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# The Noether-Lefschetz theorem and sums of 4 squares in the rational function field R($x, y$)

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**Résumé.** Il y a vingt ans, Cassels, Ellison et Pfister montrèrent que le polynôme de Motzkin, un polynôme de degré 6 de $\mathbf{Q}[x, y]$, bien que positif sur $\mathbf{R}^2$ et donc somme de 4 carrés dans le corps des fonctions $\mathbf{R}(x, y)$ (Hilbert), n'est pas somme de 3 carrés dans $\mathbf{R}(x, y)$. Dans cette note, je remarque qu'une extension d'un théorème classique de Max Noether, extension remontant à Lefschetz, permet de montrer l'existence en tout degré pair $d \geqslant 6$ de beaucoup de polynômes de $\mathbf{R}[x, y]$ positifs sur $\mathbf{R}^2$ mais non sommes de 3 carrés dans $\mathbf{R}(x, y)$.

## Introduction

Let us first briefly recall the relevant background for sums of squares in the polynomial ring $\mathbf{R}[x, y]$ and in the field of rational functions $\mathbf{R}(x, y)$ over the real field $\mathbf{R}$.

In 1888, Hilbert [12] proved that any nonnegative polynomial $P$ in $\mathbf{R}[x, y]$ of total degree at most 4 is a sum of 3 squares of polynomials (a modern proof is not available; in [5] I give a geometric proof for the weaker fact that $P$ is a sum of 3 squares in the function field $\mathbf{R}(x, y)$.) In the same article, Hilbert also showed that a nonnegative polynomial in $\mathbf{R}[x, y]$ of (even) degree $\geqslant 6$ need not be a sum of (any number) of squares of polynomials.

In 1893, Hilbert [13] showed that any nonnegative polynomial $P$ in $\mathbf{R}[x, y]$ nevertheless is a sum of squares in the *function field* $\mathbf{R}(x, y)$ and a glance at Hilbert's precise result together with an application of Euler's multiplication formula for sums of 4 squares reveals, as Landau (1906) first noticed ([15], p. 282), that $P$ actually is a sum of 4 squares in $\mathbf{R}(x, y)$.

In 1967, Motzkin produced an explicit nonnegative polynomial $P \in \mathbf{Q}[x, y]$ (of total degree 6) which is not a sum of any number of squares in $\mathbf{R}[x, y]$. More explicit examples were later given by R. M. Robinson. Choi, Lam and Reznick have since developed a systematic theory for comparing semidefinite homogeneous forms with sums of squares of forms, and they have produced examples of higher degree positive polynomials which are not sums of squares of polynomials.

In 1971, by techniques pertaining to elliptic curves, Cassels, Ellison and

Pfister [3] were able to show that the Motzkin polynomial is not a sum of 3 squares in the function field $\mathbf{R}(x, y)$. This proved that the so-called Pythagoras number of $\mathbf{R}(x, y)$ (the minimum number of squares required to express an arbitrary sum of squares), which was known to be at most 4 by Landau's remark to Hilbert's 1893 paper, and to be at least equal to 3 (since $1 + x^2 + y^2$ is not a sum of 2 squares in $\mathbf{R}(x, y)$, Cassels [2]), is actually equal to 4. More examples of the same kind were later given by Christie [4].

Let us now turn to complex algebraic geometry. A classical theorem of Max Noether says that on a "general" surface of degree $d \geqslant 4$ in 3-dimensional projective space, all curves are cut out by another surface (this fails for $d = 2, 3$). An equivalent statement is that the Picard group of $X$ is spanned by the class of a plane section.

Lefschetz, who gave new proofs of Noether's theorem, also sketched a proof ([16], p. 359) that for the double cover $X$ of projective plane $\mathbb{P}^2$, ramified along a sufficiently "general" curve of even degree $d \geqslant 6$, the Picard group $\mathrm{Pic}(X)$ is free of rank one, spanned by the inverse image of a line in the plane (this fails for $d = 4$). Various proofs and extensions of these theorems have been obtained in more recent years (Deligne [6], Griffiths-Harris [11] for Noether's case; Steenbrink [18], Buium [1], Ein [9] for more general situations including double covers). The theorem has recently been used by T. Ford [10] in the study of the Brauer group $\mathrm{Br}(X)$ of such a double cover $X$.

In this note, I remark that the precise version of the Noether-Lefschetz theorem enables one to show:

THEOREM. *For any even degree $d \geqslant 6$, there exist (many) positive polynomials $P(x, y) \in \mathbf{R}[x, y]$ of total degree $d$ such that $P$ is not a sum of 3 squares in $\mathbf{R}(x, y)$.*

(The restriction to $d \geqslant 6$, necessary in view of Hilbert's result for $d = 4$, mirrors the (necessary) geometric restriction in Lefschetz's theorem.)

We thus get a radically new proof that the so-called Pythagoras number of $\mathbf{R}(x, y)$ is 4. The proof is constructive, but produces polynomials whose coefficients are transcendental over $\mathbf{Q}$, in contrast with the Motzkin polynomial, or the polynomials in [4], whose coefficients lie in $\mathbf{Q}$.

One may wonder whether a similar approach might lead to some information on the well-known open problem: What is the exact Pythagoras number $p(\mathbf{R}(x, y, z))$ of $\mathbf{R}(x, y, z)$? It is known to be at least 5 (this follows from direct combination of a result of Cassels ([2]; [14], p. 260) and of the main result of Cassels/Ellison/Pfister [3]). That $p(\mathbf{R}(x, y, z))$ is at most 8 was shown by Ax and Pfister. Is $p(\mathbf{R}(x, y, z))$ equal to 8? Lemma 1.2 below has an analogue in terms of the kernel of the map of Galois cohomology $H^3(F, \mathbf{Z}/2) \to H^3(F(\sqrt{-f}), \mathbf{Z}/2)$, but there is no simple analogue of Lemma 1.1 at the $H^3$ level (as M. Rost has pointed out to me, one may use algebraic $K$-theory to write down some analogous exact sequence, but the $K$-cohomology groups which this sequence involves look rather intractable.)

It was M. Ojanguren who insisted that one should find a new approach to the Cassels/Ellison/Pfister result, via Lemma 1.2 below. The idea of using the Noether-Lefschetz theorem came to me while I was reading a paper of T. Ford [10] on the Brauer group of double covers of the complex projective plane.

I would like to express my hearty thanks to Prof. T.-Y. Lam for organizing the special year on real algebraic geometry and quadratic forms at Berkeley, 1990–1991, and for making our visit possible and so enjoyable.

## 1. Some Galois cohomology.

LEMMA 1.1. *Let $k$ be a field, $X/k$ be a smooth projective geometrically integral variety. Let $k(X)$ be the function field of $X$. Let $\bar{k}$ be a separable closure of $k$,* $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ and $\bar{X} = X \times_k \bar{k}$. *Then there is an exact sequence:*

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\bar{X})^{\mathfrak{g}} \to \mathrm{Br}(k) \to \mathrm{Br}(k(X)).$$

*Proof.* This is well-known, and easy to deduce from the exact sequence of Galois modules

$$1 \to \bar{k}^* \to \bar{k}(X)^* \to \mathrm{Div}(\bar{X}) \to \mathrm{Pic}(\bar{X}) \to 0$$

(the map $\bar{k}(X)^* \to \mathrm{Div}(\bar{X})$ is the divisor map from the multiplicative group of the function field $\bar{k}(X)$ of $\bar{X}$ to the divisor group; that the kernel is $\bar{k}^*$ is a consequence of the hypothesis that $X$ is proper and geometrically integral) together with Hilbert's theorem 90 and the following facts:

$H^1(\mathfrak{g}, \mathrm{Div}(\bar{X})) = 0$ (since $X$ is smooth, $\mathrm{Div}(\bar{X})$ is a sum of permutation modules, and one applies Shapiro's lemma);

$\mathrm{Div}(\bar{X})^{\mathfrak{g}} = \mathrm{Div}(X)$;

$H^2(\mathfrak{g}, \bar{k}(X)^*)$ embeds into the Brauer group $\mathrm{Br}(k(X))$ (restriction-inflation sequence).

(As a matter of fact, if smoothness is not assumed, the Leray spectral sequence for étale cohomology and the projection $X \to \mathrm{Spec}(k)$ yields an exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\bar{X})^{\mathfrak{g}} \to \mathrm{Br}(k) \to \mathrm{Br}(X)$$

where $\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$. Smoothness of $X$ comes in to guarantee (Grothendieck) that the natural map $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$ is an injection.) □

LEMMA 1.2. *Let $F$ be field,* $\mathrm{char}(F) \neq 2$. *Let $f \in F$, $f \neq 0$. The following conditions are equivalent:*

(i) *$f$ is a sum of three squares in $F$;*

(ii) $(-1)$ *is a sum of two squares in the field $F(\sqrt{-f})$;*

(iii) *the quaternion algebra* $(-1, -1)$ *is trivial over $F(\sqrt{-f})$.*

*Proof.* The implication (i) ⇒ (iii), which is the only one used below, is trivial, (ii) ⇔ (iii) is well-known, and (ii) ⇒ (i) is an easy calculation. For a generalization, see Lam [14], XI.2.6, 2.7. □

## 2. The Noether-Lefschetz theorem

I shall use the following "generic" version of the Noether-Lefschetz theorem, which can be read off from Lefschetz ([16] p. 359, a reference I found in Buium's paper), Steenbrink [18], Buium [1], Ein [9]. (References for weighted projective spaces are: Delorme [7], Dolgachev [8], Mori [17].)

THEOREM 2.1. *Let $m \geqslant 3$ be an integer, let $N = (m+2)(m+1)/2$, let $(a_i)_{1 \leqslant i \leqslant N}$ and $(b_{ij})_{1 \leqslant i \leqslant j \leqslant N}$ be independent variables. Let $K = \mathbf{Q}(a_i, b_{ij})$ be the field spanned over $\mathbf{Q}$ by these variables. Introduce the linear form*

$$l(T_1, \ldots, T_N) = \sum_{1 \leqslant i \leqslant N} a_i T_i$$

*and the quadratic form*

$$q(T_1, \ldots, T_N) = \sum_{1 \leqslant i \leqslant j \leqslant N} b_{ij} T_i T_j$$

*both defined over $K$. Let $\pi: X \to \mathbb{P}^2_K$ be the double cover defined in weighted projective space $\mathbb{P}(1, 1, 1, m)$($x, y, t$ of weight 1; $z$ of weight $m$) by the equation*

$$z^2 + z l(\ldots, x^a y^b t^c, \ldots) + q(\ldots, x^a y^b t^c, \ldots) = 0,$$

*where $(a, b, c)$ runs through the triples of nonnegative integers with $a + b + c = m$. Then $X$ is a smooth surface over $K$, and for any algebraically closed field $L$ containing $K$, the map*

$$\pi^*: \mathrm{Pic}(\mathbb{P}^2_L) \to \mathrm{Pic}(X_L)$$

*is an isomorphism.* □

## 3. Positive rational functions on the real affine plane which are not sums of 3 squares

Let $m \geqslant 3$ be an integer and $N = (m+2)(m+1)/2$. Let $l(T_1, \ldots, T_N) = \Sigma_{1 \leqslant i \leqslant N} a_i T_i$ be a linear form and $q(T_1, \ldots, T_N) = \Sigma_{1 \leqslant i \leqslant j \leqslant N} b_{ij} T_i T_j$ a quadratic form, both with coefficients in $\mathbf{C}$. Fixing an ordering on the pairs of nonnegative integers $(a, b)$ with $a + b \leqslant m$, we may define a morphism of affine spaces:

$$\varphi\colon \mathbb{A}^2 \to \mathbb{A}^N$$
$$(x, y) \to (x^a y^b).$$

Let $P(x, y) \in \mathbf{R}[x, y]$ be the polynomial

$$P(x, y) = (4q - l^2)(\varphi(x, y)) = 4q(\ldots, x^a y^b, \ldots) - l(\ldots, x^a y^b, \ldots)^2.$$

THEOREM 3.1. *With notation as above, assume that the coefficients $(a_i)_{1 \leqslant i \leqslant N}$ and $(b_{ij})_{1 \leqslant i \leqslant j \leqslant N}$ lie in* $\mathbf{R}$.

(i) *If these coefficients are algebraically independent over the rational field* $\mathbf{Q}$, *then the polynomial $P(x, y)$ is not a sum of* 3 *squares in the rational function field* $\mathbf{R}(x, y)$.

(ii) *There exists some real number $\varepsilon > 0$ such that if the coefficients satisfy the inequalities*

$$|a_i| < \varepsilon,\ |b_{ii} - 1| < \varepsilon,\ |b_{ij}| < \varepsilon \text{ (for } i \neq j),\ |b_{ii} - 1| < \varepsilon,$$

*then the polynomial $P(x, y)$ is strictly positive on* $\mathbf{R}^2$.

*Proof.* Let $K = \mathbf{Q}(a_i, b_{ij})$ be the field spanned over $\mathbf{Q}$ by the $a_i$'s and $b_{ij}$'s. Let $\pi\colon X \to \mathbb{P}^2_K$ be the double cover defined in weighted projective space $\mathbb{P}(1, 1, 1, m)$ ($x, y, t$ of weight 1; $z$ of weight $m$) by the equation

$$z^2 + zl(\ldots, x^a y^b t^c, \ldots) + q(\ldots, x^a y^b t^c, \ldots) = 0,$$

where $(a, b, c)$ runs through the triples of positive integers with $a + b + c = m$. The surface $X$ and the morphism $\pi$ are defined over $K$.

The surface $X$ appears more naturally as an intersection of two varieties in projective space $\mathbb{P}^N_K$. Let indeed $Y$ be the image of the weighted projective space $\mathbb{P}(1, 1, 1, m)$ in projective space $\mathbb{P}^N_{\mathbf{Q}}$ under the embedding given by forms of degree $m$, i.e.

$$(x, y, t; z) \to (T_0, T_1, \ldots, T_N) = (z, \ldots, x^a y^b t^c, \ldots)$$

for some fixed ordering of all natural integers $a, b, c$ satisfying $a + b + c = m$.

The variety $Y$ is a smooth projective threefold in $\mathbb{P}^N_{\mathbf{Q}}$.

Our surface $X = X_{l,q}$ is none other than the intersection $Y_K \cap Q \subset \mathbb{P}^N_K$ of the quadric $Q \subset \mathbb{P}^N_K$ given by the equation

$$T_0^2 + T_0 l(T_1, \ldots, T_N) + q(T_1, \ldots, T_N) = 0$$

in $\mathbb{P}^N$ with the threefold $Y_K = Y \times_{\mathbf{Q}} K$. Projection $\pi$ from the point $(0,0,0,1)$ makes $X$ into a double cover of the usual projective space $\mathbb{P}^2_K$, ramified along the curve given by the equation

$$4q(\ldots, x^a y^b t^c, \ldots) - (l(\ldots, x^a y^b t^c, \ldots))^2 = 0.$$

Since the quadric $Q$ is generic, a straightforward application of Bertini's theorem implies that $X$ is geometrically connected and smooth over $K$.

Since the coefficients $(a_i)_{1 \leqslant i \leqslant N}$ and $(b_{ij})_{1 \leqslant i \leqslant j \leqslant N}$ are algebraically independent over $\mathbf{Q}$, and $m \geqslant 3$, the Noether-Lefschetz theorem 2.1 implies that the map

$$\pi^*\colon \operatorname{Pic}(\mathbb{P}^2_{\mathbf{C}}) \to \operatorname{Pic}(X_{\mathbf{C}})$$

is an isomorphism. Since the coefficients $(a_i)_{1 \leqslant i \leqslant N}$ and $(b_{ij})_{1 \leqslant i \leqslant j \leqslant N}$ lie in $\mathbf{R}$, we may consider the surface $X_{\mathbf{R}}$. From the obvious commutative diagram

$$\begin{array}{ccc} \mathbf{Z}.\mathcal{O}(1) = \operatorname{Pic}(\mathbb{P}^2_{\mathbf{R}}) & \xrightarrow{\pi^*_{\mathbf{R}}} & \operatorname{Pic}(X_{\mathbf{R}}) \\ \downarrow & & \downarrow \\ \mathbf{Z}.\mathcal{O}(1) = \operatorname{Pic}(\mathbb{P}^2_{\mathbf{C}}) & \xrightarrow{\pi^*_{\mathbf{C}}} & \operatorname{Pic}(X_{\mathbf{C}}), \end{array}$$

where the left vertical map is identity on $\mathbf{Z}$, we conclude that the injective map

$$\operatorname{Pic}(X_{\mathbf{R}}) \to \operatorname{Pic}(X_{\mathbf{C}})$$

is an isomorphism.

Let $\mathbf{R}(X)$ be the function field of $X_{\mathbf{R}}$. According to Lemma 1.1, we have the exact sequence

$$0 \to \operatorname{Pic}(X_{\mathbf{R}}) \to \operatorname{Pic}(X_{\mathbf{C}})^{\operatorname{Gal}(\mathbf{C}/\mathbf{R})} \to \operatorname{Br}(\mathbf{R}) \to \operatorname{Br}(\mathbf{R}(X)),$$

and we conclude that the map $\operatorname{Br}(\mathbf{R}) \to \operatorname{Br}(\mathbf{R}(X))$ is an injection: the Hamilton quaternion algebra does not split over the function field $\mathbf{R}(X)$.

Now the field $\mathbf{R}(X)$ is $\mathbf{R}$-isomorphic to the field $\mathbf{R}(x, y)(\sqrt{-P})$. Indeed, in affine coordinates (we let $t = 1$), the equation of $X$ reads

$$z^2 + zl(\ldots, x^a y^b, \ldots) + q(\ldots, x^a y^b, \ldots) = 0,$$

i.e.

$$(z + l(\ldots, x^a y^b, \ldots)/2)^2 + q(\ldots, x^a y^b, \ldots) - (l(\ldots, x^a y^b, \ldots)^2)/4 = 0,$$

and an obvious change of variables realizes an isomorphism with the affine surface with equation

$$z^2 + P(x, y) = 0.$$

From Lemma 1.2 we conclude that the polynomial $P(x, y)$ is not a sum of three squares in the field $\mathbf{R}(x, y)$, which is claim (i).

Let us now prove (ii). The polynomial $P(x, y)$ is certainly positive on $\mathbf{R}^2$ if the quadratic form

$$4q(T_1, \ldots, T_N) - (l(T_1, \ldots, T_N))^2$$

is positive definite. Let $q_0(T_1, \ldots, T_N) = \Sigma_{1 \leqslant i \leqslant N} T_i^2$ and $l_0(T_1, \ldots, T_N) = 0$. If $\varepsilon$ in (ii) is chosen small enough, the coefficients of the quadratic form $4q - l^2$ will be very close to those of $4q_0 - l_0^2$. Since this last form clearly is positive definite, so will be $4q - l^2$.

The proof of our theorem is now complete. However, it remains to observe that the theorem is not empty. For this, we simply note that for any non-empty open interval $I$ of the real line $\mathbf{R}$, the field extension of $\mathbf{Q}$ generated by the elements of $I$ is $\mathbf{R}$, hence certainly of infinite transcendence degree over $\mathbf{Q}$. The existence of suitable $(a_i)$ and $(b_{ij})$ is thus clear. □

## 4. Remarks

4.1. The above proof gives a method for producing explicit polynomials $P(x, y)$ which are positive but not sums of three squares in $\mathbf{R}(x, y)$: it is enough to produce sufficiently many algebraically independent elements in $\mathbf{R}$. Whether such elements can be "explicitly" produced is a matter of taste.

4.2. The same proof would work if $\mathbf{R}$ was replaced by an archimedean real closed field $R$ of infinite transcendence degree over $\mathbf{Q}$ (the restrictions on $R$ being imposed in order to ensure that the theorem is not empty.)

4.3. In some of the published variants of the Noether-Lefschetz theorem, rather than using generic coefficients for the equations at hand, one deals with "general" coefficients.

The published proofs show: In the space $Z$ of coefficients, which is some Zariski open set in projective space over the complex field, there exists a countable union of closed analytic subvarieties $Z_i (i \in \mathbf{N})$, $Z_i \neq Z$, such that for $z \in Z(\mathbf{C})$ away from these varieties, the Picard group of the surface $X_z$ is reduced to $\mathbf{Z}$.

(I am told that one may actually take the $Z_i$ to be closed algebraic subvarieties.)

Granting this result in the context of double covers, it is possible to give a variant of the proof of Theorem 3.1. To prove that the theorem thus obtained is not empty, rather than using a transcendence degree argument, one here uses Baire's category theorem.

4.4. The polynomials produced by Cassels/Ellison/Pfister [3] and Christie [4] lie in $\mathbf{Q}[x, y]$. The polynomials $P(x, y)$ we exhibit certainly do not lie in $\mathbf{Q}[x, y]$, since their coefficients are transcendental. I wonder whether a combination of my approach with the technique used by Terasoma [19] (itself a combination of the Noether method as expounded by Deligne and of Hilbert's irreducibility theorem) would lead to polynomials in $\mathbf{Q}[x, y]$.

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