lambda)$ in 3.4 by the more general kind of derived functor module $\mathscr{R}^s_{\mathfrak{q}}(\lambda)$ (see [51], Section 6.3), then the proof can be carried through word for word.

For example, we consider the case of a holomorphic representation of $\widetilde{SL_2(\mathbb{R})}$ of lowest weight 1/2; this is not a discrete series representation. However, via the local theta correspondence it corresponds to an irreducible unitary representation $\pi(\frac{1}{2})$ of $O(4,3)$ (see [25]). We claim that $\pi(\frac{1}{2})|_G \cong \mathscr{R}^s_{\mathfrak{q}}(\lambda)$ is spherical. Let H_σ denote the representation space of σ. By [25], Section 5 it suffices to exhibit a Schwartz–Bruhat function φ in $\mathscr{S}(\mathbb{R}^7)$ invariant under $O(4) \times O(3)$ and an element $v \in H_\sigma$ such that

$$\int_{\widetilde{SL_2(\mathbb{R})}} (\omega(x)\varphi, \varphi)\overline{(\sigma(x)v, v)}\,\mathrm{d}x \neq 0 \tag{1}$$

We may take $\varphi = \varphi_1 \otimes \varphi_2$ where $\varphi_1 \in \mathscr{S}(\mathbb{R}^4)$ is invariant under $O(4)$ and $\varphi_2 \in S(\mathbb{R}^3)$ is invariant under $O(3)$. The trivial representation of $O(4)$ corresponds (via theta lifting) to the holomorphic representation σ_2 of $SL_2(\mathbb{R})$ of lowest weight 2, while the trivial representation of $O(3)$ (considered to be defined by a negative definite quadratic form in 3 variables) corresponds to the anti-holomorphic representation $\sigma_{-3/2}$ of highest weight 3/2. We have $(\omega(x)\varphi, \varphi) = (\sigma_2(x)\varphi_1, \varphi_1)(\sigma_{-3/2}(x)\varphi_2, \varphi_2)$ in an obvious sense. Now the contragradient representation of $\sigma_{-3/2}$ is the holomorphic representation $\sigma_{3/2}$ of lowest weight 3/2. Thus we have

$$\overline{(\sigma_{-3/2}(x)\varphi_2, \varphi_2)} = (\sigma_{3/2}(x)\bar{\varphi}_2, \bar{\varphi}_2).$$

Now $(\sigma_{3/2}(x)\bar{\varphi}_2, \bar{\varphi}_2)$ $(\sigma(x)v, v)$ is a matrix coefficient of the tensor product $\sigma_{3/2} \otimes \sigma$ which contains σ_2 with multiplicity one. If we choose v to be the lowest weight vector (of weight 1/2) then $(\sigma_{3/2}(x)\bar{\varphi}_2, \bar{\varphi}_2)(\sigma(x)v, v)$ is a constant multiple of $(\sigma_2(x)\varphi_1, \varphi_1)$. For these choices of φ and v we have that the integral (1) does not vanish. This proves our claim. The proposition in 3.5 shows that it has infinitesimal character χ_γ with $\gamma = -\alpha_1$ which lies in the W-orbit of α_1. We thus obtain

PROPOSITION. *Let $\pi(\frac{1}{2})$ denote the irreducible unitary representation of $O(4,3)$ corresponding via the local theta correspondence to the holomorphic irreducible genuine representation of $\widetilde{SL_2\mathbb{R})}$ of lowest weight $\frac{1}{2}$. Then the restriction $\pi(\frac{1}{2})|_G$ contains the unique irreducible spherical subquotient of* $\mathrm{Ind}_B^G\eta$ *where B denotes a Borel subgroup of G and $\eta = \alpha_1$. Here normalized induction is used.*

REMARKS. (1) One can show that $\pi(\frac{1}{2})|_G$ is in fact irreducible. However, the proof of this fact would take us a little too far away.

(2) One should compare this with the result of Rallis–Schiffmann [37], where they find the same parameter α_1 in the p-adic case.

3.7. THE COMPACT REAL FORM. Let G be the compact real form of the simple real Lie group G_2. Following similar arguments as given in the split case (but in substance easier) one has the following

PROPOSITION. *Let σ be an irreducible admissible genuine representation of $\widetilde{SL_2(\mathbb{R})}$ of half integral weight $k \geqslant \frac{7}{2}$. Then σ corresponds to the finite dimensional irreducible representation $A_{\mathfrak{q}}(\lambda)$ of G with extremal weight $\lambda|_{\mathfrak{t}} = \Lambda(2\alpha_1 + \alpha_2)$, $\Lambda = k - \frac{7}{2}$ (see* 2.4 *for the notation).*

REMARK. Note that in this case for $k < \frac{7}{2}$ the local lifting from $\widetilde{SL_2(\mathbb{R})}$ to $O(7)$ is identically zero (cf. [19]).

3.8. DISCRETE SERIES REPRESENTATIONS. Now we consider a (genuine) antiholomorphic discrete series representation σ of $\widetilde{SL_2(\mathbb{R})}$ of highest weight $k \leqslant -\frac{7}{2}$. Via the local theta correspondence σ corresponds to an irreducible unitary representation $\pi(k)$ of $O(4,3)$. It will turn out that the restriction $\pi(k)|_G$ of $\pi(k)$ to G is irreducible and equivalent to a discrete series representation of G. It will be convenient to realize $\pi(k)|_G$ as some $A_{\mathfrak{q}}(\lambda)$.

Let $\mathfrak{q}$ be the θ-stable parabolic subalgebra of $\mathfrak{g}$ defined by the element $x = (1, 0)$ in $\mathfrak{t}_0$; we have the Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ as in 2.1. The corresponding Levi component L is of the form $L = U(1) \times Sp(1)$ where the semi simple part $Sp(1)$ is generated by the long compact root. Given the data $\mathfrak{q}$ and an admissible one-dimensional representation $\lambda : \mathfrak{l} \to \mathbb{C}$ the irreducible representation $A_{\mathfrak{q}}(\lambda)$ belongs to the discrete series since L is compact. Observe that such a λ is determined by its restriction to $\mathfrak{t}$.

PROPOSITION. *Let $\pi(k)$ denote the irreducible unitary representation of $O(4,3)$ corresponding via the local theta correspondence to a genuine antiholomorphic discrete series representation of $\widetilde{SL_2(\mathbb{R})}$ of highest weight $k \leqslant -\frac{7}{2}$. Then the restriction $\pi(k)|_G$ of $\pi(k)$ to G is irreducible and equivalent to $A_{\mathfrak{q}}(\lambda_k)$ with λ_k defined by*

$$\lambda_{k|\mathfrak{t}} = -(2k+7)\varepsilon_1 = -(k+\tfrac{7}{2})(\alpha_1+\alpha_2)$$

The proof follows that of Proposition 3.4 almost word for word so we omit it.

3.9. Summarizing the content of 3.4, 3.8 resp. 3.7 we introduce the following notations pertaining to the irreducible (unitary) representations of the split real Lie group G of the type G_2 resp. the compact real form:

(a) As before $\pi(k)$ denotes the irreducible unitary representation of $O(4,3)$ corresponding via the local theta correspondence to a genuine *antiholomorphic* discrete series representation of $\widetilde{SL_2(\mathbb{R})}$ of highest weight $k \leqslant -\frac{7}{2}$. Then the restriction of $\pi(k)$ to G will be denoted by

$$A(-(k+\tfrac{7}{2})) := \pi(k)|_G \cong A_{\mathfrak{q}}(\lambda_k)$$

with

$$\lambda_{k|\mathfrak{t}} = -(k+\tfrac{7}{2})(\alpha_1+\alpha_2).$$

Recall that this is a discrete series representation.

(b) Let $\pi(k)$ denote the irreducible unitary representation of $O(4,3)$ corresponding via the local theta correspondence to a genuine *holomorphic* discrete series representation of $\widetilde{SL_2(\mathbb{R})}$ of lowest weight $k \geqslant \frac{7}{2}$. Then the restriction of $\pi(k)$ to G will be denoted by

$$B(k-\tfrac{7}{2}) := \pi(k)|_G \cong A_{\mathfrak{q}}(\lambda_k)$$

with

$$\lambda_{k|\mathfrak{t}} = (k-\tfrac{7}{2})(2\alpha_1+\alpha_2)$$

(c) In the case of the compact real form the irreducible representation $A_{\mathfrak{q}}(\lambda_k)$ of G with extremal weight $\lambda_{k|\mathfrak{t}} = (k-\frac{7}{2})(2\alpha_1+\alpha_2)$, $k \geqslant \frac{7}{2}$ introduced in 3.7 will be denoted by $C(k-\frac{7}{2})$.

4. Cusp forms via global theta-lifting

Let G be a simple algebraic group of type G_2 defined over a totally real algebraic number field k of degree $n=[k:\mathbb{Q}]$. Assume that G is split at the places $v_1,\dots,v_m$, $m\leqslant n$, and compact at the remaining real places. Given any integer r with $0\leqslant r\leqslant m$ and an n-tupel $(\Lambda_1,\dots,\Lambda_n)$ of non-negative integers Λ_i we show in this section that there exists an irreducible automorphic representation $\pi=\otimes\pi_v$ occurring discretely in $L^2(G(k)\backslash G(\mathbb{A}))$ such that the representation π_{v_j} of $G(k_{v_j})$ is equivalent to $A(\Lambda_j)$ for $1\leqslant j\leqslant r$, to $B(\Lambda_j)$ for $r<j\leqslant m$ and to $C(\Lambda_j)$ for $m<j\leqslant m$.

4.1. THE SPLIT SIMPLE k-GROUP OF TYPE G_2. Given an algebraic number field k there is a uniquely determined Cayley algebra over k with divisors of zero; it is called the split Cayley algebra C over k ([41], Lemma 3.16). Its norm form is non degenerate. Let G be the group of automorphisms of C; its Lie algebra, by definition, the derivation algebra $D(C)$ of C is a 14-dimensional central simple Lie algebra of type G_2. The span C_o of elements of trace zero coincides with the space spanned by the commutators $[x,y]=xy-yx$, $x,y\in C$. Hence one has $C_oD\subset C_o$ for a derivation D, and thus C_o is a 7-dimensional subspace of C which is invariant under $D(C)$. The representation in C_o is faithful and irreducible. In turn, if $g\in G$ then $g(1)=1$ and g leaves the norm form of C invariant, and hence, as an endomorphism of C_o it may be viewed as an element of the orthogonal group H_Q where Q denotes the restriction of the norm form to C_o.

4.2. GLOBAL THETA-LIFTING. Now we assume that k is a totally real algebraic number field of degree $n=[k:\mathbb{Q}]$. Let $\widetilde{SL_2(\mathbb{A})}$ be the adelic metaplectic two-fold covering of $SL_2(\mathbb{A})$. Recall that we may consider $SL_2(k)$ as a subgroup of $\widetilde{SL_2(\mathbb{A})}$. The group $\widetilde{SL_2(k)}$ is a direct product of $SL_2(k)$ and a group isomorphic to $\mathbb{Z}_2$. As usual automorphic forms on $\widetilde{SL_2(\mathbb{A})}$ are defined with respect to $SL_2(k)$ (see e.g. [8]).

Now we consider the reductive dual pair (H_Q, SL_2). The choice of a non-trivial additive character ψ of $\mathbb{A}/k$ determines a global oscillator representation ω_ψ of $\widetilde{SL_2(\mathbb{A})}\times H_Q(\mathbb{A})$ in the space $\mathscr{S}(C_o(\mathbb{A}))$ of Schwartz–Bruhat functions on $C_o(\mathbb{A})$. We may write $\psi=\otimes_{v\in V}\psi_v$, and we choose ψ so that for $v\in V_\infty$

$$\psi_v(x)=e^{a_v\sqrt{-1}x}\quad(x\in\mathbb{R}) \tag{1}$$

with $a_v>0$ holds. Given $\varphi\in\mathscr{S}(C_o(\mathbb{A}))$ we put for $g\in\widetilde{SL_2(\mathbb{A})}$ and $h\in H_Q(\mathbb{A})$

$$\Theta_\varphi(g,h)=\sum_{\xi\in C_o(k)}(\omega_\psi(h,g)\varphi)(\xi) \tag{2}$$

This series converges, defines a smooth function on $\widetilde{SL_2(\mathbb{A})} \times H_Q(\mathbb{A})$ which is left invariant under $SL_2(k) \times H_Q(k)$ and slowly increasing.

Let $\sigma = \otimes\sigma_v$ be an irreducible automorphic genuine cuspidal representation of $\widetilde{SL_2(\mathbb{A})}$ occurring discretely in $L^2(SL_2(k)\backslash\widetilde{SL_2(\mathbb{A})})$. Let $v_1, \ldots, v_n \in V_\infty$ be the archimedean places of k. Choose any integer r with $0 \leqslant r \leqslant n$. We assume that for $1 \leqslant j \leqslant r$ σ_{v_j} is an antiholomorphic discrete series representation of $\widetilde{SL_2(k_{v_j})}$ of half-integral highest weight $k_j = -(\Lambda_j + \frac{7}{2}) \leqslant -\frac{7}{2}$, and that for $r < j \leqslant n$ σ_{v_j} is a holomorphic discrete series representation of $\widetilde{SL_2(k_{v_j})}$ of half-integral lowest weight $k_j = \Lambda_j + \frac{7}{2} \geqslant \frac{7}{2}$. It will be shown in Proposition 4.4 that such cuspidal representations σ always exist. Let V_σ^∞ be the subspace of smooth functions in the space V_σ of σ. Since an element f in V_σ^∞ is rapidly decreasing the following integral makes sense

$$\Theta_\varphi^f(h) = \int_{SL_2(k)\backslash\widetilde{SL_2(\mathbb{A})}} \Theta_\varphi(g, h) f(g)\,\mathrm{d}g, \quad h \in H_Q(\mathbb{A}). \tag{3}$$

Let $\Theta(\sigma, \psi)$ be the space of all functions Θ_φ^f with $f \in V_\sigma^\infty$ and $\varphi \in S(C_o(\mathbb{A}))$. By [27], Section 5 we know that $\Theta(\sigma, \psi)$ is non zero, affords an irreducible unitary representation of $H_Q(\mathbb{A})$ and occurs discretely in $L^2(H_Q(k)\backslash H_Q(\mathbb{A}))$. Furthermore, if $\Theta(\sigma_v, \psi_v)$ denotes the local theta lifting of σ_v to $H_Q(k_v)$ then one has

$$\Theta(\sigma, \psi) = \bigotimes_{v \in V} \Theta(\sigma_v, \psi_v) \tag{4}$$

Due to our assumption on σ_{v_j} we have $\Theta(\sigma_{v_j}, \psi_{v_j}) = \pi(k_j)$, $v_j \in V_\infty$, in the notation of Section 3. Let $V(\sigma, \psi)$ be the space of functions on $G(k)\backslash G(\mathbb{A})$ obtained by restricting smooth functions in the space of $\Theta(\sigma, \psi)$ to $G(\mathbb{A})$. This defines a non-zero $G(\mathbb{A})$-invariant subspace of the space of smooth functions on $G(k)\backslash G(\mathbb{A})$. By Theorem 2, Section 2 in [38] $V(\sigma, \psi)$ is a space of smooth cusp forms if for each function f in V_σ^∞ its Fourier coefficient with respect to ψ given as

$$f_\psi(g) = \int_{k\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\psi(x)}\,\mathrm{d}x \quad (g \in \widetilde{SL_2(\mathbb{A})}) \tag{5}$$

is identically zero.

We are going to construct genuine cuspidal automorphic representations σ of $\widetilde{SL_2(\mathbb{A})}$ satisfying this condition.

4.3. SOME GENUINE CUSPIDAL REPRESENTATIONS OF $\widetilde{SL_2(\mathbb{A})}$. Let q be a quadratic form on k^3, suppose that q is negative definite at the archimedean places v_j, $1 \leqslant j \leqslant r$ and positive definite at the places v_j, $r < j \leqslant n$. Let H/k denote the isometry group of q. Then (H, SL_2) forms a reductive dual pair inside Sp_6 ([13]). Let ω_q be the corresponding oscillator representation of $\widetilde{SL_2(\mathbb{A})} \times H(\mathbb{A})$ (with respect to the given choice of ψ) in the space $\mathscr{S}(\mathbb{A}^3)$ of

Schwartz–Bruhat functions on $\mathbb{A}^3$. For each $\varphi \in S(\mathbb{A}^3)$ we put

$$\Theta_\varphi(g) = \sum_{\xi \in k^3} \omega_q(g)\varphi(\xi), \quad g \in \widetilde{SL_2(\mathbb{A})} \tag{1}$$

This series converges, defines a smooth function on $\widetilde{SL_2(\mathbb{A})}$ which is left invariant under $SL_2(k)$ and slowly increasing.

In order to choose q so that $q(\xi) \neq 1$ for all $\xi \in k^3$ we proceed as follows. Fix a finite place v_o of k. Let D be totally positive quaternion algebra over k which is ramified at v_o. Let D^o be the space of pure quaternions; we have $D = ke_o \oplus D^o$ where e_o is the neutral element. The norm form of D may be written as $a^2 + p(b)$ with $a \in k$, and p a quadratic form on $D^o \cong k^3$, $b \in k^3$. Choose an element $x \in k$ which is negative at $v_1, \ldots, v_r$, positive at $v_{r+1}, \ldots, v_n$ and such that $|x+1|_{v_o} < 1$. Then we have $-x = 1-(x+1) \in (k_{v_o}^*)^2$. Actually, this is only true if v_o is a place over an odd prime or if $|x+1|_{v_o}$ is sufficiently smaller than 1; but this is easily achieved. Putting $q := xp$ as a quadratic form on k^3 one now sees that $q(\xi) \neq 1$ for all $\xi \in k^3$. Indeed, if $q(\xi) = 1$ for $\xi \in k^3$, then we have

$$0 = 1 - q(\xi) = 1^2 + p(s\xi) \tag{2}$$

where $s^2 = -x$, $s \in k_{v_o}$. This contradicts our assumption that D remains a division algebra at v_o.

Fix a finite place v_o of k. Finally, we choose a function $\varphi = \otimes \varphi_v$ in $\mathscr{S}(\mathbb{A}^3)$ so that

> φ_{v_j}, $v_j \in V_\infty$, transforms according to the irreducible finite dimensional representation of $SO(3) \subseteq H(k_{v_j})$ of highest weight $\Lambda_j + 2$ with Λ_j a non-negative integer. (3)

> the function φ_{v_o} transforms according to a non-trivial irreducible finite dimensional representation of $H(k_{v_o})$. (4)

4.4. PROPOSITION. *Let q be a quadratic form on k^3, suppose that q is negative definite at the archimedean places v_j, $1 \leqslant j \leqslant r$ and positive definite at the places v_j, $r < j \leqslant n$, $v_j \in V_\infty$. Assume that $q(\xi) \neq 1$ for all $\xi \in k^3$* (cf. 4.3). *Then there exists a Schwartz–Bruhat function $\varphi = \otimes \varphi_v \in \mathscr{S}(\mathbb{A}^3)$ satisfying conditions* 4.3(3) *and* (4) *such that*

(1) *the function Θ_φ is a cuspidal smooth function on $\widetilde{SL_2(\mathbb{A})}$, left invariant under $SL_2(k)$, the Fourier coefficient of Θ_φ with respect to ψ is identically zero,*
(2) *the local component Θ_{φ, v_j}, $v_j \in V_\infty$ transforms under $\widetilde{SL_2(k_{v_j})}$ according to the antiholomorphic discrete series representation of highest weight $k_j = -(\Lambda_j + \frac{7}{2})$ for $1 \leqslant j \leqslant r$ resp. to the holomorphic discrete series representation of lowest weight $k_j = \Lambda_j + \frac{7}{2}$, $r < j \leqslant n$,*
(3) $\Theta_\varphi \neq 0$.

Proof. First of all, note that we have for $x \in \mathbb{A}$, and $u \in \mathbb{A}^3$

$$\omega_q\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi(u) = \psi(q(u)\cdot x)\varphi(u) \tag{4}$$

An arbitrary character of $k\backslash\mathbb{A}$ is of the form ψ_a with $a \in K$, where ψ_a is defined by $\psi_a(x) = \psi(ax)$, $x \in \mathbb{A}$. Then we have, writing $n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, by (4)

$$\Theta_\varphi(n(x)g) = \sum_{\xi\in k^3} (\omega_q(g)\varphi(\xi))\psi_{q(\xi)}(x) \tag{5}$$

and we obtain as the ψ_a-Fourier coefficient of Θ_φ

$$\begin{aligned}\int_{k\backslash\mathbb{A}} \Theta_\varphi(n(x)g)\overline{\psi_a(x)}\,\mathrm{d}x &= \sum_{\xi\in k^3} (\omega_q(g)\varphi(\xi)) \int_{k\backslash\mathbb{A}} \psi_{q(\xi)}(x)\overline{\psi_a(x)}\,\mathrm{d}x \\ &= \sum_{\substack{\xi\in k^3 \\ q(\xi)=a}} \omega_q(g)\varphi(\xi) \end{aligned} \tag{6}$$

Taking $a = o$ we find the ψ_o-Fourier coefficient of Θ_φ to be $\omega_q(g)\varphi(0)$, which is left invariant under the upper triangular unipotent subgroup of $\widetilde{SL_2(\mathbb{A})}$. Hence it has to be identically zero since its local component corresponding under the local theta lifting to φ_{v_o} generates a supercuspidal representation (cf. [37]).

Taking $a = 1$ we find the ψ_1-Fourier coefficient to be of the form

$$\sum_{\substack{\xi\in k^3 \\ q(\xi)=1}} \omega_q(g)\varphi(\xi)$$

By our assumption on q the sum ranges over the empty set. Taking into account Theorem 2, Section 2 in [38] this shows (1).

Assertion (2) is an immediate consequence of the local theta lifting (cf. [19]) from $H(k_{v_j})$ to $\widetilde{SL_2(k_{v_j})}$, $v_j \in V_\infty$. The representation generated by φ_{v_j} corresponds to the antiholomorphic discrete series representation of highest weight k_j, $1 \leqslant j \leqslant r$, resp. to the holomorphic discrete series representation of lowest weight $(\Lambda_j + 2) + \frac{3}{2} = k_j$, $r < j \leqslant n$.

In proving (3) we observe that for $\varphi = \otimes\varphi_v$ no condition is imposed on φ_v, v non-archimedean, $v \neq v_o$. Using Lemma 6 in [20] we see that there exist $\varphi \in S(\mathbb{A}^3)$ as required such that Θ_φ is non-zero.

4.5. THEOREM. *Let k be a totally real algebraic number field of degree $n = [k:\mathbb{Q}]$ with archimedean places $v_1, \ldots, v_n \in V_\infty$. Let C be a Cayley algebra over k. Let G be the simple k-group of automorphisms of C. Assume that G is split at the places $v_1, \ldots, v_m$, $m \leqslant n$, and compact at the remaining real places. Given any* *integer r with $0 \leqslant r \leqslant m$ and an n-tuple $(\Lambda_1, \dots, \Lambda_n)$ of non-negative integers there exists an irreducible automorphic representation $\pi = \otimes \pi_v$ of $G(\mathbb{A})$ occurring in the cuspidal spectrum of $L^2(G(k) \backslash G(\mathbb{A}))$ such that the representation π_{v_j} of $G(k_{v_j})$ is equivalent to*

$$A(\Lambda_j) \quad \text{for } 1 \leqslant j \leqslant r$$

$$B(\Lambda_j) \quad \text{for } r < j \leqslant m$$

$$C(\Lambda_j) \quad \text{for } m < j \leqslant n$$

(*see* 3.9 *for notations*).

Proof. Let $\sigma = \otimes \sigma_v$ be an irreducible genuine cuspidal representation of $\widetilde{SL_2(\mathbb{A})}$ such that $\sigma_{v_j}, j = 1, \dots, r$, is an antiholomorphic discrete series representation of $\widetilde{SL_2(k_{v_j})}$ of highest weight $k_j = -(\Lambda_j + \frac{7}{2})$ resp. $\sigma_{v_j}, j = r+1, \dots, n$ is a holomorphic discrete series representation of $\widetilde{SL_2(k_{v_j})}$ of lowest weight $k_j = \Lambda_j + \frac{7}{2} \geqslant \frac{7}{2}$ (see 4.4 for a construction).

Then the restriction $V(\sigma, \psi)$ of $\Theta(\sigma, \psi)$ to $G(\mathbb{A})$ is non-zero and, by 4.4(1) and 4.2, consists out of cusp forms. Its local component $\pi(k_j)|_{G(k_{v_j})}$ at an archimedean place is equivalent to $A(-(k_j + \frac{7}{2})) = A(\Lambda_j)$, for $j = 1, \dots, r$, by 3.8, to $B(k_j - \frac{7}{2}) = B(\Lambda_j)$ for $j = r+1, \dots, m$, by 3.4 and to $C(k_j - \frac{7}{2}) = C(\Lambda_j)$ for $j = m+1, \dots, n$ by 3.7.

4.6. Let us consider an irreducible genuine cuspidal representation $\sigma = \otimes \sigma_v$ of $\widetilde{SL_2(\mathbb{A})}$ such that $\sigma_{v_j}, v_j \in V_\infty$, is the holomorphic representation of $\widetilde{SL_2(k_{v_j})}$ of lowest weight $\frac{1}{2}$, and σ has zero Fourier coefficient with respect to the given non-trivial character ψ of $k \backslash \mathbb{A}$. Observe that such a σ always exists; it can be constructed by using the reductive dual pair $(O(1), SL_2)$. Using 3.6 one obtains by means of the global theta lifting from $\widetilde{SL_2(\mathbb{A})}$ to $H_Q(\mathbb{A})$ followed by restriction the following

PROPOSITION. *Let G be a split simple group of type G_2 defined over a totally real algebraic number field k. Then there exist an irreducible cuspidal representation $\pi = \otimes \pi_v$ of $G(\mathbb{A})$ such that its local component π_{v_j} at an infinite place v_j of k is the irreducible spherical unitary representation of $G(k_{v_j})$, $v_j \in V_\infty$, with infinitesimal character χ_μ where μ denotes the short root of $G(k_{v_j})$.*

REMARK. In Langlands work (see [23], App. III) the spherical representation of $G_2 \cong G(k_{v_j})$ as above is obtained via the residue of an Eisenstein series.

5. Constructing cusp cohomology classes

5.1. COHOMOLOGY OF ARITHMETIC GROUPS. Let G be a simple algebraic group of type G_2 defined over a totally real algebraic number field k of degree $n = [k : \mathbb{Q}]$. We fix a maximal compact subgroup K_∞ in the real Lie group $G_\infty := G(\mathbb{R} \otimes_{\mathbb{Q}} k)$. Given an open compact subgroup $K_f \subset G(\mathbb{A}_f)$ we consider the space

$$X_{K_f} := G(k)\backslash G(\mathbb{A})/K_\infty K_f. \tag{1}$$

A finite dimensional representation $\tau: G \times_{\mathbb{Q}} \bar{\mathbb{Q}} \to GL(E)$ where E is a $\bar{\mathbb{Q}}$-vector space provides a sheaf $\tilde{E}$ on X_{K_f} in a natural way. The objects we are interested in are the cohomology groups $H^*(X_K, \tilde{E})$ for a given choice of an open compact subgroup $K_f \subset G(\mathbb{A}_f)$. This is the adelic analogue of the cohomology of an arithmetic subgroup Γ of $G(k)$. Recall that the virtual cohomological dimension $\mathrm{vcd}(\Gamma)$ of such an arithmetic group Γ is given as

$$\mathrm{vcd}(\Gamma) = d(G) - l = n \cdot 8 - 2 \tag{2}$$

where $l = 2$ is the k-rank of G, $d(G) = n \cdot d(G(k_v))$, $v \in V_\infty$ and $d(G(k_v)) = 8$ is the real dimension of the symmetric space associated to $G(k_v)$, i.e. of the quotient of $G(k_v)$ by a maximal compact subgroup.

Given another open compact subgroup $K'_f \subset K_f$ we have a finite covering $X_{K'_f} \to X_{K_f}$ which induces an inclusion $H^*(X_{K_f}, \tilde{E}) \to H^*(X_{K'_f}, \tilde{E})$. This is a directed system of cohomology groups, and the inductive limit

$$H^*(X, \tilde{E}) := \varinjlim_{K_f} H^*(X_{K_f}, \tilde{E}) \tag{3}$$

carries a natural $G(\mathbb{A}_f)$-module structure. One may recover the cohomology of X_{K_f} by taking K_f-invariants.

Denote by $\mathfrak{g}$ the Lie algebra of G_∞; then the cohomology groups $H^*(X, \tilde{E})$ have an interpretation in terms of relative Lie algebra cohomology as

$$H^*(\mathfrak{g}, K_\infty, C^\infty(G(k)\backslash G(\mathbb{A})) \otimes E) \tag{4}$$

As usual we denote by ρ_o the representation of $G(\mathbb{A})$ acting by right translations on the space $L^2_o(G(k)\backslash G(\mathbb{A}))$ of square integrable cuspidal functions on $G(k)\backslash G(\mathbb{A})$; it decomposes into a direct Hilbert sum of irreducible admissible representations π with finite multiplicites $m_o(\pi)$, i.e. $\rho_o = \hat{\bigoplus} m_o(\pi)\pi$. Let $L^2_o(G(k)\backslash G(\mathbb{A}))^\infty$ be the space of C^∞-vectors (i.e. vectors which are K_f-finite for some K_f and C^∞ in the usual sense for the action of G_∞). Then the natural map in cohomology induced by inclusion

$$j^*: H^*(\mathfrak{g}, K_\infty : L^2_o(G(k)\backslash G(\mathbb{A}))^\infty \otimes E) \to H^*(\mathfrak{g}, K_\infty; C^\infty(G(k)\backslash G(\mathbb{A})) \otimes E) \tag{5}$$

is injective ([2], 5.5). By definition the cusp cohomology $H^*_{\mathrm{cusp}}(X, \tilde{E})$ with coefficients in E is the image of j_o^*, and one has ([6] XIII)

$$H^*_{\text{cusp}}(X, \tilde{E}) = \bigoplus_{\pi = \pi_\infty \otimes \pi_f} m_o(\pi) H^*(\mathfrak{g}, K_\infty; \pi_\infty \otimes E) \otimes \pi_f \tag{6}$$

By taking K_f-invariants one defines in the same way for a fixed open compact subgroup $K_f \subset G(\mathbb{A}_f)$ the cusp cohomology of X_{K_f}.

Starting with the discrete spectrum $L^2_d(G(k)\backslash G(\mathbb{A}))$, the square-integrable cohomology, to be denoted by $H^*_{(2)}(X, \tilde{E})$, is defined as the image of j_d^*. By [5], Section 5 there is a decomposition similar to (6).

Recall that the image of the cohomology with compact supports under the natural map is called the interior cohomology of X (or X_{K_f}) and denoted by $H^*_!(X, \tilde{E})$. It contains the cusp cohomology and it is contained in $H^*_{(2)}(X, \tilde{E})$, i.e. one has

$$H^*_{\text{cusp}}(X, \tilde{E}) \subset H^*_!(X, \tilde{E}) \subset H^*_{(2)}(X, \tilde{E})$$

The following theorem is now an immediate consequence of the construction of specific irreducible automorphic representations $\pi = \otimes \pi_v$ of $G(\mathbb{A})$ occurring in the cuspidal spectrum of $L^2(G(k)\backslash G(\mathbb{A})$ in 4.5 combined with 2.2(4).

5.2. THEOREM. *Let k be a totally real algebraic number field of degree $n = [k:\mathbb{Q}]$ with archimedean places $v_1, \ldots, v_n \in V_\infty$. Let C be a Cayley algebra over k. Let G be the simple k-group of automorphisms of C. Assume that G is split at all places $v_j \in V_\infty$. Then the cusp cohomology $H^*_{\text{cusp}}(X, \mathbb{C})$ does not vanish in degree $4r + 3(n-r)$ resp. $4r + 5(n-r)$ for any integer $0 \leqslant r \leqslant n$. In terms of relative Lie algebra cohomology these non-vanishing classes correspond to irreducible unitary automorphic representations $\pi = \otimes \pi_v$ of $G(\mathbb{A})$ occurring in the cuspidal spectrum $L^2_o(G(k)\backslash G(\mathbb{A}))$ such that the representation π_{v_j} of $G(k_{v_j})$ is equivalent to $A(0)$ for $j = 1, \ldots, r$ resp. $B(0)$ for $j = r+1, \ldots, m$. Recall that the representations π_{v_j}, $j = 1, \ldots, r$, are discrete series representations and that the π_{v_j}, $j = r+1, \ldots, n$ are non-tempered representations.*

Starting off from Theorem 4.5 one obtains an analogous non-vanishing result for the cohomology in the case of a simple group of type G_2 defined over a totally real algebraic number field k.

Observing that any lattice Γ in the simple Lie group of type G_2 is necessarily arithmetic [28] we have in particular the following consequence

5.3. PROPOSITION. *Given any irreducible lattice Γ in the simple Lie group of type G_2 there exists a subgroup Γ' of Γ of finite index such that*

$$H^i(\Gamma, \mathbb{C}) \neq 0$$

for $i = 3, 5$.

REMARKS. (1) In view of 4.5 and 2.2 these non-vanishing results are easily extended to the case of certain non-trivial coefficients. Observe that in such a case the complex conjugate contragradient representation $E_{\mathbb{C}}^*$ of $E_{\mathbb{C}}$: has to be equivalent to the representation $E_{\mathbb{C}}$ (cf. [4], 1.3).

(2) Arthur's version of the Selberg trace formula allows one to get hold of cusp cohomology classes in degree $4n$ by inserting so-called pseudo-coefficients in the trace formula and computing Euler–Poincaré characteristics. These non-vanishing classes correspond to irreducible unitary automorphic representations $\pi = \otimes\pi_v$ of $G(\mathbb{A})$ occurring in the cuspidal spectrum $L_o^2(G(k)\backslash G(\mathbb{A}))$ such that the representation π_{v_j} of $G(k_{v_j})$ is equivalent to a discrete series representation (cf. Section 2). Observe that Theorem 4.5 resp. 5.2 with $r = n$ gives a new proof of this result.

6. Eisenstein cohomology classes

In this and the following section the theory of Eisenstein series will be used to construct by analytical means certain cohomology classes in $H^*(X, \mathbb{C})$ which are represented by a regular value of a suitable Eisenstein series or a residue of such. It is not intended to give a complete description of the so-called Eisenstein cohomology though it is feasible. As our main result we exhibit residual Eisenstein cohomology classes, say for $k = \mathbb{Q}$ in $H^3(X, \mathbb{C})$, which correspond to an irreducible unitary automorphic representation $\pi = \otimes\pi_v$ of $G(\mathbb{A})$ occurring in the residual spectrum $L^2_{\text{res}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ such that the representation π_∞ is equivalent to $A_{\mathfrak{q}_1}(0)$.

We have to assume some familiarity with the procedure for the construction of Eisenstein cohomology classes as initiated by Harder in [10, 11, 12] and pursued by others [42, 43].

6.1. SOME FACTS ABOUT G_2, ROOTS AND PARABOLICS. Let C be the split Cayley algebra over $\mathbb{Q}$ and let G be the split simple $\mathbb{Q}$-group of automorphisms of C. The set of parabolic $\mathbb{Q}$-subgroups of G is denoted by $\mathscr{P}$. We fix a minimal parabolic $\mathbb{Q}$-subgroup P_o of G together with a Levi component M_o of P_o defined over $\mathbb{Q}$. Let P be a parabolic $\mathbb{Q}$-subgroup; we may (and will) assume that P is standard, i.e. we have $P \supset P_o$. The unique Levi component of P containing M_o and defined over $\mathbb{Q}$ will be denoted by $M = M_P$; the unipotent radical of P is denoted by $N = N_P$, it is defined over $\mathbb{Q}$. Let $A = A_P$ be the maximal $\mathbb{Q}$-split torus in the center of M_P. Then the real Lie algebra of A_o (resp. A_P)

$$\mathfrak{a}_o = \text{Hom}(X(M_o)_{\mathbb{Q}}, \mathbb{R}) \text{ resp. } \mathfrak{a}_P = \text{Hom}(X(M_P)_{\mathbb{Q}}, \mathbb{R}) \tag{1}$$

is a real vector space whose dual is given by

$$\mathfrak{a}_o^* = X(M_o)_{\mathbb{Q}} \otimes \mathbb{R} \text{ resp. } \mathfrak{a}_P^* = X(M_P)_{\mathbb{Q}} \otimes \mathbb{R}, \tag{2}$$

and the complex dual of $\mathfrak{a}_P$ is denoted by $\mathfrak{a}_{P,\mathbb{C}}^*$.

Let $\Phi(P, A_P)$ be the set of roots of P with respect to A_P; these are the elements in $X(A_P)_{\mathbb{Q}}$ obtained by decomposing the adjoint action of A_P on the Lie algebra $\mathfrak{n}_P$ of N_P. The set of simple roots of P with respect to A_P is denoted by $\Delta(P, A)$. These elements in $X(A_P)_{\mathbb{Q}}$ may be canonically embedded in $\mathfrak{a}_P^*$. Let $\Phi_o \subset X^*(P_o) \subset \mathfrak{a}_o^*$ be the set of roots of G with respect to A_o. Fix an ordering on Φ_o compatible with $\Phi(P_o, A_o)$, i.e. the set Φ_o^+ of positive roots is equal to $\Phi(P_o, A_o)$. Let $\Delta \subset \Phi_o^+$ be the set of simple roots. The dual root system is denoted by $\Phi_o^\vee \subset a_o$, the root in $\Phi_o^\vee$ dual to a given $\alpha \in \Phi_o$ is denoted by $\alpha^\vee$.

We want to describe the two standard maximal parabolic subgroups of G. If α_1 (resp. α_2) denotes the short (resp. long) root in Δ, then we have

$$\Phi_o^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}. \tag{3}$$

A maximal parabolic $\mathbb{Q}$-subgroup Q of G is conjugate to a standard one

$$P_i := P_{\Delta-\{\alpha_i\}} = Z(H_{\Delta-\{\alpha_i\}}) \cdot N_i \supset P_o \tag{4}$$

given as the semidirect product of the unipotent radical by the centralizer of $H_{\Delta-\{\alpha_i\}}$, $i = 1, 2$ where we denote $H_J = (\bigcap_{\alpha \in J} \ker \alpha)^o \subset M_o$ for a subset J of Δ. Note that the characters of M_o in N_i, $i = 1, 2$, are exactly the positive roots which contain at least one simple root not in $\Delta - \{\alpha_i\}$. The root spaces of $\mathfrak{n}_i$ correspond to the roots

$$\alpha_i, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2 \tag{5}$$

The Levi components M_i are isomorphic to GL_2.

Observe that the two maximal parabolic subgroups P_i, $i = 1, 2$, are not associate.

Let ρ denote half the sum of the elements in Φ_o^+, and we put $\rho_{P_i} := \rho|_{\mathfrak{a}_{P_i}}$.

6.2. COHOMOLOGY OF A STRATUM. Given a proper parabolic $\mathbb{Q}$-subgroup P of G we consider the space

$$X^P := \varprojlim_{K_f} P(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f \tag{1}$$

defined as the limit over K_f, an open compact subgroup of $G(\mathbb{A}_f)$. The cohomology group $H^*(X^P, \tilde{E})$ with the natural sheaf $\tilde{E}$ associated to the representation (τ, E) (cf. 5.1) carries a $G(\mathbb{A}_f)$-module structure in a natural way. It is known that as a $G(\mathbb{A}_f)$-module

$$H^*(X^P, \tilde{E}) = \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}[H^*(X^M, H^*(\mathfrak{n}, E))] \tag{2}$$

where $H^*(X^M, H^*(\mathfrak{n}, E)) := \varinjlim_{K_f} H^*(X^M_{K_f}, H^*(\mathfrak{n}, E))$ with $X^M_{K_f} = M(\mathbb{Q})\backslash M(\mathbb{A})/K^M_\infty K^M_f$, $K^M_\infty = K_\infty \cap M(\mathbb{R})$, $K^M_f = K_f \cap M(\mathbb{A}_f)$. The coefficient sheaf is given by the Lie algebra cohomology $H^*(\mathfrak{n}, E)$ endowed with the natural $M(\bar{\mathbb{Q}})$-module structure. The inductive limit carries a $M(\mathbb{A}_f)$-module structure; this is extended trivially to $P(\mathbb{A}_f)$ (see [42], 2.7 for a treatment in the non adelic language).

The relative Lie algebra cohomology $H^*(\mathfrak{n}, E)$ is described as an $M(\bar{\mathbb{Q}})$-module by a theorem of Kostant (cf. [21], 5.13). Let W (resp. W_M) be the Weyl group of $G \times_{\mathbb{Q}} \bar{\mathbb{Q}}$ (resp. $M \times_{\mathbb{Q}} \bar{\mathbb{Q}}$). The group W is generated by the simple reflections w_i associated to α_i, and W is isomorphic to $(\pm 1) \rtimes S_3$.

We recall that we have

$$w_i(\alpha_i) = -\alpha_i \quad \text{and} \quad \begin{cases} w_1(\alpha_2) = 3\alpha_1 + \alpha_2 \\ w_2(\alpha_1) = \alpha_1 + \alpha_2 \end{cases} \tag{3}$$

Then the set $W^{P_i} = \{w \in W | w^{-1}(\Delta_{M_i}) \subset \Phi^+\}$ is a set of representatives in the Weyl group W relative to W_M; in a given coset the element of minimal length is taken. Given the irreducible representation (τ, E) of $G \times_{\mathbb{Q}} \bar{\mathbb{Q}}$ of highest weight λ the Lie algebra cohomology $H^*(\mathfrak{n}_i, E)$ decomposes as an $M_i \times {}_{\mathbb{Q}}\bar{\mathbb{Q}}$-module

$$H^q(\mathfrak{n}_i, E) = \bigoplus_{\substack{w \in W^{P_i} \\ l(w) = q}} F_{\mu_w} \tag{4}$$

into irreducible $M_i \times_{\mathbb{Q}} \bar{\mathbb{Q}}$-modules F_{μ_w} with highest weight $\mu_w = w(\lambda + \rho) - \rho$. The sum ranges over all $w \in W^{P_i}$ with length $l(w) = q$. Note that the weights μ_w are all dominant and distinct, as w ranges through W^{P_i}. One easily checks

$$\begin{aligned} W^{P_1} &= \{1, w_1, w_1w_2, w_1w_2w_1, w_1w_2w_1w_2, w_{P_1}\} \\ W^{P_2} &= \{1, w_2, w_2w_1, w_2w_1w_2, w_2w_1w_2w_1, w_{P_2}\} \end{aligned} \tag{5}$$

where w_{P_i} denotes the unique element of length 5 in W^{P_i}.

For later use we have to determine in the case $\lambda = 0$ the parameter

$$\Lambda_w := -w(\rho)|_{\mathfrak{a}_P} \tag{6}$$

where w ranges through the set W^P. The following list gives the parameter in the form $\Lambda_w = ? \cdot \rho_{P_i}$, column I resp. II contains the values for $w \in W^{P_1}$ resp. W^{P_2}. It is convenient to identify $s \in \mathbb{C}$ with $s\tilde{\rho}_P$ where $\tilde{\rho}_P$ is defined by $\tilde{\rho}_{P_i} := \langle \rho_{P_i}, \alpha_i \rangle^{-1} \cdot \rho_{P_i}$. We have $\tilde{\rho}_{P_1} = \frac{2}{5}\rho_{P_1}$ and $\tilde{\rho}_{P_2} = \frac{2}{3}\rho_{P_2}$. The complex values s_w defined by $\Lambda_w = s_w \tilde{\rho}_P$ are given as well. The rows are indexed by the length $l(w)$ of $w \in W$.

	I		II	
$l(w)$	$\Lambda_w = ?\,\rho_{P_1}$	$s_w =$	$\Lambda_w = ?\,\rho_{P_2}$	$s_w =$
5	1	$2\frac{1}{2}$	1	$1\frac{1}{2}$
4	$\frac{4}{5}$	2	$\frac{2}{3}$	1
3	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
2	$-\frac{1}{5}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{2}$
1	$-\frac{4}{5}$	-2	$-\frac{2}{3}$	-1
0	-1	$-2\frac{1}{2}$	-1	$-1\frac{1}{2}$

6.3. The construction of Eisenstein cohomology classes (up to the actual existence of a pole) in the case of groups of $\mathbb{Q}$-rank 1 is dealt with in [10]. The main results obtained there carry over to our case in question where we deal with parabolic $\mathbb{Q}$-subgroups P of parabolic $\mathbb{Q}$-rank 1. Taking into account a more representation theoretical approach (instead of the more differential-geometric one there) the result may be reformulated in the following way:

For the cusp cohomology

$$H^*_{\text{cusp}}(X^M, H^*(\mathfrak{n}, E)) = \varinjlim_{K_f} H^*_{\text{cusp}}(X^M_{K_f}, H^*(\mathfrak{n}, E))$$

one has (5.1(6) and 6.2(4)) a decomposition as a direct sum

$$H^*_{\text{cusp}}(X^M, H^*(\mathfrak{n}, E)) = \bigoplus_{w \in W^P} \bigoplus_{\pi = \pi_\infty \otimes \pi_f} m_0(\pi) H^*(\mathfrak{m}, K_{M_\infty}, \pi_\infty \otimes F_{\mu_w}) \otimes \pi_f \qquad (1)$$

where the second sum ranges over all irreducible cuspidal automorphic representations π of $M(\mathbb{A})$ with non-trivial cohomology with respect to the fixed module $F_{\mu_w} \subset H^*(\mathfrak{n}, E)$ i.e. H_π occurs as an $M(\mathbb{A})$-submodule with multiplicity $m_0(\pi)$ in the space $L^2_0(M(\mathbb{Q})\backslash M(\mathbb{A}))$ and $H^*(\mathfrak{m}, K_{M_\infty}, \pi_\infty \otimes F_{\mu_w}) \otimes \pi_f$ does not vanish. A cohomology class in $H^*_{\text{cusp}}(X^M, H^*(\mathfrak{n}, E))$ is said to be of type (π, w) if it is an element of the summand $H^*(\mathfrak{m}, K_{M_\infty}; \pi_\infty \otimes F_{\mu_w}) \otimes \pi_f$ in the decomposition (1). Given such a cohomology class of type (π, w) there is via the Eisenstein summation depending on a parameter $\Lambda \in (\mathfrak{a}^*_P)_\mathbb{C}$ with $(\operatorname{Re}\Lambda, \alpha) > (\rho_P, \alpha)$ for each simple root of P with respect to A an intertwining operator

$$\operatorname{Eis}_{\pi,\Lambda} : \operatorname{Ind}^{G(\mathbb{A})}_{P(\mathbb{A}),\pi,\Lambda} \to A(G(\mathbb{Q})\backslash G(\mathbb{A})) \qquad (2)$$

of $G(\mathbb{A})$-modules into the space of automorphic forms on $G(\mathbb{A})$. It can be analytically continued as a meromorphic function in Λ to all of $(\mathfrak{a}_P^*)_{\mathbb{C}}$. We refer to 7.1 for a detailed treatment. In order to construct cohomology classes in $H^*(X, \tilde{E})$ one has to analyze the map (2) on the level of $(\mathfrak{g}, K_\infty)$-cohomology. By [42], 4.11 one has the following result.

6.4. PROPOSITION. *Let* $[\varphi] \in H^*_{\text{cusp}}(X^P, \tilde{E})$ *be a cohomology class of type* (π, w), $w \in W^P$, π *an irreducible cuspidal representation of* $M(\mathbb{A})$. *If the Eisenstein series assigned to* π *as in* 6.3(2) *is holomorphic at the point* $\Lambda_w = -w(\lambda + \rho)|_{\mathfrak{a}_P}$ *(which is real and uniquely determined by the datum* (π, w)*) then evaluation at* Λ_w *provides a non-trivial class* Eis$[\varphi]$ *in* $H^*(X, \tilde{E})$, *called a regular Eisenstein cohomology class.*

REMARK. Recall that the image of the class Eis$[\varphi]$ under restriction to the cohomology $H^*(X^Q, \tilde{E})$, Q a proper parabolic $\mathbb{Q}$-subgroup of G, is obtained by taking the constant Fourier coefficient of the Eisenstein series in question along Q(cf. [42], 1.10 resp. 4.7).

6.5. Given a proper parabolic $\mathbb{Q}$-subgroup P of G we consider the set W^P. Let w_H denote the longest element of W_H for $H = G$ resp. M then $w \to w' := w_M \cdot w \cdot w_G$ defines an involution on W^P. One has $l(w) + l(w') = \dim N$. Then it is easily seen by a direct computation in the case considered here that the weights μ_w occurring in the decomposition of $H^*(\mathfrak{n}, C)$ satisfy the identity:

$$w(\rho) - \rho|_{\mathfrak{a}_P} + w'(\rho) - \rho|_{\mathfrak{a}_P} = -2\rho|_{\mathfrak{a}_P}, \; w \in W^P \tag{1}$$

The irreducible ${}^0M \times_{\mathbb{Q}} \bar{\mathbb{Q}}$-module $F_{\mu_{w'}}$ is the representation contragradient to F_{μ_w} (2)

(Of course, this is a result proved in general in [45]).

Then the cusp cohomology $H^*_{\text{cusp}}(X^P, E)$ may be arranged according to isotypical components of the $M(\mathbb{A})$-modules and regrouped by pairs (w, w'), $w \in W^P$ with $l(w) > (1/2)\dim N$. We observe that $\dim N = 5$, and we may write

$$H^*_{\text{cusp}}(X^P, \mathbb{C}) = \bigoplus_{w,w')} \bigoplus_{\pi} [H^*(\pi, F_{\mu_w}) \oplus H^*(\pi', F_{\mu_{w'}})] \tag{3}$$

where $H^*(\pi, F_{\mu_w})$ denotes the induced module

$$\text{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} [H^*_{\text{cusp}}(X^M, F_{\mu_w})_\pi]$$

indexed by an irreducible cuspidal automorphic representation π of $M(\mathbb{A})$ with non-trivial cohomology with respect to the fixed module F_{μ_w}. The representation π' of $M(\mathbb{A})$ is obtained by twisting π with the opposition involution on W^P. Note that for $w \in W^P$ with $l(w) > (\frac{1}{2})\dim N$ the complex values s_w defined by

$\Lambda_w = s_w \cdot \tilde{\sigma}_P$ are *positive* real numbers (cf. Tables 6.2 and 6.4).

The main result in [10] has the following reformulation in this setting

6.6. THEOREM. *Let P be a maximal parabolic $\mathbb{Q}$-subgroup of G.*

(1) *The image of the natural restriction map*

$$r_P: H^*(X, \mathbb{C}) \to H^*_{\text{cusp}}(X^P, \mathbb{C}) = \bigoplus_{\substack{(w,w'), w \in WP \\ l(w) \| 1/2 \dim N}} \bigoplus_{\pi} [H^*(\pi, F_{\mu_w}) \oplus H^*(\pi', F_{\mu_{w'}})]$$

is compatible with the direct sum decomposition on the right-hand side, i.e.

$$\operatorname{Im} r_P = \bigoplus_{\substack{(w,w'), w \in WP \\ l(w) > 1/2 \dim N}} \bigoplus_{\pi} [\operatorname{Im} r_P \cap H^*(\pi, F_{\mu_w})] \oplus [\operatorname{Im} r_P \cap H^*(\pi', F_{\mu_{w'}})]$$

We denote by $I(\pi, w)$ resp. $J(\pi', w')$ the summand $\operatorname{Im} r_P \cap H^*(\pi, F_{\mu_w})$ *resp.* $\operatorname{Im} r_P \cap H^*(\pi', F_{\mu_{w'}})$ *in the last sum decomposition.*

(2) *There are subspaces* $\operatorname{Eis}(I(\pi, w))$ *resp.* $\operatorname{Eis}(J(\pi', w'))$ *in* $H^*(X, \mathbb{C})$ *which restrict isomorphically under r_P onto $I(\pi, w)$ resp. $J(\pi', w')$. In the first case one considers the map* Eis *induced by* 6.3(2) *on the level of* $(\mathfrak{g}, K_\infty)$*-cohomology, and* (*using* 6.4) $\operatorname{Eis}(I(\pi, w))$ *is generated by regular Eisenstein cohomology classes and always non-trivial. In the second case* $\operatorname{Eis}(J(\pi', w'))$ *is generated by residual Eisenstein cohomology classes obtained by considering the composition of* Eis *and taking the residue at Λ_w. If the subspace* $\operatorname{Eis}(J(\pi', w'))$ *is non-trivial then it consists out of square integrable cohomology classes.*

REMARK. This is a result up to the actual existence of a pole at Λ_w for the Eisenstein series attached to the datum (π, w) (cf. 6.4). Thus the main task in constructing (residual) Eisenstein cohomology classes is to discuss this problem.

7. Residual Eisenstein cohomology classes

In this section we give a description of the image of the restriction $r_P: H^*(X, \mathbb{C}) \to H^*_{\text{cusp}}(X^P, \mathbb{C})$; in particular, it will be discussed under which conditions the subspaces $\operatorname{Eis}(J(\pi', w')) \subset H^*(X, \mathbb{C})$ generated by residual Eisenstein cohomology classes are non-trivial. This is formulated in terms of a non-vanishing condition for certain partial L-series of GL_2.

We retain the notation of Section 6.

7.1. Let P be a standard maximal parabolic $\mathbb{Q}$-subgroup of G; there is a Levi decomposition $P = MN$ as in 6.1. For each place v of $\mathbb{Q}$ let $G_v = G(\mathbb{Q}_v)$. Similarly, we use P_v, M_v, N_v etc. to denote the corresponding groups of $\mathbb{Q}_v$-rational points.

For each place v there is a natural imbedding $X(M)_{\mathbb{Q}} \to X(M)_{\mathbb{Q}_v}$, and it induces an imbedding

$$\mathfrak{a}_v := \mathrm{Hom}(X(M)_{\mathbb{Q}_v}, \mathbb{R}) \to \mathfrak{a}_P = \mathrm{Hom}(X(M)_{\mathbb{Q}}, \mathbb{R});$$

the image is denoted by the same letter. For $\chi \in X(M)_{\mathbb{Q}}$, $m = (m_v)_v \in M(\mathbb{A})$ we define $\chi(m) = \Pi_{v \in V} |\chi(m_v)|_v$. The product on the right-hand side is finite since $m_v \in G(\mathcal{O}_v)$ for almost all v. We may extend this to $P(\mathbb{A})$ by putting

$$p^{\chi} := \chi(m) \quad \text{for } p = mn \text{ in } P(\mathbb{A})$$

Let $\rho_P \in \mathfrak{a}_P^*$ be half the sum of the roots of P in N. If α identifies the unique (reduced) root of A_P in N, we let $\tilde{\rho}_P = \langle \rho_P, \alpha \rangle^{-1} \rho_P$; it belongs to $\mathfrak{a}_P^*$. We shall now identify $\mathbb{C}$ with a subspace of $\mathfrak{a}_{\mathbb{C}}^*$ by identifying $s \in \mathbb{C}$ with $s\tilde{\rho}_P \in \mathfrak{a}_{\mathbb{C}}^*$. Then $s \in \mathbb{C}$ can also be realized as an element in $(\mathfrak{a}_v)_{\mathbb{C}}^*$ for each $v \in V$ via $\mathfrak{a}^* \to \mathfrak{a}_v^*$. For a given Λ in $\mathfrak{a}_{P,\mathbb{C}}^*$ of the form $s\tilde{\rho}_P$ we may write

$$m^{\Lambda} = |\tilde{\rho}_P(m)|_{\mathbb{A}}^{s}, \quad m \in M(\mathbb{A}).$$

We shall extend Λ to $G(\mathbb{A})$ by making it trivial on $N(\mathbb{A})$ and $K = \Pi K_v$ where $K_v = G(\mathcal{O}_v)$ when G is unramified over a place v and otherwise K_v is a special maximal compact subgroup $K_v \subset G_v$.

Let $\pi = \otimes \pi_v$ be a cuspidal automorphic representation on $M(\mathbb{A})$. Given a K-finite function φ in the space of π there is an associated. Eisenstein series defined by

$$E(\varphi, \Lambda, g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \varphi_{\Lambda}(\gamma g), \quad g \in G(\mathbb{A})$$

where $\varphi_{\Lambda}(g) = \varphi(g) g^{\Lambda + \rho_P}$, $g \in G(\mathbb{A})$.

For a given Λ with $\mathrm{Re}(\Lambda) \in \rho_P + (\mathfrak{a}_P^*)^+$ the series converges absolutely; it defines a C^{∞}-function on $G(\mathbb{A}) \times \{\Lambda \in \mathfrak{a}_{P,\mathbb{C}}^* | \mathrm{Re}(\Lambda) \in \rho_P + (\mathfrak{a}_P^*)^+\}$ which is holomorphic in Λ. The function $E(\varphi, \Lambda, g)$ can be analytically continued as a meromorphic function in Λ to $\mathfrak{a}_{P,\mathbb{C}}^*$. The Eisenstein series has only a finite number of simple poles in the real interval

$$\{\Lambda \in \mathfrak{a}_{P,\mathbb{C}}^* | \operatorname{Im} \Lambda = 0, \rho_P(\alpha^{\vee}) \geqslant \mathrm{Re}(\Lambda)(\alpha^{\vee}) > 0 \quad \text{for all } \alpha \in \Delta(P, A)\}.$$

and all other possible poles of $E(\varphi, \Lambda, g)$ lie in the region $\{\Lambda \in \mathfrak{a}_{P,\mathbb{C}}^* | \mathrm{Re}(\Lambda)(\alpha^{\vee}) < 0$ for all $\alpha \in \Delta(P, A)\}$.

For a given $\Lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ let $I_{P,\pi,\Lambda}$ be the induced representation

$$I_{P,\pi,\Lambda} = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi \otimes (\Lambda + \rho_P)$$
$$= \{f \in C^\infty(G(\mathbb{A}), H_\pi) \mid f(pg) = \pi(p)p^{\Lambda+\rho_P} f(g))\}$$

acted upon by $G(\mathbb{A})$ via right translations. Having written the representation π as a restricted tensor product $\pi = \otimes \pi_v$ of irreducible representation (π_v, H_v) of G_v the induced representation is the restricted tensor product of the local induced representation.

Given an element w in the Weyl group W of (G, A_0) we fix a representative denoted by the same letter in the intersection of $G(\mathbb{Q})$ and the normalizer of A_0; this element is determined up to $M_0(\mathbb{Q})$. If P, Q are two parabolic $\mathbb{Q}$-subgroups of G, let $W(A_P, A_Q)$ be the set of distinct isomorphisms from $\mathfrak{a}_P$ to $\mathfrak{a}_Q$ obtained by restricting elements in W to $\mathfrak{a}_P$. If $W(A_P, A_Q)$ is not empty the parabolic $\mathbb{Q}$-subgroups P and Q are called to be associate. This defines an equivalence relation on the set of parabolic $\mathbb{Q}$-subgroups of G.

Recall that for a given $t \in W(A_Q, A_P)$ there is a uniquely determined element w_t in W such that t is obtained by restricting w_t to $\mathfrak{a}_P$ and such that $w_t^{-1}\alpha$ is a root of P_0 with respect to A_0 for every simple root α of $(M_P \cap P_0, A_0)$. Thus, $W(A_P, A_Q)$ may be regarded as a subset of W; note that w_t is an element in $G(\mathbb{Q})$ for a given t in $W(A_P, A_Q)$.

Let Q be a parabolic $\mathbb{Q}$-subgroup of G, associate to the given maximal parabolic $\mathbb{Q}$-subgroup P; we may assume that also Q is a standard parabolic $\mathbb{Q}$-subgroup. Let t be an element in $W(A_P, A_Q)$ and let w_t be a representative in W chosen as above. There are global intertwining operators

$$M(\Lambda, \pi, t)\colon I_{P,\pi,\Lambda} \to I_{Q,w_t\pi,w_t\Lambda}$$

defined for a given φ as above and $\Lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ with $\mathrm{Re}(\Lambda) \in \rho_P + (\mathfrak{a}^*_P)^+$ by

$$M(\Lambda, \pi, t)\varphi_\Lambda(g) = \int_{N_Q(\mathbb{A}) \cap w_t N_P(\mathbb{A}) w_t^{-1} \backslash N_Q(\mathbb{A})} \varphi_\Lambda(w_t^{-1}ng)(w^{-1}ng)^{\Lambda+\rho_P} g^{\Lambda+\rho_P}\,\mathrm{d}n$$

This integral converges absolutely in that region; it can be continued as a meromorphic function in Λ to $\mathfrak{a}^*_{P,\mathbb{C}}$. For Λ with $\mathrm{Re}(\Lambda)(\alpha^\vee) > 0$ for all $\alpha \in \Delta(P, A_P)$ there is only a finite number of simple poles, all on the real axis. There are the local intertwining operators $A(\Lambda, \pi_v, t)$, $v \in V$, defined in an analogous way, and one has for the given cuspidal automorphic representation $\pi = \otimes \pi_v$ of $M(\mathbb{A})$

$$M(\Lambda, \pi, t) = \bigotimes_v A(\Lambda, \pi_v, t)$$

The properties of $M(\Lambda, \pi, t)$ that we will need are all contained, at least implicitly, in [23]; a more recent treatment is given in [31].

7.2. For every connected algebraic group H defined over $\mathbb{Q}$ we denote by ${}^LH$ its L-group. It is the semidirect product of a complex group ${}^LH^o$ and the Weil group $W(\bar{\mathbb{Q}}/\mathbb{Q})$. For every place v of $\mathbb{Q}$ let ${}^LH_v$ denote the L-group of H viewed as a $\mathbb{Q}_v$-group. There is a natural homomorphism ${}^LH_v \to {}^LH$. Note that in the case of the Levi component $M \cong GL_2$ of P one has ${}^LM^o = GL_2(\mathbb{C})$. Using the map $\eta_v: {}^LM_v \to {}^LM$ alluded to one attaches to a given finite dimensional complex representation r of ${}^LM$ the representation $r_v = r \circ \eta_v$ of ${}^LM_v$.

Given a place v of $\mathbb{Q}$ such that π_v (cf. 7.1) and G_v are both unramified there is the local Langlands L-function $L(s, \pi_v, r_v)$ attached to π_v and r_v; for a definition we refer to [3], [24]. If $S \supset S_\infty$ is a finite set of places of $\mathbb{Q}$ such that for every $v \notin S$ the representation π_v and the group G_v are both unramified then there is the partial L-function

$$L_S(s, \pi, r) := \prod_{v \notin S} L(s, \pi_v, r_v) \tag{1}$$

Fix the maximal parabolic $\mathbb{Q}$-subgroup P of type i, let ${}^LN$ be the unipotent radical of ${}^LP$; its Lie algebra is denoted by ${}^L\mathfrak{n}$. The group ${}^LM$ acts on ${}^L\mathfrak{n}$ by the adjoint action. If $\beta^\vee$ ranges through the set of dual roots $\beta^\vee$, $\beta \in \Phi_0^+$, for which $X_{\beta^\vee} \in {}^L\mathfrak{n}$ holds then the numbers $\langle \tilde{\rho}_{P_i}, \beta \rangle$ take a string of integers from 1 to a positive integer m. For a given j, $1 \leqslant j \leqslant m$, we define

$$V_j := \{X_{\beta^\vee} \in {}^L\mathfrak{n} \mid \langle \widetilde{\rho_{P_i}}, \beta \rangle = j\} \tag{2}$$

Then the adjoint action of ${}^LM$ on ${}^L\mathfrak{n}$ leaves V_j for each j stable. The restriction of this action of ${}^LM$ to V_j is denoted by r_j, the contragradient representation is denoted by $\tilde{r}_j$.

Let $P^- = MN^-$ be the maximal parabolic $\mathbb{Q}$-subgroup opposite to the given one P. Then it is shown in [24] that in the case $Q = P^-$ the value of the intertwining operator $M(\Lambda, \pi, t)$ with $\Lambda = s\tilde{\rho}_P$ on a function $f = \otimes f_v$ with f_v the unique K_v-fixed function normalized by $f_v(e_v) = 1$ for each $v \notin S$, and $\tilde{f}_v$ is the K_v-fixed function in the space of $I_{Q, w\pi_v, -\Lambda}$ is given by the expression on the right-hand side

$$M(s, \pi, t)f = \bigotimes_{v \in S} A(s, \pi_v, t) f_v \otimes \bigotimes_{v \notin S} \tilde{f}_v \cdot \prod_{j=1}^{m} \frac{L_S(js, \pi, \tilde{r}_j)}{L_S(js+1, \pi, \tilde{r}_j)} \tag{3}$$

In the two cases at hand the term of ratios of partial L-series may be described in more detail in the following way. Let ρ_2 denote the 2-dimensional standard representation of $GL_2(\mathbb{C})$, then $\Lambda^2\rho_2$ is the 1-dimensional representation of

$GL_2(\mathbb{C})$ given by the central character ω. Next we need the 4-dimensional representation r_3^o, called the adjoint cube, and defined by $r_3^o = S^3\rho_2 \otimes (\Lambda^2\rho_2)^{-1}$ where $S^k\rho_2$ denotes the kth symmetric power representation of ρ_2. Then we have:

The adjoint action of ${}^LM_1 \cong GL_2(\mathbb{C})$ on ${}^L\mathfrak{n}_1$ is given as $r_3^o \oplus \Lambda^2\rho_2 = r_1 \oplus r_2$ (4)

The adjoint action of ${}^LM_2 \cong GL_2(\mathbb{C})$ on ${}^L\mathfrak{n}_2$ is given as

$$\rho_2 \oplus \Lambda^2\rho_2 \oplus \rho_2 \otimes \Lambda^2\rho_2 = r_1 \oplus r_2 \oplus r_3 \tag{5}$$

This is achieved by a computation carried through e.g. in [24]; see also [48], p. 268 resp. [49], Section 6.

7.3. The poles of the Eisenstein series assigned to a cohomology class $[\varphi] \in H^*_{\text{cusp}}(X^P, E)$ of type (π, w), $w \in W^P$, π an irreducible cuspidal representation of $M(\mathbb{A})$, as in Section 6 are determined by the poles of the intertwining operator $M(s, \pi, t)$, t the non-trivial element in $W(A_P, A_P)$ as defined in 7.2. Depending on the type of $[\varphi]$ we have to study the analytic behaviour at the point $\Lambda_w = -w(\lambda+\rho)|_{\mathfrak{a}_P}$. In view of 6.4–6.6 we may restrict ourselves to cohomology classes of type (π, w), with $l(w) > \frac{1}{2}\dim N$. For simplicity we assume $E \cong \mathbb{C}$ (or equivalently, $\lambda = 0$); compare Remarks 7.7 for the general case.

Since the given irreducible cuspidal representation (π, H_π) of $M(\mathbb{A}) \cong GL_2(\mathbb{A})$ has non-trivial cohomology with respect to the fixed module $F_{\mu_w} \subset H^{l(w)}(\mathfrak{n}, \mathbb{C})$, $w \in W^P$, $l(w) > \frac{1}{2}\dim N$, its infinite component π_∞ is a discrete series representation. The Ramanujan conjecture as proved by Deligne [7] (see [40] for a group representation theoretical formulation) then asserts that each local component π_v of π is a tempered representation of M_v. In such a case Theorems 5.3, 5.4 in [46] imply that the local intertwining operator $A(s, \pi_v, t)$ is holomorphic for $\mathrm{Re}(s) > 0$. For a given $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ one knows that there is a function f_v such that $A(s, \pi_v, t) f_v$ is non-zero (see e.g. [50], Prop. 3.1, p. 332).

Now we choose a finite set S of places of $\mathbb{Q}$ such that S contains the set S_∞ of archimedean places and such that for every $v \notin S$ the representation π_v and G_v are both unramified. If the Eisenstein series assigned to $[\varphi]$ of type (π, w) as above has a pole at the point $\Lambda_w = -w(\lambda+\rho)|_{\mathfrak{a}_P}$ then the ratio of partial L-series

$$\prod_{j=1}^{m} \frac{L_S(js, \pi, \tilde{r}_j)}{L_S(js+1, \pi, \tilde{r}_j)} \tag{1}$$

has a pole at $s = s_w$ (cf. the list in 6.2).

7.4. PROPOSITION. *Given a maximal parabolic* $\mathbb{Q}$*-subgroup* P *of* G *let* $[\varphi]\in H^*_{\mathrm{cusp}}(X^P,\mathbb{C})$ *be a cohomology class of type* (π, w), $w\in W^P$, $l(w)=5,4$, π *an irreducible cuspidal representation of* $M(\mathbb{A})$. *Then the associated Eisenstein series is holomorphic at* $\Lambda_w=-w(\rho)|_{\mathfrak{a}_P}$, *and evaluation at* Λ_w *provides a regular Eisenstein cohomology class* $\mathrm{Eis}[\varphi]$ *in* $H^{l(w)+1}(X,E)$.

Proof. First we consider the case of a maximal parabolic $\mathbb{Q}$-subgroup of type P_2; we may assume $P=P_2$. Taking into account 7.2(5) we have to analyze the analytic behaviour of

$$\frac{L_S(s,\pi,\tilde\rho_2)}{L_S(s+1,\pi,\tilde\rho_2)}\cdot\frac{L_S(2s,\pi,\tilde\omega)}{L_S(2s+1,\pi,\tilde\omega)}\cdot\frac{L_S(3s,\pi,\tilde\rho_2\otimes\tilde\omega)}{L_S(3s+1,\pi,\tilde\rho_2\otimes\tilde\omega)} \tag{1}$$

at $s=3/2$ for w with $l(w)=5$ and $s=1$ for w with $l(w)=4$.

Recall that $L_S(s,\pi,\tilde\rho_2)$ converges absolutely for $\mathrm{Re}(s)>1$ (cf. e.g. [17]). Moreover, $L_S(s,\pi,\tilde\rho_2)$ is non-zero [18] for $\mathrm{Re}(s)\geqslant 1$, and the fact that $L_S(s,\pi,\tilde\rho_2)$ is holomorphic for $\mathrm{Re}(s)=1$ is proved in [9]. Together with the known analytic properties of the partial L-series $L_S(s,\pi,\tilde\omega)$ (see [53]) these facts imply that the ratio of partial L-series in (1) does not have a pole at $s=3/2$ and $s=1$ respectively. In view of the arguments given in 7.3 it follows that the Eisenstein series attached to $[\varphi]$ of type (π,w) does not have a pole at $\Lambda_w=-w(\rho)|_{\mathfrak{a}_P}$. The final assertion is provided by 6.4.

In the case of a maximal parabolic $\mathbb{Q}$-subgroup of type P_1 we have to analyze, following 7.2(4), the analytic behaviour of

$$\frac{L_S(s,\pi,\tilde r_3^0)}{L_S(s+1,\pi,\tilde r_3^0)}\cdot\frac{L_S(2s,\pi,\tilde\omega)}{L_S(2s+1,\pi,\tilde\omega)} \tag{2}$$

at $s=5/2$ for w with $l(w)=5$ and $s=2$ for w with $l(w)=4$.

Observe $L_S(s,\pi,\tilde r_3^0)$ is absolutely convergent in the half-plane $\mathrm{Re}(s)>1$; in particular, the function $L_S(s,\pi,\tilde r_3^0)$ does not vanish for $\mathrm{Re}(s)>1$ (cf. [17], Theorem 5.3). Again, this implies that the ratio of partial L-series in (2) does not have a pole at $s=5/2$ and $s=2$ respectively. As above, this proves the claim.

7.5. PROPOSITION. *Given a maximal parabolic* $\mathbb{Q}$*-subgroup* P *of* G *let* $[\varphi]\in H^4_{\mathrm{cusp}}(X^P,\mathbb{C})$ *be a cohomology class of type* (π,w), $w\in W^P$, $l(w)=3$, π *an irreducible cuspidal representation of* $M(\mathbb{A})$. *In order that the associated Eisenstein series does have a pole at* $\Lambda_w=-w(\rho)|_{\mathfrak{a}_P}$ *it is necessary that the central character* ω *of* π *is trivial and that the partial L-series* $L_S(s,\pi,\tilde\rho_2)$ *in case* P *is of type* 2 *resp.* $L_S(s,\pi,\tilde r_3^0)$ *in case* P *is of type* 1 *(with S large enough) does not vanish at* $s=1/2$. *In turn if this is not the case there is a regular Eisenstein cohomology class* $\mathrm{Eis}[\varphi]$ *in* $H^4(X,\mathbb{C})$.

Proof. In analyzing the analytic behaviour of 7.4(1) resp. (2) at $s = 1/2$ (cf. the list in 6.2 for this value) one sees that the term

$$\frac{L_S(2s, \pi, \tilde{\omega})}{L_S(2s+1, \pi, \tilde{\omega})}$$

does have a pole at $s = 1/2$ if and only if w is trivial. In both cases of maximal parabolic $\mathbb{Q}$-subgroups of G to be considered this pole can be possibly compensated for by a zero of the first L factor in 7.4(1) resp. (2). Observe that the third L-factor in 7.4(1) does not have pole at $s = 1/2$. By the arguments given in 7.3 it follows that the non-vanishing condition as stated together with the condition that the central character ω of π is trivial are necessary ones in order to have a pole.

7.6. COROLLARY. *Let P be a maximal parabolic $\mathbb{Q}$-subgroup of G, and let r_P: $H^*(X, \mathbb{C}) \to H^*(X^P, \mathbb{C})$ be the natural restriction map (cf. 6.5, 6.6 for notation).*

(1) *Then the spaces $H^*(\pi, F_{\mu_w}) \subset H^*_{\mathrm{cusp}}(X^P, \mathbb{C})$ indexed by an irreducible cuspidal automorphic representation π of $M(\mathbb{A}) \cong GL_2(\mathbb{A})$ with non-trivial cohomology with respect to the fixed module F_{μ_w}, $w \in W^P$, $l(w) = 5, 4$, are in the image of r^*_P, i.e. one has $I(\pi, w) = H^*(\pi, F_{\mu_w})$ and $J(\pi', w') = \{0\}$ in this case. The corresponding subspaces*

$$\mathrm{Eis}(I(\pi, w)) \subset H^{l(w)+1}(X, \mathbb{C})$$

are generated by regular Eisenstein cohomology classes.

(2) *Let π be an irreducible cuspidal automorphic representation of $M(\mathbb{A}) \cong GL_2(\mathbb{A})$ with non-trivial cohomology with respect to the fixed module F_{μ_w}, $w \in W^P$, $l(w) = 3$. In order that $I(\pi, w) \neq H^*(\pi, F_{\mu_w})$ and $J(\pi', w') \neq \{0\}$, it is necessary*
   (a) *in case P is of type 1 that the central character ω of π is trivial and that the partial L-series $L_S(s, \pi, \tilde{r}^0_3)$, r^0_3 the 4-dimensional adjoint cube representation of $GL_2(\mathbb{C})$, does not vanish at $s = 1/2$*
   (b) *in case P is of type 2 $L_S(s, \pi, \tilde{\rho}_2)$, ρ_2 the 2-dimensional standard representation of $GL_2(\mathbb{C})$, (with S large enough) does not vanish at $s = 1/2$.*

The corresponding subspaces

$$\mathrm{Eis}(I(\pi, w)) \subset H^4(X, \mathbb{C})$$

resp.

$$\mathrm{Eis}(J(\pi', w')) \subset H^3(X, \mathbb{C})$$

are generated by regular Eisenstein cohomology classes in the first case and by residual ones in the second.

7.7. REMARKS. (1) Observe that it may very well happen that the partial L-series $L_S(s, \pi, \tilde{\rho}_2)$ vanishes at $s = 1/2$. Numerical examples may be given. Following a remark of D. Rohrlich it is very likely, that examples for the vanishing of $L_S(s, \pi, \tilde{r}_3^0)$ at $s = 1/2$ may be constructed as well. This deserves further study.

(2) The condition given in 7.6(2) is not sufficient to ensure that $J(\pi', w')$ is non-trivial. It may very well happen that the pole of the ratio of partial L-series in 7.4(1) resp. (2) is compensated for by a zero of one of the local intertwining operators $A(s, \pi_v, t)$ with $v \in S$. Recall that for these operators there exists a function f_v for which $A(s, \pi_v, t) f_v$ is non-zero. Thus, a suitable choice of the components at these places provides a residual Eisenstein cohomology class i.e. $J(\pi', w') \neq (0)$. However, this discussion leads to a careful study of the reducibility of local representations at places $v \in S$ and the local intertwining operators alluded to.

(3) The non-vanishing cohomology classes in $\mathrm{Eis}(J(\pi', w')) \subset H^3(X, \mathbb{C})$ are square-integrable and correspond to irreducible unitary automorphic representations $\tau = \otimes \tau_v$ of $G(\mathbb{A})$ occurring in the discrete spectrum $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ such that the representation π_∞ is equivalent to $A_{\mathfrak{q}_1}(0)$ in the case P is of type 2 and to $A_{\mathfrak{q}_2}(0)$ in the case P is of type 1. This follows from the fact that $A_{\mathfrak{q}}(0)$ is the Langlands quotient of the principal series representation $\mathrm{Ind}_{P(\mathbb{R}),\delta,\lambda}$ with $\delta = \pi_\infty$ and $\Lambda = \Lambda_w$ as given in 6.2, i.e. $\Lambda_w = \frac{1}{5}\rho_{P_1}$ and $\Lambda_w = \frac{1}{3}\rho_{P_2}$ respectively.

(4) If the highest weight of the given representation (τ, E) of $G \times_{\mathbb{Q}} \bar{\mathbb{Q}}$ is regular then there are no residual Eisenstein cohomology classes contributing to $H^*(X, \tilde{E})$. The remaining cases can be dealt with as above taking into account a shift in the parameter s to the right.

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