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Configuration space of 8 points on the projective line and a 5-dimensional Picard modular group

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Dedicated to Professor Shoshichi Kobayashi on his sixtieth birthday

Let X_n be the configuration space of $n+3$ points on the complex projective line $\mathbb{P}^1$. The problem of modular representation of the space, namely the problem on how to find a discrete subgroup Λ acting on the complex n-ball $\mathbb{B}_n$ such that $X_n \cong \mathbb{B}_n/\Lambda$, has been studied by many authors (e.g. [Pic], [Ter], [DM]). Their idea is to consider a family of curves presented as covers of $\mathbb{P}^1$ branching at $n+3$ points and to study a suitable set of periods, which turns out to form a solution system of the so-called Appell hypergeometric system E_D^n defined on X_n. When $n=1$ the modular interpretation is classically known by the name of Schwarz theory. When $n \geqslant 6$ no one has ever succeeded to construct such a theory. In this paper we study the case $n=5$; we express the discrete subgroup in question, first as a reflection group and second as a congruence subgroup. Several properties of the group are described in terms of combinatorics of the $n+3$ points.

Consider the family of curves

$$S_x: s^4 = \prod_{j=1}^{8} (t - x_j),$$

and their six periods

$$u_k(x) = \int_{c_k(x)} \prod_{j=1}^{8} (t - x_j)^{-1/4}\, \mathrm{d}t \qquad (k = 1, \dots, 6),$$

where the $c_k(x)$'s are six suitable linearly independent cycles on S_x. The period map $x \mapsto (u_1(x), \dots, u_6(x))$ is multi-valued with monodromy Λ, a subgroup of $\mathrm{GL}(6, \mathbb{C})$; the group Λ acts properly discontinuously on a domain $\mathbb{D}$ isomorphic to the 5-dimensional ball $\mathbb{B}_5$. The period map induces an injection of the configuration space

$$X_8 = \{(x_1, \dots, x_8) \in (\mathbb{P}^1)^8 \mid x_i \neq x_j (i \neq j)\}/\mathrm{PGL}(2)$$

of 8 distinct points on $\mathbb{P}^1$ into the quotient space $\mathbb{D}/\Lambda$. In Section 1 we recall the notion of admissible sequences introduced in [MSY] and get in Section 2 a combinatorial description of the boundary of X_8 and its compactification (some wrong statements in [MSY] are corrected here). The monodromy group Λ is known to be a reflection group, whose root vectors of generating reflections are given in Section 4. The group Λ is presented explicitly as a principal congruence subgroup of the modular group given in Section 3; a proof is given in Sections 5, 6 and 7. Since the space X_8 obviously admits the action of the symmetry group S_8, the group Λ should have an extension by S_8; this extension is given as a reflection group and also as a modular group.

The facts stated in Sections 2 and 3 are essentially known ([Ter], [DM]); so we do not give any proof; there we present a combinatorial explanation in terms of admissible sequences.

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1. Admissible sequences

Let $\mu_1, \ldots, \mu_{n+3}$ be $n+3$ rational numbers satisfying

$$0 < \mu_j < 1, \sum_{j=1}^{n+3} \mu_j = 2.$$

A sequence of rational numbers $\mu_1, \ldots, \mu_{n+3}$ is said to be n-admissible if there are integers n_{ij} greater than 1 or equal to ∞ such that

$$|\mu_i + \mu_j - 1| = \frac{1}{n_{ij}}$$

for any distinct i and j. For such a sequence $\mu_1, \ldots, \mu_{n+3}$, if there are indices i and j $(i < j)$ such that $\mu_i + \mu_j < 1$ then the sequence

$$\mu_1, \ldots, \mu_{i-1}, \mu_i + \mu_j, \mu_{i+1}, \ldots \mu_j, \ldots, \mu_{n+3}$$

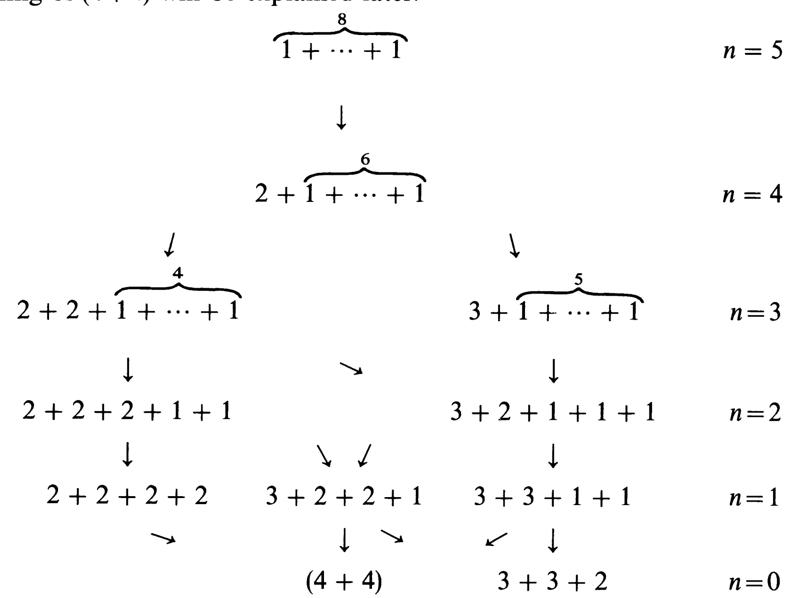
is $(n-1)$-admissible, which will be called the restriction with respect to i and j. If $n \geqslant 2$, there are only finitely many such sequences; it also turns out that n must be $\leqslant 5$. The complete list of them can be found in [Ter], [DM] and [MSY]. In case $n = 5$, there is a unique admissible sequence

$$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}.$$

The following diagram shows its restrictions; in the diagram, the sequence of rational numbers

$$\frac{m_1}{4}, \ldots, \frac{m_{n+3}}{4} \quad \text{is expressed by} \quad m_1 + \cdots + m_{n+3};$$

two sequences connected by $\rightarrow$ mean that the latter is a restriction of the former. The meaning of $(4+4)$ will be explained later.



2. Families of curves and their periods

Let $x_1, \ldots, x_{n+3}$ be $n + 3$ ($n \geqslant 2$) distinct points on $\mathbb{P}^1$. For each admissible sequence $\mu = \{\mu_1, \ldots, \mu_{n+3}\}$, consider the family $S(\mu)$ of curves

$$S_x\colon s^d = \prod_{j=1}^{n+3} (t - x_j)^{d\mu_j},$$

where d is the smallest common denominator of the μ_j's and the following $n + 1$

periods of S_x:

$$u_k(x) = \int_{c_k(x)} \prod_{j=1}^{n+3} (t - x_j)^{-\mu_j} \mathrm{d}t \qquad (k = 1, \ldots, n+1),$$

where the $c_k(x)$'s are $n+1$ linearly independent cycles on S_x depending continuously on $x = (x_1, \ldots, x_{n+3})$. Let $\Gamma(\mu)$ be the monodromy group of the (multi-valued) vector function $(u_1, \ldots, u_{n+1})$; it turns out that the group acts properly discontinuously on $\mathbb{B}_n$. Let $\mathbb{B}_n^0(\mu)$ be the maximal open subset of $\mathbb{B}_n$ on which the group $\Gamma(\mu)$ acts freely; the complement of $\mathbb{B}_n^0(\mu)$ in $\mathbb{B}_n$ is the union of countably many hyperplanes passing through the ball. Let X_n be the space of $n+3$ distinct points on the projective line:

$$X_n = \{(x_1, \ldots, x_{n+3}) \in (\mathbb{P}^1)^{n+3} \mid x_i \neq x_j (i \neq j)\}/\mathrm{PGL}(2),$$

which is a Zariski open set of $\mathbb{C}^n$. Then the period map

$$x \mapsto (u_1(x), \ldots, u_{n+1}(x)) \in \mathbb{P}^n$$

gives an isomorphism

$$X_n \to \mathbb{B}_n^0(\mu)/\Gamma(\mu).$$

If μ has restriction with respect to i and j, i.e. if

$$\mu_1, \ldots, \mu_{i-1}, \mu_i + \mu_j, \mu_{i+1}, \ldots, \widehat{\mu_j}, \ldots, \mu_{n+3}$$

is $(n-1)$-admissible, then attach to X_n the manifold

$$\{(x) \in (\mathbb{P}^1)^{n+3} \mid \text{all distinct but } x_i = x_j\}/\mathrm{PGL}(2),$$

which is isomorphic to X_{n-1} in the obvious way. If μ has a restricted $(n-2)$-sequence, i.e. if

$$\mu_1, \ldots, \mu_i + \mu_j, \ldots, \mu_k + \mu_l, \ldots, \widehat{\mu_j}, \ldots, \widehat{\mu_l} \ldots, \mu_{n+3}$$

or

$$\mu_1, \ldots, \mu_i + \mu_j + \mu_k, \ldots \widehat{\mu_j}, \ldots \widehat{\mu_k}, \ldots, \mu_{n+3}$$

is $(n-2)$-admissible, then attach

$$\{(x) \in (\mathbb{P}^1)^{n+3} \mid \text{all distinct but } x_i = x_j \text{ and } x_k = x_l\}/\mathrm{PGL}(2),$$

or

$$\{(x) \in (\mathbb{P}^1)^{n+3} \mid \text{all distinct but } x_i = x_j = x_k\}/\mathrm{PGL}(2),$$

respectively. In general, for any restricted r-sequence ($r = 0, 1, \ldots, n-1$)

$$\lambda = (\lambda_1, \ldots \lambda_{r+3}), \qquad \lambda_i = \sum_{j=1}^{r_i} \mu_{k_i(j)}$$

of μ, we attach to X_n the manifold

$$S_\lambda = \{(x) \in (\mathbb{P}^1)^{n+3} \mid \text{all distinct but } x_{k_i(1)} = \cdots = x_{k_i(r_i)}\ (i = 1, \ldots, r)\}/\mathrm{PGL}(2),$$

which is isomorphic to X_r. In this way we get a quasi-projective variety $X'(\mu)$. The period map extends to

$$X'_n(\mu) \to \mathbb{B}_n/\Gamma(\mu).$$

The variety $X'(\mu)$ can be compactified to a projective variety $\bar{X}_n(\mu)$ by adding a finite number of points: the points to be added correspond to the ways of dividing the set $\{1, \ldots, n+3\}$ into two subsets I_1 and I_2 so that

$$\sum_{i \in I_1} \mu_i = \sum_{i \in I_2} \mu_i = 1.$$

The Satake compactification $\overline{\mathbb{B}_n/\Gamma(\mu)}$ of the quotient space $\mathbb{B}_n/\Gamma(\mu)$ is obtained by adding the cusps, a finite number of points; the period map extends to the isomorphism

$$\bar{X}_n(\mu) \to \overline{\mathbb{B}_n/\Gamma(\mu)}.$$

If $x^i (i = 1, \ldots, n)$ denotes the cross-ratio of $(x_i, x_{n+1}, x_{n+2}, x_{n+3})$, or equivalently, if we normalize the x_j's as $x_i = x^i$ $(i = 1, \ldots, n)$, $x_{n+1} = 0$, $x_{n+2} = 1$, $x_{n+3} = \infty$ then the periods $u_1(x), \ldots, u_{n+1}(x)$ are linearly independent solutions of Appell's hypergeometric system $E_D^n(a, b_1, \ldots, b_n, c)$ of differential equations where $a = \alpha_{n+2}$, $b_j = 1 - \alpha_j$ $(1 \leqslant j \leqslant n)$, $c = \alpha_{n+1} + \alpha_{n+2}$, $(\alpha_k = 1 - \mu_k)$.

If an admissible sequence $\mu_1, \ldots, \mu_{n+3}$ has an r-restriction λ, then the restriction of a linear combination of the periods $u_1(x), \ldots, u_{n+1}(x)$ which is holomorphic along $S_\lambda (\cong X_\lambda)$ satisfies the hypergeometric system E_D^r with the corresponding parameters; the system can be thought of as the restriction of the system E_D^n along S_λ in the sense that any solution of the restricted system is the restriction of a solution of the system E_D^n which is holomorphic along S_λ.

3. The monodromy group $\Gamma(1/4)$ as a reflection group

Let us introduce some terminology of reflections. Let A be a Hermitian form with signature $(n-, 1+)$ on an $(n+1)$-dimensional vector space V; the form $(u, v) := A(u, v)$ is supposed to be $\mathbb{C}$-linear in v and anti-$\mathbb{C}$-linear in u. Let

$$V^+ = \{v \in V \mid (v, v) > 0\}, \quad V^0 = \{v \in V \mid (v, v) = 0\},$$
$$V^- = \{v \in V \mid (v, v) < 0\}.$$

Notice that $D := V^+/\mathbb{C}^\times$ is isomorphic to the unit ball $\mathbb{B}_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 < 1\}$ and that the group $\mathrm{Aut}(D)$ of automorphisms of D is given by the projectivization of the group $\{g \in \mathrm{GL}(V) \mid (gu, gv) = (u, v)\}$. Notice also that $\partial D = V^0/\mathbb{C}^\times$. For $\alpha \in V^-$ and an r-th root of unity $\varepsilon \neq 1$, we define the following transformation $R_{\alpha,\varepsilon}$, called the reflection with respect to a root α and an exponent ε, by

$$v \longmapsto v - (1 - \varepsilon)\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha,$$

which is of order r and keeps the Hermitian form A invariant; in particular, it defines an automorphism of D. Notice that R_α pointwisely keeps the subspace $\alpha^\perp = \{v \in V \mid (\alpha, v) = 0\}$, which is called the mirror of the reflection. When $r = 2$ i.e. $\varepsilon = -1$, we write R_α in place of $R_{\alpha,\varepsilon}$. A group generated by reflections is called a reflection group.

Let us fix a basis of a 6-dimensional linear space V and a Hermitian form $A = (a_{jk})$ $(1 \leqslant j, k \leqslant 6)$ as follows:

$$a_{jj} = -2, \quad a_{jk} = \bar{a}_{kj} = -1 + i \ (j < k), \text{ where } i = \sqrt{-1}.$$

Let $\Gamma(1/4)$ denote the monodromy group $\Gamma(\mu)$ which corresponds to the admissible sequence $\mu = \{1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4\}$. Generators of the monodromy group $\Gamma(1/4)$ can be obtained by tracing the homotopic change of the cycles c_k appeared in Section 2. Actual computation can be found in many literatures e.g. [Ter]; so we recall the result.

PROPOSITION 3.1. *The monodromy group* $\Gamma(1/4)$ *is generated by the* $21 = \binom{7}{2}$ *reflections* $R(jk) = R(kj) := R_{\alpha(jk)}$ *where*

$$\alpha(01) = (1, 0, \ldots, 0), \ldots, \alpha(06) = (0, \ldots, 0, 1),$$
$$\alpha(jk) = \alpha(0j) - \alpha(0k) \ (1 \leqslant j < k \leqslant 6).$$

Each reflection $R(jk)$, $0 \leqslant j < k \leqslant 6$, corresponds to a path in X_5 caused by a travel of the point x_j following a loop going once around the point x_k. For symmetry sake let us define seven other reflections $R(j7) = R(7j) := R_{\alpha(j7)}$, $0 \leqslant j \leqslant 6$, with the following roots $\alpha(j7)$:

$$\alpha(07) = (1,\ -i,\ -1,\ i,\ 1,\ -i),$$
$$\alpha(17) = (1+i,\ -i,\ -1,\ i,\ 1,\ -i), \quad \alpha(27) = (-1,\ 1+i,\ -i,\ -1,\ i,\ 1),$$
$$\alpha(37) = (-i,\ -1,\ 1+i,\ -i,\ -1,\ i), \quad \alpha(47) = (1,\ -i,\ -1,\ 1+i,\ -i,\ -1),$$
$$\alpha(57) = (i,\ 1,\ -i,\ -1,\ 1+i,\ -i), \quad \alpha(67) = (-1,\ i,\ 1,\ -i,\ -1,\ 1+i).$$

These reflections can be expressed by $R(jk)$ $0 \leqslant j < k \leqslant 6$; indeed we have

$$R(j7) = -iR((j+1)j) \cdots R(6j)R(0j)R(1j) \cdots R((j-1)j) \quad (0 \leqslant j \leqslant 6);$$

an expression of iI_6 will be given in Lemma 6.5. We thus defined $28 = \binom{8}{2}$ reflections.

REMARK 3.2. The fundamental group of X_n is the colored $(n+3)$-braid group and the system E_D^n gives an $(n+1)$-dimensional representation of the group.

4. Results: the monodromy group as a congruence subgroup

Let $\mathbb{Z}[i]$ be the ring of Gauss integers; the full modular group Γ is defined by

$$\Gamma = \{X \in \mathrm{GL}(6, \mathbb{Z}[i]) \mid X^*AX = A\}.$$

The principal congruence subgroup $\Gamma(1-i)$ with respect to the ideal $(1-i) \subset \mathbb{Z}[i]$ is defined by

$$\Gamma(1-i) = \{X \in \Gamma \mid X \equiv I_6 \bmod (1-i)\}.$$

An integral root of norm -2 is a vector $\alpha \in V^-$ whose entries are in $\mathbb{Z}[i]$ such that $(\alpha, \alpha) = -2$. By definition, for every integral root α of norm -2, the reflection R_α belongs to $\Gamma(1-i)$ and the reflection $R_{\alpha,i}$ belongs to Γ, which imply that the monodromy group $\Gamma(1/4)$ is a subgroup of the group $\Gamma(1-i)$, and the group generated by reflections with norm -2 and exponent i is a subgroup of Γ. Let $\Gamma(1/\!/4)$ be the group generated by the reflections with the roots $\alpha(jk)$ $(0 \leqslant j < k \leqslant 7)$ and exponents i; by the definition of reflections it is obvious that $\Gamma(1/\!/4)$ is a subgroup of Γ.

The following are the main theorems of the present paper.

THEOREM 4.1. $\Gamma(1/4) = \Gamma(1-i)$.

THEOREM 4.2. $\Gamma(1//4) = \Gamma$.

To show these theorems we study the cusps and the corresponding parabolic parts of the reflection groups and the congruence groups. The following sections are devoted to the proof of the theorems. The theorems, the argument in the preceeding sections and the proof of the theorems lead to the following facts:

(1) The group $\Gamma(1-i)$ is a lattice of $\mathrm{Aut}(D)$; it has 35 cusps.

(2) Let D_{reg} be the subset of D consisting of points where the group $\Gamma(1-i)$ acts freely. Then we have

$$D_{\mathrm{reg}} = D - \cup\{\alpha^{\perp} \cap D \mid \alpha\text{: integral of norm } -2\}.$$

(3) The period map $X_8 \ni x \mapsto u(x) \in D$ given by the integrals

$$u_k(x) = \int_{c_k(x)} \prod_{j=1}^{8} (t - x_j)^{-1/4}\, \mathrm{d}t \quad 1 \leqslant k \leqslant 6,$$

(where the c_j are suitable linearly independent cycles) induces the isomorphisms

$$X_8 \to D_{\mathrm{reg}}/\Gamma(1-i), \quad X_8'(1/4) \to D/\Gamma(1-i),$$

$$\bar{X}_8(1/4) \to \overline{D/\Gamma(1-i)}.$$

(4) The group Γ is a lattice of $\mathrm{Aut}(D)$; it has only one cusp.

(5) The quotient group $\Gamma/\Gamma(1-i)$ is isomorphic to the symmetric group S_8 of degree 8 acting transitively on the set of 35 cusps of $\Gamma(1-i)$.

(6) The set of mirrors of the reflection in Γ coincides with that of $\Gamma(1-i)$.

(7) The quotient space D_{reg}/Γ is isomorphic to the quotient space X_8/S_8 of the configuration space X_8 by the group S_8 acting as a full symmetric group of the 8 points; the space X_8/S_8 can be thought of as the configuration space of distinct 8 (unordered) points on the complex projective line.

5. Combinatorial structure of the cusps of $\Gamma(1/4)$

By the argument in Section 2, we have the following description of the cusps, which are by definition the equivalence classes of the rational boundary components.

PROPOSITION 5.1. *The cusps of* $\Gamma(1/4)$ *are described in terms of the way of dividing the set* $\{0, 1, \ldots, 7\}$ *into two subsets with the same cardinality.*

It is now clear that the number of $\Gamma(1/4)$-cusps is $35 = \binom{8}{4}/2$. Let us denote by

$P(I_1, I_2) = P(I_2, I_1)$ the cusp corresponding to the division

$$I_1 \cup I_2 = \{0, 1, \ldots, 7\}, \ |I_1| = |I_2| = 4.$$

PROPOSITION 5.2. *A reflection $R(jk) \in \Gamma(1/4)$ fixes a cusp $P(I_1, I_2)$ if and only if*

$$\{j, k\} \subset I_1 \quad \textit{or} \quad \{j, k\} \subset I_2.$$

Proof. By virtue of the geometric meaning of the reflection $R(jk)$ mentioned at the end of section 4 and of Proposition 5.1, the assertion is clear. □

Since $\Gamma(1/4)$ is a subgroup of $\Gamma(1-i)$, the number of cusps of $\Gamma(1-i)$ is not greater than 35. We can easily check the following

LEMMA 5.3. *The following* 35 $\Gamma(1/4)$*-rational boundary components are not mutually* $\Gamma(1-i)$ *equivalent. Twenty of them are given by coordinates* $(x_1, \ldots, x_6)$ *such that three of* x_j*'s are* 0 *and the remaining three are* $1, -i, -1$ *in this order. Fifteen of them are given by coordinates* $(x_1, \ldots, x_6)$ *such that two of* x_j*'s are* 0 *and the remaining four are* $1, -i, -1, i$ *in this order.*

COROLLARY 5.4. *The number of cusps of* $\Gamma(1-i)$ *is equal to that of* $\Gamma(1/4)$.

The following lemma is also easy to check.

LEMMA 5.5. *The group* $\Gamma(1//4)$ *acts transitively on the* 35 *cusps given in Lemma* 5.3. *Thus the group* $\Gamma(1//4)$ *as well as the group* Γ *has exactly one cusp.*

6. Parabolic parts of Γ(1/4) and Γ(1 − i)

Since the 35 cusps of the group $\Gamma(1/4)$ have the same structure (Lemma 5.5), let us study the parabolic part of the group $\Gamma(1-i)$ corresponding to the Γ-rational boundary point $(-1, i, 1, 0, 0, 0)$, which represents the cusp $P(\{0, 1, 2, 3\}, \{4, 5, 6, 7\})$. As usual we make a linear change of coordinates $z \mapsto w = Qz$ in order to send the boundary point to a point at infinity. Put

$$Q := \begin{pmatrix} -1 & 1 & 1 \\ i & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} i & 1 & -1 \\ 0 & -1 & i \\ -i & 0 & 1 \end{pmatrix} \in \mathrm{GL}(6, \mathbb{Z}[i])$$

$$B := Q^* A Q = -\begin{pmatrix} 0 & 0 & 1-i \\ 0 & H & 0 \\ 1+i & 0 & 0 \end{pmatrix} : 6 \times 6 \text{ Hermitian form of sign } (5-, 1+)$$

where

$H := h \oplus h : 4 \times 4$ positive definite Hermitian form,

$$h := \begin{pmatrix} 2 & 1-i \\ 1+i & 2 \end{pmatrix}.$$

Accordingly the group $\Gamma(1-i)$ is changed into the principal congruence subgroup

$$\Gamma_B(1-i) = \{X \in \Gamma \mid X \equiv I \bmod (1-i)\}.$$

of the group

$$\Gamma_B = \{X \in GL(6, \mathbb{Z}[i]) \mid X^* B X = B\}.$$

and the boundary point into $w = (1, 0, 0, 0, 0, 0)$, which will be called ∞. Let us define the parabolic part $P(1-i)$ to be the subgroup of $\Gamma_B(1-i)$ consisting of transformations which keep the point ∞.

LEMMA 6.1. *The group $P(1-i)$ consists of the following matrices*:

$$[X, c, r] := \begin{pmatrix} 1 & \frac{1}{2}(i-1)c^* H X & \frac{i}{2} N[c] + r(1+i) \\ 0 & X & c \\ 0 & {}^t 0 & 1 \end{pmatrix} \in GL(6, \mathbb{Z}[i]),$$

where $c \in (1-i)\mathbb{Z}[i]^4$, $r \in \mathbb{Z}$, $N[c] = c^* H c$, and

$$X \in U(H, \mathbb{Z}[i])(1-i)$$
$$:= \{X \in GL(4, \mathbb{Z}[i]) \mid X^* H X = H,\ X \equiv I_4 \bmod (1-i)\}.$$

Proof. A transformation $Y \in P(1-i)$ is by definition of the following form:

$$\begin{pmatrix} 1 & b & d \\ 0 & X & c \\ 0 & {}^t 0 & 1 \end{pmatrix} \in GL(6, \mathbb{Z}[i]);$$

computing $Y^* B Y = B$, we have the assertion. □

From this Lemma the center of $P(1-i)$ can be seen to be equal to

$$\{[I_4, 0, r] \in P(1-i)\} \cong \mathbb{Z}.$$

Let us define a homomorphism π of $P(1-i)$ into the group of affine transformations, by forgetting the first line and the first column:

$$\pi : P(1-i) \ni [X, c, r] \mapsto [X, c] := \begin{pmatrix} X & c \\ {}^t0 & 1 \end{pmatrix}$$

and define a normal subgroup of $P(1-i)$:

$$P_1(1-i) := \{[I_4, c, r] \in P(1-i)\};$$

the image under π of $P_1(1-i)$ is

$$\{[I_4, c] \mid c \in (1-i)\mathbb{Z}[i])^4\},$$

called the lattice $L(P_1(1-i))$. Then we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & P_1(1-i) & \longrightarrow & L(P_1(1-i)) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & P(1-i) & \xrightarrow{\pi} & \pi(P(1-i)) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & \longrightarrow & G(P(1-i)) & \longrightarrow & U(H, \mathbb{Z}[i])(1-i) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \qquad \text{(CD)}$$

where the first row is given by the homomorphisms

$$r \mapsto [I_4, 0, r], \quad \text{and} \quad [1, a, r] \mapsto a;$$

and the third column, given by

$$a \mapsto [I_4, a], \quad \text{and} \quad [X, a] \mapsto X,$$

shows that the crystallographic group $\pi(P(1-i))$ is the extension of the lattice $L(P_1(1-i))$ by the point group $U(H, \mathbb{Z}[i])(1-i)$. One can readily see

LEMMA 6.2. (1) *Every exact sequence in the above table splits. In particular, the crystallographic group* $\pi(P(1-i))$ *is the semi-direct product of the lattice and the point group; the group* $P(1-i)$ *is the semi-direct product of the center and* $\pi(P(1-i))$.

(2) $U(H, \mathbb{Z}[i])(1-i)$ *is the direct product of two copies of a reflection group generated by the three reflections, with respect to the Hermitian form h, with exponents* -1 *and with the following roots*:

$$(1, 0), (0, 1), (1, -1);$$

this reflection group is isomorphic to the imprimitive 2-dimensional unitary reflection group $G(4, 2, 2)$ *of order* 16.

Now we prove

PROPOSITION 6.3. *The parabolic part of the group* $\Gamma(1/4)$ *coincides with that of the group* $\Gamma(1-i)$.

This proposition together with Corollary 5.4 leads to Theorem 4.1. To prove Proposition 6.3, it is enough to show the following

PROPOSITION 6.4. *The group* $P(1-i)$ *is a reflection group generated by* 12 *reflections with the roots*

$$\beta(jk) := Q^{-1}\alpha(jk),\ \{j, k\} \subset \{0, 1, 2, 3\} \quad \textit{or} \quad \{j, k\} \subset \{4, 5, 6, 7\}.$$

For notational simplicity, let us call the reflection $Q^{-1}R(jk)Q$ with root $\beta(jk)$ by the same name $R(jk)$.

LEMMA 6.5. $iI_6 \in \Gamma(1/4)$.

Proof. We have

$$\mathrm{diag}(1, i, i, 1, 1, 1) = R(23)R(13)R(12),$$

$$\mathrm{diag}(1, 1, 1, i, i, 1) = R(56)\ R(46)R(45)$$

Put

$$r_1 := R(04)R(05)R(06)R(45)R(46)R(56),$$

$$r_2 := R(47)R(57)R(67)R(56)R(46)R(45) = [I_4, 0, 1]$$

then we have

$$R(07)r_2^{-1}r_1 = \mathrm{diag}(i, 1, 1, 1, 1, i).$$

So that the product of the three diagonal matrices is iI_6. Recall that, by definition, $iR(j7)$ is in the group; to prove the lemma, we have only to notice that in the product the $R(j7)$'s appear just 4 times. □

LEMMA 6.6. (1) *A generator of the center of* $P(1-i)$ *is given by* r_2.

(2) *The group* $\{[X, 0, 0] \mid X \in U(H, \mathbb{Z}[i])(1-i)\}$, *a lift of the point group, is generated by the following* 6 *reflections*:

$$R(12),\ R(13),\ R(23),\ R(45),\ R(46),\ R(56).$$

(3) *The group* $\{[I_4, c, 0] \mid c \in L(P_1(1-i))\}$, *a lift of the lattice, is contained in the reflection group given in* 6.4.

Proof. (1) and (2) can be proved by straightforward computation. (3) Since the point group $U(H, \mathbb{Z}[i])(1-i)$ acts on the lattice, it suffices to see

$$R(02)R(13) = [I_4, (0, -1+i, 0, 0), -1]$$

$$R(45)R(46)R(57)R(45)\mathrm{diag}(1, 1, 1, -1, -1, 1) = [I_4, (0, 0, -1-i, 0), -1].$$

□

This lemma together with Lemma 6.2 lead to Proposition 6.4.

7. Parabolic parts of Γ(1//4) and Γ

To prove Theorem 4.2, in view of Lemma 5.5, we have only to show that the parabolic parts of the groups Γ(1//4) and Γ are the same. We transform the groups by the transformation Q in Section 6. Accordingly the group Γ is changed into the group Γ_B, and the boundary point into $w = (1, 0, 0, 0, 0, 0)$, which will be called ∞. Let us define the parabolic part P as the subgroup of Γ_B consisting of transformations which keep the point ∞.

LEMMA 7.1. *The group P consists of the matrices*: $[X, c, r] \in \mathrm{GL}(6, \mathbb{Z}[i])$, *where* $c \in (\mathbb{Z}[i])^4$, $r \in \mathbb{Z}$, *and*

$$X \in U(H, \mathbb{Z}[i]) := \{X \in \mathrm{GL}(4, \mathbb{Z}[i]) \mid X^*HX = H,\}.$$

Proof. A transformation $Y \in P$ is by definition of the following form:

$$\begin{pmatrix} 1 & b & d \\ 0 & X & c \\ 0 & {}^t0 & 1 \end{pmatrix} \in \mathrm{GL}(6, \mathbb{Z}[i]);$$

computing $Y^*BY = B$, we have the assertion. □

This Lemma shows that the center of P is equal to that of $P(1-i)$. Define a normal subgroup of P:

$$P_1 := \{[I_4, c, r] \in P\};$$

the image under π of P_1 is

$$\{[I_4, c] \mid c \in (\mathbb{Z}[i])^4\},$$

called the lattice $L(P_1)$. Then we have a commutative diagram $(CD)'$ almost exactly the same to (CD), from which $(1-i)$'s should now be removed. The crystallographic group $\pi(P)$ is the extension of the lattice $L(P_1)$ by the point group $U(H, \mathbb{Z}[i])$.

LEMMA 7.2. (1) *Every exact sequence in* $(CD)'$ *splits. In particular, the crystallographic group* $\pi(P)$ *is the semi-direct product of the lattice and the point group; the group* P *is the semi-direct product of the center and* $\pi(P)$.

(2) $U(H, \mathbb{Z}[i])(1-i)$ *admits the following exact sequences*

$$1 \to U(H, \mathbb{Z}[i])(1-i) \to U(H, \mathbb{Z}[i]) \to (S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z} \to 1$$

$$1 \to (G(4,2,2) \rtimes S_3)^2 \to U(H, \mathbb{Z}[i]) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

The group $G(4,2,2) \rtimes S_3$ *is the primitive* 2*-dimensional unitary reflection group of order* 96*, which has the Shephard-Todd registration number* 8.

PROPOSITION 7.3. *The parabolic part of the group* $\Gamma(1//4)$ *coincides with that of the group* Γ.

This proposition together with Lemma 5.5 leads to Theorem 4.2. To prove Proposition 7.3, it is enough to show the following

PROPOSITION 7.4. *The group* P *is a reflection group generated by* 12 *reflections* $\sqrt{R(jk)}$ *with exponents* i *and with the roots*

$$\beta(jk),\ \{j,k\} \subset \{0,1,2,3\} \quad \textit{or}\ \{j,k\} \subset \{4,5,6,7\}.$$

LEMMA 7.5. (1) *The group* $\{[X, 0, 0] \mid X \in U(H, \mathbb{Z}[i])\}$*, a lift of the point group, is generated by the following* 6 *reflections*:

$$\sqrt{R(12)},\ \sqrt{R(13)},\ \sqrt{R(23)},\ \sqrt{R(45)},\ \sqrt{R(46)},\ \sqrt{R(56)}.$$

(2) *The group* $\{[I_4, c, 0] \mid c \in L(P_1)\}$*, a lift of the lattice, is contained in the reflection group given in* 7.4.

Proof. (2) Since the point group $U(H, \mathbb{Z}[i])$ acts on the lattice, it suffices to see

$$\sqrt{R(02)}\sqrt{R(13)} = [I_4, (0, i, 0, 0), -1]$$

$$\sqrt{R(45)}\sqrt{R(46)}\sqrt{R(57)}\sqrt{R(45)}\,\mathrm{diag}(1, 1, 1, -i, -i, 1) = [I_4, (0, 0, -1, 0), 0].$$

□

This lemma and Lemma 7.1 lead to Proposition 7.4. The following is a corollary to Lemma 7.2.

PROPOSITION 7.6. $\Gamma/\Gamma(1-i) \cong \Gamma(1//4)/\Gamma(1/4) \cong S_8$.

Proof. The following equalities

$$|U(H, \mathbb{Z}[i])/U(H, \mathbb{Z}[i])(1-i)| = |(S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z}| = 6^2 \cdot 2$$

$$|L(P_1)/L(P_1(1-i))| = |(\mathbb{Z}/2\mathbb{Z})^4| = 2^4$$

$$35 \cdot 6^2 \cdot 2 \cdot 2^4 = 8!$$

imply $|\Gamma/\Gamma(1-i)| = 8!$. On the other hand, the group $\Gamma(1//4)/\Gamma(1/4)$ acts holomorphically on the configuration space X_8, of which group of automorphisms is known to be S_8. □

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