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Introduction

Let $\mathbf{G}$ be a connected reductive quasi-split algebraic group defined over a nonarchimedean local field F of characteristic 0. Let $G = \mathbf{G}(F)$. Harish-Chandra's philosophy of cusp forms describes the classification of the irreducible admissible representations of G in two steps: determine the supercuspidal representations of all Levi components of G, and decompose those representations which are parabolically induced from supercuspidals. With regard to this second step we ask when these parabolically induced representations are reducible. We restrict ourselves to the case where the inducing representation is unitary.

Suppose P is a parabolic subgroup of G with split component A, Levi component M, and unipotent radical N. Let $W(A) = N_G(A)/Z_G(A)$ be the Weyl group of A. Let π be a unitary irreducible supercuspidal representation of M. If $w \in W(A)$ then we define π^w by the formula $\pi^w(m) = \pi(w^{-1}mw)$. This defines an action of $W(A)$ on the equivalence classes of irreducible unitary supercuspidal representations of M. Let $W(\pi)$ be the elements of $W(A)$ which fix π. Bruhat theory shows that if $W(\pi) = \{1\}$, then the unitarily induced representation $\mathrm{Ind}_P^G(\pi) = \mathrm{Ind}_P^G(\pi \otimes 1_N)$ is irreducible. This reduces us to studying the case where $W(\pi) \neq \{1\}$.

We study the group $G = U(2, 2)$ defined with respect to a quadratic extension E/F. Then there is a maximal parabolic subgroup P of G with Levi factor $M \simeq \mathrm{GL}(2, E)$. If π is an irreducible unitary supercuspidal representation of M, then $W(\pi)$ is non-trivial if and only if π is fixed by the automorphism of $\mathrm{GL}(2, E)$ given by $g \mapsto {}^t\bar{g}^{-1}$. Let $s \in \mathbb{C}$ and $I(s, \pi) = \mathrm{Ind}_P^G(\pi \otimes |\det(\ )|_E^{s/2})$. Suppose $W(\pi)$ is nontrivial. Then there is a standard intertwining operator [12, 20] $A(s, \pi): I(s, \pi) \to I(-s, \pi)$, which is meromorphic in s as an operator valued function. Since P is maximal, the theory of intertwining operators implies $\mathrm{Ind}_P^G(\pi)$ is reducible if and only if $W(\pi) \neq \{1\}$ and 0 is not a pole of $A(s, \pi)$ [14].

A lemma of Rallis (Lemma 2.1) allows us to interpret the intertwining operator as a sum of twisted orbital integrals of the type that appear in [11]. By decomposing the domain of the orbital integral (Lemma 2.4) we are able to detect the possible pole of the intertwining operator and show that it is related to base change from $U(2)$. More explicitly, Proposition 2.6 shows that π being a base change lift is a necessary condition for $A(s, \pi)$ to have a pole at $s = 0$.

We then use orthogonality relations for twisted characters and an expansion of the Plancherel formula in terms of stable characters to show that a sufficient condition for $A(s, \pi)$ to have a pole at $s = 0$ is that π be a *non standard*, in the sense of [11], base change lift from $U(2)$ (Theorem 2.9 and Corollary 2.10). Therefore, the two base change maps from $U(2)$ to $\mathrm{GL}(2, E)$ described by Rogawski are distinguished by poles of these intertwining operators. This is the local analogue of the global result of Asai [1]. Namely, the poles of the global Asai L-function, associated to a Hilbert modular form, are related to two base change liftings, and these poles distinguish the liftings from one another.

Since the poles of the intertwining operator allow us to compute reducibility criteria we do so in Theorem 2.11. We then apply the theory of Shahidi [14] to compute the length of the complementary series in Theorems 2.12 and 2.13.

In recent work Shahidi [15] has shown that if $G = \mathrm{SO}(2n+1)$, $\mathrm{Sp}(2n)$, or $\mathrm{SO}(2n)$, and $P = MN$ with $M \simeq \mathrm{GL}(n)$, then the poles of the intertwining operators can be interpreted in terms of the theory of twisted endoscopy. Our result gives another example where the theory of transfer of twisted orbital integrals plays a role in determining the reducibility of induced representations.

In Section 1 we briefly describe the results on base change which are necessary for our argument. These results all appear in [11]. In Section 2 we carry out the computation of the pole of $A(s, \pi)$ and computation of reducibility criteria. In [18] Tamir computed a normalizing factor for $A(s, \pi)$. Our calculations duplicate this result, but also give the interpretation of the pole of the operator in terms of base change and the location of the complementary series.

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1. Preliminaries on base change

Let F be a nonarchimedean local field of characteristic 0. Let $\mathscr{R}_F$ be its ring of integers and $\mathfrak{p}_F$ the unique maximal ideal in $\mathscr{R}_F$. Let ϖ_F be a uniformizer in F, i.e. $\mathfrak{p}_F = \varpi_F \mathscr{R}_F$. Let $q_F = |\mathscr{R}_F/\mathfrak{p}_F|$ be the residual characteristic of F. Let $\bar{F}$ be a (separable) algebraic closure of F.

Let E be a quadratic extension of F. We choose an element u of F, which is not a square in F, so that $E = F(\beta)$, with $\beta^2 = u$. Let $\mathcal{R}_E$, $\mathfrak{p}_E$, ϖ_E, and q_E be the appropriate objects in E. Let $\sigma: E \to E$ be the nontrivial Galois automorphism. We also denote the action of σ by $\sigma(x) = \bar{x}$. Let $N: E^* \to F^*$ be the norm map, $N(x) = x\bar{x}$.

Let $\mathbf{H} = \mathbf{U}(2)$ as an algebraic group over F. $\mathbf{H}$ is defined as follows. Let

$$\delta_1 = \begin{pmatrix} & \beta \\ -\beta & \end{pmatrix}.$$

Then

$$\mathbf{H} = \{g \in \mathbf{GL}(2) | g\delta_1 {}^t\bar{g} = \delta_1\}.$$

Let $\tilde{\mathbf{H}} = \mathrm{Res}_{E/F}(\mathbf{H})$. Then $\tilde{\mathbf{H}}$ is an algebraic group such that

$$\tilde{\mathbf{H}}(F) = \mathbf{H}(E) = \mathrm{GL}(2, E).$$

Over E, $\tilde{\mathbf{H}} \simeq \mathbf{H} \times \mathbf{H}$. Let $H = \mathbf{H}(F)$ be the F-rational points of $\mathbf{H}$. Let $\tilde{H} = \tilde{\mathbf{H}}(F)$ be the F-rational points of $\tilde{\mathbf{H}}$. We define the automorphism $\varepsilon: \tilde{H} \to \tilde{H}$ by $g \mapsto {}^t\bar{g}^{-1}$.

DEFINITION 1.1. An element δ of $\tilde{H}$ is said to be *ε-semisimple* if (δ, ε) is semisimple in the non-connected group $\tilde{H} \rtimes \langle\varepsilon\rangle$.

DEFINITION 1.2. Two elements δ and δ' of $\tilde{H}$ are said to be *ε-conjugate* if there is a $g \in \tilde{H}$ such that $\delta' = g^{-1}\delta\varepsilon(g)$.

Let δ be an ε-semisimple element of $\tilde{H}$. Let $\tilde{H}_{\delta\varepsilon} = \{g \in \tilde{H} \,|\, g^{-1}\delta\varepsilon(g) = \delta\}$, and let $\tilde{H}'_{\delta\varepsilon} = \{g \in \tilde{H} \,|\, g^{-1}\delta\varepsilon(g)\delta^{-1} \in F^*\}$. Similarly, for γ a semisimple element of H, we define $H_\gamma = \{g \in H \,|\, g^{-1}\gamma g = \gamma\}$.

DEFINITION 1.3. An element $\delta \in \tilde{H}$ is stably ε-conjugate to δ' if there is a $g \in \tilde{\mathbf{H}}(\bar{F})$ so that $\delta = g^{-1}\delta'\varepsilon(g)$. In this case $\sigma(g)g^{-1} \in \tilde{\mathbf{H}}_{\delta\varepsilon}$ [11]. Two elements of H are *stably conjugate* if they are conjugate by some g in $\mathbf{H}(\bar{F})$. This implies that $\sigma(g)g^{-1} \in \mathbf{H}_\gamma$.

LEMMA 1.4.

(1) *A stable conjugacy class in H is a union of conjugacy classes.*
(2) *A stable ε-conjugacy class in $\tilde{H}$ is the union of ε-conjugacy classes.* □

DEFINITION 1.5. For $\gamma \in H$ we let $\mathcal{O}_{\mathrm{st}}(\gamma)$ be the stable conjugacy class of γ in H. For $\delta \in \tilde{H}$ we let $\mathcal{O}_{\varepsilon-\mathrm{st}}(\delta)$ be the ε-stable conjugacy class of δ in $\tilde{H}$.

Let E^1 be the norm 1 elements in E, i.e. $E^1 = \{z \in E^* \,|\, z\bar{z} = 1\}$. Note that

$Z = Z(H) \simeq E^1$ and $\tilde{Z} = Z(\tilde{H}) \simeq E^*$. Let $\omega: E^1 \to \mathbb{C}^*$ be a character. Let $\tilde{\omega}: E^* \to \mathbb{C}^*$ be the character defined by $\tilde{\omega}(z) = \omega(z/\bar{z})$.

Let $C(H, \omega)$ be the space of locally constant functions, compactly supported modulo Z, such that $f(zg) = \omega^{-1}(z)f(g)$ for all $z \in Z$, and $g \in H$. Similarly, we let $C(\tilde{H}, \tilde{\omega})$ be the space of locally constant functions, compactly supported modulo $\tilde{Z}$, such that $f(zg) = \tilde{\omega}^{-1}(z)f(g)$ for all $z \in \tilde{Z}$, and $g \in \tilde{H}$.

For γ a semisimple element of H and $f \in C(H, \omega)$, we define

$$\Phi(\gamma, f) = \int_{H_\gamma \backslash H} f(g^{-1}\gamma g)\, \mathrm{d}g^*,$$

where $\mathrm{d}g^*$ is the right invariant measure on the quotient coming from Haar measure $\mathrm{d}g$ on H. This is referred to as the *orbital integral* of f at γ. Similarly, for δ an ε-semisimple element of $\tilde{H}$ and $f \in C(\tilde{H}, \tilde{\omega})$, we define

$$\Phi_\varepsilon(\delta, f) = \int_{\tilde{H}'_{\delta\varepsilon} \backslash \tilde{H}} f(g^{-1}\delta\varepsilon(g))\, \mathrm{d}g^*,$$

where again the measure $\mathrm{d}g^*$ is the right invariant one coming from Haar measure. This is called the *ε-twisted orbital integral* of f at δ.

DEFINITION 1.6. We define the norm map for $\delta \in \tilde{H}$ by $N(\delta) = \delta\varepsilon(\delta)$. Note that $N(g^{-1}\delta\varepsilon(g)) = g^{-1}N(\delta)g$. Thus, N defines an injection $\mathcal{N}: [\delta] \mapsto N([\delta])$ from ε-stable conjugacy classes of $\tilde{H}$, to the set of stable conjugacy classes of H [11, §3.10].

LEMMA 1.7 (Rogawski [11, Proposition 3.12.1(e)]). *If $N(\delta)$ is scalar then for any $\delta' \in \mathcal{O}_{\varepsilon-\mathrm{st}}(\delta)$, $\det(\delta'\delta^{-1}) \in F^*$. In this case there are two ε-classes in $\mathcal{O}_{\varepsilon-\mathrm{st}}(\delta)$, and the ε-class of δ' is in correspondence with* $\det(\delta'\delta^{-1}) \in NE^* \backslash F^*$. □

We will apply this to the following situation. Recall that $\delta_1 = \begin{pmatrix} & \beta \\ -\beta & \end{pmatrix}$. Then δ_1 is hermitian, and ε-conjugacy is the equivalence of hermitian forms. If δ_2 is stably ε-conjugate to δ_1 then δ_2 is also hermitian. Note that $N\delta_1 = 1$, so the lemma applies. Therefore, $\mathcal{O}_{\varepsilon-\mathrm{st}}(\delta_1)$ consists of two ε-classes. Since there are only two classes of hermitian forms [8, 10], $\mathcal{O}_{\varepsilon-\mathrm{st}}(\delta_1)$ consists of all hermitian matrices in $\tilde{H}$. If δ_2 is a representative of the hermitian class which is not ε-equivalent to δ_1, then

$$\det(\delta_2\delta_1^{-1}) = \det(\delta_2)\beta^{-2} = \det(\delta_2)u^{-1} \not\equiv 1 \pmod{NE^*}. \tag{1.1}$$

For any connected reductive group G defined over F we let $q(G)$ be the F-rank of the commutator subgroup $[G, G]$. Let G' be the quasi-split form of G and let

$$e(G) = (-1)^{q(G)-q(G')} \quad [11, \text{pg. } 40].$$

For an ε-semisimple $\delta \in \tilde{H}$ we let $e(\delta) = e(\tilde{H}_{\delta\varepsilon})$ [11, pg. 57]. Let $\{\delta^v\}$ be a collection of representatives for the ε-conjugacy classes in $\mathcal{O}_{\varepsilon-\mathrm{st}}(\delta)$. It is possible to define a character κ on the collection of ε-conjugacy classes in $\mathcal{O}_{\varepsilon-\mathrm{st}}(\delta)$ [11, §4.10]. We denote the value of this character on the ε-class of δ^v by $\kappa(v)$. If $N(\delta)$ is scalar then κ is the nontrivial character on $NE^*\backslash F^*$. In any case κ is either the trivial character, or the nontrivial character on $NE^*\backslash F^*$ [11, pg. 59].

DEFINITION 1.8. Let δ be an ε-semisimple element of $\tilde{H}$. Let $\varphi \in C(\tilde{H}, \tilde{\omega})$. Define

$$\Phi_\varepsilon^\kappa(\delta, \varphi) = \sum_v \kappa(v) e(\delta^v) \Phi_\varepsilon(\delta^v, \varphi).$$

Also define

$$\Phi_\varepsilon^{\mathrm{st}}(\delta, \varphi) = \sum_v e(\delta^v) \Phi_\varepsilon(\delta^v, \varphi).$$

REMARK. Note that if $N(\delta)$ is scalar then

$$\Phi_\varepsilon^\kappa(\delta, \varphi) = \sum_{NE^*\backslash F^*} \kappa(v) e(\delta^v) \Phi_\varepsilon(\delta^v, \varphi).$$

Since $\tilde{H}_{\delta_1\varepsilon}$ is the quasi-split form of $U(2)$, $e(\delta_1) = 1$. For δ_2 representing the other hermitian class, $\tilde{H}_{\delta_2\varepsilon}$ is the anisotropic form of $U(2)$. Therefore, $e(\delta_2) = -1$. Moreover, if v_i is the norm class corresponding to δ_i then $\kappa(v_1) = -\kappa(v_2)$. So from definition 1.8

$$\begin{aligned}
\Phi_\varepsilon^\kappa(\delta_1, \varphi) &= \kappa(v_2) e(\delta_2) \Phi_\varepsilon(\delta_2, \varphi) + \kappa(v_1) e(\delta_1) \Phi_\varepsilon(\delta_1, \varphi) \\
&= -\kappa(v_2)(\Phi_\varepsilon(\delta_2, \varphi) + \Phi_\varepsilon(\delta_1, \varphi)) \\
&= \pm(\Phi_\varepsilon(\delta_2, \varphi) + \Phi_\varepsilon(\delta_1, \varphi)). \qquad (1.2)
\end{aligned}$$

DEFINITION 1.9. For $f \in C(H, \omega)$ and $\gamma \in H$ semisimple, we let

$$\Phi^{\mathrm{st}}(\gamma, f) = \sum_{\{\gamma'\}} e(\gamma') \Phi(\gamma', f),$$

where γ' runs over a set of representatives of conjugacy classes in $\mathcal{O}_{\mathrm{st}}(\gamma)$ [11, pg. 39]. If γ is regular, all the signs $e(\gamma') = 1$.

Let $\omega_{E/F}$ be the local class field theory character of F^* attached to E^*,

$$\omega_{E/F}(x) = \begin{cases} 1 & x \in NE^* \\ -1 & x \notin NE^*. \end{cases}$$

We let μ be a character of E^* whose restriction to F^* is $\omega_{E/F}$.

THEOREM 1.10 (Rogawski [11, Proposition 4.11.1]). *Let $\varphi \in C(\tilde{H}, \tilde{\omega})$. Then there exist $f_1 \in C(H, \omega)$ and $f_2 \in C(H, \omega\mu^{-1})$ such that, for any regular ε-semisimple $\delta \in \tilde{H}$ with $\gamma = N\delta \in H$,*

$$\Phi_\varepsilon^{st}(\delta, \varphi) = \Phi^{st}(\gamma, f_1), \tag{1.3}$$

and

$$\mu(\det \delta)\Phi_\varepsilon^\kappa(\delta, \varphi) = \Phi^{st}(\gamma, f_2). \tag{1.4}$$

□

THEOREM 1.11 (Rogawski [11, Proposition 8.4.4]). *Suppose $\delta \in \tilde{H}$ and $\gamma = N\delta$ is central in H. Then for any $\varphi \in C(\tilde{H}, \tilde{\omega})$:*

$$\Phi_\varepsilon^{st}(\delta, \varphi) = f_1(\gamma), \tag{1.5}$$

$$\Phi_\varepsilon^\kappa(\delta, \varphi) = \mu(\det \delta) f_2(\gamma), \tag{1.6}$$

where f_1 and f_2 are given in Theorem 1.10. □

We briefly recall the construction of the Weil form of the L-group. For more details the reader should consult [3]. Let K be a nonarchimedean local field and let $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$. The Weil group, W_K, is a topological group endowed with a continuous homomorphism, $\varphi: W_K \to \Gamma_K$, whose image is dense. For each finite extension K'/K we let $W_{K'} = \varphi^{-1}(\Gamma_{K'})$. Then $W_{K'}$ is a Weil group for K'. Moreover, $W_K^{ab} \simeq K^*$, and for each finite extension K'/K, $W_K/W_{K'} \simeq \mathrm{Hom}_K(K', \bar{K})$. The group W_K satisfies further number theoretic and algebraic properties which are described in [19].

If G is a connected reductive algebraic group defined over K, then G is given by a root datum $\psi(G) = (X^*(T), \Delta, X_*(T), \check{\Delta})$, where T is a maximal torus in G, $X^*(T)$ the group of characters of T, $X_*(T)$ the 1-parameter subgroups of T, Δ a choice of simple roots of T in G, and $\check{\Delta}$ the simple coroots [17]. Note that the choice T, Δ defines a Borel subgroup B of G.

The group Γ_K acts on root data as follows. If $\tau \in \Gamma_K$ then $\tau(T)$ is another maximal torus of $G(\bar{K})$ and $\tau(B)$ a Borel. Therefore, we define

$$\tau(\psi(G)) = (X^*(\tau(T)), \tau(\Delta), X_*(\tau(T)), \tau(\check{\Delta})).$$

The root datum dual to $\psi(G)$ is given by $\psi(G)^\vee = (X_*(T), \check{\Delta}, X^*(T), \Delta)$.

To define the L-group we first define its connected component. Let ${}^LG^0$ be the complex group with canonical root datum $\psi(G)^\vee$. Then we let ${}^LG = {}^LG^0 \rtimes W_K$, where W_K acts on ${}^LG^0$ through the map $\varphi\colon W_K \to \Gamma_K$, and Γ_K acts on ${}^LG^0$ by the action on root data.

We return now to the setting $\mathbf{H} = \mathrm{U}(2)$, $\tilde{\mathbf{H}} = \mathrm{Res}_{E/F}(\mathbf{H})$. Since $\mathbf{H}(E) = \mathrm{GL}(2, E)$, ${}^L\mathbf{H}^0 = \mathrm{GL}(2, \mathbb{C})$. Since $\tilde{\mathbf{H}}(E) \simeq \mathbf{H}(E) \times \mathbf{H}(E)$

$$ {}^L\tilde{H}^0 \simeq {}^LH^0 \times {}^LH^0 = \mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C}). \tag{1.7}$$

The Weil group actions are determined by the action of $\sigma \in \mathrm{Gal}(E/F)$. If $x \in {}^LH^0$, then $\sigma(x) = \Phi_2 {}^tx^{-1}\Phi_2$, where $\Phi_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. If $(x, y) \in {}^L\tilde{H}^0$, then $\sigma((x, y)) = (\sigma(y), \sigma(x))$ [11, pg. 47].

There are two base change maps from H to $\tilde{H}$. These maps are parameterized by two maps, ψ_H, $\psi'_H\colon {}^LH \to {}^L\tilde{H}$. These L-homomorphisms are described by Rogawski [11, pg. 50]. The maps ψ_H and ψ'_H are related by a cocycle $\alpha\colon W_F \to {}^L\tilde{H}$. In the Langlands correspondence this cocycle defines the character $\chi_\mu = \mu \circ \det$ over F. In order to describe the images of ψ_H and ψ'_H we need some more vocabulary.

DEFINITION 1.12. A distribution on H is called *stable* if it vanishes on every f such that $\Phi^{\mathrm{st}}(\gamma, f) = 0$ for every regular semisimple $\gamma \in H$.

Let $\mathscr{E}(H)$ be the set of equivalence classes of admissible irreducible representations of H, and $\mathscr{E}_2(H) \subset \mathscr{E}(H)$ is the collection of square integrable equivalence classes. Let ${}^\circ\mathscr{E}(H)$ be the supercuspidal equivalence classes. We make no notational distinction between a class $[\pi]$ and its representative π. For any $\pi \in \mathscr{E}(H)$ with central character ω, we define the distribution character of π on $C(H, \omega)$ by $\chi_\pi(f) = \mathrm{Tr}(\pi(f))$, where

$$\pi(f) = \int_{Z(H)\backslash H} f(g)\pi(g)\,\mathrm{d}g.$$

For $H = U(2)$ an L-packet is defined to be a $\mathrm{PGL}(2, F)$ orbit in $\mathscr{E}(H)$ [11, pg. 161]. We denote the collection of tempered L-packets of H by $\bar{\mathscr{E}}(H)$. We let $\bar{\mathscr{E}}_2(H)$ be the subset of discrete series L-packets. If $\Pi \in \bar{\mathscr{E}}(H)$, then we can choose non zero integers $m(\rho)$ for each $\rho \in \Pi$ such that

$$\chi_\Pi = \sum_{\rho \in \Pi} m(\rho)\chi_\rho$$

is stable. The distribution χ_Π is called the stable tempered character attached to Π.

For each $\Pi \in \bar{\mathscr{E}}(H)$ there should be two base change lifts, $\psi_H(\Pi), \psi'_H(\Pi) \in \mathscr{E}(\tilde{H})$, coming from the two maps of L-groups described above. The existence of the base change maps is an example of the Langlands correspondence [5]. That is if $\Psi: W_F \to {}^L H$ is an admissible homomorphism defining Π then $\psi_H \circ \Psi: W_F \to {}^L \tilde{H}$ should define $\psi_H(\Pi)$. Since the two L-maps differ by α, the two lifts are related by

$$\psi_H(\Pi) = \psi'_H(\Pi) \otimes \chi_\mu \qquad [11, \text{pg. } 50]. \tag{1.8}$$

We need the following observation [11, pp. 161–162]. Let $H_0 = U(1) \times U(1)$. Let $\theta = \theta_1 \otimes \theta_2$ be a character of H_0. Let $\theta' = \theta_2 \otimes \theta_1$. Then there is an L-packet $\rho(\theta)$ of $U(2)$ corresponding to the set $\{\theta, \theta'\}$, and each $\rho(\theta)$ has two elements. Some of the properties of these L-packets are given below.

THEOREM 1.13 (Rogawski) [11, Proposition 11.1.1].

(1) *An L-packet of H has more than one element if and only if it is of the form $\rho(\theta)$ for some θ.*
(2) *If $\rho(\theta) = \{\pi_1, \pi_2\}$ then $\chi_{\pi_1} + \chi_{\pi_2}$ is stable, i.e. $\chi_{\rho(\theta)} = \chi_{\pi_1} + \chi_{\pi_2}$.*
(3) *$\theta \neq \theta'$ if and only if $\rho(\theta)$ is supercuspidal, i.e. both π_1 and π_2 are supercuspidal.*
(4) *If $\theta = \theta'$ and $\rho(\theta)$ is unitary, then $\pi_1 \oplus \pi_2 = \mathrm{Ind}_B^H(\tilde{\theta}\mu^{-1})$, where $\tilde{\theta}(z) = \theta(z/\bar{z})$.* □

Let (π, V) be an irreducible admissible representation of $\tilde{H}$. Suppose that $\pi^\varepsilon \simeq \pi$ where $\pi^\varepsilon(g) = \pi(\varepsilon(g))$. We let $\pi(\varepsilon)$ be an equivalence;

$$\pi(\varepsilon)\pi(g)\pi(\varepsilon)^{-1} = \pi(\varepsilon(g)) \, \forall g \in \tilde{H}.$$

We define a distribution $\chi_{\pi\varepsilon}$ on $C(\tilde{H}, \tilde{\omega})$ by $\chi_{\pi\varepsilon}(\varphi) = \mathrm{Tr}(\pi(\varphi)\pi(\varepsilon))$. Let $\mathscr{E}'(\tilde{H})$ be the set of ε-invariant irreducible admissible representations of $\tilde{H}$ with trivial central character on F^*. Then the images of ψ_H and ψ'_H lie in $\mathscr{E}'(\tilde{H})$ [11].

Suppose Π is a tempered L-packet in $\bar{\mathscr{E}}(H)$, with central character ω. Then

$$\pi = \psi_H(\Pi) \text{ if and only if } \chi_{\pi\varepsilon}(\varphi) = \chi_\Pi(f_1) \tag{1.9}$$

for every $\varphi \in C(\tilde{H}, \tilde{\omega})$, where χ_Π is the stable tempered character of Π and $f_1 \in C(H, \omega)$ is given by (1.3).

$$\pi = \psi'_H(\Pi) \text{ if and only if } \chi_{\pi\varepsilon}(\varphi) = \chi_\Pi(f_2), \tag{1.10}$$

for every $\varphi \in C(\tilde{H}, \tilde{\omega})$, where $f_2 \in C(H, \omega\mu^{-1})$ is given by (1.4) [11, pg. 164].

THEOREM1.14 (Rogawski) [11, Proposition 11.4.1(c)]. *Every square integrable $\pi \in \mathscr{E}'(\tilde{H})$ is a base change lift of the form $\psi_H(\Pi)$ or $\psi'_H(\Pi)$ for a unique square* *integrable L-packet* Π, *which is not of the form* $\rho(\theta)$. *The images* $\psi_H(\bar{\mathscr{E}}_2(H))$ *and* $\psi'_H(\bar{\mathscr{E}}_2(H))$ *are disjoint.* □

We will use these results to describe the pole of the intertwining operator for a particular parabolic of $U(2,2)$.

2. Relation of reducibility in $U(2,2)$ to base change

Let E/F be as in Section 1. Recall that $E = F(\beta)$. Let $G = U(2,2)$ with respect to the form $J = \begin{pmatrix} & \beta I_2 \\ -\beta I_2 & \end{pmatrix}$. Then $G = \{g \in \mathrm{GL}(4,E) | gJ{}^t\bar{g} = J\}$. Let T be the maximal torus of diagonal elements:

$$T = \left\{ \begin{pmatrix} x & & & \\ & y & & \\ & & \bar{x}^{-1} & \\ & & & \bar{y}^{-1} \end{pmatrix} \middle| x, y \in E^* \right\}.$$

Let T_d be the maximal F-split sub-torus of T:

$$T_d = \left\{ \begin{pmatrix} x & & & \\ & y & & \\ & & x^{-1} & \\ & & & y^{-1} \end{pmatrix} \middle| x, y \in F^* \right\}.$$

The restricted root system $\Phi(G, T_d)$ is of type C_2. Let A be the subtorus of T_d given by $\{e_1 - e_2\}$.

$$A = \left\{ \begin{pmatrix} x & & & \\ & x & & \\ & & x^{-1} & \\ & & & x^{-1} \end{pmatrix} \middle| x \in F^* \right\}.$$

Let M be the centralizer of A in G. Then

$$M = \left\{ \begin{pmatrix} g & \\ & {}^t\bar{g}^{-1} \end{pmatrix} \middle| g \in \mathrm{GL}(2,E) \right\} \simeq \mathrm{GL}(2,E) = \tilde{H}.$$

The Weyl group $W(A)$ is of order two, with the non trivial element w represented by $\begin{pmatrix} & -I_2 \\ I_2 & \end{pmatrix}$. Let $P = MN$ with

$$N = \left\{ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \middle| X \in M(2, E), {}^t\bar{X} = X \right\}.$$

Then P is a maximal parabolic of G.

Let $X(M)_F$ denote the F-rational characters of M. Let $\mathfrak{a}$ be the real Lie algebra of A. Then $\mathfrak{a} = \mathrm{Hom}(X(M)_F, \mathbb{R})$ [6]. Let $\mathfrak{a}^* = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R}$ be its dual, and let $\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$. There is a homomorphism [6] $H_P : M \to \mathfrak{a}$ defined by

$$q_F^{\langle \chi, H_P(m) \rangle} = |\chi(m)|_F, \ \forall \chi \in X(M)_F.$$

Let $\mathfrak{z}$ be the center of $\mathfrak{a}$. Let ρ be half the sum of the positive roots in N. Let α be the unique simple root in N. Let $\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \rho$. Here $\langle \gamma_1, \gamma_2 \rangle$ is defined as follows [13]. Let γ_1' and γ_2' be non-restricted roots of T in G, which restrict to γ_1 and γ_2 respectively. Let (,) be the standard Euclidean inner product on $\Phi(G, T)$. Then

$$\langle \gamma_1, \gamma_2 \rangle = \frac{2(\gamma_1', \gamma_2')}{(\gamma_2', \gamma_2')}.$$

A straightforward calculation shows that if $m = \begin{pmatrix} g & \\ & {}^t\bar{g}^{-1} \end{pmatrix}$, then

$$q_F^{\langle \tilde{\alpha}, H_P(m) \rangle} = |\det g|_E^{1/2}.$$

We identify $\mathfrak{a}_{\mathbb{C}}^*/\mathfrak{z}$ with $\mathbb{C}$ via the map $s \mapsto s\tilde{\alpha}$.

Let $(\pi, V) \in {}^\circ\mathscr{E}(M)$. Let $\tilde{\omega}$ be the central character of π. Let $s \in \mathbb{C}$ and let

$$V(s, \pi) = \{ f \in C^\infty(G, V) \mid f(mng) = \pi(m) q_F^{\langle s\tilde{\alpha}, H_P(m) \rangle} \delta_P^{1/2}(m) f(g) \forall g \in G, m \in M, n \in N \}.$$

Then G acts on $V(s, \pi)$ by right translations. We denote this action by

$$I(s, \pi) = \mathrm{Ind}_P^G(\pi \otimes q_F^{\langle s\tilde{\alpha}, H_P(\,) \rangle}) = \mathrm{Ind}_P^G(\pi \otimes |\det(\ )|_E^{s/2}).$$

We write $\mathrm{Ind}_P^G(\pi)$ for $I(0, \pi)$.

Note that by Bruhat theory [6] $\mathrm{Ind}_P^G(\pi)$ is irreducible if $\pi^w \not\simeq \pi$. If $m = \begin{pmatrix} g & \\ & {}^t\bar{g}^{-1} \end{pmatrix}$ with $g \in \mathrm{GL}(2, E)$, then $wmw^{-1} = \begin{pmatrix} {}^t\bar{g}^{-1} & \\ & g \end{pmatrix}$. Therefore, $\pi^w = \pi^\varepsilon$.

We formally define an operator $A(s, \pi)$ on $V(s, \pi)$ by

$$(A(s, \pi)f)(g) = \int_N f(w^{-1}ng)\,\mathrm{dn}$$

for $f \in V(s, \pi)$, $g \in G$. If $A(s, \pi)$ converges, then it defines an intertwining operator between $I(s, \pi)$ and $I(-s, \pi^w)$. It is a theorem of Harish-Chandra [12] that, for π supercuspidal, $A(s, \pi)$ converges for $\mathrm{Re}\, s > 0$. Moreover, $s \mapsto A(s, \pi)$ is meromorphic as an operator valued function, and has a meromorphic continuation to the whole plane. This means that there is some fixed polynomial $P(t)$ so that $s \mapsto P(q_F^{-s}) < A(s, \pi)f(g), \tilde{v}>$ is holomorphic for each $g \in G$, $\tilde{v} \in \tilde{V}$, $f \in V(s, \pi)$.

Harish-Chandra's completeness theorem [16] implies $\mathrm{Ind}_P^G(\pi)$ is reducible if and only if $\pi \simeq \pi^w$ and 0 is not a pole of $s \mapsto A(s, \pi)$.

LEMMA 2.1 (Rallis, Shahidi [15]). *Let*

$$V(s, \pi)_0 = \{f \in V(s, \pi) \mid \mathrm{supp} f \subset P\bar{N} \text{ and is compact } \mathrm{mod}\, P\}.$$

Then every pole of $s \mapsto A(s, \pi)$ *is a pole of* $s \mapsto A(s, \pi)f(e)$ *for some* $f \in V(s, \pi)_0$. □

Thus, we study poles of $s \mapsto A(s, \pi)f(e)$ for $f \in V(s, \pi)_0$ and $\pi \simeq \pi^\varepsilon$. Let $L = M(2, \mathfrak{p}_E^m)$ for some $m \in \mathbb{Z}^+$. Let $L' \subset \bar{N}$ be given by

$$L' = \left\{ \begin{pmatrix} I & 0 \\ x & I \end{pmatrix} \middle| x \in L \right\}.$$

Let $f \in V(s, \pi)_0$. We assume that there is a $v \in V$ so that for $\bar{n} \in \bar{N}$,

$$f(\bar{n}) = \begin{cases} v & \bar{n} \in L' \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.2. *If* $n \in N$, *then* $w^{-1}n \in P\bar{N}$ *if and only if* $n = \begin{pmatrix} I & a \\ 0 & I \end{pmatrix}$ *with* $a \in \mathrm{GL}(2, E)$ *and* $a = {}^t\bar{a}$.

Proof. Suppose $a, b, c \in M(2, E)$, and $g \in \mathrm{GL}(2, E)$. If

$$\begin{pmatrix} & -I \\ I & \end{pmatrix}\begin{pmatrix} I & a \\ & I \end{pmatrix} = \begin{pmatrix} g & b \\ 0 & {}^t\bar{g}^{-1} \end{pmatrix}\begin{pmatrix} I & \\ c & I \end{pmatrix},$$

then

$$\begin{pmatrix} & -I \\ I & a \end{pmatrix} = \begin{pmatrix} g + bc & b \\ {}^t\bar{g}^{-1} & {}^t\bar{g}^{-1} \end{pmatrix}.$$

Therefore, $a = {}^t\bar{g}^{-1}$, which implies $a \in \mathrm{GL}(2, E)$. Moreover, if $n \in N$ then $a = {}^t\bar{a}$. Conversely, if $a \in \mathrm{GL}(2, E)$ with ${}^t\bar{a} = a$, then the above calculation shows

$$\begin{pmatrix} & -I \\ I & a \end{pmatrix} = \begin{pmatrix} a^{-1} & -I \\ & a \end{pmatrix} \begin{pmatrix} I & \\ a^{-1} & I \end{pmatrix}. \qquad \square$$

Let $\tilde{v} \in \tilde{V}$. By Lemma 2.2

$$\begin{aligned}
\langle A(s, \pi) f(e), \tilde{v} \rangle &= \int_N \langle f(w^{-1} n), \tilde{v} \rangle \, \mathrm{d}n \\
&= \int_{\substack{\det a \neq 0 \\ \bar{a} = {}^t a}} \left\langle f\left(\begin{pmatrix} \varepsilon(a) & -I \\ 0 & a \end{pmatrix} \begin{pmatrix} I & 0 \\ a^{-1} & I \end{pmatrix} \right), \tilde{v} \right\rangle \mathrm{d}a \\
&= \int_{\substack{\det a \neq 0 \\ \bar{a} = {}^t a}} \left\langle \pi(\varepsilon(a)) f\left(\begin{pmatrix} I & 0 \\ a^{-1} & I \end{pmatrix} \right), \tilde{v} \right\rangle |\det a|_E^{-s/2 - \langle \rho, \tilde{\alpha} \rangle} \, \mathrm{d}a \\
&= \int_{\substack{\det a \neq 0 \\ \bar{a} = {}^t a \\ a^{-1} \in L}} \langle \pi(\varepsilon(a)) v, \tilde{v} \rangle \, |\det a|_E^{-s/2 - \langle \rho, \tilde{\alpha} \rangle} \, \mathrm{d}a.
\end{aligned}$$

LEMMA 2.3. *If* $d^\times a = |\det a|_E^{-\langle \rho, \tilde{\alpha} \rangle} \mathrm{d}a$ *then* $d^\times a^{-1} = d^\times a$.

Proof. Let M act on N by conjugation. If ${}^t\bar{X} = X \in M(2, E)$ and $g \in \mathrm{GL}(2, E)$ then

$$\begin{pmatrix} g^{-1} & \\ & {}^t\bar{g} \end{pmatrix} \begin{pmatrix} I & X \\ & I \end{pmatrix} \begin{pmatrix} g & \\ & {}^t\bar{g}^{-1} \end{pmatrix} = \begin{pmatrix} I & g^{-1} X {}^t\bar{g}^{-1} \\ & I \end{pmatrix}.$$

Therefore, on N, $d(g^{-1} X {}^t\bar{g}^{-1})/dX = |\det X|_E^{-2\langle \rho, \tilde{\alpha} \rangle}$.

Now suppose $g = X$. Then

$$|\det X^{-1}|_E^{-\langle \rho, \tilde{\alpha} \rangle} \mathrm{d}X^{-1} = |\det X|_E^{-\langle \rho, \tilde{\alpha} \rangle} \mathrm{d}X,$$

and therefore $d^\times a^{-1} = d^\times a$. $\square$

Let $\varphi(g) = \langle \pi(g) v, \tilde{v} \rangle$. For $x \in M(2, E)$ let

$$\xi_L(x) = \begin{cases} 1 & x \in L \\ 0 & x \notin L. \end{cases}$$

Then, using the relation $\bar{a} = {}^t a$ and Lemma 2.3, we rewrite (2.1) as

$$\langle A(s, \pi)f(e), \tilde{v}\rangle = \int_{\substack{\det a \neq 0 \\ \bar{a} = {}^t a}} \varphi(a^{-1}) |\det a|_E^{-s/2} \xi_L(a^{-1})\, \mathrm{d}^\times a$$

$$= \int_{\substack{\det a \neq 0 \\ \bar{a} = {}^t a}} \varphi(a) |\det a|_E^{s/2} \xi_L(a)\, \mathrm{d}^\times a. \tag{2.2}$$

REMARK. Since v and $\tilde{v}$ were arbitrary, any matrix coefficient φ of π can appear in (2.2).

Let δ_2 represent the equivalence class of hermitian forms in $\mathrm{GL}(2, E)$ which are not equivalent to δ_1. By Lemma 1.7 $\det(\delta_2 \delta_1^{-1}) \notin NE^*$. If $a \in \mathrm{GL}(2, E)$ is hermitian then $a = g^{-1}\delta_i \varepsilon(g)$ for some g and a unique $i = 1$ or 2. Thus, using the notation in Section 1, we can rewrite (2.2) as

$$\langle A(s, \pi)f(e), \tilde{v}\rangle =$$
$$\sum_{i=1}^{2} \int_{\tilde{H}_{\delta_i\varepsilon} \backslash \tilde{H}} \varphi(g^{-1}\delta_i \varepsilon(g)) |\det(g^{-1}\delta_i \varepsilon(g))|_E^{s/2}\, \xi_L(g^{-1}\delta_i\varepsilon(g))\, \mathrm{d}^\times g, \tag{2.3}$$

where $\mathrm{d}^\times g$ is the invariant measure on the quotient coming from $\mathrm{d}^\times a$.

Note that the orbital integral in (2.3) is over $\tilde{H}_{\delta_i\varepsilon}\backslash\tilde{H}$, while Rogawski's theorem (Theorem 1.10) gives transfer of orbital integrals over $\tilde{H}'_{\delta_i\varepsilon}\backslash\tilde{H}$. We wish to rewrite (2.3) to include such an integral, allowing us to use the transfer described in Theorem 1.11.

LEMMA 2.4. *For* $i = 1$ *or* 2, $\tilde{H}_{\delta_i\varepsilon}\backslash\tilde{H}'_{\delta_i\varepsilon} \simeq F^*$.

Proof. Note that $\psi: \tilde{H}'_{\delta_i\varepsilon} \to F^*$ given by $\psi(g) = g^{-1}\delta_i\varepsilon(g)\delta_i^{-1}$ is a homomorphism. By its definition $\tilde{H}_{\delta_i\varepsilon}$ is the kernel of ψ. Let $z \in F^*$. Then, by Landherr's theorem [10], $z\delta_i$ and δ_i are equivalent hermitian forms. Since ε-conjugacy is equivalence of hermitian forms, there is some $g \in \tilde{H}$ so that $g^{-1}\delta_i\varepsilon(g) = z\delta_i$. Therefore, g is in $\tilde{H}'_{\delta_i\varepsilon}$. Since $\psi(g) = z$, we see ψ is surjective. □

Note that if $g \in \tilde{H}$ is a representative of a coset of $\tilde{H}'_{\delta_i\varepsilon}\backslash\tilde{H}$, then we have just shown that for any $z_0 \in F^*$ there is some $g_0 \in \tilde{H}'_{\delta_i\varepsilon}$ so that

$$(g_0 g)^{-1}\delta_i\varepsilon(g_0 g) = \begin{pmatrix} z_0 & \\ & z_0 \end{pmatrix} g^{-1}\delta_i\varepsilon(g).$$

Thus, we can choose coset representatives $\{g\}$ for $\tilde{H}'_{\delta_i\varepsilon}\backslash\tilde{H}$ so that the supremum norm $\|g^{-1}\delta_i\varepsilon(g)\|_\infty = q_E^{-m}$. We fix these representatives. Then for all $g \in \tilde{H}'_{\delta_i\varepsilon}\backslash\tilde{H}$ and $z \in F^*$,

$$g^{-1}z\delta_i\varepsilon(g) \in L \text{ if and only if } |z|_E \leqslant 1. \tag{2.4}$$

We now can rewrite (2.3) as

$$\begin{aligned}&\langle A(s,\pi)f(e),\tilde{v}\rangle\\&=\sum_{i=1}^{2}\int_{F^*}\int_{\tilde{H}'_{\delta_i\varepsilon}\backslash\tilde{H}}\varphi(g^{-1}(z\delta_i)\varepsilon(g))|\det(g^{-1}(z\delta_i)\varepsilon(g))|_E^{s/2}\xi_L(g^{-1}(z\delta_i)\varepsilon(g))d^\times g\,\mathrm{d}^\times z\\&=\int_{\mathscr{R}_F}\tilde{\omega}(z)|z|_E^s\,\mathrm{d}^\times z\sum_{i=1}^{2}\int_{\tilde{H}'_{\delta_i\varepsilon}\backslash\tilde{H}}\varphi(g^{-1}\delta_i\varepsilon(g))|\det(g^{-1}\delta_i\varepsilon(g))|_E^{s/2}\,\mathrm{d}^\times g.\qquad(2.5)\end{aligned}$$

LEMMA 2.5. *$\{g\in\tilde{H}'_{\delta_i\varepsilon}\backslash\tilde{H}\mid g^{-1}\delta_i\varepsilon(g)\in\operatorname{supp}\varphi\}$ is compact.*

Proof. Let $L_0=\{x\mid\|x\|_\infty=q_E^{-m}\}\subset M(2,E)$. We have already chosen representatives g with $g^{-1}\delta_i\varepsilon(g)\in L_0$. Let $S\subset M(2,E)$ be the set of hermitian matrices. Since transposition and Galois conjugation are continuous S is closed in $M(2,E)$. Define $\psi:\tilde{H}\to S$ by $\psi(g)=g^{-1}\delta_i\varepsilon(g)$. If $g\in\tilde{H}$ then $\det(\psi(g))\equiv\det\delta_i \bmod NE^*$. Since both NE^* and its complement are closed in F^*, and the determinant is continuous, $\operatorname{Im}(\psi)$ is closed in S, and hence is closed in $\tilde{H}$.

If $s\in S$ and $\psi(g)=s$, then $\psi^{-1}(\{s\})=\tilde{H}_{\delta_i\varepsilon}g$. Note that $\psi^{-1}(Z_Fs)=\tilde{H}'_{\delta_i\varepsilon}g$, where $Z_F\subset\tilde{Z}$ is the set of F-scalars. Let C be a compact subset of $\tilde{H}$ such that $\operatorname{supp}\varphi\subset C\tilde{Z}$. Since $x\mapsto\|x\|_\infty$ and $x\mapsto|\det x|_E$ are continuous we can choose integers j, k, l, n so that $q_E^k\leqslant\|c\|_\infty\leqslant q_E^j$ and $q_E^l\leqslant|\det c|_E\leqslant q_E^n$ for all $c\in C$. Therefore, if $c\in C$ and $cz\in C\tilde{Z}\cap L_0$ then $q_E^{-m-j}\leqslant|z|_E\leqslant q_E^{-m-k}$. We let

$$\Omega=\{z\mid q_E^{-m-j}\leqslant|z|_E\leqslant q_E^{-m-k}\}.$$

Then $C\tilde{Z}\cap L_0\subset C\Omega$, which is compact. Therefore, $\operatorname{Im}\psi\cap\operatorname{supp}\varphi\cap L_0$ is compact in $\tilde{H}$. Hence the lemma holds. □

By Lemma 2.5

$$\begin{aligned}&\lim_{s\to0}\sum_{i=1}^{2}\int_{\tilde{H}'_{\delta_i\varepsilon}\backslash\tilde{H}}\varphi(g^{-1}\delta_i\varepsilon(g))\,|\det(g^{-1}\delta_i\varepsilon(g))|_E^{s/2}\,\mathrm{d}g^\times\\&=\sum_{i=1}^{2}\int_{\tilde{H}'_{\delta_i\varepsilon}\backslash\tilde{H}}\varphi(g^{-1}\delta_i\varepsilon(g))\,\mathrm{d}^\times g=\pm\Phi_\varepsilon^\kappa(\delta_1,\varphi).\qquad(2.6)\end{aligned}$$

Consequently, taking the residue at 0 of $A(s,\pi)f(e)$ we get

$$\lim_{s\to0}s\langle A(s,\pi)f(e),\tilde{v}\rangle=\pm\left(\lim_{s\to0}s\int_{\mathscr{R}_F}\tilde{\omega}(z)|z|_E^s\,\mathrm{d}^\times z\right)\Phi_\varepsilon^\kappa(\delta_1,\varphi).\qquad(2.7)$$

Thus, if $A(s, \pi)f(e)$ has a pole at $s = 0$, then

$$s \mapsto \int_{\mathcal{R}_F} \tilde{\omega}(z)|z|_E^s \, d^\times z$$

has a pole at $s = 0$.

$$\int_{\mathcal{R}_F} \tilde{\omega}(z)|z|_E^s \, d^\times z = \int_{\mathcal{R}_F} \tilde{\omega}(z)|z|_F^{2s} \, d^\times z$$
$$= \left(1 - \frac{1}{q_F}\right) \sum_{n=0}^{\infty} \tilde{\omega}(\varpi_F)^n q_F^{-2sn} \int_{\mathcal{R}_F^*} \tilde{\omega}(x) \, d^\times x. \tag{2.8}$$

Therefore, if $\tilde{\omega}|_{F^*}$ is ramified, then the sum (2.8) is 0. Consequently, if $\tilde{\omega}$ is ramified, $A(s, \pi)$ cannot have a pole. If $\tilde{\omega}|_{F^*}$ is unramified then (2.7) is proportional to

$$\left(\sum_{n=0}^{\infty} (\tilde{\omega}(\varpi_F)q_F^{-2s})^n\right) \Phi_\varepsilon^\kappa(\delta_1, \varphi) = \left(\frac{1}{1 - \tilde{\omega}(\varpi_F)q_F^{-2s}}\right) \Phi_\varepsilon^\kappa(\delta_1, \varphi). \tag{2.9}$$

Therefore, there is no pole at $s = 0$ unless $\tilde{\omega}(\varpi_F) = 1$. Since $\tilde{\omega}|_{F^*}$ is unramified and $\omega(\varpi_F) = 1$, $\tilde{\omega}|_{F^*} \equiv 1$.

REMARK. With our normalization of $I(s, \pi)$, the normalizing factor $(1 - \tilde{\omega}(\varpi_F)q_F^{-2s})$ is the same one determined by Tamir [18].

PROPOSITION 2.6. *If* $\pi \in {}^\circ\mathcal{E}(\mathrm{GL}(2, E))$ *and* $A(s, \pi)$ *has a pole at* $s = 0$ *then* π *is a base change lift from* $U(2)$, *i.e.* $\pi = \psi_H(\Pi)$ *or* $\psi'_H(\Pi)$ *for some discrete series L-packet* Π *of* $U(2)$.

Proof. Suppose $A(s, \pi)$ has a pole. Then π is ramified in G and therefore $\pi \simeq \pi^\varepsilon$. We have just shown that the central character $\tilde{\omega}$ of π is trivial on F^*. Since π is supercuspidal π is a base change lift from $U(2)$ by Theorem 1.14. □

REMARK. One can show that every ε-invariant supercuspidal representation of $\mathrm{GL}(2, E)$ is a base change lift from $U(2)$. That is, every such π has central character whose restriction to F^* is trivial. This follows from Theorem 11.5.2 of [11] and Proposition 5.1 of [14].

Fix a character ω of E^1, and let $\tilde{\omega}$ be the character of E^* given by $\tilde{\omega}(z) = \omega(z/\bar{z})$. Let π be an irreducible unitary supercuspidal representation of $\mathrm{GL}(2, E)$ with central character $\tilde{\omega}$. From (2.9) we see that if π is a base change lift from $U(2)$ to $\mathrm{GL}(2, E)$, then $A(s, \pi)$ has a pole at $s = 0$ if and only if $\Phi_\varepsilon^\kappa(\delta_1, \varphi) \neq 0$ for some matrix coefficient φ of π.

Since $\varphi \in C(\tilde{H}, \tilde{\omega}^{-1})$, Theorem 1.11 implies $\Phi_\varepsilon^\kappa(\delta_1, \varphi) = \mu(\det \delta_1) f_2(e)$ where $f_2 \in C(H, \omega^{-1}\mu^{-1})$ is given by Theorem 1.10. We intend to show, by use of the Plancherel formula, that if π is in the image of ψ_H then, for each matrix coefficient φ of π, $f_2(e) = 0$. On the other hand we will show that if π is in the image of ψ'_H, then in fact there is a pole of $A(s, \pi)$.

LEMMA 2.7. *Let (π_1, V_1), (π_2, V_2) be two inequivalent irreducible admissible representations with central character $\tilde{\omega}$. Further suppose that π_2 is supercuspidal. Then for any matrix coefficient φ of π_2, $\tilde{\pi}_1(\varphi) = 0$. Therefore, if π_1 and π_2 are ε-invariant, $\chi_{\tilde{\pi}_1 \varepsilon}(\varphi) = 0$.*

Proof. This follows from Theorem 2.42 of [2], and the fact that any matrix coefficient of π_2 is compactly supported modulo $\tilde{Z}$. □

Let $\mathscr{M}$ be the collection of standard Levi components of H. Let $\mathscr{E}_2(M)_{\omega^{-1}}$ be the collection of $\rho \in \mathscr{E}_2(M)$ with central character ω^{-1}. Let $\hat{\rho} = \mathrm{Ind}_P^H(\rho)$. By the Plancherel formula [7],

$$f_2(e) = \sum_{M \in \mathscr{M}} C(M) \int_{\rho \in \mathscr{E}_2(M)_{\omega^{-1}}} \mu(\rho) d(\rho) \chi_{\hat{\rho}}(f_2)\, \mathrm{d}\rho, \tag{2.10}$$

where $C(M) > 0$, $\mu(\rho)$ is the Plancherel measure [14], $\mathrm{d}(\rho)$ is the formal degree [4], and $d\rho$ is the measure described in [7].

We intend to collect terms corresponding to L-packets as follows. By Theorem 1.13, an L-packet $\hat{\Pi}$ of $U(2)$ is either a singleton, or of the form $\rho(\theta) = \{\hat{\rho}_1, \hat{\rho}_2\}$. If $\hat{\Pi} = \rho(\theta)$ then $\chi_{\rho(\theta)} = \chi_{\hat{\rho}_1} + \chi_{\hat{\rho}_2}$. If $\hat{\rho}_1$ and $\hat{\rho}_2$ are supercuspidal, then they have the same formal degree since they are PGL(2) conjugate. Since $\hat{\rho}_i$ are not induced, $\mu(\rho_i) = 1$. We let $\lambda_H(\Pi) = \mathrm{d}(\rho_i)$, for $i = 1, 2$. Then, combining the terms in (2.10) for $\hat{\rho}_1$ and $\hat{\rho}_2$, we obtain $\lambda_H(\Pi)\chi_{\hat{\Pi}}$. If $\rho(\theta)$ is not supercuspidal, then Theorem 2.13 (4) implies $\hat{\rho}_1 \oplus \hat{\rho}_2 = \widehat{\tilde{\theta}\mu^{-1}}$. The formal degree of the inducing representation $\tilde{\theta}\mu^{-1}$ is 1. Therefore, the stable character $\chi_{\hat{\Pi}}$ appears with coefficient $\lambda_H(\Pi) = \mu(\tilde{\theta}\mu^{-1})$. If $\hat{\Pi} = \{\hat{\rho}\}$ is a singleton L-packet then $\chi_{\hat{\rho}}$ must be stable. Therefore, $\chi_{\hat{\Pi}}$ appears with coefficient $\lambda_H(\Pi) = \mu(\rho)\mathrm{d}(\rho)$. Consequently, we rewrite (2.10) as

$$f_2(e) = \sum_{M \in \mathscr{M}} C(M) \int_{\Pi \in \bar{\mathscr{E}}_2(M)_{\omega^{-1}}} \lambda_H(\Pi) \chi_{\hat{\Pi}}(f_2)\, \mathrm{d}\Pi. \tag{2.11}$$

By (1.10) $\chi_{\hat{\Pi}}(f_2) = \chi_{\pi'\varepsilon}(\varphi)$ where $\pi' = \psi'_H(\hat{\Pi})$. By Lemma 2.7

$$\chi_{\pi'\varepsilon}(\varphi) = 0 \text{ unless } \pi' \simeq \tilde{\pi}. \tag{2.12}$$

DEFINITION 2.8. Let π be an irreducible admissible ε-invariant discrete series representation of $\tilde{H}$. A function $\varphi_\pi \in C(\tilde{H}, \tilde{\omega}^{-1})$ is called an *ε-pseudo-coefficient* of

π if: (i) for every irreducible tempered ε-invariant representation $\tau \not\simeq \pi$ with central character $\tilde{\omega}$, $\chi_{\tilde{\tau}\varepsilon}(\varphi_\pi) = 0$, and (ii) $\chi_{\tilde{\pi}\varepsilon}(\varphi_\pi) \neq 0$.

Kazhdan [9] proved the existence of pseudo-coefficients for discrete series representations, and non twisted characters. That is for every discrete series representation π there is a function f_π so that $\chi_{\tilde{\tau}}(f_\pi) = \delta_{\tau\pi}$. In the twisted case for unitary groups, the existence of ε-psuedo-coefficients for the discrete series is discussed in [11]. If π is ε-invariant, and supercuspidal, then Lemma 5 of [7] and Lemma 2.7 show that there must be matrix coefficients of π which are ε-pseudo coefficients.

THEOREM 2.9. *Let* $\pi \in {}^\circ\mathscr{E}(\tilde{H})$. *Then* $A(s,\pi)$ *has a pole at* $s = 0$ *if and only if* $\tilde{\pi} = \psi'_H(\Pi)$ *for some L-packet* Π *of* H.

Proof. Proposition 2.6 implies that if $A(s,\pi)$ has a pole at $s = 0$, then π is a base change lift. By Theorem 1.14 $\tilde{\pi}$ must also be a base change lift. Let φ be any matrix coefficient of π. If $\tilde{\pi} = \psi_H(\hat{\Pi})$ then, by Theorem 1.14, $\tilde{\pi} \not\simeq \psi'_H(\hat{\Lambda})$ for all L-packets $\hat{\Lambda}$. Therefore, by (2.12), $\chi_{\pi'\varepsilon}(\varphi) = 0$ for any $\pi' = \psi'_H(\hat{\Lambda})$. Consequently, each term in (2.11) vanishes, and thus $f_2(e) = 0$. Hence there can be no pole of $A(s,\pi)$ at $s = 0$.

On the other hand, suppose $\tilde{\pi} = \psi'_H(\hat{\Pi})$. Suppose that φ is a matrix coefficient for π which is also an ε-pseudo coefficient. By Theorems 1.3 and 1.14 $\hat{\Pi} = \{\hat{\rho}\}$ is a singleton. Then, by (2.12), there is some $c \neq 0$ so that $f_2(e) = c\chi_{\tilde{\pi}\varepsilon}(\varphi)$. Thus, for such a φ, we have $f_2(e) \neq 0$. Hence, by (2.9), $A(s,\pi)$ has a pole at $s = 0$. □

COROLLARY 2.10. *Let* $\pi \in {}^\circ\mathscr{E}(\tilde{H})$. *Then* $\pi = \psi'_H(\Lambda)$ *for some L-packet* Λ *of* $U(2)$ *if and only if* $\tilde{\pi} = \psi'_H(\Pi)$ *for some L-packet* Π *of* $U(2)$.

Proof. By the remark following Proposition 2.6, and Theorem 2.9, $\tilde{\pi} = \psi'_H(\Pi)$ if and only if $\Phi^\kappa_\varepsilon(\delta_1, \varphi) \neq 0$ for some matrix coefficient of π. Let $\bar{\varphi}(g) = \varphi(g^{-1})$. Then $\bar{\varphi}$ is a matrix coefficient of $\tilde{\pi}$. Let $\delta \in \{\delta_1, \delta_2\}$. Then

$$\Phi_\varepsilon(\delta, \bar{\varphi}) = \int_{\tilde{H}'_{\delta\varepsilon}\backslash H} \bar{\varphi}(g^{-1}\delta\varepsilon(g))\, d^\times g = \int_{\tilde{H}'_{\delta\varepsilon}\backslash \tilde{H}} \varphi(\varepsilon(g)^{-1}\delta^{-1}g)\, d^\times g.$$

Since $\det\delta \equiv \det\delta^{-1} (\operatorname{mod} NE^*)$, the hermitian forms δ and δ^{-1} are equivalent. Thus,

$$\Phi_\varepsilon(\delta, \bar{\varphi}) = \int_{\tilde{H}'_{\delta\varepsilon}\backslash \tilde{H}} \varphi(\varepsilon(g)^{-1}\delta g).$$

Since ε is measure preserving, $\Phi_\varepsilon(\delta, \bar{\varphi}) = \Phi_\varepsilon(\delta, \varphi)$. Thus, $\Phi^\kappa_e(\delta_1, \bar{\varphi}) = \Phi^\kappa_\varepsilon(\delta_1, \varphi)$. Therefore, $\tilde{\pi} = \psi'_H(\Pi)$ if and only if $\Phi^\kappa_\varepsilon(\delta_1, \bar{\varphi}) \neq 0$, for some matrix coefficient $\bar{\varphi}$ of $\tilde{\pi}$. By Theorem 2.9, this is equivalent to $\pi = \psi'_H(\Lambda)$ for some L-packet Λ of $U(2)$. □

THEOREM 2.11. *Let $G = U(2, 2)$, $M \simeq \mathrm{GL}(2, E)$, and $\pi \in {}^\circ\mathscr{E}(M)$. Then $\mathrm{Ind}_P^G(\pi)$ is reducible if and only if $\pi = \psi_H(\Pi)$, for some L-packet Π of $U(2)$.*

Proof. Since P is a rank 1 parabolic subgroup, $\mathrm{Ind}_P^G(\pi)$ is reducible if and only if π is ramified in G and $A(s, \pi)$ does not have a pole at $s = 0$. By the remark following Proposition 2.6, π is ramified if and only if π is in the image of ψ_H or ψ'_H. By Theorem 2.9 and Corollary 2.10, $A(s, \pi)$ has a pole at $s = 0$ if and only if $\pi = \psi'_H(\Pi)$ for some L-packet Π of $U(2)$. Thus, π is ramified and $A(s, \pi)$ is holomorphic at $s = 0$ if and only if $\pi = \psi_H(\Pi)$ for some L-packet Π of $U(2)$. □

Based on the work of Shahidi [14], we can determine when the representation $I(s, \pi)$ is reducible, for $s \notin i\mathbb{R}$. We compute the constituents of the adjoint representation of ${}^LM$ on ${}^L\mathfrak{n}$, where ${}^LM$ is the L-group of M, ${}^L\mathfrak{n}$, is the Lie algebra of ${}^LN$, and ${}^LP = {}^LM{}^LN$ [3].

Since $G(\bar{F}) = \mathrm{GL}(4, \bar{F})$, we have ${}^LG^0 = \mathrm{GL}(4, \mathbb{C})$. If $g \in {}^LG^0$, then $\sigma(g) = \Phi_4{}^tg^{-1}\Phi_4$, where $\Phi_4 = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$ [11, pg. 47]. G is a group of type A_3, while the restricted roots $\Phi(G, A_0)$ are of type C_2. Recall that $P = MN$ is generated by the short root $e_1 - e_2$ in $\Phi(G, A_0)$. Note that $e_1 - e_2$ is the restriction of two roots of A_3. Namely $e_1 - e_2$ and $e_3 - e_4$. (This clear from the automorphism of A_3 which gives rise to G.) Thus, ${}^LP$ is the parabolic subgroup of ${}^LG$ corresponding to $\check{\theta} = \{e_1 - e_2, e_3 - e_4\}^\vee$. Therefore,

$$ {}^LM^0 = \left\{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \middle| g, h \in \mathrm{GL}(2, \mathbb{C}) \right\} \simeq \mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C}). $$

Note that the action of $\mathrm{Gal}(E/F)$ on ${}^LM^0$ is given by

$$ \begin{aligned} \sigma(g, h) &= \Phi_4 \begin{pmatrix} {}^tg & 0 \\ 0 & {}^th \end{pmatrix}^{-1} \Phi_4^{-1} \\ &= \begin{pmatrix} \Phi_2{}^th^{-1}\Phi_2^{-1} & 0 \\ 0 & \Phi_2{}^tg^{-1}\Phi_2^{-1} \end{pmatrix} = (\sigma(h), \sigma(g)). \end{aligned} $$

Note that this is consistent with the description given in the discussion prior to Definition 1.12.

Now ${}^LM = {}^LM^0 \rtimes W_F$ with this action. The unipotent radical ${}^LN = {}^LN^0$ is given by ${}^LN = \left\{ \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} \middle| X \in M(2, \mathbb{C}) \right\}$. Thus, ${}^L\mathfrak{n} = \left\{ \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \middle| X \in M(2, \mathbb{C}) \right\}$. Let ${}^LM$ act on ${}^L\mathfrak{n}$ by the adjoint representation. We denote this representation by r.

Then $r(m)Y = mYm^{-1}$. Let $(g,h) \in {}^LM^0$. Then

$$r((g,h))\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & gXh^{-1} \\ 0 & 0 \end{pmatrix}.$$

Therefore, $r|_{L_{M^0}} \simeq \rho_2 \otimes \tilde{\rho}_2$, where ρ_2 is the standard representation of $\mathrm{GL}(2,\mathbb{C})$. Since $r|_{L_{M^0}}$ is irreducible, r must also be irreducible.

THEOREM 2.12. *Let* $G = U(2,2)$ *and suppose* $P = MN$, *with* $M \simeq \mathrm{GL}(2,E)$. *Let* $\pi \in {}^\circ\mathscr{E}(M)$. *Suppose that* $\pi = \psi'_H(\Pi)$ *for some discrete series L-packet of* $H = U(2)$.

(a) *For* $0 < s < 1$ *the representation* $I(s,\pi) = \mathrm{Ind}_P^G(\pi \otimes |\det(\ )|_E^{s/2})$ *is irreducible and unitarizable (i.e. in the complementary series).*
(b) $I(1,\pi)$ *is reducible. It has a unique generic non-supercuspidal discrete series subrepresentation. Its Langlands quotient is degenerate (non-generic) pre-unitary, and nontempered.*
(c) *If* $s > 1$ *then* $I(s,\pi)$ *is irreducible and never unitarizable.*

Proof. By Theorem 2.11 $\pi \simeq \pi^w$ and $\mathrm{Ind}_P^G(\pi)$ is irreducible. Since r is irreducible, Corollary 7.6 of [14] implies that the polynomial $P_{\pi,1}(t)$ has a zero at $t = 1$. (See [14] for the definition of this polynomial.) Therefore, (a), (b) and (c) follow immediately from Theorem 8.1 of [14]. □

THEOREM 2.13. *Suppose* $\pi \in \psi_H(\bar{\mathscr{E}}_2(H)) \cap {}^\circ\mathscr{E}(M)$. *Then for all* $s > 0$, *the representation* $I(s,\pi)$ *is irreducible and not unitarizable.*

Proof. This follows from Theorem 2.11 and Theorem 8.1 (d) of [14]. □

Thus, we have completely described the complementary series coming from this parabolic subgroup of G. Notice that the technique of rewriting the intertwining operator as a sum of twisted orbital integrals is valid for the group $G = U(n,n)$ and the parabolic subgroup $P = MN$, with $M \simeq \mathrm{GL}(n,E)$. Therefore, one hopes to interpret the poles of these intertwining operators in terms of the theory of transfer of such integrals, once this theory is understood.

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