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Generalization of the Moore exact sequence and the wild kernel for higher K-groups

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Introduction

Let F be a number field, l be an odd prime number and n be a positive integer. It is a consequence of a result of C. Moore [Mil] p. 157 that we have the following exact sequence:

$$K_2(F) \to \bigoplus_v \mu(F_v) \to \mu(F) \to 0,$$

where F_v denotes the completion of F at a finite and real infinite primes and $\mu(F)$ ($\mu(F_v)$ resp.) denotes the group of roots of unity in F (F_v resp.). In this paper among other results we attempt to generalize this exact sequence to higher K-groups for the odd torsion part. Let us now discuss the organization of the paper. First using results of Dwyer and Friedlander [D-F] and Soulé [S1], [S2] we obtain the following.

THEOREM 1. *There is the following surjective map:*

$$K_{2n}(F)_l \to H^2_{\mathrm{cont}}(F;\, \mathbf{Z}_l(n+1))_l$$

Next we connect our Theorem 1 to an exact sequence of Schneider [Sch] satz 8 to obtain the following.

THEOREM 2. *There is the following exact sequence:*

$$K_{2n}(F)_l \xrightarrow{\lambda_l} \bigoplus_v W^n(F_v) \longrightarrow W^n(F) \longrightarrow 0,$$

where $W^n = \mathbf{Q}_l/\mathbf{Z}_l(n)$ and $W^n(F)$ ($W^n(F_v)$ resp.) denotes the fixed points of $G(\bar{F}/F)$ ($G(\bar{F}_v/F_v)$ resp.) acting on W^n.

We define the wild kernel (or more precisely its l-torsion part) $K^w_{2n}(O)_l$ to be the kernel of λ_l in the above exact sequence.

In Section II we first prove the following Theorem 3 which is our joint observation with Manfred Kolster:

THEOREM 3. (i) *Let* $D(n) = \bigcap_{r \geqslant 1} K_{2n}(F)^r$ *denote the group of divisible elements in* $K_{2n}(F)$ *and let* $D_{n+1}(F)$ [Sch] *denote the group of divisible elements in* $H^1(F; W^{n+1})/\mathrm{Div}$. *For any number field* F, *any integer* $n > 0$ *and any odd prime number* l *we have the following canonical isomorphism:*

$$D(n)_l \approx D_{n+1}(F)$$

(ii) *Let* F *be a totally real number field,* n *be an odd positive integer and* l *be an odd prime number. Let* $\zeta_F(s)$ *be the Dedekind zeta function of* F *and* $w_n(k)$ *be the biggest number* m *such that the exponent of* $G(k(\mu_m)/k)$ *divides* n *for a field* k. *Then:*

$$\left|\frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|l} w_n(F_v)}\right|_l^{-1} = \# D(n)_l$$

We also observe the following propositions:

PROPOSITION 1. *For any totally real number field, any positive, odd* n *and any odd prime number* l *the following are equivalent:*

(a) $\left|\dfrac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|l} w_n(F_v)}\right|_l^{-1} = 1$

(b) $K_{2n}(F)_l \approx K_{2n}(O)_l \oplus \bigoplus\limits_v K_{2n-1}(k_v)_l$

PROPOSITION 2. *For a number field* F, *prime number* l (*if* F *contains* 4th *root of unity than* l *can be also equal to* 2) *and* $n > 0$ *we have the following non canonical isomorphism:*

$$K_{2n}(O)_l = K_{2n}^{\mathrm{et}}\left(O\left[\frac{1}{l}\right]\right) \oplus C(n)$$

where $C(n)$ *is the kernel of the map from Theorem* 1.

Section III compares $K_{2n}^{w}(O)_l$ and $K_{2n}(O)_l$ for any number field. More precisely we get the following.

THEOREM 4. *For any number field F, any positive integer n and any odd prime number l we have the following equality.*

$$\frac{\# K_{2n}(O)_l}{\# K_{2n}^{w}(O)_l} = \left|\frac{\prod_{v|l} w_n(F_v)}{w_n(F)}\right|_l^{-1}$$

In addition we reformulate the conjecture of Quillen-Lichtenbaum in terms of $K_{2n}^{w}(O)_l$.

In Section IV we give another evidence for Coates-Sinnott conjecture [C-S]. Namely we prove the following.

THEOREM 5. *Let F be an abelian extension of $\mathbf{Q}$, n be an odd positive integer and l be an odd prime. Then S_n annihilates $\lambda_l(K_{2n}(O)_l)$, where S_n denotes the Stickelberger ideal* [B].

Section I

1. NOTATION. Let F be a number field, O be its ring of integers. In addition let l be an odd prime number and $S \subset T$ denote two finite sets of prime ideals of O which contain these over l. Moreover O_S (O_T resp.) denotes the ring of S (T resp.) integers. If G is an abelian group, then G_l denotes the l-torsion part of G. For a commutative ring with identity A; $K_n(A)$ ($K_n^{\text{et}}(A)$ resp.) denotes the Quillen K-group (the étale K-group resp.) [D-F], [Q1].

2. ETALE K-THEORY. Let $X.$ be a simplicial scheme (written s-scheme) and let (F_j) be a diagram of locally constant étale sheaves on $X.$. Then the continuous cohomology is defined in the following way [D-F], [J]:

$$H^n_{\text{cont}}(X.; (F_j)) = \pi_0\left(\varprojlim_j \varinjlim_{U..} \operatorname{Hom}(\pi\Delta U..; K(\mathbf{M}_j, n))_{NG_j}\right).$$

In the above definition $U..$ denotes a hypercovering of $X.$ for which F_j is constant on U_{00}, Δ denotes the diagonal of $U..$, and π is the set of connected components of the simplicial scheme $\Delta U..$. In addition $\mathbf{M}_j$ denotes the coefficient system associated to F_j and $K(\mathbf{M}_j, n)$ is the Eilenberg-Mac Lane simplicial space. For A as above we write the continuous cohomology of spec A as $H^n_{\text{cont}}(A; (F_j))$ and as usual $H^n(A; \mathbf{Z}_l(i))$ denotes the l-adic cohomology $\varprojlim_m H^n(\operatorname{spec} A; \mathbf{Z}/l^m(i))$. There is an alternative definition of continuous cohomology [J]. Namely $H^n_{\text{cont}}(X.; (F_j))$ is the right derived functor of the functor

$\varprojlim_j H^0(X.; F_j)$ in the category of diagrams of abelian sheaves (F_j) on $X.$. As shown in [J] p. 223, these two definitions agree. Moreover the comparison map between these two definitions is functorial in $X.$ and (F_j).

Let us consider the following commutative diagram.

$$\begin{array}{ccc}
K_{2n}(O_S)_l & \xrightarrow{nat} & K_{2n}(O_T)_l \\
\downarrow & & \downarrow \\
K_{2n}^{et}(O_S) & \xrightarrow{nat} & K_{2n}^{et}(O_T) \\
\downarrow \approx & & \downarrow \approx \\
H^2_{cont}(O_S; \mathbf{Z}_l(n+1)) & \xrightarrow{nat} & H^2_{cont}(O_T; \mathbf{Z}_l(n+1)) \\
\downarrow \approx & & \downarrow \approx \\
H^2(O_S; \mathbf{Z}_l(n+1)) & \xrightarrow{nat} & H^2(O_T; \mathbf{Z}_l(n+1))
\end{array}$$

Diagram 1.1

The upper vertical maps are surjective [D-F]. The commutativity of the middle square is a consequence of the morphism of Dwyer-Friedlander spectral sequences [D-F].

$$\begin{array}{ccc}
H^p_{cont}(O_S; \mathbf{Z}_l(q/2)) & \Rightarrow & K^{et}_{q-p}(O_S) \\
\downarrow & & \downarrow \\
H^p_{cont}(O_T; \mathbf{Z}_l(q/2)) & \Rightarrow & K^{et}_{q-p}(O_T)
\end{array}$$

Diagram 1.2

This morphism is obviously induced by the map spec $O_T \to$ spec O_S. The middle vertical maps are isomorphisms [D-F] p. 276. Let us now discuss the commutativity of the lowest square in diagram 1.1. In this square the upper horizontal arrow is naturally induced (see the definition of continuous cohomology above) by the map spec $O_T \to$ spec O_S. The commutativity follows because of the functoriality of the comparison map mentioned above. Also the lowest vertical arrows are isomorphisms because the étale cohomology groups $H^n(O_S; \mathbf{Z}/l^m(i))(H^n(O_T; \mathbf{Z}/l^m(i))$ resp.) are finite for all $n \geqslant 0$, $i \geqslant 0$, $m > 0$.

3. ETALE COHOMOLOGY. Let us denote $W^i = \mathbf{Q}_l/\mathbf{Z}_l(i)$ the étale sheaf for

spec O_S (spec O_T resp.). Because $H^2(O_S; W^{n+1}) = 0$ for $n > 0$ [S1] Theorem 5, the exact sequence:

$$0 \to \mathbf{Z}/l^m(n+1) \to W^{n+1} \to W^{n+1} \to 0$$

gives the following isomorphism:

$$H^1(O_S;\ W^{n+1})/l^m \xrightarrow{\sim} H^2(O_S;\ \mathbf{Z}/l^m(n+1))$$

In addition $H^1(O_S;\ W^{n+1}) = \mathrm{Div} \oplus$ (finite l-torsion group) [cf. S2] p. 376, where Div denotes the maximal divisible subgroup of $H^1(O_S;\ W^{n+1})$. Now we obtain (upon taking inverse limit with respect to m in the above isomorphisms) the following isomorphism:

$$H^1(O_S;\ W^{n+1})/\mathrm{Div} \xrightarrow{\sim} H^2(O_S;\ \mathbf{Z}_l(n+1))$$

Hence we have the following commutative diagram:

$$\begin{tikzcd}
H^2(O_S; \mathbf{Z}_l(n+1)) \arrow[r, "nat"] \arrow[d, "\approx"] & H^2(O_T; \mathbf{Z}_l(n+1)) \arrow[d, "\approx"] \\
H^1(O_S; W^{n+1})/Div \arrow[r, "nat"] & H^1(O_T; W^{n+1})/Div
\end{tikzcd}$$

Diagram 1.3

Let us now consider the following morphism of spectral sequences:

$$\begin{tikzcd}
H^p(O_S; R^q j_* W^{n+1}) \arrow[r, Rightarrow] \arrow[d] & H^{p+q}(F; W^{n+1}) \arrow[d, "id"] \\
H^p(O_T; R^q \alpha_* W^{n+1}) \arrow[r, Rightarrow] & H^{p+q}(F; W^{n+1})
\end{tikzcd}$$

Diagram 1.4

In the above diagram W^{n+1} is the corresponding étale sheaf on spec F and j (α resp.) is the map spec $F \to$ spec O_S (spec $F \to$ spec O_T resp.). This morphism gives the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^1(O_S; j_* W^{n+1}) \arrow[r] \arrow[d] & H^1(F; W^{n+1}) \arrow[r] \arrow[d, "id"] & \bigoplus_{v \notin S} H^0(k_v; W^n) \arrow[d, "proj"] \\
0 \arrow[r] & H^1(O_T; \alpha_* W^{n+1}) \arrow[r] & H^1(F; W^{n+1}) \arrow[r] & \bigoplus_{v \notin T} H^0(k_v; W^n)
\end{tikzcd}$$

Diagram 1.5

In the above diagram k_v denotes the corresponding residue field.

From the above diagram we obtain the following isomorphism:

$$\varinjlim_{S} H^1(O_S;\, W^{n+1})/\mathrm{Div} \tilde{\to} H^1(F;\, W^{n+1})/\mathrm{Div}$$

We also used the following isomorphism: $j_* W^{n+1} \approx W^{n+1}$ as sheaves on $\operatorname{spec} O_S$. We also observe that the quotient of the left vertical arrow in the diagram 1.5 by Div is the same as the bottom horizontal arrow in diagram 1.3. It is shown in [T] Proposition 2.3 that:

$$H^1(F;\, W^{n+1})/\mathrm{Div} \tilde{\to} H^2_{\mathrm{cont}}(F;\, \mathbf{Z}_l(n+1))_l$$

Taking direct limit with respect to S in diagram 1.1 we get the following:

THEOREM 1. *For a number field F, an odd prime number l and a positive integer n, there is the following surjective map:*

$$K_{2n}(F)_l \to H^2_{\mathrm{cont}}(F;\, \mathbf{Z}_l(n+1))_l$$

REMARK 1. Computing Dwyer-Friedlander spectral sequence [D-F]:

$$H^p_{\mathrm{cont}}(F;\, \mathbf{Z}_l(q/2)) \Rightarrow K^{\mathrm{et}}_{q-p}(F)$$

we can check that:

$$K^{\mathrm{et}}_{2n}(F) \approx H^2_{\mathrm{cont}}(F;\, \mathbf{Z}_l(n+1)).$$

Hence Theorem 1 says that there is surjective map:

$$K_{2n}(F)_l \to K^{\mathrm{et}}_{2n}(F)_l.$$

The surjectivity of the above map is easily obtained in different way from the following diagram 1.6 upon taking direct limit with respect to l^k over the right vertical maps in this diagram:

$$\begin{tikzcd}
K_{2n+1}(F;Z/l^k) \arrow[r] \arrow[d] & K_{2n}(F)[l^k] \arrow[r] \arrow[d] & 0 \\
K^{et}_{2n+1}(F;Z/l^k) \arrow[r] & K^{et}_{2n}(F)[l^k] \arrow[r] & 0
\end{tikzcd}$$

Diagram 1.6

The left vertical arrow is surjective by [D-F] Theorem 8.5. So the right vertical arrow is also surjective. Nevertheless our construction above indicates how to construct some exact sequences and commutative diagrams involving the map from Theorem 1 which we will use later.

4. GENERALIZATION OF THE MOORE EXACT SEQUENCE. Now we are going to connect Theorem 1 to a result of Schneider [Sch] satz 8. Let us first introduce some notation. Let Σ be the set of all the primes of F which are either finite over l or infinite primes. Let F_Σ denote the maximal extension of F unramified outside Σ and $G_\Sigma = G(F_\Sigma/F)$. Schneider [see loc. cit.] defines the following numbers:

$$i_n(F) = \dim H^2(G_\Sigma;\ W^n),$$

$$d_n(F) = \dim H^1(G_\Sigma;\ W^n),$$

where the dimension denotes the divisible dimension. He conjectures [see loc. cit. p. 192] that $i_n(F) = 0$ for $n \neq 1$. The following lemma (proved by Soulé [S1] Th. 5) is a partial verification of Schneider's conjecture.

LEMMA 1. *For an arbitrary number field F*, an odd prime number l and for $n > 1$ *we have* $i_n(F) = 0$.

Proof. Let $O_l = O\left[\frac{1}{l}\right]$ and let $j\colon \operatorname{spec} F \to \operatorname{spec} O_l$ denote the natural map. We know [Sch] p. 203, that for $i \geqslant 0$:

$$H^i(O_l;\ j_* W^n) = H^i(G_\Sigma;\ W^n).$$

Soulé shows [S1] Theorem 5, that $H^2(O_l;\ j_* W^n) = 0$ for $n > 1$. □

Observe that the lemma follows also by the following results of Soulé and Schneider. Indeed Soulé [S2], p. 376 proves that:

$$H^1(O_l;\ W^n) = (\mathbf{Q}_l/\mathbf{Z}_l)^{r_1+r_2} + (\text{finite group}) \text{ if } n \text{ is odd and } n > 1,$$

$$H^1(O_l;\ W^n) = (\mathbf{Q}_l/\mathbf{Z}_l)^{r_2} + (\text{finite group}) \text{ if } n \text{ is even and } n > 0.$$

We used above the following isomorphism $j_* W^n \approx W^n$ [S1] Lemma 4. On the other hand Schneider proves [see loc. cit. satz 6] that:

$$d_n(F) = i_n(F) + r_1 + r_2 \quad \text{if } n \text{ is odd},$$

$$d_n(F) = i_n(F) + r_2 \quad \text{if } n \text{ is even and } n \neq 0.$$

By [Sch] satz 8 and Lemma 1 above we have the following exact sequence ($n \geqslant 1$):

$$H^1(F;\, W^{n+1})/\mathrm{Div} \to \bigoplus_v (H^1(F_v;\, W^{n+1})/\mathrm{Div}) \to W^{-n}(F)^* \to 0$$

Upper star denotes the Pontrjagin dual. Also by [Sch] satz 4 we have:

$$H^1(F_v;\, W^{n+1})/\mathrm{Div} \approx W^{-n}(F_v)^*.$$

In addition we have obvious isomorphisms:

$$W^{-n}(F)^* \approx W^n(F) \quad \text{and} \quad W^{-n}(F_v)^* \approx W^n(F_v).$$

This exact sequence, Theorem 1 and the discussion above give:

THEOREM 2. *For a number field F, an odd prime number l and a positive integer n, there is the following exact sequence:*

$$K_{2n}(F)_l \xrightarrow{\lambda_l} \bigoplus_v W^n(F_v) \to W^n(F) \to 0.$$

REMARK 2. Observe that $W^1(F) = (\mu(F))_l$, and $W^1(F_v) = (\mu(F_v))_l$ where $\mu(F)$ ($\mu(F_v)$ resp.) is the group of roots of unity in F (F_v resp.). We can find out in [Mil] Theorem 16.1 that we have the following exact sequence:

$$K_2(F) \to \bigoplus_v \mu(F_v) \to \mu(F) \to 0.$$

DEFINITION 1. The kernel of λ_l in the exact sequence of Theorem 2 is called the wild kernel (more precisely l-torsion part of the wild kernel) and we denote it $K_{2n}^w(O)_l$.

REMARK 3. It is obvious from Theorem 2 that the group of divisible elements in $K_{2n}(F)_l$ is contained in the wild kernel. Theorem 3 in the next chapter shows that the group of divisible elements is often quite big, so the wild kernel is in many cases non trivial. The appropriate definition of the wild kernel for the case $l = 2$ is still non clear for us.

Section II

1. DIVISIBLE ELEMENTS. Let $D(n) = \bigcap_{r \geqslant 1} K_{2n}(F)^r$ denote the group of divisible elements in $K_{2n}(F)$. Let $D_{n+1}(F)$ [Sch] denote the group of divisible elements in $H^1(F;\, W^{n+1})/\mathrm{Div}$. Theorem 3 below is our joint observation with Manfred Kolster.

THEOREM 3. (i) *For any number field F, any $n > 0$ and any odd prime number l we have the following canonical isomorphism:*

$$D(n)_l \approx D_{n+1}(F).$$

(ii) *Let F be a totally real number field, n be an odd positive integer and l be an odd prime number. Let $\zeta_F(s)$ be the Dedekind zeta function of F and $w_n(k)$ be the biggest number m such that the exponent of $G(k(\mu_m)/k)$ divides n for a field k. Then:*

$$\left|\frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|l} w_n(F_v)}\right|_l^{-1} = \# D(n)_l.$$

Proof. Let us prove (i). Observe that for $n = 1$ the map from Theorem 1 is an isomorphism. It is so because the upper vertical maps in diagram 1.1 are isomorphisms by Proposition 8.2 [D-F]. So $D_2(F) \approx D(1)_l$. Take $n > 1$ and take l^k to be divisible by the exponent of $K_{2n}(O)_l$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
K_{2n}(O)_l \arrow[r] \arrow[d] & K_{2n}(F)/l^k \arrow[d] \\
K_{2n}(O; Z/l^k) \arrow[r] \arrow[d] & K_{2n}(F; Z/l^k) \arrow[d] \\
K_{2n-1}(O)[l^k] \arrow[r] \arrow[d] & K_{2n-1}(F)[l^k] \arrow[d] \\
0 & 0
\end{tikzcd}$$

Diagram 2.1

By [Q1] corollary of Theorem 5, [Q2] Theorem 8, p. 583 and [S1] Theorem 3, p. 574 the maps:

$$K_{2n}(O) \to K_{2n}(F) \quad \text{and} \quad K_{2n-1}(O) \to K_{2n-1}(F)$$

are injective for $n > 0$. The second map is actually an isomorphism for $n > 1$. In addition the kernel of the upper horizontal arrow is $K_{2n}(O)_l \cap K_{2n-1}(F)^{l^k}$. Because $K_{2n}(O)$ is finite we can take l^k to be big enough so the kernel is equal to $D(n)_l$. Hence by diagram 2.1 the kernel of the middle horizontal arrow is also

$D(n)_l$ for l^k as above. Similarly we can consider the following commutative diagram (l^k as before):

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K_{2n}^{et}(O[\frac{1}{l}]) & \longrightarrow & K_{2n}^{et}(O_S)/l^k \\
\downarrow & & \downarrow \\
K_{2n}^{et}(O[\frac{1}{l}]; Z/l^k) & \longrightarrow & K_{2n}^{et}(O_S; Z/l^k) \\
\downarrow & & \downarrow \\
K_{2n-1}^{et}(O[\frac{1}{l}])[l^k] & \longrightarrow & K_{2n-1}^{et}(O_S)[l^k] \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Diagram 2.2

We know from Diagrams 1.1, 1.3 and 1.5 that the arrow:

$$K_{2n}^{\mathrm{et}}\left(O\left[\frac{1}{l}\right]\right) \to K_{2n}^{\mathrm{et}}(O_S)$$

is an injection. In the same way and by the same methods (going down to étale cohomology) as in Sections I.2 and I.3 we can check that the arrow:

$$K_{2n-1}^{\mathrm{et}}\left(O\left[\frac{1}{l}\right]\right) \to K_{2n-1}^{\mathrm{et}}(O_S)$$

is an isomorphism for $n > 1$. Let us take the direct limit with respect to S in the Diagram 2.2. Taking into account the computations in Section I and the localization exact sequence in étale K-theory with coefficients we get the following commutative diagram.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K_{2n}^{et}(O[\frac{1}{l}]) & \longrightarrow & K_{2n}^{et}(F)_l/l^k \\
\downarrow & & \downarrow \\
K_{2n}^{et}(O[\frac{1}{l}]; Z/l^k) & \longrightarrow & K_{2n}^{et}(F; Z/l^k) \\
\downarrow & & \downarrow \\
K_{2n-1}^{et}(O[\frac{1}{l}])[l^k] & \longrightarrow & K_{2n-1}^{et}(F)[l^k] \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Diagram 2.3

In the same way as before we conclude that for l^k big enough the kernel of the upper horizontal arrow in Diagram 2.3 is equal to the kernel of the middle horizontal arrow in this diagram. In addition the kernel of the upper horizontal arrow (l^k big enough) is equal to the group of divisible elements in $K_{2n}^{\mathrm{et}}(F)_l$. But we know that:

$$K_{2n}^{\mathrm{et}}(F)_l = H^1(F;\, W^{n+1})/\mathrm{Div}.$$

Hence the kernel equals to $D_{n+1}(F)$; the divisible elements in $H^1(F;\, W^{n+1})/\mathrm{Div}$. Consider the following commutative diagram:

$$\begin{tikzcd}
K_{2n+1}(F;\, Z/l^k) \arrow[r] \arrow[d] & \bigoplus_{v \nmid l} K_{2n}(k_v;\, Z/l^k) \arrow[r] \arrow[d] & D(n)_l \arrow[r] \arrow[d] & 0 \\
K_{2n+1}^{et}(F;\, Z/l^k) \arrow[r] & \bigoplus_{v \nmid l} K_{2n}^{et}(k_v;\, Z/l^k) \arrow[r] & D_{n+1}(F) \arrow[r] & 0
\end{tikzcd}$$

Diagram 2.4

The horizontal exact rows come from respective localization sequences and our previous considerations. By [D-F] Theorem 8.5 the left vertical arrow is a surjection and by [D-F] Corollary 8.6 the middle vertical arrow is an isomorphism. Hence the right vertical arrow is an isomorphism. (ii) By [Sch] Section 8 and Section 5 satz 5 we have (n odd and $n > 0$ and F totally real):

$$\# D_{n+1}(F) = \left| \frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|l} w_n(F_v)} \right|_l^{-1}$$

because the following Lichtenbaum conjecture:

$$|\zeta_F(-n)|_l^{-1} = \frac{\# H^1(O_l;\, j_* W^{n+1})}{\# H^0(O_l;\, j_* W^{n+1})}$$

follows from the main conjecture in Iwasawa theory [C] p. 335 proven by Wiles. Hence (ii) follows immediately from (i). □

REMARK 4. By Diagram 3.2 it is clear that for any number field F, the group of divisible elements $D_{n+1}(F)$ in $H^1(F;\, W^{n+1})/\mathrm{Div}$ is finite. Obviously we do not know the number of elements of $D_{n+1}(F)$ in general. In addition we could also use Diagram 3.2 instead of 2.4 to finish the proof of part (i) of Theorem 3.

REMARK 5. The integrality of the number:

$$\left| \frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|l} w_n(F_v)} \right|_l^{-1}$$

is not trivial. It follows from the main conjecture in Iwasawa theory proven by Wiles [C] p. 335.

Now we will observe that there is remarkable similarity between the class group and the group of divisible elements in $K_{2n}(F)$. For any number field F we have the following commutative diagram.

$$\begin{array}{ccc} K_{2n+1}(F; Z/l^k) & \xrightarrow{\delta} & \bigoplus_v K_{2n}(k_v; Z/l^k) \\ \downarrow B & & \approx \downarrow B \\ K_{2n}(F)[l^k] & \xrightarrow{\delta} & \bigoplus_v K_{2n-1}(k_v)[l^k] \end{array}$$

Diagram 2.5

In the above diagram B denotes the Bockstein homomorphism. Also if G is an abelian group, then $G[m]$ denotes the subgroup of elements in G which have order dividing m. For l^k divisible by the exponent of $K_{2n}(O)_l$ and taken to be big enough we see from Diagram 2.4 that $D(n)_l$ is the cokernel of the upper horizontal arrow. Hence by Diagram 2.5 it is also the cokernel of the bottom horizontal arrow because the left vertical arrow is surjective. So we have the following exact sequence:

$$K_{2n}(F)[l^k] \xrightarrow{\delta} \bigoplus_v K_{2n-1}(k_v)[l^k] \to D(n)_l \to 0.$$

Observe that there is similar exact sequence:

$$F^* = K_1(F) \xrightarrow{\delta} \bigoplus_v Z \to Cl(O) \to 0,$$

where $Cl(O)$ denotes the class group of O.

Let F be abelian over $\mathbf{Q}$. The classical Stickelberger Theorem says that $\Theta_0(b)$ annihilates $Cl(O)$ for $(b, 2f) = 1$. See Section IV.1 for the definition of $\Theta_n(b)$. We can easily check that this theorem is equivalent to the existence of a homomorphism:

$$\Lambda: \bigoplus_v Z \to K_1(F)$$

such that:

$$\delta \circ \Lambda = \text{multiplication by } \Theta_0(b).$$

In [B] Definition 2 we constructed explicitly a homomorphism: (F abelian over $\mathbf{Q}$)

$$\Lambda\colon \bigoplus_v K_{2n-1}(k_v)[l^k] \to K_{2n}(F)[l^k]$$

such that for $(b, w_{n+1}(\mathbf{Q}(\mu_f)) = 1$:

$$\delta \circ \Lambda = \text{multiplication by } n\Theta_n(b).$$

As for the case of $n = 0$ it is easy to check that the existence of such a map is equivalent to saying that $n\Theta_n(b)$ annihilates $D(n)_l$ which we proved before [B] Theorem A.

Let F be an arbitrary number field. Let us consider again the exact sequence:

$$K_{2n}(F)[l^k] \xrightarrow{\delta} \bigoplus_v K_{2n-1}(k_v)[l^k] \to D(n)_l \to 0.$$

Consider l such that $D(n)_l = 0$. Then for each place v we can take k so big that:

$$K_{2n-1}(k_v)[l^k] = K_{2n-1}(k_v)_l.$$

The assumptions on l and k and the exact sequence give us a homomorphism:

$$\bar{\Lambda}_v\colon K_{2n-1}(k_v)_l \to K_{2n}(F)_l$$

such that $\delta_v \circ \bar{\Lambda}_v = \mathrm{id}$ where $\delta = \bigoplus_v \delta_v$. These considerations imply the following:

COROLLARY 1. *For any number field F, any $n > 0$ and any odd prime number l the following are equivalent:*

(a) $D(n)_l = 0$,

(b) $K_{2n}(F)_l \approx K_{2n}(O)_l \oplus \bigoplus_v K_{2n-1}(k_v)_l$.

The above corollary and Theorem 3 imply the following:

PROPOSITION 1. *For any totally real number field, any positive, odd n and any odd prime number l the following are equivalent:*

(a) $$\left| \frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|l} w_n(F_v)} \right|_l^{-1} = 1$$

(b) $K_{2n}(F)_l \approx K_{2n}(O)_l \oplus \bigoplus_v K_{2n-1}(k_v)_l$.

EXAMPLE 1. Take $F = \mathbf{Q}$ and n odd. Then $w_n(\mathbf{Q}_l) = 1$. Hence in this case the number of divisible elements in $K_{2n}(\mathbf{Q})_l$ is equal to $|w_{n+1}(\mathbf{Q})\zeta_F(-n)|_l^{-1}$. This gives:

COROLLARY 2. *Let n be an odd positive integer and l be an odd prime number. Then the exact sequence:*

$$0 \to K_{2n}(\mathbf{Z})_l \to K_{2n}(\mathbf{Q})_l \to \bigoplus_v K_{2n-1}(k_v)_l \to 0$$

splits if and only if l does not divide $w_{n+1}(\mathbf{Q})\zeta_F(-n)$.

REMARK 6. We constructed [B] Theorem 1, a homomorphism:

$$\Lambda: \bigoplus_v K_{2n-1}(k_v)_l \to K_{2n}(\mathbf{Q})_l$$

such that its composition $(\delta \circ \Lambda)$ with the natural map:

$$\delta: K_{2n}(\mathbf{Q})_l \to \bigoplus_v K_{2n-1}(k_v)_l$$

equals to the multiplication by the number $n(b^{n+1} - 1)\zeta_F(-n)$ where b is an integer used in the construction of Λ. In [B] we proved Corollary 2 up to irregular primes l dividing n. Nevertheless we think that the map Λ can be still interesting for doing more arithmetic in K-theory. Indeed, our work in progress shows that the construction of the map Λ indicates immediately how to construct an Euler system for algebraic K-theory.

EXAMPLE 2. Now consider the exact sequence:

$$0 \to K_{134}(\mathbf{Z})_l \to K_{134}(\mathbf{Q})_l \to \bigoplus_v K_{133}(k_v)_l \to 0.$$

We can check that $37 \| w_{68}(\mathbf{Q})\zeta_{\mathbf{Q}}(-67)$. Hence in this case the exact sequence does not split for the $l = 37$ torsion. Observe that $l = 37 < 67 = n$ in this case. This case was not resolved by our previous results [B].

EXAMPLE 3. Consider again the group $K_{134}(\mathbf{Q})$. By Examples 1 and 2 we know that the subgroup of divisible elements in $K_{134}(\mathbf{Q})_{37}$ is equal to $\mathbf{Z}/37$.

2. REMARKS ABOUT QUILLEN-LICHTENBAUM CONJECTURE

LEMMA 2. *For any number field, any $n > 0$ and any odd prime l the map:*

$$K_{2n}(F)_l \to H^2_{\mathrm{cont}}(F; \mathbf{Z}_l(n+1))_l$$

has finite kernel.

Proof. Consider first the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_{2n}(O_S)_l \arrow[r] \arrow[d] & K_{2n}(O_T)_l \arrow[r] \arrow[d] & \bigoplus_{v \in T-S} K_{2n-1}(k_v)_l \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & H^1(O_S; W^{n+1})/Div \arrow[r] & H^1(O_T; W^{n+1})/Div \arrow[r] & \bigoplus_{v \in T-S} H^0(k_v; W^n) \arrow[r] & 0
\end{tikzcd}$$

Diagram 2.6

The left square commutes by Diagrams 1.1 and 1.3. Observe also that the groups $K_{2n-1}(k_v)$ and $H^0(k_v; W^n)$ are finite and isomorphic. In addition we define the right vertical map to be the quotient of the middle vertical map by the left vertical map. It is clear that the right vertical map is an isomorphism. Hence the kernels of the left and the middle vertical arrows are equal. But the map:

$$K_{2n}(F)_l \to H^2_{\text{cont}}((F; \mathbf{Z}_l(n+1))_l$$

is obtained by taking the direct limit over S over the maps

$$K_{2n}(O_S)_l \to H^1(O_S; W^{n+1})/\text{Div}. \qquad \square$$

REMARK 7. If S contains precisely all the primes over l then $O_S = O\left[\frac{1}{l}\right]$ and we get from the lemma that the maps:

$$K_{2n}\left(O\left[\frac{1}{l}\right]\right)_l \to H^2\left(O\left[\frac{1}{l}\right]; \mathbf{Z}_l(n+1)\right)$$

and

$$K_{2n}(F)_l \to H^2_{\text{cont}}(F; \mathbf{Z}_l(n+1))_l$$

have the same kernel. It was conjectured by Quillen that the first of these maps is an isomorphism.

Let $C(n)$ denote the kernel of the homomorphisms from Remark 7. The following proposition is a direct corollary of a result of Dwyer and Friedlander [D-F]. We proved this proposition during reading the preprint of M. Kurihara [K] who proved this proposition for special number fields. This proposition gives us better understanding of the exponent of the group $K_{2n}(O)$.

PROPOSITION 2. *For a number field F, prime number l (if F contains 4th root of* *unity than l can be also equal to* 2) *and* $n > 0$ *we have the following isomorphism:*

$$K_{2n}(O)_l = K_{2n}^{\text{et}}\left(O\left[\frac{1}{l}\right]\right) \oplus C(n).$$

Proof. Let $f: G \to H$ be a homomorphism of finite, l-torsion abelian groups such that the induced homomorphism $G[l^k] \to H[l^k]$ is surjective for every k. Then it is easy to observe that f is split surjective. Indeed, H is finite hence:

$$H = \mathbf{Z}/l^{k_1} \oplus \mathbf{Z}/l^{k_2} \oplus \cdots \oplus \mathbf{Z}/l^{k_r}.$$

Let $y_1, y_2, \dots, y_r$ be the generators of H with respect to the above decomposition. The assumptions show that for each y_i there is x_i in $G[l^{k_i}]$ mapping onto y_i via f. In this way we can define a homomorphism on generators as follows:

$$H \to G, \qquad y_i \to x_i.$$

It is obvious that this is well defined homomorphism which splits f.

Observe that $K_{2n}\left(O\left[\frac{1}{l}\right]\right)_l = K_{2n}(O)_l$. Consider the following commutative diagram (for every $k > 0$):

$$\begin{tikzcd}
K_{2n+1}(O[\frac{1}{l}]; \mathbf{Z}/l^k) \arrow[r] \arrow[d] & K_{2n}(O[\frac{1}{l}])[l^k] \arrow[r] \arrow[d] & 0 \\
K_{2n+1}^{et}(O[\frac{1}{l}]; \mathbf{Z}/l^k) \arrow[r] & K_{2n}^{et}(O[\frac{1}{l}])[l^k] \arrow[r] & 0
\end{tikzcd}$$

Diagram 2.7

The exact rows come from the Bockstein sequences. The left vertical arrow is surjective by [D-F] Theorem 8.5. Hence the right vertical arrow is also surjective. The groups $K_{2n}\left(O\left[\frac{1}{l}\right]\right)_l$ and $K_{2n}^{\text{et}}\left(O\left[\frac{1}{l}\right]\right)$ are finite so we can apply the observation from the beginning of the proof to the natural map:

$$K_{2n}\left(O\left[\frac{1}{l}\right]\right)_l \to K_{2n}^{\text{et}}\left(O\left[\frac{1}{l}\right]\right).$$

Note that the splitting homomorphism we obtain is obviously non canonical. $\square$

Section III

1. DIAGRAMS. Before stating our main result in this section we will consider few diagrams. Let $S \subset T \subset U$ be three finite sets of prime ideals in O containing these over l. Then we have the following commutative diagram.

$$\begin{tikzcd}[column sep=small]
 & K_{2n}(O_S)_l \arrow[rr] \arrow[dl] \arrow[dd] & & K_{2n}(O_T)_l \arrow[rr] \arrow[dl] \arrow[dd] & & \bigoplus_{v \in T-S} K_{2n-1}(k_v)_l \arrow[rr] \arrow[dd, "\approx"] & & 0 \\
K_{2n}(O_S)_l \arrow[rr] \arrow[dd] & & K_{2n}(O_U)_l \arrow[rr] \arrow[dd] & & \bigoplus_{v \in U-S} K_{2n-1}(k_v)_l \arrow[rr] \arrow[ur] \arrow[dd, "\approx"] & & 0 & \\
 & H^1(O_S; W^{n+1})/Div \arrow[rr] \arrow[dl] & & H^1(O_T; W^{n+1})/Div \arrow[rr] \arrow[dl] & & \bigoplus_{v \in T-S} H^0(k_v; W^n) \arrow[rr] & & 0 \\
H^1(O_S; W^{n+1})/Div \arrow[rr] & & H^1(O_U; W^{n+1})/Div \arrow[rr] & & \bigoplus_{v \in U-S} H^0(k_v; W^n) \arrow[rr] \arrow[ur] & & 0 &
\end{tikzcd}$$

Diagram 3.1

The commutativity of the right vertical square follows from the commutativity of all other squares and by the surjectivity of the appropriate horizontal arrows. From this diagram we conclude that we can take the direct limit with respect to T to obtain the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & K_{2n}(O_S)_l \arrow[r] \arrow[d] & K_{2n}(F)_l \arrow[r] \arrow[d] & \bigoplus_{v \notin S} K_{2n-1}(k_v)_l \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & H^1(O_S; W^{n+1})/Div \arrow[r] & H^1(F; W^{n+1})/Div \arrow[r] & \bigoplus_{v \notin S} H^0(k_v; W^n) \arrow[r] & 0
\end{tikzcd}$$

Diagram 3.2

Observe that the right vertical arrow is the direct sum of the isomorphic arrows for each v. It is clear from Diagram 3.1. Let us notice that the lower horizontal exact sequence in Diagram 3.2 can be written as follows:

$$0 \to K_{2n}^{et}(O_S) \to K_{2n}^{et}(F)_l \to \bigoplus_{v \notin S} K_{2n-1}^{et}(k_v) \to 0.$$

Observe that there is not any localization exact sequence for étale K-theory without coefficients.

Let O_v^h denote the henselization of O with respect to the prime which corresponds to the valuation v. Let F_v^h be its field of fractions. We have the obvious cartesian, commutative diagram.

$$\begin{tikzcd} spec\ F_v^h \arrow[r, "j_v"] \arrow[d] & spec\ O_v^h \arrow[d] \\ spec\ F \arrow[r, "j"] & spec\ O \end{tikzcd}$$

Diagram 3.3

This diagram gives us the following morphism of spectral sequences.

$$\begin{tikzcd} H^p(O_S; R^q j_* W^{n+1}) \arrow[r, Rightarrow] \arrow[d] & H^{p+q}(F; W^{n+1}) \arrow[d, "res"] \\ H^p(O_v^h; R^q j_{v*} W^{n+1}) \arrow[r, Rightarrow] & H^{p+q}(F_v^h; W^{n+1}) \end{tikzcd}$$

Diagram 3.4

From this morphism of spectral sequences we get the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & H^1(O_S; j_* W^{n+1}) \arrow[r] \arrow[d] & H^1(F; W^{n+1}) \arrow[r] \arrow[d, "res"] & \bigoplus_{v \notin S} H^0(k_v; W^n) \arrow[r] \arrow[d, "proj"] & 0 \\ 0 \arrow[r] & H^1(O_v^h; j_{v*} W^{n+1}) \arrow[r] & H^1(F_v^h; W^{n+1}) \arrow[r] & H^0(k_v; W^n) \arrow[r] & 0 \end{tikzcd}$$

Diagram 3.5

Let us observe that $H^*(F_v^h; W^{n+1}) = H^*(F_v; W^{n+1})$. Hence Diagram 3.5 gives us the following commutative diagram.

$$\begin{tikzcd} H^1(F; W^{n+1})/Div \arrow[rr] \arrow[dr] & & \bigoplus_{v \notin S} H^0(k_v; W^n) \\ & \bigoplus_v H^1(F_v; W^{n+1})/Div \arrow[ur] & \end{tikzcd}$$

Diagram 3.6

2. COMPARISON OF $K_{2n}(O)$ AND $K_{2n}^w(O)$. Combining Diagrams 3.2, 3.6 and the exact sequence from Theorem 2 we obtain the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & K_{2n}^w(O)_l \arrow[r] \arrow[d] & K_{2n}(F)_l \arrow[r, "\lambda_l"] \arrow[d, "id"] & \bigoplus_v W^n(F_v) \arrow[r] \arrow[d, "proj"] & W^n(F) \arrow[r] & 0 \\ 0 \arrow[r] & K_{2n}(O)_l \arrow[r] & K_{2n}(F)_l \arrow[r, "\delta_l"] & \bigoplus_v K_{2n-1}(k_v)_l \arrow[r] & 0 & \end{tikzcd}$$

Diagram 3.7

We call the right vertical map *proj* because it is an isomorphism for v which does not divide l and it is trivial for $v \mid l$ because $K_{2n-1}(k_v)_l = 0$ in this case.

Using Diagram 3.7 and applying few times the snake lemma as it was done in the paper [C-L] p. 534, we get the following theorem.

THEOREM 4. *For any number field F, any positive integer n and any odd prime number l we have the following equality.*

$$\frac{\# K_{2n}(O)_l}{\# K_{2n}^w(O)_l} = \left| \frac{\Pi_{v|l}\, w_n(F_v)}{w_n(F)} \right|_l^{-1}.$$

REMARK 8. For the case $n = 1$ Theorem 4 was first pointed out by Tate [C-L] Proposition 5.1.

REMARK 9. Let F be totally real and n be positive and odd. Then $w_n(F) = 1$. The following conjecture:

CONJECTURE 1 (QUILLEN-LICHTENBAUM). For F totally real, n odd, positive and l an odd prime number the following equality holds.

$$\# K_{2n}(O)_l = |w_{n+1}(F)\zeta_F(-n)|_l^{-1},$$

is equivalent by Theorem 4 to the following:

CONJECTURE 2. For F totally real, n odd, positive and l an odd prime number the following equality holds.

$$\# K_{2n}^w(O)_l = \left| \frac{w_{n+1}(F)\zeta_F(-n)}{\Pi_{v|l}\, w_n(F_v)} \right|_l^{-1}.$$

REMARK 10. The above conjectures are true for $n = 1$.

REMARK 11. Nowadays it is presumed that in Conjecture 1 the group $K_{2n}(O)$ should be replaced by $\mathrm{Gr}_\gamma^{n+1} K_{2n}(O)$, where Gr_γ^* is the gradation with respect to γ-operations on K-theory.

Section IV

1. NOTATION. In this section F will denote an abelian extension of Q, n will denote a non negative integer and l is as usual an odd prime. Let $G = G(F/Q)$. Let f be the conductor of F and let b denote an integer relatively prime to f. In addition if we have $(a, f) = 1$, we write σ_a to be the automorphism of $Q(\mu_f)$ such

that its action on μ_f is just raising to the a power. Let (a, F) = restriction of σ_a to F. The Stickelberger element $\Theta_n(b)$ is defined in the following way:

$$\Theta_n(b) = (b^{n+1} - (b, F)) \sum_{(a,f)=1; 1 \leqslant a < f} \zeta_f(a, -n)(a, F)^{-1},$$

where

$$\zeta_f(a, s) = \sum_{n>0; n \equiv a \bmod f} n^{-s}$$

is the partial zeta function. If b is an integer relatively prime to $w_{n+1}(\mathbf{Q}(\mu_f))$, then Coates and Sinnott proved that $\Theta_n(b) \in \mathbf{Z}[G]$. Let us denote S_n to be the ideal of $\mathbf{Z}[G]$ generated by the elements $\Theta_n(b)$ for b relatively prime to $w_{n+1}(\mathbf{Q}(\mu_f))$. This ideal is called the Stickelberger ideal.

2. EVIDENCE FOR COATES-SINNOT CONJECTURE. Coates and Sinnott [C-S] conjectured that S_n annihilates $K_{2n}(O)$, where O is the ring of integers in F and n is an odd positive integer. In this chapter we will give some more evidence to this conjecture. More precisely we will prove the following.

THEOREM 5. *S_n annihilates $\lambda_l(K_{2n}(O)_l)$.*

As usual we will do some additional work before proving this theorem. It is obvious from Diagram 3.7 that

$$\lambda_l(K_{2n}(O)_l) \subset \bigoplus_{v|l} W^n(F_v).$$

LEMMA 3. *$W^n(F_v) = 0$ if $v \mid l$ and $(l, f) = 1$.*

Proof. If $Q(\mu_f)$ is the ray class field for F and if v_1 denotes a prime of $Q(\mu_f)$ over v, then it is enough to observe that $W^n(Q(\mu_f)_{v_1}) = 0$. Assume on contrary that $W^n(Q(\mu_f)_{v_1})$ contains an element of order l^k. Denote v_2 to be a place of $Q(\xi)$ over l, where ξ is a root of order l^k. Consider $Q(\xi)_{v_2}$. Then obviously we have:

$$Q(\xi)_{v_2} \cap Q(\mu_f)_{v_1} = Q_l$$

where the intersection takes place in $\bar{Q}_l$. Hence we can find a number $a \equiv 1 \bmod f$ such that a generates the group $(\mathbf{Z}/l^k)^*$. In addition we can see that σ_a acts trivially on $Q(\mu_f)_{v_1}$, where σ_a is considered to act on $Q(\mu_f)_{v_1}$ in the obvious way. But the group $W^n(Q(\mu_f)_{v_1})$ contains an element of order l^k hence $\sigma_a^n(\xi) = \xi^{a^n} = \xi$. Hence: $a^n \equiv 1 \bmod l^k$. So $l^{k-1}(l-1) \mid n$. But it cannot happen if $k > 0$, because n is odd. $\square$

Hence we have to only prove that for any odd $l \mid f$, S_n annihilates $\bigoplus_{v|l} W^n(F_v)$.

LEMMA 4. *Let $l \mid f$. Let β denote a prime ideal over l in the ring of integers of the field $Q(\mu_f)$. Let D_β be the decomposition group of β. Then:*

$$\sum_{\sigma_a \in \sigma_c D_\beta} \zeta_f(a; 0) = 0.$$

Proof. Let $f = l^e f_0$, where $(f_0; l) = 1$. Let β_1 be a prime over l in $Q(\mu_{f_0})$. Let in addition D_{β_1} be the decomposition group of β_1 in $Q(\mu_{f_0})$. Let K (H resp.) be the fields of invariants under $D_\beta (D_{\beta_1}$ resp.). We observe that $K = H$. Let us define $G_0 = G(K/Q)$ and $\hat{G}_0$ the character group of G_0. Observe that the Frobenius automorphism σ_l is trivial when restricted to K. So for every $\chi \in \hat{G}_0$; $\chi(\sigma_l) = 1$. We can consider every character mod f_0 as a character mod f. Let us define:

$$L_f(\chi; s) = \prod_{(p,f)=1} (1 - \chi(p)/p^s)^{-1}.$$

Observe that we have:

$$L_f(\chi; s) = (1 - 1/l^s) L_{f_0}(\chi; s).$$

Considering each character of $\hat{G}_0$ mod f we get:

$$L_f(\chi; s) = \sum_{(a,f)=1; 1 \leqslant a < f} \chi(a) \zeta_f(a, s).$$

In the following we will denote $D = D_\beta$. Observe that:

$$\chi(a) = \chi(a') \quad \text{if } (a; K) = (a'; K) \quad \text{iff } \sigma_{a'} \sigma_a^{-1} \in D_\beta.$$

Hence:

$$L_f(\chi; s) = \sum_{\sigma_c} \sum_{\sigma_a \in \sigma_c D} \chi(c) \zeta_f(a, s) = \sum_{\sigma_c} \left\{ \sum_{\sigma_a \in \sigma_c D} \zeta_f(a, s) \right\} \chi(c),$$

where the summation Σ_{σ_c} is over some chosen set of representatives mod D. But for every $\chi \in \hat{G}_0$ we have:

$$L_f(\chi; 0) = (1 - 1/l^0) L_{f_0}(\chi; 0) = 0.$$

So

$$\sum_{\sigma_c} \left\{ \sum_{\sigma_a \in \sigma_c D} \zeta_f(a; 0) \right\} \chi(c) = 0.$$

Hence by independence of characters we get our Lemma 4. □

COROLLARY 3. *Lemma 4 is valid if we put any abelian F/Q (with conductor f) instead of $Q(\mu_f)$ and put β to be a prime of F over l. More precisely:*

$$\sum{}' \zeta_f(ac; 0) = 0$$

where the summation is over $(a; f) = 1$, $1 \leqslant a \leqslant f$, $(a; F)\beta = \beta$. In addition c is a fixed integer number such that $(c, f) = 1$, $1 \leqslant c \leqslant f$.

Denote:

$$\Delta_{n+1}(a, b, f) = b^{n+1}\zeta_f(a, -n) - \zeta_f(ab, -n).$$

Then it is easy consequence of Corollary 3 that:

$$\sum{}' \Delta_1(ac, b, f) = 0,$$

where Σ' and c are as in Corollary 3.

3. PROOF OF THEOREM 5. Let $\# W^n(F_v) = l^r$. Let us remind that $v \mid l$. Observe that $l^r \mid f_n$ where $f_n = f \prod_{p \mid f} p^{v_p(n)}$. It is so because $l \mid f$ and in this case: $f_n = |w_n(Q(\mu_f))|_l^{-1} = |w_n(Q_l(\mu_f))|_l^{-1}$ and $w_n(F_v) \mid w_n(Q_l(\mu_f))$. Consider the following diagram:

$$\begin{tikzcd}
 & Q(\mu_{f_n}) \arrow[r, dash] \arrow[d, dash] & Q_l(\mu_{f_n}) \arrow[d, dash] \\
Q(\mu_f) \arrow[dr, dash] & F(\mu_{l^r}) \arrow[r, dash] \arrow[d, dash] & F_v(\mu_{l^r}) \arrow[d, dash] \\
 & F \arrow[r, dash] \arrow[d, dash] & F_v \arrow[d, dash] \\
 & Q \arrow[r, dash] & Q_l
\end{tikzcd}$$

Diagram 4.1

Let $(a, f) = (b, f) = 1$. Let also $\sigma \in D_v \subset G(F/\mathbf{Q})$ and in addition let $\sigma = (a, F) = (b, F)$. We let D_v be the decomposition group of the valuation v, corresponding to the prime ideal β over l. Hence $\sigma \in G(F_v/Q_l) = D_v$. Let $\tilde{\sigma}_a$, $\tilde{\sigma}_b$ denote the liftings of σ_a and σ_b to the field $Q_l(\mu_{f_n})$ respectively. Because $l^r = \# W^n(F_v)$, we have $(\tilde{\sigma}_a)^n = (\tilde{\sigma}_b)^n$ on $\mu_{l^r} \subset Q_l(\mu_{f_n})$. In addition $(a, f_n) = (b, f_n) = 1$ because $l \mid f$. Hence we can consider $\tilde{\sigma}_a$ acting on μ_{f_n} by raising to the a power. Let us explain now how we consider the action of $G(F/\mathbf{Q})$ on $\bigoplus_{v|l} W^n(F_v)$.

We define it in the following way.

(1) If $(a, F) \in D_v$ then we consider it to act as $\tilde{\sigma}_a$ on $W^n(F_v)$ for each v and it acts by raising to the a^n power. Hence the action of σ does not depend on the lifting.

(2) If $(a, F) \notin D_v$ (say (a, F) shifts v to v'), then we can lift it to an isomorphism of the fields:

$$\tilde{\sigma}_a\colon F_v(\mu_{l^r}) \to F_{v'}(\mu_{l^r})$$

which will induce the action:

$$\tilde{\sigma}_a\colon W^n(F_v) \to W^n(F_{v'}).$$

As in (1) it is also very easy to see that this action does not depend on the lifting $\tilde{\sigma}_a$. Observe that for every $v \mid l$ all D_v are equal and we simply denote it D. Now we are ready to finish the proof of the theorem. Observe that:

$$\begin{aligned}\Theta_n(b) &= \sum_{(a,f)=1;1\leqslant a<f} \Delta_{n+1}(a, b, f)(a, F)^{-1}\\ &= \sum_{(c,F)} \left\{ \sum_{(a,F)\in D} \Delta_{n+1}(ac, b, f)(a, F)^{-1} \right\} (c, F)^{-1}.\end{aligned}$$

The summation $\Sigma_{(c,F)}$ is over (c, F) such that $1 \leqslant c \leqslant f$, $(c, f) = 1$ and the cosets $(c, F)D$ are all, different cosets of D in G.

Coates and Sinnott proved that:

$$\Delta_{n+1}(a, b, f) \equiv a^n b^n \Delta_1(a, b, f) \bmod f_n.$$

But $\# W^n(F_v) \mid f_n$ as we mentioned before so:

$$\Theta_n(b) \equiv \sum_{(c,F)} \left\{ \sum_{(a,F)\in D} a^n b^n c^n \Delta_1(ac, b, f)(a, F)^{-1} \right\} (c, F)^{-1} \bmod l^r.$$

Also:

$$\sum_{(a,F)\in D} a^n b^n c^n \Delta_1(ac, b, f)(a, F)^{-1}$$

acts on each $W^n(F_v)$ via multiplication by:

$$\sum_{(a,F)\in D} a^n b^n c^n \Delta_1(ac, b, f) a^{-n} = b^n c^n \sum_{(a,F)\in D} \Delta_1(ac, b, f) = 0$$

by Corollary 3. The first equality above is considered on $W^n(F_v)$. □

4. REMARKS ABOUT THE COATES-SINNOTT CONJECTURE

LEMMA 5. *Let $F/\mathbf{Q}$ be Galois. Then the map*

$$\lambda_l\colon K_{2n}(F)_l \to \bigoplus_v W^n(F_v)$$

is $G(F/\mathbf{Q})$ invariant.

Proof. λ_l is the composition of

$$K_{2n}(F)_l \to H^1(F;\, W^{n+1})/\mathrm{Div}$$

and

$$H^1(F;\, W^{n+1})/\mathrm{Div} \to \bigoplus_v W^n(F_v).$$

To see that the upper map commutes with Galois action, it is enough to look at Diagrams 1.1, 1.2, 1.3, 1.4 and the left square of 1.5. Then take S and T to be $G(F/\mathbf{Q})$ invariant and observe that all the maps in these diagrams are $G(F/\mathbf{Q})$ invariant. On the other hand to check that the lower map is $G(F/\mathbf{Q})$ invariant is enough to observe that:

$$H^1(F;\, W^{n+1})/\mathrm{Div} \to \bigoplus_v H^1(F_v;\, W^{n+1})/\mathrm{Div}$$

is $G(F/Q)$ invariant and that the action of $G(F/Q)$ on $\bigoplus_v H^1(F_v;\, W^{n+1})/\mathrm{Div}$ agrees with the action on $\bigoplus_v W^n(F_v)$ which we defined in the proof of Theorem 5. □

COROLLARY 4. $\{K_{2n}(O)_l\}^{S_n} \subset K_{2n}^{w}(O)_l$.

LEMMA 6. *If $f\colon G \to H$ is a surjective homomorphism of l-torsion abelian groups and it has finite kernel then the induced map $f\colon \mathrm{div}(G) \to \mathrm{div}(H)$ is surjective where* $\mathrm{div}(G)$ ($\mathrm{div}(H)$ resp.) *denotes the group of divisible elements in G (H resp.).*

Proof. [B] Lemma 11.

COROLLARY 5. *Let F be totally real abelian over $\mathbf{Q}$, n be an odd positive integer and l be an odd prime number. Then S_n annihilates $D(n)_l$.*

Proof. By Diagrams 1.1, 1.3 and 3.2 we observe that $D_{n+1}(F) \subset K_{2n}^{\mathrm{et}}\left(O\left[\frac{1}{l}\right]\right)$ canonically. By [L] $H^1\left(O\left[\frac{1}{l}\right];\, W^{n+1}\right) \approx (A_\infty \otimes \tau(n))^{G_\infty}$ where A_∞ is the direct limit with respect to m over the l torsion parts of the class groups of $F(\mu_{l^m})$, $\tau(n)$ is the Tate module twisted with itself n-times and $G_\infty = G(F(\mu_{l^\infty})/F)$. By [C-S] Theorem 2.1 S_n annihilates $(A_\infty \otimes \tau(n))^{G_\infty}$. Hence S_n annihilates $D_{n+1}(F)$. By Theorems 1 and 3 and Lemma 6; $D(n)_l$ is isomorphic with $D_{n+1}(F)$ via:

$$K_{2n}(F)_l \to H^1(F; W^{n+1})/\mathrm{Div}$$

which is $G(F/\mathbf{Q})$ invariant by Lemma 5. So S_n annihilates $D(n)_l$. □

REMARK 12. We proved [B] that nS_n annihilates $D(n)_l$ for any abelian extension of **Q**.

REMARK 13. The Lichtenbaum-Quillen conjecture implies the Coates-Sinnott conjecture.

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