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Mean curvature comparison for tubular hypersurfaces in Kähler manifolds and some applications

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Abstract. Given a pair (P, M), where M is a Kähler manifold of real dimension $2n$ and P is a complex submanifold of M of real dimension $2q$, we give lower bounds for the mean curvature of a tubular hypersurface around P in terms of the bounds of some curvatures of M. We prove (and this is the fundamental result) that if the equality is attained for every tubular hypersurface around P, then (P, M) is holomorphically isometric to $(\mathbb{C}P^q(\lambda), \mathbb{C}P^n(\lambda))$. We give some applications to get estimates for the first Dirichlet eigenvalue and the mean exit time of certain domains of M.

1. Introduction

Getting comparison theorems for the mean curvature of tubular hypersurfaces is a main tool to get comparison theorems for other geometric quantities, such as focal distances and volumes ([HK], [Gr1, 2, 3], [GM1], [Gi1], ...), eigenvalues of the laplacian ([Ch], [Ga], [Le], [Ks], [GM2], ...), mean exit time ([DGM], [Ma], [Pa], [MP], ...), heat kernel ([CY], [Ch], ...) and others. The initial aim of this paper was to continue the studies in [Pa] and [MP] on bounds for the mean exit time. However, after the work has been completed, the central point has been to obtain bounds for the mean curvature of tubular hypersurfaces around a complex submanifold P of a Kähler manifold M with bounded curvature (Theorem 2.1) and to characterize the pair $(\mathbb{C}P^q(\lambda), \mathbb{C}P^n(\lambda))$ as the pair (P, M) on which the bounds are attained (Theorem 3.3).

Theorem 2.1 is a stronger version of [Gr3, Lemma 8.31] and [Gi1, Lemma 3.1] as [MP, Theorem 3.1] is a stronger version of [GM1, Theorem 5.1] and [GM2, Theorem 2.3]. As in [MP] and in [Na], this statement has been possible thanks to using Jacobi fields and the index lemma to compare the Weingarten map of the tubular hypersurface instead of getting this comparison directly from the Riccati equation that it satisfies (cfr. [Gr3]). These two methods resulted equivalent in Riemannian geometry (compare [HK] with [Gr1] and [Ro] or [EH]) but in Kähler geometry the Jacobi fields method gives more general results than Riccati equation method when in the hypothesis we have bounds on partial sums of sectional curvatures and not on every sectional curvature

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(compare the results in [Gi1] with the corresponding ones in section 2 of this paper). We do not know if there are some tricks in Riccati equation method which allow us to get results with the same degree of generality that Jacobi fields method. Then we have used the latter method, which gives, not only a result stronger than [Gi, Lemma 3.1], but also new results of this kind. Moreover, the trick used (inspired by [Na]) suggests that these are all the possible bounds we can get on the mean curvature for tubular hypersurfaces around complex submanifolds starting from bounds on the curvature of the ambient Kähler manifold.

The main part of this paper is the proof of Theorem 3.3. It is modelled on the proof of the corresponding theorem for the riemannian case given by E. Heintze and H. Karcher ([HK, section 4 and 5]). There are two main significant, interesting and nice differences between the Riemannian and Kähler cases. The first one is that in the Kähler case we do not characterize directly the submanifold P, but through a certain fibre bundle S on P with fibre S^1 which is diffeomorphic to S^{2q+1}. The other one is that in the Kähler case the normal bundle to P is not flat, but, as we shall show, it is still possible to determine its normal connection.

An immediate consequence of Theorem 3.3 is the generalization of [Gi1, Theorem 3.4] for every complex dimension q of the complex submanifold P, solving a problem posed in [Gi1, Remark 3.5] in the more general context of our Theorem 2.1.

Finally, in section 4, we apply theorems 2.1 and 3.3 to get sharp comparison theorems on the mean exit time and the Dirichlet first eigenvalue of a tube about a complex submanifold in a Kähler manifold. The results on this last problem complete the work of [Le] and [Ga] for Riemannian manifolds and of [Gi2] on geodesic balls of Kähler manifolds. We also complete the works [GM2] and [MP] by comparing arbitrary domains with tubes about $\mathbb{C}P^{n-1}(\lambda)$ instead of with geodesic spheres. The paragraph finishes with Theorem 4.8, a result really striking for us, because it says that the relative behaviour of the Dirichlet first eigenvalue on a tube and its complementary is just in the opposite sense that the relative behaviour of the volume (cfr. [Gi1, Theorem 4.1 and Corollary 4.2]).

Now, some comments on the notation we shall use. From now on M will denote a connected complete Kähler manifold of real dimension $2n$, with Riemannian metric $<, >$, and almost complex structure J. By P we shall denote a connected closed complex submanifold of M of real dimension $2q$.

Given any connected (real or complex) closed submanifold Q of M, Q_r will denote the tube of radius r around Q, and ∂Q_r will denote its boundary, i.e., the tubular hypersurface of radius r.

For the curvature and the Riemann Christoffel tensor we shall adopt the following convention sign

$$R(X, Y)Z = -[\nabla_X, \nabla_Y]Z + \nabla_{[X,Y]}Z \quad \text{and} \quad R_{X,Y,Z,W} = \langle R(X, Y)Z, W\rangle.$$

Given a point $p \in M$, a vector $X \in T_pM$ and a holomorphic subspace Π of T_pM of real dimension $2r$ and orthogonal to X, the antiholomorphic r-sectional curvature $K(X, \Pi)$ of X at Π is defined by

$$K(X, \Pi) = \sum_{i=1}^{2r} R_{Xe_iXe_i},$$

where $\{e_1, Je_1 = e_2, \ldots, e_{2r-1}, Je_{2r-1} = e_{2r}\}$ is a J-orthonormal basis of Π. This concept is just the restriction of the $2r$-mean curvature defined in [BC, page 253] to the holomorphic planes. Then $K(X, \Pi)$ depends only on the $2r$-plane Π and on X. When $r = n - 1$, the plane Π is uniquely determined, $K(X, \Pi)$ depends only on X and is called the *antiholomorphic Ricci curvature* $\rho_A(X)$ of X. We remark that $\rho(X, X) = \rho_A(X) + K_H(X)|X|^2$, where ρ is the Ricci curvature of M and $K_H(X)$ is the holomorphic sectional curvature of the plane generated by X and JX.

We shall denote by $(\mathscr{S}\mathscr{N}P)$ $\mathscr{N}P$ the (unit) normal bundle of P in M, and by $\mathscr{N}_pP$ (resp. $\mathscr{S}\mathscr{N}_pP$) the fibre of $\mathscr{N}P$ (resp. $\mathscr{S}\mathscr{N}P$) over $p \in P$. For every $N \in \mathscr{S}\mathscr{N}P$, L_N will denote the Weingarten map of P associated to N. For every $e \in T_pM$, e^{T}, $e^{\perp}$ and $\{e\}^{\perp}$ will denote the component of e in T_pP, the component of e in $\mathscr{N}_pP$, and the orthogonal complement of the vector space generated by e in T_pM respectively.

Given any fibre bundle B on P and $p \in P$, B_p will denote the fibre of B over $p \in P$.

Given any Riemannian vector bundle V on any manifold Q, we shall denote by $V(t)$ the set $\{\zeta \in V / |\zeta| = t\}$.

We shall use $'$ to denote indistinctly the ordinary and the covariant derivative. Its exact meaning will be clear from the context.

2. The comparison theorem for the mean curvature of a tubular hypersurface

First we recall some necessary background. Given $p \in P$, $N \in \mathscr{S}\mathscr{N}_pP$, and the geodesic $\gamma_N(t)$ of M satisfying $\gamma_N(0) = p$, $\gamma_N'(0) = N$, the Jacobi operator $\mathscr{A}(t)$ is defined as the map

$$\mathscr{A}(t)\colon \{N\}^{\perp} \to \{N\}^{\perp} \text{ such that } \mathscr{A}(t)e = \tau_t^{-1} Y(t),$$

where $Y(t)$ is the P-Jacobi field along $\gamma_N(t)$ such that $Y(0) = e^{\mathrm{T}}$ and $\nabla_N Y + L_N e^{\mathrm{T}} = e^{\perp}$, and τ_t is the parallel transport along $\gamma_N(t)$.

If $f(N) = \inf\{t > 0 / \gamma_N(t) \text{ is a focal point of } P\}$, then

$$f(N) = \inf\{t > 0 / \operatorname{rank} \mathscr{A}(t) < 2n - 1\} \tag{2.1}$$

and

$$\operatorname{rank}\mathscr{A}(t) = \operatorname{rank}\exp_{\mathscr{N}P*tN} - 1 = \operatorname{rank}\exp_{\mathscr{N}P(t)*tN} \tag{2.2}$$

where $\exp_B$ denotes the restriction of the exponential map to the subset $\mathscr{B}$ of TM.

$S(t)$ will denote the Weingarten map of the tubular hypersurface of radius t about P with respect to the unit normal vector $\gamma'_N(t)$. The operators $\mathscr{A}(t)$ and $S(t)$ are related by

$$S(t) = -\mathscr{A}'(t)\mathscr{A}^{-1}(t) \tag{2.3}$$

as follows from [Ka, (1.2.6)] and is explicitly written in [CV] for geodesic spheres.

If $Z(t)$ is a vector field along $\gamma_N(t)$ such that $Z(0) \in T_pP$, the index form $I_0^t(Z)$ of Z is defined by

$$I_0^t(Z) = -\langle Z, L_N Z\rangle(0) + \int_0^t (\langle Z', Z'\rangle - \langle R(s)Z, Z\rangle)\,\mathrm{d}s, \tag{2.4}$$

where Z' denotes the covariant derivative of Z along $\gamma_N(t)$ and $R(s)Z \equiv R(\gamma'_N(s), Z(s))\gamma'_N(s)$. The index lemma for submanifolds (cfr. [BC, page 228]) says that if $Y(s)$ is a P-Jacobi field along $\gamma_N(s)$ such that $Y(t) = Z(t)$, then

$$I_0^t(Z) \geqslant I_0^t(Y), \tag{2.5}$$

and the equality holds iff $Y(s) = Z(s)$ for every s in $[0, t]$.

If $\{Y_i(s)\}_{1 \leqslant i \leqslant 2n-1}$ are P-Jacobi fields along $\gamma_N(s)$ such that $\{Y_i(t)\}_{1 \leqslant i \leqslant 2n-1}$ is an orthonormal basis of $\{\gamma'_N(t)\}^\perp$, then, applying (2.3), we have

$$\begin{aligned}\operatorname{tr} S(t) &= \sum_{i=1}^{2n-1} \langle S(t)Y_i(t), Y_i(t)\rangle \\ &= -\sum_{i=1}^{2n-1} \langle Y_i'(t), Y_i(t)\rangle = -\sum_{i=1}^{2n-1} I_0^t(Y_i)\end{aligned} \tag{2.6}$$

When $P = \mathbb{C}P^q(\lambda)$ and $M = \mathbb{C}P^n(\lambda)$, the operators $S(t)$ and $\mathscr{A}(t)$ along $\gamma_N(t)$ will be denoted by $S_\lambda^q(t)$ and $\mathscr{A}_\lambda(t)$ respectively. If $E(t)$ and $U(t)$ are parallel unit vector fields along $\gamma_N(t)$ such that $E(0) \in T_pP$ and $U(0) \in \mathscr{N}_pP \cap \{N, JN\}^\perp$, then (cfr [Gi1])

$$\begin{aligned}&\mathscr{A}_\lambda(t)E(T) = \cos(\sqrt{\lambda}t)E(t), \\ &\mathscr{A}_\lambda(t)U(t) = \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\,U(t), \\ &\mathscr{A}_\lambda(t)J\gamma'_N(t) = \frac{\sin(2\sqrt{\lambda}t)}{2\sqrt{\lambda}}\,J\gamma'_N(t)\end{aligned} \tag{2.7}$$

and

$$
\begin{aligned}
S_\lambda^q(t)E(T) &= \sqrt{\lambda}\tan(\sqrt{\lambda}t)E(t),\\
S_\lambda^q(t)U(t) &= -\sqrt{\lambda}\cot(\sqrt{\lambda}t)U(t), \qquad (2.8)\\
S_\lambda^q(t)J\gamma_N'(t) &= -2\sqrt{\lambda}\cot(2\sqrt{\lambda}t)J\gamma_N'(t)
\end{aligned}
$$

We shall denote the *trace of* $S_\lambda^q(t)$ by $\sigma(t)$ and the *determinant of* $\mathscr{A}_\lambda(t)$ by $\alpha(t)$. By s_λ and c_λ we shall denote the functions

$$s_\lambda(s, t) = \frac{\sin(\sqrt{\lambda}s)}{\sin(\sqrt{\lambda}t)} \quad \text{and} \quad c_\lambda(s, t) = \frac{\cos(\sqrt{\lambda}s)}{\cos(\sqrt{\lambda}t)}.$$

Let us observe that $s_{2\lambda}(s, t) = s_\lambda(s, t)c_\lambda(s, t)$.

We shall consider the following orthogonal direct sum decomposition

$$T_{\gamma_N(t)}M = H_t \oplus \langle\{\gamma_N'(t), J\gamma_N'(t)\}\rangle \oplus V_t,$$

where $H_t = \tau_t T_pP$ and $\langle\{\gamma_N'(t), J\gamma_N'(t)\}\rangle$ is the vector space generated by $\gamma_N'(t)$ and $J\gamma_N'(t)$.

2.1. THEOREM. *Let us assume that, for every $p \in P$ and $N \in \mathscr{SN}_pP$, one of the following conditions holds for every $t \in [0, R]$, $R \leqslant f(N)$,*

(a) $\rho(\gamma_N'(t), \gamma_N'(t)) \geqslant (2n+2)\lambda$, $K(\gamma_N'(t), V_t) \leqslant 2(n-q-1)\lambda$ *and* $K_H(\gamma_N'(t)) \leqslant 4\lambda$;
(b) $\rho(\gamma_N'(t), \gamma_N'(t)) \geqslant (2n+2)\lambda$, $K(\gamma_N'(t), H_t) \geqslant 2q\lambda$ *and* $K_H(\gamma_N'(t)) \geqslant 4\lambda$;
(c) $\rho(\gamma_N'(t), \gamma_N'(t)) \geqslant (2n+2)\lambda$, $K(\gamma_N'(t), V_t) \leqslant 2(n-q-1)\lambda$ *and* $K(\gamma_N'(t), H_t) \geqslant 2q\lambda$;

then

$$\operatorname{tr} S(t) \geqslant \sigma(t) \textit{ for every } t \in [0, R] \qquad (2.1.1)$$

and $R \leqslant \pi/2\sqrt{\lambda}$.

Proof. Let $\{E_i(s)\}_{i=1}^{2n-1}$ be a parallel orthonormal J-frame of $\{\gamma_N'(s)\}^\perp$ along $\gamma_N(s)$ such that $\{E_1(0), \ldots, E_{2q}(0)\}$ is a J-basis of T_pP and $E_{2n-1}(s) = J\gamma_N'(s)$. Let $Y_1(s), Y_2(s), \ldots, Y_{2n-1}(s)$ be the P-Jacobi fields along $\gamma_N(s)$ satisfying $Y_i(t) = E_i(t)$. Let us define the vector fields $Z_i(s) = f_i(s, t)E_i(s)$ along $\gamma_N(s)$ with

$f_i(s, t) = c_\lambda(s, t)$ if $i = i = 1, \ldots, 2q$;

$f_i(s, t) = s_\lambda(s, t)$ if $i = 2q+1, \ldots, 2n-2$; and

$f_{2n-1}(s, t) = s_{2\lambda}(s, t)$.

Then, from (2.5), (2.6), (2.4) and the fact that complex submanifolds are minimal submanifolds, we get

$$\begin{aligned}\operatorname{tr} S(t) &= -\sum_{i=1}^{2n-1} I_0^t(Y_i) \geqslant -\sum_{i=1}^{2n-1} I_0^t(Z_i)\\ &= -\int_0^t (2qc_\lambda'^2(s,t) + 2(n-q-1)s_\lambda'^2(s,t) + s_{2\lambda}'^2(s,t))\,\mathrm{d}s\\ &\quad + \int_0^t (c_\lambda^2(s,t)\sum_{i=1}^{2q} R_{\gamma_N' E_i \gamma_N' E_i} + s_\lambda^2(s,t)\sum_{i=2q+1}^{2n-2} R_{\gamma_N' E_i \gamma_N' E_i}\\ &\quad + s_{2\lambda}^2(s,t) R_{\gamma_N' J\gamma_N' \gamma_N' J\gamma_N'})\,\mathrm{d}s.\end{aligned}$$

Now, for $0 \leqslant s \leqslant t \leqslant \pi/(2\sqrt{\lambda})$, $c_\lambda - s_\lambda \geqslant 0$, $c_\lambda - s_{2\lambda} \geqslant 0$ and $s_\lambda - s_{2\lambda} \leqslant 0$. Then, from (2.7), we have that:

If (a) holds,

$$\begin{aligned}\operatorname{tr} S(t) &\geqslant -\int_0^t (2qc_\lambda'^2 + 2(n-q-1)s_\lambda'^2 + s_{2\lambda}'^2)\,\mathrm{d}s\\ &\quad + \int_0^t (c_\lambda^2 \rho(\gamma_N'(s), \gamma_N'(s)) + (s_\lambda^2 - c_\lambda^2)K(\gamma_N'(s), V_s)\\ &\quad + (s_{2\lambda}^2 - c_\lambda^2)K_H(\gamma_N'(s))\,\mathrm{d}s\\ &\geqslant -\int_0^t (2qc_\lambda'^2 + 2(n-q-1)s_\lambda'^2 + s_{2\lambda}'^2)\,\mathrm{d}s\\ &\quad + \int_0^t (c_\lambda^2(2n+2)\lambda + (s_\lambda^2 - c_\lambda^2)2(n-q-1)\lambda + (s_{2\lambda}^2 - c_\lambda^2)4\lambda)\,\mathrm{d}s = \sigma(t);\end{aligned}$$

if (b) holds,

$$\begin{aligned}\operatorname{tr} S(t) &\geqslant -\int_0^t (2qc_\lambda'^2 + 2(n-q-1)s_\lambda'^2 + s_{2\lambda}'^2)\,\mathrm{d}s\\ &\quad + \int_0^t (s_\lambda^2 \rho(\gamma_N'(s), \gamma_N'(s)) + (c_\lambda^2 - s_\lambda^2)K(\gamma_N'(s), H_s)\\ &\quad + (s_{2\lambda}^2 - s_\lambda^2)K_H(\gamma_N'(s))\,\mathrm{d}s \geqslant \sigma(t);\end{aligned}$$

and, if (c) holds,

$$\begin{aligned}\operatorname{tr} S(t) &\geqslant -\int_0^t (2qc_\lambda'^2 + 2(n-q-1)s_\lambda'^2 + s_{2\lambda}'^2)\,\mathrm{d}s\\ &\quad + \int_0^t (s_{2\lambda}^2 \rho(\gamma_N'(s), \gamma_N'(s)) + (c_\lambda^2 - s_{2\lambda}^2)K(\gamma_N'(s), H_s)\\ &\quad + (s_\lambda^2 - s_{2\lambda}^2)K(\gamma_N'(s), V_s)\,\mathrm{d}s \geqslant \sigma(t);\end{aligned}$$

From (2.3), $\operatorname{tr} S(t) \geqslant \sigma(t)$ implies $-\Sigma_{i=1}^{2n-1} \langle \mathscr{A}'\mathscr{A}^{-1}E_i, E_i\rangle \geqslant \sigma(t) = -\mathrm{d}/\mathrm{d}t \ln \alpha(t)$, and, since $-\Sigma_{i=1}^{2n-1} \langle \mathscr{A}'\mathscr{A}^{-1}E_i, E_i\rangle = \mathrm{d}/\mathrm{d}t(\ln \det \mathscr{A}(t))$ and $\lim_{t\to 0} \det \mathscr{A}(t) = \lim_{t\to 0} \alpha(t)$ (as follows from the initial conditions for the Jacobi fields), we have that $\det \mathscr{A}(t) \leqslant \alpha(t)$. Then, from (2.1) and (2.2), we have $f(N) \leqslant \pi/(2\sqrt{\lambda})$.

From this theorem, the arguments used in [Gi1] give the following generalization of [Gi1, Theorem 3.3].

2.2. COROLLARY. *Under the hypothesis of Theorem* 2.1, *we have*

$$\frac{\operatorname{vol}(P)}{\operatorname{vol}(M)} \geqslant \frac{\operatorname{vol}(\mathbb{C}P^q(\lambda))}{\operatorname{vol}(\mathbb{C}P^n(\lambda))}$$

3. Characterizing the equality

First, we shall give two well-known lemmas, with the proof of the first one because it is not very easy to find it in the books under this form

3.1. LEMMA. *For every $(p, N) \in \mathscr{S}\mathscr{N}P$ and every $r \in \mathbb{R}$, the kernel of $\exp_{\mathscr{N}P*(p,rN)}$ is the set of vectors $(c'(0), r\xi'(0)) \in T_{(p,rN)}\mathscr{N}P$ tangent to curves $(c(s), r\xi(s))$ in $\mathscr{N}P$ with $(c(0), \xi(0)) = (p, N)$ and such that the P-Jacobi field $Y(t)$ along $\gamma_N(t)$ satisfying $Y(0) = c'(0)$ and $Y'(0) = (\nabla/dt)\xi(0)$ also satisfies $Y(r) = 0$.*

Proof. Let $f(t, s) = \exp_{c(s)} t\xi(s)$. Every P-Jacobi field $Y(t)$ along a geodesic $\gamma_N(t)$ starting from P has the form $Y(t) = \partial f/\partial s(t, 0)$ for some ξ and c such that $\xi(0) = N$. Then, a simple computation gives that

$$Y'(0) = (\nabla_{c'(0)}\xi)(0) \tag{3.1}$$

On the other hand,

$$\exp_{\mathscr{N}P*(c(s),r\xi(s))}(c'(s), r\xi'(s)) = \frac{\partial}{\partial s}\exp_{\mathscr{N}P}(c(s), r\xi(s)) = \frac{\partial}{\partial s} f(r, s)$$

and

$$\exp_{\mathscr{N}P*(c(0),r\xi(0))}(c'(0), r\xi'(0)) = \frac{\partial}{\partial s} f(r, 0) = Y(r), \tag{3.2}$$

hence

$$\operatorname{Ker} \exp_{\mathscr{N}P*(c(0),r\xi(0))} = \{(c'(0), r\xi'(0)) \in T_{(c(0),\xi(0))}\mathscr{N}P / Y(r) = 0, \tag{3.3}$$

$$\text{where } Y(t) = \frac{\partial f}{\partial s}(t, 0)\}.$$

The combination of (3.1) and (3.3) gives the lemma.

3.2. LEMMA ([HK, Lemma 5.6] and [Gi1, Proposition 2.5]). *The Kähler structure (metric and almost complex structure) of M is determined by the Kähler structure of P, the normal connection on $\mathscr{N}P$, the almost-complex structure of $\mathscr{N}P$, and the Jacobi operators $\mathscr{A}(t) \equiv \mathscr{A}(t, p, N)$ for every $p \in P$, $N \in \mathscr{S}\mathscr{N}_pP$ and $t \in [0, c(N)[$, where $c(N) = \sup\{t/d(\gamma_N(t), P) = t\}$.*

3.3. THEOREM. *Let M and P be as in 2.1(a), (b) or (c). If $\operatorname{tr} S(t) = \sigma(t)$ for every $p \in P$, $N \in \mathscr{S}\mathscr{N}_pP$ and $t \in [0, f(N)[$, then there is a holomorphic isometry $i: M \to \mathbb{C}P^n(\lambda)$ which, restricted to P, gives a holomorphic isometry between P and $\mathbb{C}P^q(\lambda)$.*

Proof. We shall do it in several steps. First we shall suppose that $0 < q < n - 1$.

Step 1. We shall show that P is a totally geodesic submanifold of M, $f(N) = \pi/(2\sqrt{\lambda})$ (for every $N \in SN_pP$ and $p \in P$), the normal connection on $\mathscr{N}P$ is just the restriction of the Levi-Civita connection on TM, and $\mathscr{A}(t)$ is given by the formulae (2.7). We shall refer to this last fact by writing $\mathscr{A}(t) = \mathscr{A}_\lambda(t)$.

In fact, $\operatorname{tr} S(t) = \sigma(t)$ for every $t \in [0, f(N)[$ implies that $I_0^t(Y_i) = I_0^t(Z_i)$ for every $t \in [0, f(N)[$, which implies, by the index lemma for submanifolds, $Z_i(t) = Y_i(t)$ and $\mathscr{A}(t) = \mathscr{A}_\lambda(t)$ for every $t \in [0, f(N)[$. Since $f(N)$ is the first zero of $\det \mathscr{A}(t)$, the precedent equality implies $f(N) = \pi/(2\sqrt{\lambda})$. Then, from (2.3), we also have $S(t) = S_\lambda^q(t)$. Since $L_N = \lim_{t \to 0} S(t)|_{H_t}$ (see [Gr3, page 38]), we have $L_N = 0$.

Step 2. Let U be a simply connected open neighbourhood of p in P, $N \in \mathscr{S}\mathscr{N}_pP$ and Π the holomorphic plane containing N. Then we shall prove that: (a) The parallel transport of Π along a curve in U does not depend on the curve, and hence it defines a complex line bundle on U, which will also be denoted by Π, and is an holomorphic vector subbundle of $\mathscr{N}U \equiv \mathscr{N}P|_U$; and (b) $\exp(\Pi(\pi/2\sqrt{\lambda}))$ is a fixed point $y \in M$.

In fact, since $\mathscr{A}(t) = \mathscr{A}_\lambda(t)$, $\{Z_i(t)\}_{i=1}^{2n-1}$ is a basis of the P-Jacobi fields along $\gamma_N(t)$. Then $\{Z_1, \ldots, Z_{2q}, Z_{2n-1}\}$ is a basis of the P-Jacobi fields along $\gamma_N(t)$ vanishing at $\pi/(2\sqrt{\lambda})$. Then, from Lemma 3.1, a vector is in $\operatorname{Ker} \exp_{\mathscr{N}P*(p, \pi/(2\sqrt{\lambda})N)}$ if and only if it can be decomposed as the sum of two vectors $(c_i'(t_0), \pi/(2\sqrt{\lambda})N_i'(t_0))$, $i = 1, 2$, such that

$$c_1'(t_0) \in T_{c_1(t_0)}P \quad \text{and} \quad \frac{\nabla N_1}{dt}(t_0) = 0$$

and

$$c_2'(t_0) = 0 \quad \text{and} \quad \frac{\nabla N_2}{dt}(t_0) = \nu JN \text{ for some } \nu \in \mathbb{R}.$$

Now, let c be a closed curve starting from $p \in P$, contained in U. Since U is simply connected, there is a differentiable homotopy $H(t, \beta)$ from the constant curve c_p to c. For every β, H defines a curve $\delta_\beta : t \to H(t, \beta)$, and we can consider the unit normal vector field $N_\beta(t)$ parallel along δ_β and satisfying $N_\beta(0) = N$. Then

$$\left(\delta'_\beta(t), \frac{\pi}{2\sqrt{\lambda}} N'_\beta(t)\right) \in \operatorname{Ker} \exp_{\mathcal{N}P*(\delta_\beta(t), \pi/(2\sqrt{\lambda})N_\beta(t))}$$

and $\exp(\delta_\beta(t), \pi/(2\sqrt{\lambda})N_\beta(t))$ is a constant point $y \in M$ for every t. But $(\delta_\beta(0), N_\beta(0)) = (p, N)$ for every β, then

$$\exp\left(\delta_\beta(t), \frac{\pi}{2\sqrt{\lambda}} N_\beta(t)\right) = y \text{ for every } (t, \beta) \tag{3.3.1}$$

In particular, when $t = 1$, $\exp(p, \pi/(2\sqrt{\lambda})N_\beta(1)) = y$ for every β, and this implies that

$$\left(0, \frac{\pi}{2\sqrt{\lambda}} \frac{\mathrm{d}}{\mathrm{d}\beta} N_\beta(1)\right) \in \operatorname{Ker} \exp_{\mathcal{N}P*(p, \pi/(2\sqrt{\lambda})N_\beta(1))}, \tag{3.3.2}$$

which implies that, for every β, there is a curve $\xi_\beta(s)$ in $\mathcal{SN}_pP$ satisfying $\xi_\beta(0) = N_\beta(1)$ and $\mathrm{d}/\mathrm{d}s\,\xi_\beta(0) = vJN_\beta(1)$ such that $\mathrm{d}/\mathrm{d}\beta\, N_\beta(1) = \eta(\beta)(\mathrm{d}/\mathrm{d}s)\xi_\beta(0)$ for some function $\eta(\beta)$, i.e., taking $v\eta(\beta) = \mu(\beta)$, we have that $N_\beta(1)$ must satisfy the equation

$$\frac{\mathrm{d}N_\beta(1)}{\mathrm{d}\beta} = \mu(\beta) J N_\beta(1) \tag{3.3.3}$$

with the initial condition $N_0(1) = N$, since $N_0(t)$ is the parallel transport of N along the curve $\delta_0(t) = H(t, 0) = c_p(t) = p$. Then, by the unicity of the solutions of a differential equation, $N_\beta(1)$ must be of the form

$$N_\beta(1) = f(\beta)N + g(\beta)JN, \quad \text{with } f^2 + g^2 = 1.$$

This proves (a). (b) is a consequence of (a) and (3.3.2). In fact, since $\Pi(\pi/2\sqrt{\lambda})$ is connected and $\exp(p, (\pi/2\sqrt{\lambda})N) = y$, we have only to prove that for any $(x, N) \in S = \{(x, N) \in \Pi / |N| = 1\}$, $\exp_{\mathcal{N}P*(x, \pi/(2\sqrt{\lambda})N)}|_{T\Pi(\pi/(2\sqrt{\lambda}))} = 0$. Given $W \in T_{(x,N)}\Pi \cong T_xP \oplus T_N\Pi(\pi/2\sqrt{\lambda})_x$, we can decompose $W = Y + Z$, with $Y \in T_xP$ and $Z \in T_N\Pi(\pi/2\sqrt{\lambda})_x$. Obviously $Y \in \operatorname{Ker} \exp_{\mathcal{N}P*(x, \pi/(2\sqrt{\lambda})N)}$. Moreover, since $\Pi(\pi/2\sqrt{\lambda})_x$ is a circle, Z is orthogonal to N, and, then, tangent to JN, whence $Z \in \operatorname{Ker} \exp_{\mathcal{N}P*(x,(\pi/2\sqrt{\lambda})N)}$, and (b) is proved.

Step 3. We claim that P is holomorphically isometric to $\mathbb{C}P^q(\lambda)$.

The idea for proving this step is the following: to treat U and the exp-image of its bundle Π in M as if they were $\mathbb{C}P^q \subset \mathbb{C}P^{q+1}$; then, as the curvature along the normal geodesics behaves as in this standard situation, this eventually will give that the S^1-bundle of Π is isometric to an open set of S^{2q+1}, hence every point p in P will have a neighbourhood isometric to an open in $\mathbb{C}P^q$, and therefore P will have constant holomorphic sectional curvature.

In fact, let Π, U, and S be as in step 2. S is a fibre bundle on U with fibre S^1. From step 2(b), the map

$$\Phi: S \to S^{2n-1} \subset T_y M \text{ such that } \Phi(N) = \gamma_N'\left(\frac{\pi}{2\sqrt{\lambda}}\right)$$

is well defined. On the other hand, the equality $\mathcal{A}(t) = \mathcal{A}_\lambda(t)$ implies, by (2.2), that $\operatorname{rank} \exp_{\mathcal{N}U(\pi/2\sqrt{\lambda})*(p,\pi/(2\sqrt{\lambda})N)} = 2(n-q-1)$ for every $(p,N) \in S$. Then, from the constant rank theorem ([Bo, page 70]), there is an open set W of $\mathcal{N}U(\pi/2\sqrt{\lambda})$ such that $(p, \pi/(2\sqrt{\lambda})N) \in W$, $\exp(W)$ is a submanifold of M of real dimension $2(n-q-1)$ and $y \in \exp(W)$. Moreover, from (3.3) and the expression of $\mathcal{A}(t)$, it follows that $\exp(W)$ is a complex submanifold of M. From the generalized Gauss lemma, which states the orthogonality between the geodesics γ_N' and the tubular hypersurfaces (see [Gr3, page 28], [HK, page 468] and [Gi2, page 19]) we have that $\gamma_N'(\pi/2\sqrt{\lambda})$ is orthogonal to $T_y \exp W$ for every $N \in S$. Then $\Phi(S)$ is contained in a complex subspace E of $T_y M$ of real dimension $2q+2$, and $\Phi(S) \subset S^{2q+1} = E \cap S^{2n+1}$.

Now, let us study Φ_*. Given $(p,N) \in S$, let $\{e_1, \ldots, e_{2q}, e_{2n-1} \equiv JN\}$ be an orthonormal basis of $T_{(p,N)}S \equiv T_pP \oplus T_N S_p$. Let $(c_i(t), N_i(t))$ be curves on S such that $(c_i(t_0), N_i(t_0)) = (p,N)$, $c_i'(t_0) = e_i$ and $(\nabla N_i/\mathrm{d}t)(t_0) = 0$ for $i = 1, \ldots, 2q$ and $c_2'(t_0) = 0$ and $(\nabla N_{2n-1}/\mathrm{d}t)(t_0) = 2\sqrt{\lambda} JN$. If $E_i(t) = \tau_t e_i$, and $Z_i(t)$ is defined from E_i as in the proof of Theorem 2.1, then

$$\Phi_{*(p,N)}(e_i) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \gamma_{N_i(t)}'\left(\frac{\pi}{2\sqrt{\lambda}}\right) = \frac{\nabla}{dt}\frac{\partial}{\partial s}\exp_{c_i(t)} sN_i(t)|_{t=t_0, s=(\pi/2\sqrt{\lambda})} \tag{3.3.4}$$

$$= Z_i'\left(\frac{\pi}{2\sqrt{\lambda}}\right) = \begin{cases} -\sqrt{\lambda}E_i\left(\dfrac{\pi}{2\sqrt{\lambda}}\right), & \text{for } i = 1, \ldots, 2q \\ -E_{2n-1}\left(\dfrac{\pi}{2\sqrt{\lambda}}\right) = J\gamma_{N_{2n-1}(t_0)}'\left(\dfrac{\pi}{2\sqrt{\lambda}}\right), & \text{for } i = 2n-1. \end{cases}$$

Let $N(t)$ be a curve in S_x. Then $N'(t)$ must be perpendicular to $N(t)$, and there is a function $\mu(t)$ such that $N'(t) = \mu(t)JN(t)$, and a computation similar to the one done just before gives that $\mathrm{d}/\mathrm{d}t\,\Phi(x, N(t)) = Y'(\pi/2\sqrt{\lambda})$, $Y(s)$ being the Jacobi field along $\gamma_{N(t)}(s)$ such that $Y(0) = 0$ and $Y'(0) = \mu(t)JN(t)$. Then $Y(s) = \alpha(t)Z_{2n-1}(s)$ and $Y'(\pi/2\sqrt{\lambda}) = \alpha(t)J\gamma'_{N_{2n-1}(t)}(\pi/2\sqrt{\lambda}) = \alpha(t)J\Phi(x, N(t))$. Then, from the unicity of the solutions of a differential equation, $\Phi(x, N(t))$ is a linear combination of $\Phi(x, N(0))$ and $J\Phi(x, N(0))$. This means that, if π_2 is the canonical projection from S^{2q+1} onto $\mathbb{C}P^q(1)$, then $\pi_2 \circ \Phi(x, N(t)) = \pi_2 \circ \Phi(x, N(0))$ for every t; i.e., if π_1 is the canonical projection between S and P, then $\pi_1(N) = \pi_1(\xi)$ implies $\pi_2 \circ \Phi(N) = \pi_2 \circ \Phi(\xi)$. That allows us to define the map $\psi: U \subset P \to \mathbb{C}P^q(1)$ by requiring that

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & S^{2q+1} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ U & \xrightarrow{\psi} & \mathbb{C}P^q(1) \end{array}$$

be a commutative diagram. Since $\{E_i(\pi/2\sqrt{\lambda})\}_{i=1}^{2q}$ are horizontal vectors for the Riemannian submersion $\pi_2: S^{2q+1} \to \mathbb{C}P^q(1)$, it follows that ψ_*, which takes $E_i(0)$ into $\pi_{2*}(-\sqrt{\lambda}E_i(\pi/2\sqrt{\lambda}))$, is a biholomorphic isometry up to the constant factor $\sqrt{\lambda}$. This proves that P has a constant holomorphic sectional curvature 4λ and finishes the proof of step 3.

Step 4. Because P is $\mathbb{C}P^q(\lambda)$, P is simply connected. Then we can take $U = P$ in step 2. Here we claim that, with this choice, $\exp(\{tN/N \in S, 0 \leqslant t \leqslant \pi/2\sqrt{\lambda}\})$ is holomorphically isometric to $\mathbb{C}P^{q+1}(\lambda)$.

In fact, we know that $\exp(\Pi(\pi/2\sqrt{\lambda})) = y$, and that $\Phi(S) \subset S^{2q+1} \subset T_yM$. Since S is compact, $\Phi(S)$ is closed in S^{2q+1}, and, since Φ is a local diffeomorphism (as follows from (3.3.4)), $\Phi(S)$ is open in S^{2q+1}. Then $\Phi(S) = S^{2q+1}$ and $Q \equiv \exp(\{tN/N \in S, \quad 0 \leqslant t \leqslant \pi/2\sqrt{\lambda}\}) = \exp_y(\{t\xi/\xi \in S^{2q+1}, \quad 0 \leqslant t \leqslant \pi/2\sqrt{\lambda}\})$. Since the P-Jacobi fields along γ_N vanish at y, they are also $\{y\}$-Jacobi fields along $\gamma_\xi(t) = \gamma_N(\pi/2\sqrt{\lambda} - t)$. Then the Jacobi operators on Q for the geodesics starting from y are the same that those in $\mathbb{C}P^{q+1}(\lambda)$. Then, from Lemma 3.2, Q is holomorphically isometric to $\mathbb{C}P^{q+1}(\lambda)$.

Step 5. According to step 2(a), we can write $\mathcal{N}P = \Pi_1 \oplus \cdots \oplus \Pi_{n-q}$, each Π_i being a complex line bundle. Analogously, we can write $\mathcal{N}\mathbb{C}P^q(\lambda) = \Pi_1^\lambda \oplus \cdots \oplus \Pi_{n-q}^\lambda$. Let us identify P and $\mathbb{C}P^q(\lambda)$, let $\{\xi_i, J\xi_i\}$ (respec. $\{\xi_i^\lambda, J\xi_i^\lambda\}$) be a local orthonormal frame of Π_i (respec. Π_i^λ)). Then we can define a local biholomorphism by the correspondence $\xi_i \to \xi_i^\lambda$ for every $i = 1, \ldots, n-q$. This biholomorphism preserves the normal connections on $\mathcal{N}P$ and $\mathcal{N}\mathbb{C}P^q(\lambda)$.

In fact, from the constructions of the Π_i (resp. Π_i^λ), the connections on these vector bundles are the restrictions to them of the normal connection on $\mathcal{N}P$ (respec. $\mathcal{N}\mathbb{C}P^q(\lambda)$). On the other hand, from step 4, Π_i (resp. Π_i^λ) is the normal bundle of P in a certain $\mathcal{N}\mathbb{C}P^{q+1}(\lambda)$, then the connection we are considering on it is just the normal connection of P in $\mathbb{C}P^{q+1}(\lambda)$, and it is the same on Π_i than on Π_i^λ.

Step 6. The theorem is now a consequence of Lemma 2.2, steps 3 and 5, and the fact (step 1) that $\mathscr{A}(t) = \mathscr{A}_\lambda(t)$.

When $q = 0$, the theorem has been proved by Nayatani ([Na]), and a stronger version is given in [Pa]. When $q = n - 1$ the theorem has been essentially proved by Giménez ([Gi1]). In fact, in this case the theorem follows from step 1 and the arguments in [Gi1, Theorem 3.4].

From this theorem, the arguments used in [Gi1] give the following generalization of [Gi1, Theorem 3.4], which was posed as a question in that paper

3.4. COROLLARY. *If we have an equality in Corollary* 2.2, *then there is a holomorphic isometry* $i: M \to \mathbb{C}P^n(\lambda)$ *such that* $i(P) = \mathbb{C}P^q(\lambda)$.

4. Comparison theorems for the mean exit time and the first Dirichlet eigenvalue

To begin with, we recall a fundamental fact about the Laplacian. Let Q be any (real or complex) submanifold of M of real dimension m. By a radial function we shall mean a real function depending only on the distance to Q. For such a function the Laplacian has the expression

$$\Delta f = -f'' - \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial t}\sqrt{g}\right] f' \tag{4.1}$$

where t is the distance to Q, $'$ is the derivative respect to t, and g is the determinant of the metric tensor in a polar Fermi coordinate system (see (GKP]).

On the other hand, it is easy to see (using [GKP, Theorem 2] and [Gr3, page 47]) that $(1/\sqrt{g})[\partial/\partial t\sqrt{g}] = -\operatorname{tr} S(t)$, where $S(t)$ is the Weingarten map of the tubular hypersurface of radius t about Q. Then

$$\Delta f = -f'' + \operatorname{tr} S(t) f' \tag{4.2}$$

Given $N \in \mathcal{S}\mathcal{N}Q$, we shall denote $c(N) = \sup\{t \geqslant 0/d(Q, \gamma_N(t)) = t\}$, $c(Q) = \inf\{c(N)/N \in \mathcal{S}\mathcal{N}Q\}$ and $\operatorname{cut}(Q) = \{\gamma_N(c(N))/N \in \mathcal{S}\mathcal{N}Q\}$.

4.1 LEMMA. *If $P = \mathbb{C}P^q(\lambda)$, $M = \mathbb{C}P^n(\lambda)$, the mean exit time function $E^\lambda_{q,r}$ of $\mathbb{C}P^q(\lambda)_r$ is a radial function satisfying $E^\lambda_{q,r}(t) \geqslant 0$ and $E^{\lambda\prime}_{q,r}(t) \leqslant 0$.*

Proof. $E^\lambda_{q,r}$ is a solution of the problem (cfr. [Dy, vol. 2, page 51])

$$\Delta E^\lambda_{q,r} = 1, \quad E^\lambda_{q,r}|_{\partial \mathbb{C}P^q(\lambda)_r} = 0 \tag{4.1.1}$$

Let us define the operator A acting on smooth functions on M by

$$A(f)(t) = \frac{\int_{\partial P_t} f(x)\,dx}{\operatorname{vol}(\partial P_t)}.$$

Then, $A(f)$ is a radial function. An argument similar to [Sz, Lemma 1] shows that $A\Delta = \Delta A$. Then, if $E^\lambda_{q,r}$ is a solution of (4.1.1), we have

$$\Delta A(E^\lambda_{q,r}) = A\Delta E^\lambda_{q,r} = A(1) = 1 \text{ and } A(E^\lambda_{q,r})|_{\partial \mathbb{C}P^q(\lambda)_r} = 0$$

Therefore, by the unicity of the solutions of (4.1.1), $A(E^\lambda_{q,r}) = (E^\lambda_{q,r})$, and $(E^\lambda_{q,r})$ is radial. Then, from (4.2), the conditions (4.1.1) can be written as

$$-(E^\lambda_{q,r})''(t) + (\operatorname{tr} S^q_\lambda(t))(E^\lambda_{q,r})'(t) = 1, \quad (E^\lambda_{q,r})(r) = 0. \tag{4.1.2}$$

Moreover $E^{\lambda\prime}_{q,r}(0) = 0$, because $E^\lambda_{q,r}$ is a radial function. From (4.1.2), we have that $E^{\lambda\prime\prime}_{q,r}(0) = -1 < 0$, so that $E^{\lambda\prime}_{q,r}$ is decreasing in a neighbourhood of 0, and hence negative. Therefore, $E^\lambda_{q,r}$ is decreasing in this neighbourhood. Let $t_0 = \inf\{t/E^\lambda_{q,r}(t) = 0\}$, then $E^{\lambda\prime\prime}_{q,r}(t_0) = -1 < 0$, and $E^\lambda_{q,r}(t_0)$ would be a maximum, which is a contradiction with the fact that $E^\lambda_{q,r}$ is decreasing before t_0. Then $E^{\lambda\prime}_{q,r} \leqslant 0$. Since $E^\lambda_{q,r}$ is decreasing for every t, and positive at $t = 0$ and 0 at $t = r$, we have $E^\lambda_{q,r} \geqslant 0$.

4.2. THEOREM *Let M and P be as in Theorem 2.1, $r \leqslant \pi/2\sqrt{\lambda}$, E_r the mean exit time function of P_r and $\mathscr{E}_r\colon P_r \to \mathbb{R}$ the function defined by $\mathscr{E}_r(x) = E^\lambda_{q,r}(d(P, x))$. Then*

$$\mathscr{E}_r(x) \leqslant E_r(x).$$

If the equality holds for every $x \in P_r$ and every $r \leqslant f(P) \equiv \inf\{f(N)/N \in \mathscr{SN}P\}$, then there is a holomorphic isometry $i\colon M \to \mathbb{C}P^n(\lambda)$ such that $i(P) = \mathbb{C}P^q(\lambda)$.

Proof. We shall prove first the inequality for $r < c(P)$. From 2.1 and (4.2) we have that

$$\begin{aligned} \Delta\mathscr{E}_r(x) &= -E^{\lambda\prime\prime}_{q,r}(t) + \operatorname{tr} S(t) E^{\lambda\prime}_{q,r}(t) \leqslant -E^{\lambda\prime\prime}_{q,r}(t) + \sigma(t) E^{\lambda\prime}_{q,r}(t) \\ &= \Delta E^\lambda_{q,r}(t) = 1, \quad \text{and } \mathscr{E}_r|_{\partial P_r} = 0. \end{aligned}$$

Then $\Delta(\mathscr{E}_r - E_r) \leqslant 0$, and $(\mathscr{E}_r - E_r)|_{\partial P_r} = 0$; and, from the minimum principle, $\mathscr{E}_r \leqslant E_r$ on P_r.

If $r \geqslant c(P)$, we have that $\mathscr{E}_r$ could not be C^1 on $\mathrm{cut}(P) \cap P_r$, and then we cannot apply the minimum principle to the function $\mathscr{E}_r - E_r$ on P_r. However, $\mathscr{E}_r - E_r$ is continuous on P_r, smooth on $P_r - \mathrm{cut}(P)$ and subharmonic. Then, applying an approximation theorem by Green and Wu ([GW, corollary 1 of theorem 3.1]), we can approximate $\mathscr{E}_r - E_r$ by subharmonic functions which are C^2 on $P_r - \partial P_r$ and then apply the above argument to these functions.

The equality $\mathscr{E}_r = E_r$ implies the equality $\mathrm{tr}\, S(t) = \sigma(t)$, $t \in [0, r]$ and the second assertion of the theorem follows from Theorem 3.3.

4.3. LEMMA. *Let $\mu^{\lambda}_{q,r}$ be the first eigenvalue of the Dirichlet problem*

$$\Delta f = \mu f, \quad f|_{\partial \mathbb{C}P^q(\lambda)_r} = 0$$

Then there is an eigenfunction $f^{\lambda}_{q,r}$ with eigenvalue $\mu^{\lambda}_{q,r}$ which is radial and satisfies $f^{\lambda}_{q,r}|_{[0,r[} > 0$ and $f^{\lambda\prime}_{q,r}|_{]0,r]} < 0$.

Proof. It follows from arguments similar to those in the proof of 4.1.

4.4 THEOREM. *Let P and M be as in 2.1. Let μ_1 be the first eigenvalue of the Dirichlet eigenvalue problem on P_r, then*

$$\mu_1 \leqslant \mu^{\lambda}_{q,r}$$

And, if the equality holds for every $r \in]0, f(N)[$, then there is a holomorphic isometry $i: M \to \mathbb{C}P^n(\lambda)$ such that $i(P) = \mathbb{C}P^q(\lambda)$.

Proof. Let us define $\tilde{f}^{\lambda}_{q,r} P_r: \to \mathbb{R}$ by $\tilde{f}^{\lambda}_{q,r}(x) = f^{\lambda}_{q,r}(d(x, P))$. We have $\tilde{f}^{\lambda}_{q,r} \in H_1(P_r)$ (where $H_1(P_r)$ denotes the Sobolev space of order 1 in P_r), since the distance function is Lipschitz. Then the proof of the inequality follows from Theorem 2.1 in the same way that the proof of Theorem 1 in [Le, pages 845–846)].

If $\mu_1 = \mu^{\lambda}_{q,r}$, then $\mathrm{tr}\, S(t) = \sigma(t)$ and the second assertion of the theorem follows from theorem 3.3.

When $q = 0$, it is possible to get stronger versions of theorems 4.2 (see [Pa]) and 4.4 (see [Gi2]).

To get bounds for the first Dirichlet eigenvalue on a domain Ω we shall use the following version of Barta's Lemma:

4.5 LEMMA (cfr. [Ks, Lemma 1.1]). *Let M be a Riemannian manifold. Let Ω be a connected compact domain in M with smooth boundary $\partial\Omega$. Let W be an open set of Ω such that $\bar{W} = \Omega$. Let $\Psi \in C^0(\Omega) \cap C^\infty(W)$ such that*

$$\Psi|_{\Omega} > 0, \qquad \Psi|_{\partial\Omega} = 0.$$

Then

$$\inf_\Omega\left(\frac{\Delta\Psi}{\Psi}\right) \leqslant \mu_1(\Omega) \quad \textit{(in the sense of distributions)},$$

where $\mu_1(\Omega)$ is the first eigenvalue of the Dirichlet problem in Ω.

If equality holds, then Ψ is the first Dirichlet eigenfunction (i.e. $\Delta\Psi = \mu_1(\Omega)\Psi$).

In [GM2] and [MP] comparison theorems for μ_1 and the mean exit time between domains of M and geodesic balls of $\mathbb{C}P^n(\lambda)$ are given. Here we complete that work by comparing with tubes around $\mathbb{C}P^{n-1}(\lambda)$ in $\mathbb{C}P^n(\lambda)$. To do it we need some definitions.

Let Ω be a domain in M, with smooth boundary $\partial\Omega$, then we can take a unit normal vector field N on $\partial\Omega$ pointing inward (that is, for every $p \in \partial\Omega$ the geodesic $\gamma_{N(p)}(t)$ lies in $\Omega - \partial\Omega$ for small $t \geqslant 0$). L will denote the Weingarten map associated to N. We define the JN-normal curvature k_{JN} of $\partial\Omega$ at p in the direction JN as

$$k_{JN}(p) = \langle LJN, JN\rangle(p).$$

We define the JN-mean curvature H_{JN} of $\partial\Omega$ at p as the real number:

$$H_{JN}(p) = \frac{1}{2n-2}(\operatorname{tr} L - k_{JN}) = \frac{(2n-1)H - k_{JN}}{2n-2}$$

where H is the mean curvature of $\partial\Omega$ at p. Let $h > 0$, k be real numbers, and let r_h and r_k be defined by

$$\begin{aligned} h &= \frac{-1}{2n-1}\operatorname{tr} S_\lambda^{n-1}(r_h) \\ &= \frac{1}{2n-1}(2\sqrt{\lambda}\cot(2\sqrt{\lambda}r_h) - (2n-2)\sqrt{\lambda}\tan(\sqrt{\lambda}r_h)) \\ &= \frac{-1}{2n-1}\sigma(r_h), \end{aligned}$$

$$k = 2\sqrt{\lambda}\cos(2\sqrt{\lambda}r_k) \quad \text{and} \quad 0 < r_k,\, r_h < \frac{\pi}{2\sqrt{\lambda}}.$$

Take $r = \max\{r_k, r_h\}$.

4.6 LEMMA. *Suppose that, on M, $\rho \geqslant (2n+2)\lambda$ and $K_H \geqslant 4\lambda$ and, on $\partial\Omega$, $H \geqslant h$ and $k_{JN} \geqslant k$. Then, for every $p \in \partial\Omega$, we have*

$$\begin{aligned}\operatorname{tr} S(t) &\geqslant -\operatorname{tr} S_\lambda^{n-1}(r-t)\\ &= 2\sqrt{\lambda}\cot(2\sqrt{\lambda}(r-t)) - (2n-2)\sqrt{\lambda}\tan(\sqrt{\lambda}(r-t)),\end{aligned}$$

for every $t \in [0, f(N(p))[$.

Proof. Let $\{E_i(s)\}_{1 \leqslant i \leqslant 2n-1}$ be a parallel J-orthonormal frame of $\{\gamma_N'(s)\}^\perp$ along $\gamma_N(s)$ such that $E_{2n-1}(s) = J\gamma_N'(s)$.

Given $t \in [0, r[$, let $\{Y_i(s)\}_{1 \leqslant i \leqslant 2n-1}$ be $\partial\Omega$-Jacobi fields along $\gamma_N(s)$ such that $Y_i(t) = E_i(t)$. Let us define, for every $t \in (0, r)$, the fields $Z_i(s)$ along $\gamma_N(s)$ by $Z_i(s) = f_i(s)E_i(s)$, where:

$$f_i(s) = f(s) \equiv c_\lambda(r-s, r-t), \quad \text{for } i = 1, \dots, 2n-2$$

and

$$f_{2n-1}(s) = s_{2\lambda}(r-s, r-t), \quad 0 \leqslant s \leqslant t.$$

Then:

$$\begin{aligned}\operatorname{tr} S(t) &= -\sum_{i=1}^{2n-1} I_0^t(Y_i) \geqslant -\sum_{i=1}^{2n-1} I_0^t(Z_i)\\ &= f_2(0)\operatorname{tr} L + (f_{2n-1}^2(0) - f^2(0)) < E_{2n-1}, LE_{2n-1} > (0)\\ &\quad - \int_0^t \left\{\sum_{i=1}^{2n-1} f_i'^2(s) - f^2(s)\rho - (f_{2n-1}^2(s) - f^2(s))K_H\right\} ds\\ &\geqslant f^2(0)(2\sqrt{\lambda}\cot(2\sqrt{\lambda}r) - (2n-2)\sqrt{\lambda}\tan(\sqrt{\lambda}r))\\ &\quad + (f_{2n-1}^2(0) - f^2(0))2\sqrt{\lambda}\cot(2\sqrt{\lambda}r)\\ &\quad - \int_0^t \left\{\sum_{i=1}^{2n-1} f_i'^2(s) - f^2(s)(2n+2)\lambda - (f_{2n-1}^2(s) - f^2(s))4\lambda\right\} ds\\ &= -\sigma(r-t).\end{aligned}$$

4.7 THEOREM. *Under the hypothesis of* 4.6, *if E_Ω denotes the mean exit time function of the domain Ω and $\mu_1(\Omega)$ denotes the first eigenvalue of the Dirichlet problem in Ω, we have*:

$$E_\Omega(x) \leqslant \mathbf{E}_r(x), \tag{4.7.1}$$

$$\mu_1(\Omega) \geqslant \mu_{q,r}^\lambda, \tag{4.7.2}$$

where $\mathbf{E}_r(x) = E^{\lambda}_{q,r}(r - d(x, \partial\Omega))$ *and* $E^{\lambda}_{q,r}$, $\mu^{\lambda}_{q,r}$ *have the same meaning than in Theorem* 2.3 *and Lemma* 4.3, *but with* $q = n - 1$.

Moreover, if $M = \mathbb{C}P^n(\lambda)$, *the equality in* (4.7.1) *or* (4.7.2) *implies that* Ω *is holomorphically isometric to* $\mathbb{C}P^{n-1}(\lambda)_r$.

Proof. We shall give the proof of (4.7.2). Let us define $T:\Omega \to \mathbb{R}$ by $T(x) = f^{\lambda}_{q,r}(r - d(x, \partial\Omega))$. We have $T|_{\Omega - \partial\Omega} > 0$, $T|_{\partial\Omega} = 0$ and, from (4.2) and Theorem 2.1,

$$\begin{aligned} \Delta T &= -f^{\lambda\prime\prime}_{q,r}(r - t) - \operatorname{tr} S(t) f^{\lambda\prime}_{q,r}(r - t) \\ &\geqslant -f^{\lambda\prime\prime}_{q,r} + \operatorname{tr} S^{n-1}_{\lambda}(r - t) f^{\lambda\prime}_{q,r} = \mu^{\lambda}_{q,r} f^{\lambda}_{q,r}(r - t) \\ &= \mu^{\lambda}_{q,r} T(x), \end{aligned}$$

and, applying lemma 4.5, we have $\mu_1 \geqslant \mu^{\lambda}_{q,r}$.

Also from lemma 4.5 we get that if $\mu_1 = \mu^{\lambda}_{q,r}$, then $\Delta T = \mu_1 T = \mu^{\lambda}_{q,r} T$, and, therefore, $\operatorname{tr} S(t) = -\sigma(r - t)$ and, as in the proof of theorem 3.3, we have $S(t) = -S^{n-1}_{\lambda}(r - t)$. Then, the Weingarten map of $\partial\Omega$ is the same that the Weingarten map of $\mathbb{C}P^{n-1}(\lambda)_r$, and, if $M = \mathbb{C}P^n(\lambda)$, we get that $\partial\Omega = \partial\mathbb{C}P^{n-1}(\lambda)_r$ from the classification theorem of real hypersurfaces of $\mathbb{C}P^n(\lambda)$ with constant principal curvatures (cfr. [Ki]).

The proof of (4.7.1) follows arguments similar to those given in the proof of Theorem 4.2, and the discussion of the equality is as before.

In the following theorem, $r < \pi/2\sqrt{\lambda}$, F_r will denote the mean exit time function of $M - P_r$ and $\mathscr{F}_r: P_r \to \mathbb{R}$ will be the function defined by $\mathscr{F}_r(x) = E^{\lambda}_{n-q-1,\pi/2\sqrt{\lambda}-r}(\pi/2\sqrt{\lambda} - r - d(\partial P, x))$.

4.8 THEOREM. *Let M and P be as in Theorem* 2.1, *then*

$$\mu_1(M - P_r) \geqslant \mu^{\lambda}_{n-q-1,\pi/2\sqrt{\lambda}-r} \tag{4.8.1}$$

$$F_r(x) \leqslant \mathscr{F}_r(x). \tag{4.8.2}$$

Moreover the equality in (4.8.1) *or* (4.8.2) *implies that there is a holomorphic isometry* $i: M \to \mathbb{C}P^n(\lambda)$ *which, restricted to P, gives a holomorphic isometry between P and* $\mathbb{C}P^q(\lambda)$.

Proof. From Theorem 2.1 we have that

$$\operatorname{tr} S(r + t) \geqslant \operatorname{tr} S^q_{\lambda}(r + t) = -\operatorname{tr} S^{n-q-1}_{\lambda}(\pi/2\sqrt{\lambda} - r - t).$$

Let us define $T(x) = f^{\lambda}_{n-q-1,\pi/2\sqrt{\lambda}-r}((\pi/2\sqrt{\lambda}) - r - d(x, \partial P))$. From this point, the proof of (4.8.1) follows as that of (4.7.2). The proof of (4.8.2) uses similar ideas.

The equality in (4.8.1) or (4.8.2) implies that $\operatorname{tr} S(t) = \sigma(t)$ for every t, and the holomorphic isometry follows from Theorem 3.3.

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