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MARCEL BÖKSTEDT
AMNON NEEMAN

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Homotopy limits in triangulated categories

MARCEL BÖKSTEDT[2] and AMNON NEEMAN[1,2,3]
[1]*Department of Mathematics, University of Virginia, Charlottesville, VA 22903, USA*
[2]*Fakultät für Mathematik, Universität Bielefeld, D-4800 Bielefeld, Germany*

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0. Introduction

Topologists and algebraists have been independently studying the homotopy category for a long time. To a large extent the theories they have developed are parallel. But the topologists have had a number of insights which have eluded the algebraists, and what we have tried to do in this paper is to expose these insights, giving some algebraic applications.

Most basic is the notion of homotopy limits and colimits. The topologists make frequent and systematic use of the *mapping telescope*, and perhaps the key point of this article is that this construction can be done in any triangulated category. Let $0 \to X_0 \to X_1 \to X_2 \to \cdots$ be a sequence of objects and morphisms in a triangulated category. We define the homotopy colimit hocolim (X_i) as the third edge of the triangle

$$\begin{tikzcd} \oplus X_i \arrow[rr, "\text{1-shift}"] & & \oplus X_i \\ & \operatorname{hocolim}(X_i) \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{1}$$

and we will attempt to systematically show that this is a natural and useful construction. We give two major applications.

THEOREM 2.14 Let $\mathscr{A}$ be an abelian category satisfying $AB4$ and $AB4^*$, with enough projectives and injectives. Then there exist functors

$${}^L\otimes : D(\mathscr{A}) \times D(\mathscr{A}) \to D(\text{Abelian groups})$$

and

$$\mathrm{RHom}\, D(\mathscr{A})^{op} \times D(\mathscr{A}) \to D(\text{Abelian groups})$$

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which satisfy all reasonable good properties. □

In short, the tensor product and RHom functors extend to the unbounded derived category under very weak hypotheses. This result was first obtained by Spaltenstein [S], and the point here is that our proof is shorter and simpler.

The second application is

COROLLARY 5.5. *Let X be a separated quasi-compact scheme. Let $D(qc/x)$ be the derived category of chain complexes of quasi-coherent sheaves over X, and let $D_{qc}(X)$ be the derived category of chain complexes of arbitrary modules over X, with quasi-coherent cohomology. Then the natural map*

$$D(qc/x) \to D_{qc}(X)$$

is an equivalence of categories. □

Once again, Corollary 5.5 is not startlingly new. If X is Noetherian, this may be deduced from the study of indecomposable injectives on X. For D^+, the result was known to Verdier, using the adjoint of the inclusion of quasi-coherents in all modules. The main point is again that the proof is so trivial.

Sections 1 and 2 develop the basic properties of homotopy colimits of sequences, and Theorem 2.14 falls out as an immediate corollary. Sections 3 and 4 are the conjectural sections: there we discuss totalizations of complexes and arbitrary colimits, both subjects which are poorly understood. As an illustration of the importance of the questions, we prove in Section 3 that the category $D^b(R)$ is closed under splitting idempotents.

But somehow the most interesting part of the paper is Section 4, where we discuss the notion of Bousfield localization, and show that everything we did in Section 2 should really be viewed as a special case. We also introduce Ravenel's notion of smashing subcategory, illustrating it with the examples from Section 2.

Section 5 is devoted to the proof of Corollary 5.5. In Section 6 we briefly discuss other examples of localization and smashing subcategories. Perhaps most remarkable is the fact that Grothendieck's local cohomology functor is nothing other than a Bousfield localization.

When we wrote this article we were not aware of Spaltenstein's work. We thought that we had been very clever to find such a simple proof that standard operations can be lifted to the unbounded derived category. After the article was already completely typed up, it was pointed out to us that our results were obtained by Spaltenstein five years ago. Needless to say, we were deflated to discover that Spaltenstein is every bit as clever as us, and faster. With some reluctance, we agreed to publish our results anyway. Spaltenstein's proof of Theorem 2.14 is essentially identical with ours, with one minute difference: whereas we study homotopy colimits in the derived category, Spaltenstein studies colimits of fibrations of actual complexes – he works in a model category. The reader is encouraged to read Spaltenstein's work, to see what trouble can be caused by working with the wrong notion of limits. Of course, we had an advantage over Spaltenstein there, in that our work is based on the approach of the topologists.

1. Direct sums in triangulated categories

LEMMA 1.1. *Let $\mathscr{A}$ be an abelian category satisfying AB3 (there exist arbitrary direct sums). Then the category $K(\mathscr{A})$ of chain complexes over and chain homotopy equivalence classes of maps also has direct sums, and direct sums of triangles are triangles.*

Proof. It is trivial to show that the direct sum of chain complexes is a categorical direct sum in $K(\mathscr{A})$. □

DEFINITION 1.2. A triangulated category is said to have direct sums if it has categorical direct sums, and direct sums of triangles are triangles.

DEFINITION 1.3. Let $\mathscr{S}$ be a triangulated category with arbitrary direct sums. Then a full triangulated subcategory $L \subset \mathscr{S}$ is called *localizing* if

Every direct summand of an object in L is in L. (1.3.1)

Every direct sum of objects of L is in L. (1.3.2)

REMARK 1.4. We will see later that (1.3.1) is superfluous; (1.3.2) $\Rightarrow$ (1.3.1). By Rickard's criterion for épaisse subcategories, ([Ri], proposition 1.4) L is épaisse and one may form the quotient category $\mathscr{S}/_L$, where the objects of L are identified with 0.

LEMMA 1.5. *If L is a localizing subcategory of the triangulated category, then the triangulated category $\mathscr{S}/_L$ has direct sums. In fact, the functor $\mathscr{S} \to \mathscr{S}/_L$ preserves direct sums.*

Proof. Let $\{X_i, i \in I\}$ *be a family of objects of* $\mathscr{S}$. Then in $\mathscr{S}$ we have for each i morphisms $X_i \to \oplus_{i \in I} X_i$ and we need to show

(1) Any collection of maps $X_i \xrightarrow{f_i} Y$ in $\mathscr{S}/_L$ can be lifted to a map in $\mathscr{S}/_L$ $\oplus_{i \in I} X_i \xrightarrow{f} Y$.

(2) Given a map $\oplus_{i \in I} X_i \xrightarrow{f} Y$ in $\mathscr{S}/_L$ such that all the composites $X_i \to \oplus_{i \in I} X_i \xrightarrow{f} Y$ are zero, then f is zero.

Proof of (1). A map $X_i \to Y$ in $\mathscr{S}/_L$ is a diagram in $\mathscr{S}$

$$\begin{tikzcd} & X_i' \arrow[dl, "\alpha_i"'] \arrow[dr, "\beta_i"] & \\ X_i & & Y \end{tikzcd}$$

where α_i is a quasi-isomorphism, i.e. in the triangle $X'_i \to X_i \to Z_i \to \Sigma X'_i$, $Z_i \in L$. Thus we get a diagram

$$\begin{tikzcd} & \oplus X'_i \arrow[dl, "\oplus\alpha_i"'] \arrow[dr] & \\ \oplus X_i & & Y_i \end{tikzcd}$$

Because direct sums of triangles are triangles, $\oplus X'_i \xrightarrow{\oplus\alpha_i} \oplus X_i \to \oplus Z_i \to \Sigma(\oplus X'_i)$ is a triangle. Because L is localizing, $\oplus Z_i \in L$, and so the map $\oplus\alpha_i\colon \oplus X'_i \to \oplus X_i$ is a quasi-isomorphism. □

Proof of (2). Given a map $\oplus_i X_i \xrightarrow{f} Y$ in $\mathscr{S}/_L$, it corresponds to a diagram

$$\begin{tikzcd} \oplus_i X_i \arrow[dr, "\alpha"'] & & Y \arrow[dl, "\beta"] \\ & Y' & \end{tikzcd}$$

where β is a quasi-isomorphism. If the composite $X_i \to \oplus X_i \to Y$ is zero in $\mathscr{S}/_L$, then in $\mathscr{S}$ we have a diagram

$$\begin{tikzcd} X_i \arrow[dr] & & & \\ & \oplus X_i \arrow[dr] & & Y \arrow[dl, "\beta"] \\ & & Y' & \end{tikzcd}$$

which corresponds to the zero morphism in $\mathscr{S}/_L$. This means that the composite $X_i \to \oplus X_i \to Y'$ must factor as $X_i \to Z_i \to Y'$ with $Z_i \in L$. Thus f factorizes as

$$\begin{tikzcd} \oplus X_i \arrow[dr] & & & \\ & \oplus Z_i \arrow[dr] & & Y \arrow[dl, "\beta"] \\ & & Y' & \end{tikzcd}$$

But because L is localizing, $\oplus_i Z_i \in L$ and $f = 0$. □

EXAMPLE 1.6. Let $L \subset K(\mathscr{A})$ be the subcategory of homologically trivial complexes of objects in the abelian category $\mathscr{A}$. If $\mathscr{A}$ satisfies $AB4$ (i.e. direct sums of exact sequences are exact) then L is localizing.

COROLLARY 1.7 *If $\mathscr{A}$ satisfies $AB4$ then $D(\mathscr{A})=K(\mathscr{A})/_L$ has direct sums.* □

Needless to say, all the lemmas so far are self-dual; so if $\mathscr{A}$ is an abelian category satisfying $AB4^*$, then $D(\mathscr{A})$ has direct products (i.e. categorical direct products exist, and products of triangles are triangles). In the cases we will consider in the rest of this article, $\mathscr{A}$ will be the category of modules over a ring R, which satisfies both $AB4$ and $AB4^*$.

2. Countable direct limits

Let $\mathscr{S}$ be a triangulated category with direct sums. Suppose $\{X_i, i \in \mathbb{N}\}$ is a sequence of objects in $\mathscr{S}$, together with maps $X_i \to X_{i+1}$. We wish to define the homotopy colimit of the sequence.

DEFINITION 2.1. The homotopy colimit of the sequence above is the third edge of the triangle

$$\begin{tikzcd} \oplus_i X_i \arrow[rr, "1\text{-shift}"] & & \oplus_i X_i \arrow[dl] \\ & \mathrm{hocolim}\,(X_i) \arrow[ul, "(1)"] & \end{tikzcd}$$

where the map (shift) above is the shift map, whose coordinates are the natural maps $X_i \to X_{i+1}$.

REMARK 2.2. This is nothing more than the usual "mapping telescope" construction of topology. If $\mathscr{S} = D(\mathscr{A})$, and $\mathscr{A}$ is an abelian category satisfying $AB5$ (filtered direct limits of exact sequences are exact), the reader will easily prove:

$$H^i\left(\underset{j}{\mathrm{hocolim}}\,(X_j)\right) = \underset{j}{\mathrm{colim}}\, H^i(X_j). \tag{2.2.1}$$

If we choose actual chain maps of chain complexes $X_i \to X_j$ (not merely homotopy equivalence classes of such maps), then one can prove easily:

$$\text{There is a natural quasi-isomorphism } \underset{i}{\mathrm{hocolim}}\,(X_i) \to \underset{i}{\mathrm{colim}}\,(X_i). \tag{2.2.2}$$

REMARK 2.3. Of course, the dual is also true. However, the dual of (2.2.2) is not so useful. In our applications $\mathscr{A}$ will be something like the category of modules over a ring R, and this does not satisfy $AB5^*$. We will use a slight modification:

Let X be an object of $D(\mathscr{A})$, where $\mathscr{A}$ is an abelian category satisfying $AB4^*$, and let $\to X_n \to X_{n-1} \to \cdots \to X_0$ be a sequence of objects in $D(\mathscr{A})$, together with maps $X \to X_i$ compatible with the sequence maps. Then the composite $X \to \Pi X_i \xrightarrow{\text{1-shift}} \Pi X_i$ is zero, so there is a deduced map $X \to \operatorname{holim}(X_i)$. If for every n, the map $H^n(X) \to H^n(X_i)$ is eventually an isomorphism, then we get a short exact sequence

$$0 \to H^n(X) \to H^n\left(\prod_i X_i\right) \xrightarrow{\text{1-shift}} H^n\left(\prod_i X_i\right) \to 0$$

and one immediately deduces that the morphism $X \to \operatorname{holim}(X_i)$ is a homology isomorphism, hence an isomorphism. (2.3.1)

APPLICATION 2.4. Let $\mathscr{A}$ be an abelian category satisfying $AB4^*$ with enough injectives. Then every object of $D(\mathscr{A})$ is quasi-isomorphic to a complex of injectives.

Proof. Let $X \in D(\mathscr{A})$ be arbitrary. Then $D(\mathscr{A})$ has a natural t-structure, and we denote by $X^{\geqslant n}$ the truncation of X above dimension n. There is a natural map $X \to X^{\geqslant n}$ which is a homology isomorphism in dimension $\geqslant n$, and $X^{\geqslant n}$ vanishes in dimensions $\leqslant n-1$.

Because $\mathscr{A}$ has enough injectives, we can choose a quasi-isomorphism $X^{\geqslant n} \to I_n$ (we use the fact that the complex $X^{\geqslant n}$ is bounded below). The diagram

$$\begin{array}{ccc} X^{\geqslant n-1} & \to & X^{\geqslant n} \\ \downarrow & & \downarrow \\ I_{n-1} & & I_n \end{array}$$

defines in the derived category a morphism $I_{n-1} \to I_n$, but as these are bounded below complexes of injective objects, we can choose a chain map realizing this morphism. Now we have morphisms

$$X = \varprojlim_n (X_{-n}) \xrightarrow{\alpha} \underset{\longleftarrow n}{\operatorname{holim}}(X_{-n}) \xrightarrow{\beta} \underset{\longleftarrow n}{\operatorname{holim}}(I_{-n})$$

α is a quasi-isomorphism because of (2.3.1), and β is a quasi-isomorphism because it is a holim of quasi-isomorphisms. But $\underset{\longleftarrow n}{\operatorname{holim}}(I_{-n})$ is, by construction, a complex of injectives. □

REMARK 2.5. Although everything that has preceded is completely trivial, the sequence of triviality has got us someplace. Application 2.4 seems new. The classical constructions, such as in [Ha], Section 1, permit the reader to find for every object in $D(\mathscr{A})$ a quasi-isomorphic complex of injectives only under very stringent hypotheses of finite injective dimension, and the argument used there is not as trivial as ours.

More significant then the statement of Application 2.4 is the proof. We proved a little more than we stated. What we actually have is:

APPLICATION 2.4′. Let K be the colocalizing subcategory of $D(\mathscr{A})$ generated by bounded below complexes of injectives. That is, K is a full subcategory containing the bounded below complexes of injectives, and is closed under direct products and the formation of triangles (i.e. the dual of a localizing subcategory; see Definition 1.3). Then every object of $D(\mathscr{A})$ is isomorphic to an object of K. $\square$

DEFINITION 2.6. Let $\mathscr{S}$ be a triangulated category, $L \subset \mathscr{S}$ an épaisse subcategory (see Definition 1.3). An object $Y \in \mathscr{S}$ is called *L-local* if, for every $X \in L$, $\mathrm{Hom}(X, Y) = 0$.

REMARK 2.7. This definition is initially due to Sullivan, but was used by Adams and very extensively by Bousfield. We will return to this definition in Section 4. For now, the example we want the reader to have in mind is Example 1.6: $\mathscr{S} = K(\mathscr{A})$ is the category of complexes of objects in $\mathscr{A}$ with homotopy equivalence classes of maps, and L is the subcategory of acyclic objects. We have:

LEMMA 2.8. *With $\mathscr{S} = K(\mathscr{A})$, L the category of acyclic complexes, if I is a bounded below complex of injectives, then I is L-local.*

Proof. This is just the standard fact that any map from an acyclic complex to a bounded below complex of injectives is null homotopic. $\square$

LEMMA 2.9 (well-known). *Let $\mathscr{S}$ be a triangulated category, L a localizing subcategory and Y an L-local object. Suppose X is any object of $\mathscr{S}$. Then*

$$\mathrm{Hom}_{\mathscr{S}}(X, Y) = \mathrm{Hom}_{\mathscr{S}/_L}(X, Y).$$

Proof. (Included only for the convenience of the reader). A map in $\mathscr{S}/_L$ from X to Y is a diagram in:

$$\begin{tikzcd} & X' \arrow[dl, "\alpha"'] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

where α is a quasi-isomorphism. That is, in the triangle $X' \xrightarrow{\alpha} X \to Z \to \Sigma X$, $Z \in L$. But then

$$0 = \mathrm{Hom}(\Sigma^{-1}Z, Y) \to \mathrm{Hom}(X', Y) \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}(Z, Y) = 0$$

is exact; i.e. there is a (unique) factorization

$$\begin{tikzcd} & X' \arrow[dl, "\alpha"'] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

which proves that $\mathrm{Hom}_{\mathscr{S}}(X, Y) \to \mathrm{Hom}_{\mathscr{S}/_L}(X, Y)$ is surjective. But if $f: X \to Y$ is a morphism in $\mathscr{S}$ which maps to zero in $\mathscr{S}/_L$, then it factors as $X \to Z \to Y$ with $Z \in L$, but as Y is local, the map $Z \to Y$ is zero. □

Now we note:

LEMMA 2.10. *Let $\mathscr{S}$ be a triangulated category, $L \subset \mathscr{S}$ an épaisse subcategory. Then the full subcategory of all L-local objects in $\mathscr{S}$ is épaisse.*

Proof. Trivial. □

LEMMA 2.11. *Let $\mathscr{S}$ be a triangulated category with products, L an épaisse subcategory. Then the subcategory of L-local objects is colocalizing. (i.e. it is closed under arbitrary direct products).*

Proof. Clear. □

Let $\mathscr{A}$ be an abelian category satisfying $AB4^*$. Then set $K(I)$ to be the smallest colocalizing subcategory of $K(\mathscr{A})$ containing the bounded below complexes of injectives. By Lemma 2.11, $K(I)$ consists of L-local objects in $K(\mathscr{A})$. If $\mathscr{A}$ has enough injectives, then we know from Application 2.4′ that every object in $D(\mathscr{A})$ is quasi-isomorphic to a complex in $K(I)$. Thus we deduce:

PROPOSITION 2.12. *Let $\mathscr{A}$ be an abelian category with enough injectives satisfying $AB4^*$. Then the composite functor:*

$$K(I) \hookrightarrow K(\mathscr{A}) \to D(\mathscr{A})$$

is an equivalence of categories. □

Of course, everything we have done is easily dualizable. If $\mathscr{A}$ is an abelian category with enough projectives satisfying $AB4$, then the composite

$$K(P) \hookrightarrow K(\mathscr{A}) \to D(\mathscr{A})$$

is an equivalence of categories, where $K(P)$ is the smallest localizing subcategory containing the bounded above projectives.

REMARK 2.13. The complexes in $K(I)$ are, among other things, complexes of injective objects. We call them *special complexes of injectives.* Similarly the objects of $K(P)$ will be called *special complexes of projectives.* In fact, the entire point of this section is that they offer the right framework for doing homological algebra in the unbounded derived category.

THEOREM 2.14. *Let $\mathscr{A}$ be an abelian category with a tensor product satisfying AB4 and AB4*, with enough injectives and projectives. Then there exist functors:*

$$^L\otimes : D(\mathscr{A}) \times D(\mathscr{A}) \to D(\textit{ab groups})$$

and

$$\mathrm{RHom}: D(\mathscr{A})^{op} \times D(\mathscr{A}) \to D(\textit{Ab groups})$$

which satisfy all reasonable good properties.

Proof. The point is that $^L\otimes$ and RHom are easy to define on $K(\mathscr{A})$. But now we know that $D(\mathscr{A})$ is isomorphic to subcategories of $K(\mathscr{A})$, and this permits us to define $^L\otimes$ as a map

$$K(P) \times K(P) \to D(Ab)$$

and RHom as a map

$$K(P)^{op} \times K(I) \to D(Ab)$$

and the fact that this is reasonable is left largely to the reader. □

REMARK 2.15. If $\mathscr{A}$ has an internal Hom functor, or an internal tensor product, then of course one can define on the derived category

$$^L\otimes : D(\mathscr{A}) \times D(\mathscr{A}) \to D(\mathscr{A})$$

$$\mathrm{RHom}: D(\mathscr{A})^{op} \times D(\mathscr{A}) \to D(\mathscr{A}).$$

The following is a list of some good properties of these constructions. All proofs are left to the reader.

(2.16.1) The tensor is symmetric and associative; there are natural isomorphisms

$$X^L \otimes Y \cong Y^L \otimes X$$

and

$$(X^L \otimes Y)^L \otimes Z \cong X^L \otimes (Y^L \otimes Z).$$

(2.16.2) The tensor commutes with triangles and direct sums in either variables. In particular, it commutes with hocolim.

(2.16.3) For objects in $D^-(\mathscr{A})$, it reduces to the usual tensor.

(2.16.4) RHom commutes with triangles in each factor.

(2.16.5) RHom sends direct sums in the first factor, and direct products in the second, to direct products.

The following lemma is so crucial in [N2], that we feel it deserves to be proved here.

LEMMA 2.17. *Let R be a commutative ring, $\alpha: R \to k$ a homomorphism of R into a field k. Let X be an arbitrary object of $D(R)$. Then $X \otimes k$ is a direct sum of suspensions of k.*

Proof. Put $X = \underrightarrow{\mathrm{hocolim}}_i X_i$, X_i bounded above. Then $X \otimes k = \underrightarrow{\mathrm{hocolim}}_i X_i \otimes k$. But this hocolim is really being taken in $D(k)$, and the result is an object in $D(k) \subset D(R)$; i.e. $X \otimes k$ is a direct sum of suspensions of k. □

3. Totalizing a complex

Let $\mathscr{S}$ be a triangulated category. Let $\cdots \to X_n \to X_{n-1} \to \cdots \to X_0 = 0$ be a sequence of maps in $\mathscr{S}$. Then sometimes it may happen that one can totalize the complex. (Precisely, one tries to copy the construction of passing from a double complex to its total, single complex). What this entails is the following. Complete $X_2 \to X_1$ to a triangle $X_2 \to X_1 \to Y_1 \to \Sigma X_2$. Because the composite $X_3 \to X_2 \to X_1$ is zero, we can lift to $\Sigma X_3 \to Y_1$. If we are lucky, the composite $\Sigma X_4 \to \Sigma X_3 \to Y_1$ will be zero, and then we can iterate. Assuming that the iteration works, we get a diagram, which we will schematically indicate

$$\begin{tikzcd}
 & & \cdots & Y_3 \arrow[l] \arrow[d, "(-2)"] & Y_2 \arrow[l] \arrow[d, "(-1)"] & Y_1 \arrow[l] \arrow[d] & & \\
\cdots \arrow[r] & X_n \arrow[r] & {} \arrow[r] & X_4 \arrow[r] \arrow[ur] & X_3 \arrow[r] \arrow[ur] & X_2 \arrow[r] & X_1 \arrow[r] \arrow[ul] & 0
\end{tikzcd}$$

If everything so far has gone without a hitch, one defines the *totalization* of the sequence $\{X_n\}$, denoted by $|\{X_n\}|$, by the formula:

$$|\{X_n\}| = \underrightarrow{\mathrm{hocolim}}_n\, (Y_n).$$

There is already some literature about this construction, which has been studied in the topological setting by [C] and [W], and in the derived category by [K].

There are well-known obstructions to the lifting process, the so-called Toda classes. It is perhaps simplest to illustrate with an example.

Let us choose any triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{S}$. Consider the complex

$$\cdots \to \Sigma^{-1}Z \to X \to Y \to Z \to 0.$$

We assert that for almost all triangles in just about any triangulated category, this complex cannot be totalized.

The first step of the totalization process is to complete $Y \to Z$ to a triangle. Then we look for a map α

$$\begin{tikzcd}
 & & \Sigma X \arrow[d, "f"] & & \\
\Sigma^{-1}Z \arrow[r, "g"'] & X \arrow[ur, "\alpha"] \arrow[r, "f"'] & Y \arrow[r] & Z \arrow[ul] \arrow[r] & 0
\end{tikzcd}$$

where $f \circ \alpha = f$ and $\alpha \circ g = 0$.

Because $\alpha \circ g = 0$, $\alpha = \varphi \circ f$. Because $f \circ (1 - \alpha) = 0$, $1 - \alpha = g \circ \Theta$. Thus $1_X = g \circ \Theta + \varphi \circ f$, and so if α exists it would follow that the triangle $X \to Y \to Z \to \Sigma X$ is contractible. (See N1). This is extremely rare.

What we have shown here, is that for the triangle $\Sigma^{-1}Z \to X \to Y \to Z$, the first Toda class is the identity map $\Sigma(\Sigma^{-1}Z) = Z \to Z$. Also, for very formal reasons that is rarely zero. The way Toda classes have traditionally been used by topologists is to construct non-zero maps in homotopy. Thus the main interest has centered around constructing examples where the totalization process is obstructed.

What we would like to suggest is that a very interesting question, which ought to be pursued further, is to find useful sufficient conditions for the existence, and maybe also the uniqueness of the totalization. We do not have such a useful criterion. But to illustrate what we want, let us give the following, nearly useless result:

PROPOSITION 3.1. *Let us be given the sequences of objects of $\mathcal{S}$ and maps:*

$$\begin{tikzcd}
{} \arrow[r, shift left] & X_n \arrow[l, shift left] \arrow[r, shift left, "i_{n-1}"] & X_{n-1} \arrow[l, shift left, "j_{n-1}"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left, "i_1"] & X_1 \arrow[l, shift left, "j_1"] \arrow[r, "i_0"] & X_0 \arrow[r] & 0
\end{tikzcd}$$

and suppose that $i_{k+1} \circ i_k = 0$ and $i_k \circ j_k \circ i_k = i_k$. Then there is a way to "functorially" totalize the complex, in particular the complex has a totalization. Given a morphism of complexes (i.e. maps of complexes commuting with both *the i's and j's) then it induces a (non-unique) morphism of totalizations. And an isomorphism of complexes induces an isomorphism of totalizations.*

Proof. The point is that after choosing

$$\begin{tikzcd} & & & Y_1 \arrow[dl, "g"'] & \\ X_3 \arrow[r, "i_2"'] & X_2 \arrow[urr, "\alpha"] \arrow[r, "i_1"'] & X_1 \arrow[rr] & & X_0 \arrow[ul, "h"'] \end{tikzcd}$$

such that $g\circ\alpha = i_1$, we can replace α by $\alpha\circ j_1\circ i_1$. Then we still have $g\circ\alpha = i_1$, but now clearly $(\alpha\circ j_1\circ i_1)\circ i_2 = 0$. And of course, we can then iterate.

Now what about the functoriality? Given maps of complexes $\{X'_n, i, j\}\to\{X_n, i, j\}$ we can choose a morphism of triangles

$$\begin{tikzcd} X'_1 \arrow[r] \arrow[d] & X'_0 \arrow[r] \arrow[d] & Y'_1 \arrow[r] \arrow[d, "\beta"] & \Sigma X'_1 \arrow[d] \\ X_1 \arrow[r] & X_0 \arrow[r] & Y_1 \arrow[r] & \Sigma X_1 \end{tikzcd}$$

Of course, for this choice of β we have no assurance that the diagram

$$\begin{tikzcd} \Sigma X'_2 \arrow[r, "\alpha'"] \arrow[d, "f_2"'] & Y'_1 \arrow[d, "\beta"] \\ \Sigma X_2 \arrow[r, "\alpha"] & Y_1 \end{tikzcd}$$

will commute. However, if we compose with the projection $Y_1\to\Sigma X_1$, we must get equality. That means that the map $\beta\circ\alpha' - \alpha\circ f_2$ is a composite

$$\Sigma X'_2 \xrightarrow{p} X_0 \xrightarrow{h} Y_1.$$

By construction, $\alpha' = \alpha'\circ j'_1\circ i'_1$ and $\alpha = \alpha\circ j_1\circ i_1$ (after all, we replaced our arbitrary α by $\alpha\circ j_1\circ j_1$, and $(j_1\circ i_1)\circ(j_1\circ i_1) = j_1\circ i_1$.

Therefore we deduce

$$\begin{aligned}(h\circ p)\circ(j'_1\circ i'_1) &= (\beta\circ a' - \alpha\circ f_2)(j'_1\circ i'_1)\\ &= \beta\circ(\alpha'\circ j'_1\circ i'_1) - \alpha\circ f_2\circ j'_1\circ i'_1\\ &= \beta\circ a' - \alpha\circ j_1\circ f_1\circ i'_1\\ &= \beta\circ a' - \alpha\circ j_1\circ i_1\circ f_2\\ &= h\circ p.\end{aligned}$$

Thus we may replace p by $p \circ j'_1 \circ i'_1$. But β may be replaced by $\beta - h \circ (p \circ j') \circ g'$, and an easy computation establishes that for the new β the square

$$\begin{tikzcd}
\Sigma X'_3 \arrow[r] \arrow[d] & Y'_1 \arrow[d] \\
\Sigma X_3 \arrow[r] & Y_1
\end{tikzcd}$$

actually commutes. Once again, one may iterate. □

As we said, this should not be viewed as an interesting criterion in its own right, but rather as an indication of the sort of result one should be looking for. With this aid, let us nevertheless exhibit an application of Proposition 3.1.

PROPOSITION 3.2. *Let $\mathscr{S}$ be a triangulated category with direct sums. Suppose $e: X \to X$ is an idempotent in $\mathscr{S}$. Then e is split in $\mathscr{S}$.*

Proof. The idea is to totalize the complex

$$\cdots \longrightarrow X \xrightarrow{e} X \xrightarrow{1-e} X \xrightarrow{e} X \longrightarrow 0.$$

Precisely, by Proposition 3.1, the following three complexes may be totalized

$$\cdots \underset{1}{\overset{1-e}{\rightleftarrows}} X \underset{1}{\overset{e}{\rightleftarrows}} X \underset{1}{\overset{1-e}{\rightleftarrows}} X \xrightarrow{e} X \tag{1}$$

$$\cdots \underset{1}{\overset{e}{\rightleftarrows}} X \underset{1}{\overset{1-e}{\rightleftarrows}} X \underset{1}{\overset{e}{\rightleftarrows}} X \xrightarrow{1-e} X \tag{2}$$

$$\underset{1}{\overset{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}{\rightleftarrows}} X \oplus X \underset{1}{\overset{\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}{\rightleftarrows}} X \oplus X \xrightarrow{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)} X \oplus X \tag{3}$$

And it is trivial to show that (1) $\oplus$ (2) is isomorphic to (3). Thus the same is true on the totalizations. Let the totalization of (1) be Y, the totalization of (2), Z. Then $X = Y \oplus Z$, and the reader can check that $1 - e$ is zero on Z, e is zero on Y. □

REMARK 3.3. The reader may observe that there are other ways to prove Proposition 3.2, which avoid using Proposition 3.1. In this Remark, we will outline such a proof. Nevertheless, the proof above is the "right" proof, in a sense that will be made precise soon.

Observe that we have three sequences

(1) $X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots$

(2) $X \xrightarrow{1-e} X \xrightarrow{1-e} X \xrightarrow{1-e}$

(3) $X \oplus X \xrightarrow{\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)} X \oplus X \xrightarrow{\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)} X \oplus X \xrightarrow{\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}$

and (3) $\cong$ (1) $\oplus$ (2). Thus $\operatorname{hocolim}(1) \oplus \operatorname{hocolim}(2) = \operatorname{hocolim}(3) = X$ gives a direct sum decomposition of X into the kernels of $1 - e$ and e.

PROPOSITION 3.4. *Let R be a ring, $D^b(R)$ the derived category of finite complexes of finitely generated projective R-modules. Then every idempotent in $D^b(R)$ is split.*

Proof. Let X be an object of $D^b(R)$, $e: X \to X$ idempotent. As in the proof of Lemma 3.2, we can construct totalizing sequences

$$\begin{tikzcd}
 & Y_3 \arrow[d, "(-2)"] & Y_2 \arrow[l, "d"'] \arrow[d, "(-1)", "+"'] & Y_1 \arrow[l, "d"'] \arrow[d, "d", "+"'] & \\
\cdots \arrow[r] & X \arrow[r, "e"] \arrow[ur] & X \arrow[r, "1-e"] & X \arrow[r, "e"] \arrow[ul] & X \arrow[ul]
\end{tikzcd} \tag{1}$$

and

$$\begin{tikzcd}
 & Z_3 \arrow[d, "(-2)"] & Z_2 \arrow[l, "d"'] \arrow[d, "(-1)"] & Z_1 \arrow[l, "d"'] \arrow[d, "d", "+"'] & \\
\arrow[r] & X \arrow[r, "1-e"] \arrow[ur] & X \arrow[r, "e"] \arrow[ur] & X \arrow[r, "1-e"] & X \arrow[ul]
\end{tikzcd} \tag{2}$$

and we know that

$$X = \operatorname{hocolim}(Y_i) \oplus \operatorname{hocolim}(Z_i)$$

is the required splitting of X.

But there is a map $\operatorname{colim}(Y_i) \to \operatorname{hocolim}(Y_i)$ which is a quasi-isomorphism by 2.2.1, and the sequence Y_i is a sequence of objects in $D^b(R)$, and in each degree the sequence is ultimately stable. It follows that $\operatorname{colim}(Y_i) \in D^-(R)$, but more specifically it is a bounded above complex of *finitely generated* projectives. Thus $X = Y \oplus Z$ where Y and Z may be chosen bounded above complexes of finitely generated projectives.

Suppose $X = X^{\geqslant n}$ is a complex of projective objects in degree $\geqslant n$. Then $X^{\geqslant n} = Y^{\geqslant n} \oplus Z^{\geqslant n}$. Furthermore,

$$\begin{aligned} 0 &= \operatorname{Hom}(X^{\geqslant n}, D(R)^{\leqslant n-1}) (\text{by the projectivity of } X) \\ &= \operatorname{Hom}(Y^{\geqslant n}, D(R)^{\leqslant n-1}) \oplus \operatorname{Hom}(Z^{\geqslant n}, D(R)^{\leqslant n-1}) \end{aligned}$$

and from this it is easy to deduce that $Y^{\geqslant n}$ and $Z^{\geqslant n}$ are complexes of projectives, and the finite generation follows immediately from the finite generation in each degree for Y and Z. □

4. Arbitrary colimits and localizations

Until now, we have been very modest in our constructions; the only homotopy colimits we have so far dealt with are direct sums and the colimit of a countable sequence. There is of course an extensive literature on homotopy colimits. We will not even try to sketch what is known. Let us just say what happens if we try to copy the standard constructions of Bousfield and Kan in a triangulated category.

Let $\mathscr{S}$ be a triangulated category with direct sums. Let $F: I \to \mathscr{S}$ be a functor from the small category I to $\mathscr{S}$. To construct $\underrightarrow{\operatorname{hocolim}} F$, one considers the simplicial set (I), the nerve of the category I, and the chain complex $\tilde{F}$:

$$\longrightarrow \bigoplus_{s \in \mathscr{N}_{i+1}(I)} F(\partial_0 s) \xrightarrow{\partial} \bigoplus_{s \in \mathscr{N}_i(I)} F(\partial_0 s) \longrightarrow$$

where the map ∂ is the alternating sum of the differentials. The homotopy colimit of F is $|\tilde{F}|$, the totalization of the chain complex $\tilde{F}$. It would be very nice to know whether $\tilde{F}$ can always be totalized, preferably in a functorial way. Perhaps one needs to make some hypothesis on I; in applications, I is nearly always a totally ordered, or even a well ordered set.

What we want to observe is the following:

LEMMA 4.1. *Let $\mathscr{T}$ and $\mathscr{S}$ be triangulated categories with enough direct sums, and let $G: \mathscr{T} \to \mathscr{S}$ be a triangulated functor which preserves direct sums. If $F: I \to \mathscr{T}$ is a functor from a small category I into $\mathscr{T}$, and if F has a hocolim in $\mathscr{T}$, then $G(\underrightarrow{\operatorname{hocolim}}(F))$ is a hocolim of $G \circ F$ in $\mathscr{S}$.* □

REMARK 4.2. In the case where $G: \mathscr{T} \to \mathscr{S}$ is an inclusion, one gets a little more; namely all $\mathscr{S}$-colimits of functors into $\mathscr{T}$ are in $\mathscr{T}$ (note that with hocolim as we defined it, neither existence *nor uniqueness* is clear).

Let L be an épaisse subcategory of the triangulated category $\mathscr{S}$. We recall (Definition 2.6) that Y is *L-local* if for every $X \in L$, $\operatorname{Hom}(X, Y) = 0$.

DEFINITION 4.3 (Bousfield). With $\mathscr{S}$ and L as above, a morphism $X \to Y$ is called a *localization* if

(1) Y is L-local
(2) For any L-local object Z, the natural map $\operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$ is an isomorphism.

THEOREM 4.4 (Bousfield). *If $\mathscr{S}$ is sensible (e.g. the category of spectra, or $D(R)$ for a ring R) and L is localizing, then up to increasing the universe, every object of has a localization.* □

The idea of the proof, which was actually due to Adams, is very simple. One constructs by transfinite induction a sequence of objects X_α of $\mathscr{S}$. For every α, one considers the set of all maps $f\colon Y_f \to X_\alpha$, where $Y_f \in L$. Then construct $X_{\alpha+1}$ from the triangle $\amalg_f Y_f \to X_\alpha \to X_{\alpha+1} \to \Sigma(\amalg_f Y_f)$. For limit ordinals, define $X_\alpha = \mathrm{hocolim}_{\beta<\alpha} X_\beta$. Then if one chooses an enormous enough cardinal γ, X_γ must be L-local. This is because for decent triangulated, any homomorphism from an object Y to X_γ must factor through a X_β for $\beta < \gamma$. The reason we bring up the proof is because it is so formal that it ought to work in any reasonable triangulated category.

REMARK 4.5. The localization $X \to Y$ of X is unique up to canonical isomorphism in $\mathscr{S}$. Thus it defines a functor $X \mapsto X_L$. It is clear that the localization functor takes triangles to triangles. However, in general it *does not commute* with either products or coproducts.

The localization functor $(\ )_L\colon \mathscr{S} \to \mathscr{S}$ factors through the category $\mathscr{S}/_L$; the objects of L localize to zero. In fact, the localization functor gives an equivalence of categories $\mathscr{S}/_L \to \mathscr{T} \subset \mathscr{S}$, where $\mathscr{T}$ is the full subcategory consisting of L-local objects. By Lemma 1.5 $\mathscr{S}/_L$ has direct sums. This means that $\mathscr{T}$ is a category with direct sums. In general, however, the inclusion $\mathscr{T} \hookrightarrow \mathscr{S}$ *does not respect direct sums.*

DEFINITION 4.6 (Ravenel). The localizing subcategory $L \subset \mathscr{S}$ is called *smashing* if localization commutes with direct sums. Equivalently, L is localizing if the inclusion $\mathscr{T} \hookrightarrow \mathscr{S}$ of Remark 4.5 preserves direct sums. Yet another equivalent formulation is that $\mathscr{S}$-direct sums of objects of $\mathscr{T}$ are in $\mathscr{T}$; i.e. direct sums of L-local objects are local.

EXAMPLE 4.7. Let $\mathscr{S} = K(\ )$ and L be the subcategory of acyclic complexes, as in Example 1.6. Then by Lemma 2.8 any bounded below complex of injectives is L-local. And what Application 2.4′ says is that given $X \in K(A)$ there is an object J in $K(\mathscr{A})$, with $J = \underleftarrow{\mathrm{holim}}\, I_j$, I_j bounded below complexes of injectives, and a quasi-isomorphism $X \to J$. It is clear that $X \to J$ is simply a localization; $J = X_L$. Thus the category $K(I)$ of Proposition 2.12 is nothing more than the category of L-local objects. And Lemma 2.10 can be restated as follows: if L is a localizing category, then the category of L-local objects is closed under $\mathscr{S}$-direct products. The (completely trivial) Lemma 1.5 is the statement that if L is colocalizing (as well as localizing), then the localization functor also commutes with direct products.

The interesting remark is that $L \subset K(\mathscr{A})$ is in general *not* smashing. To see this, observe that under the inclusion $\mathscr{A} \hookrightarrow K(\mathscr{A})$, the objects whose image is local are precisely the injectives in $\mathscr{A}$. Thus a necessary condition for L to be smashing is that arbitrary direct sums of injectives be injective. And, for a general abelian category, this is not true.

5. A comparison of derived categories

Let R be a commutative ring. Let $X = \mathrm{Spec}(R)$ be its spectrum. There is a well known isomorphism of abelian categories:

$$(R\text{-modules}) = (\text{quasi-coherent } O_X \text{ sheaves on } X).$$

Thus we obtain a map of derived categories

$$D(R) = D(\text{Quasi coherent}/_X) \to D_{qc}(X)$$

where $D_{qc}(X)$ is the category of chain complexes of sheaves on X with quasi-coherent cohomology. We will prove:

THEOREM 5.1. *The map $D(R) \to D_{qc}(X)$ is an equivalence of categories.*

LEMMA 5.2. *Let X be a topological space. Let $K(X)$ be the homotopy category of the category of sheaves on X. That is, it is the category of complexes of sheaves, with morphisms the homotopy equivalence classes of chain maps. Then this category contains all small products.*

Proof. The product of sheaves is a sheaf. □

Clearly, the derived category is the quotient of $K(X)$ by the full subcategory $E(X) \subset K(X)$ containing all the complexes with vanishing cohomology. That is, $E(X)$ is the subcategory of all chain complexes of sheaves which are exact. $E(X)$ is a localising subcategory; a coproduct of exact sequences of sheaves is exact. However, it is not in general colocalising. A product of exact sequences of sheaves need not be exact. It follows from the results of Section 1 that the quotient category $D(X)$ admits coproducts, but not necessarily products.

Any bounded below complex i of injective sheaves is $E(X)$ local. That is, for $e \in E(X)$, the group $\mathrm{Hom}(e, i) = 0$. Every object of $K^+(X)$ has an $E(X)$ localisation, in the sense of Bousfield; there is a map $x \to i_x$ from x to a bounded below complex of injectives, where the mapping cone is in $E(X)$. This is classical.

LEMMA 5.3. *Let X be a scheme. Suppose Z is a complex of sheaves on X with quasi-coherent cohomology. Then $Z \in K(X)$ has an $E(X)$ Bousfield localisation.*

Proof. Choose an object Z in $K(X)$, as above. Then write $Z^{\geqslant n}$ for the t-structure truncation of Z above n. Of course, it is not unique as an object of $K(X)$; it is only defined up to quasi-isomorphism, that is as an object of $D(X)$. Nevertheless, we can choose a representative which is a bounded below complex of injectives. Because the bounded below complexes of injectives are $E(X)$-local, these representatives are defined up to canonical isomorphism in $K(X)$, and moreover, in $K(X)$ we get canonically defined maps $Z^{\geqslant n} \to Z^{\geqslant (n+1)}$.

Consider now the map in $K(X)$

$$\prod_n Z^{\geqslant n} \xrightarrow{\text{1-shift}} \prod_n Z^{\geqslant n}$$

We would like to compute the cohomology of the mapping cone on this map. Because products do not preserve exactness in the category of sheaves, this is a computation best done in the category of presheaves.

On any open set $U \subset X$, the ith cohomology group of the complex $Z^{\geqslant n}$ is precisely the ith hypercohomology group $\mathbb{H}^i(U, Z^{\geqslant n})$. But because Z had quasi-coherent cohomology groups, so does $Z^{\geqslant n}$. There is a spectral sequence for the hypercohomology of $\mathbb{H}^{p+q}(U, Z^{\geqslant n})$, whose (p, q) terms are $H^p(U, H^q(Z^{\geqslant n}))$. If U happens to be affine, then $H^q(Z^{\geqslant n}) = 0$ for $q > 0$. This is EGA III, 1.3.1. Thus the spectral sequence must degenerate. It follows that for $n \ll 0$, the map

$$\mathbb{H}^i(U, Z^{\geqslant n}) \to \mathbb{H}^i(U, Z^{\geqslant (n+1)})$$

induces an isomorphism. In other words, the cohomology of $Z^{\geqslant n}$ stabilises not only in the category of sheaves, but also in the category of presheaves. (Almost; on open affines it stabilises).

But then the ith cohomology of the mapping cone C on the map

$$\prod_n Z^{\geqslant n} \xrightarrow{\text{1-shift}} \prod_n Z^{\geqslant n}$$

is a presheaf whose value on any affine open set is just the ith cohomology of Z. It follows immediately that the map from Z to the mapping cone (or rather its desuspension) is a homology isomorphism on affine opens. Since the open affines give a basis for the topology, the map is an isomorphism in the derived category $D(X)$ of the category of *sheaves* on X. But products of $E(X)$-local objects are local, and mapping cones on maps of $E(X)$-local objects are local. This forces C to be local, and $X \to C$ is a Bousfield localisation. □

LEMMA 5.4. *The map $D(R) \to D(X)$ is fully faithful.*

Proof. Let Y and Z be objects of $D(R)$. We need to show that the natural map

$$\mathrm{Hom}_{D(R)}(Y, Z) \to \mathrm{Hom}_{D(X)}(Y, Z)$$

is an isomorphism.

First, by application 2.4′, every object Z of $D(R)$ is quasi-isomorphic to $\underset{\longleftarrow n}{\mathrm{holim}}\, Z^{\geqslant n}$ where $Z^{\geqslant n}$ are bounded below complexes of injectives. There is no problem here, since the category of R-modules has exact products. That is, we get a triangle

$$\Sigma^{-1}\left(\prod_n Z^{\geqslant n}\right) \xrightarrow{\text{1-shift}} \Sigma^{-1}\left(\prod_n Z^{\geqslant n}\right) \to Z \to \prod_n Z^{\geqslant n}$$

What we have just done in the proof of Lemma 5.3 is show that the same can be done in $D(X)$, although some care is in order because products of sheaves do not in general preserve exactness. The precise correct statement is that Z is quasi-isomorphic to $\underset{\longleftarrow n}{\mathrm{holim}}\, Z_l^{\geqslant n}$, where $Z_l^{\geqslant n}$ is the Bousefield localisation of $Z^{\geqslant n}$ in the category $K(X)$. Furthermore, the homotopy limit $\underset{\longleftarrow n}{\mathrm{holim}}\, Z_l^{\geqslant n}$ is itself Bousfield local. Let us call it Z_l, to indicate that it is the Bousefield localisation of Z. We have a triangle in $K(X)$

$$\Sigma^{-1}\left(\prod_n Z_l^{\geqslant n}\right) \xrightarrow{\text{1-shift}} \Sigma^{-1}\left(\prod_n Z_l^{\geqslant n}\right) \to Z_l \to \prod_n Z_l^{\geqslant n}$$

and hence a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\longrightarrow \mathrm{Hom}_{D(R)}\left(Y, \Sigma^{-1}\left(\prod_n Z^{\geqslant n}\right)\right) & \longrightarrow & \mathrm{Hom}_{D(R)}(Y, Z) & \longrightarrow & \mathrm{Hom}_{D(R)}\left(Y, \prod_n Z^{\geqslant n}\right) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\longrightarrow \mathrm{Hom}_{K(X)}\left(Y, \Sigma^{-1}\left(\prod_n Z_l^{\geqslant n}\right)\right) & \longrightarrow & \mathrm{Hom}_{K(X)}(Y, Z_l) & \longrightarrow & \mathrm{Hom}_{K(X)}\left(Y, \prod_n Z_l^{\geqslant n}\right) & \longrightarrow
\end{array} \tag{$*$}$$

Of course, by definition of Bousfield localisations $\mathrm{Hom}_{K(X)}(Y, Z_l) = \mathrm{Hom}_{D(X)}(Y, Z_l) = \mathrm{Hom}_{D(X)}(Y, Z)$. Combining this with the five lemma, it suffices to prove that

$$\mathrm{Hom}_{D(R)}(Y, Z^{\geqslant n}) \to \mathrm{Hom}_{K(X)}(Y, Z_l^{\geqslant n}) = \mathrm{Hom}_{D(X)}(Y, Z^{\geqslant n})$$

is an isomorphism for each n. That is, we may assume Z is bounded below.

Similarly, we may assume Y is bounded above. This argument is slightly easier, since $D(X)$ has coproducts (the category of sheaves satisfies $AB4$). Now Y is quasi-isomorphic to a bounded above complex of free modules, and Y is therefore the direct limit of its finite subcomplexes; using essentially the same argument as before (but for arbitrary hocolims; see the discussion in Section 4) we deduce that it suffices to consider Y a *finite* complex of free R-modules. Y can be built up successively as a finite extension of suspensions of R; i.e. it suffices to consider the case $Y = R$.

Thus we are reduced to showing that the map

$$\mathrm{Hom}_{D(R)}(R, Z) \to \mathrm{Hom}_{D(X)}(R, Z)$$

is an isomorphism, where Z is bounded below.

Now Z can be devisaged. For each n, we have a triangle

$$Z^{\geqslant n} \to Z^{\geqslant n-1} \to \Sigma^{-n+1} M \to \Sigma Z^{\geqslant n}$$

where M is an ordinary R module (viewed as a complex concentrated in degree zero). If $n > 0$, then

$$\mathrm{Hom}_{D(R)}(R, Z^{\geqslant n}) = 0 = \mathrm{Hom}_{D(X)}(R, Z^{\geqslant n})$$

whereas if $n \ll 0$, $Z^{\geqslant n} = Z$.

Thus it suffices to show that for every R-modules M and for every n,

$$\mathrm{Hom}_{D(R)}(R, \Sigma^n M) = \mathrm{Hom}_{D(X)}(R, \Sigma^n M).$$

If $n < 0$, both sides are clearly 0. If $n = 0$, both sides are M. If $n > 0$, the left hand side is $\mathrm{Ext}_R(R, M) = 0$, whereas the right hand side is nothing other than $H^n(X, M)$, which is zero by EGA III, 1.3.1. $\square$

Proof of Theorem 5.1. We know therefore that the functor $F: D(R) \to D_{qc}(X)$ is fully faithful. It remains to show that every object in $D_{qc}(X)$ is isomorphic to an object in the image of F.

STEP 1. Every object in $D^b_{qc}(X)$ is in the image of F. Proof is by induction on the length of the complex. Clearly this is true for complexes of length 0. Suppose it is true for complexes of length $\leqslant n$. Pick a complex Y in $D^b_{qc}(X)$ of length $n + 1$, and for definiteness suppose $H^i(Y) = 0$ unless $0 \leqslant i \leqslant n + 1$. Then there is a triangle

$$Y^{\geqslant n} \to Y \to Y^{\geqslant n+1} \xrightarrow{\partial} \Sigma Y^{\geqslant n}.$$

By induction, $Y^{\leqslant n+1}$ and $Y^{\geqslant n}$ are in the image of $D^b(R)$. Because F is fully faithful, the map $\partial: Y^{\geqslant n+1} \to \Sigma Y^{\leqslant n}$ is also in $D^b(R)$. Therefore so is Y, which is the third edge of the triangle on ∂.

STEP 2. Every object of $D^+_{qc}(X)$ is in the image of F. For if Z is a bounded below complex of sheaves with quasi-coherent cohomology, then $Z = \underset{\overrightarrow{m}}{\text{hocolim}}\, Z^{\leqslant m}$ expresses Z as a hocolim of objects in $D^b_{qc}(X) = D^b(R)$. Because the functor $D(R) \to D(X)$ respects coproducts, it respects homotopy colimits and we may already form the homotopy colimit in $D(R)$. This shows that Z is isomorphic to something in the image of $D(R)$.

STEP 3. Every object of $D_{qc}(X)$ is in the image of F. Given Z an object of $D_{qc}(X)$, we would like to say that Z is a holimit of bounded below objects. Because the category of sheaves does not have exact products, some care is needed. What is true is

$$Z_l = \underset{\overleftarrow{n}}{\text{holim}}\, Z_l^{\geqslant -n}$$

expresses Z_l, the Bousfield localisation of Z, as a holim in the category $K(X)$. But each $Z_l^{\geqslant -n}$ is isomorphic to an object in $D^+(R)$, call it Y^{-n}. Of course, because $Z_l^{\geqslant -n}$ is Bousfield local, the isomorphisms $Y^{-n} \simeq Z_l^{\geqslant -n}$, which begins its life as a map in $D^+_{qc}(X)$, is actually given by a map (unique up to homotopy) in $K(X)$. Furthermore, if we choose the Y^{-n} to be complexes of injective R-modules, that is Bousfield local objects for the projection $K(R) \to D(R)$, then the maps $Y^{-n} \to Y^{-n+1}$ in $D(R)$ which are guaranteed to exist because the map $D(R) \to D_{qc}(X)$ is fully faithful, are also liftable to $K(R)$. Let Y be the homotopy limit, in $K(R)$, of the sequence Y^{-n}. Then we get a morphism of triangles in $K(X)$

$$\begin{tikzcd}
\Sigma^{-1}\left(\prod_n Y^{-n}\right) \arrow[r, "\text{1-shift}"] \arrow[d, "\Sigma^{-1}\alpha"'] & \Sigma^{-1}\left(\prod_n Y^{-n}\right) \arrow[r] \arrow[d, "\Sigma^{-1}\alpha"'] & Y \arrow[r] \arrow[d, "\beta"'] & \prod_n Y^{-n} \arrow[d, "\alpha"'] \\
\Sigma^{-1}\left(\prod_n Z_l^{\geqslant -n}\right) \arrow[r, "\text{1-shift}"] & \Sigma^{-1}\left(\prod_n Z_l^{\geqslant -n}\right) \arrow[r] & Z_l \arrow[r] & \prod_n Z_l^{\geqslant -n}
\end{tikzcd}$$

In the proof of Lemma 5.3 we computed the cohomology of Z_l. We showed that the cohomology of $Z_l^{\geqslant -n}$ stabilises, even (almost) as a presheaf, and the cohomology of Z_l is this stable value. But then it is easy to compute from the above that the map $Y \to Z_l$ is a homology isomorphism. Thus Z is isomorphic in $D_{qc}(X)$ to Y, which is in the image of $D(R)$. □

COROLLARY 5.5. If X is an arbitrary scheme, there is a natural functor

$$F: D(qc/_X) \to D_{qc}(X)$$

where $D(qc/X)$ is the derived category of chain complexes of quasi-coherent sheaves, while $D_{qc}(X)$ is, as above, the full subcategory of $D(X)$ whose objects have coherent cohomology. If X is quasi-compact and separated, F is an equivalence of categories.

REMARK 5.6. The proof is an easy induction on the number of open affines in a cover for X, but we delay it till the end of Section 6 where it will be even more trivial. Traditionally such statements are proved by showing that the category of quasi-coherent O_X-sheaves has enough objects which are injective even in the category of all O_X-sheaves. This involves one in studying injective enevelopes, and one traditionally needs to assume X to be Noetherian. The point here is that the statement is very general, and the proof completely trivial.

6. Some algebraic examples

In this section we will attempt to show that Bousfield localization, and that Ravenel's notion of smashing subcategories, are really very natural notions which have a history going back to Grothendieck's work on local cohomology. We will not attempt any generality; instead, we will treat the simplest case, where our point can be made most clearly. Let R be a commutative ring, $X = \mathrm{Spec}(R)$. Let $Y \subset X$ be a closed subset, $U = X - Y$ its complement. We denote the inclusions by $j: U \hookrightarrow X$, $i: Y \to X$. Then for any object $A \in D(X)$, there is a triangle:

$$i_* i^!(A) \to A \to j_* j^*(A) \to \Sigma i_* i^!(A).$$

The functor $i_* i^!$ is well known as Grothendieck's local cohomology functor, and the triangle above has extensively been studied in the literature. We will now see how to view it as a Bousfield localization.

Let $L \subset D(X)$ be the category of all sheaves of O_X-modules on X whose support is contained in $Y \subset X$. It is well known that $i_*: D_Y(X) \to D(X)$ is an isomorphism of $D_Y(X)$ with L. Also, i_* commutes with products and coproducts, and hence L is localizing and colocalizing.

Clearly, any object of the form $j_* B$ is L-local; this is because:

$$\mathrm{Hom}(i_* A, j_* B) = \mathrm{Hom}(j^* i_* A, B) = 0.$$

Thus in the triangle

$$i_*i^!(A) \to A \to j_*j^*A \to \Sigma i_*i^!(A)$$

the term $i_*i^!(A)$ is in L, while $j_*j^*(A)$ is L-local. Thus the map $A \to j_*j^*(A)$ is nothing more than a Bousfield localization with respect to $L = i_*D_Y(X)$.

When is the localization smashing? Clearly, j^* commutes with direct sums, in fact j^* has a right adjoint. Thus the reader will easily see that j_*j^* commutes with direct sums if and only if j_* does. If U is quasi-compact, j_* always commutes with direct sums. For general rings R, the compactness of U probably is not necessary. However, for many interesting R it is. From here on, we assume U is quasi-compact.

Of course, U is a union of affine open subsets. If U is quasi-compact, this union is finite; i.e. $U = \cup_{i=1}^n \operatorname{Spec}(R[1/f_i])$. This is an especially happy case, because then the functor $j_*j^*: D(X) \to D(X)$ takes $D_{qc}(X)$ into itself. Because $D_{qc}(X) = D(R)$ (Theorem 5.1), we need only show that for $X \in D(R)$ $j_*j^*(X)$ is in $D_{qc}(X)$. But every object of $D(R)$ is a direct limit of finite complexes of free R-modules. As j_*j^* commutes with direct limits, it suffices to show that $j_*j^*(X)$ is in $D_{qc}(X)$ when $X \in D^b(R)$. By devisage, it suffices to consider the case $X = R$, but then $j_*j^*(R)$ is the complex

$$0 \to \prod_{i=1}^n R\left[\frac{1}{f_i}\right] \to \prod_{1 \leq i < j \leq n} R\left[\frac{1}{f_i f_j}\right] \to \cdots$$

is clearly in $D(R) = D_{qc}(X)$.

Let us observe the easy fact:

PROPOSITION 6.1. Let $U = \bigcup_{i=1}^n X_{f_i}$. Then $A \in D_{qc}(X)$ is local if and only if, for all n,

$$\operatorname{Hom}(\Sigma^n B, A) = 0$$

where B is the complex

$$B = \bigotimes_{i=1}^n \left(R \xrightarrow{f_i} R\right).$$

Proof. $\operatorname{Hom}(\Sigma^n B, A) = H^n \operatorname{RHom}(B, A)$, and it therefore suffices to prove that $\operatorname{RHom}(B, A)$ is zero if and only if A is local.

STEP 1. If A is local, then $\operatorname{RHom}(B, A) = 0$. This is because A is local $\Leftrightarrow A = j_*C$ for some C in $D(U)$, and

$$\mathrm{RHom}(B, j_*C) = \mathrm{RHom}(j^*B, C) = 0$$

as $j^*B = 0$.

STEP 2. If $j^*A = 0$, then $\mathrm{RHom}(B, A) = 0$ if and only if $A = 0$.

Proof. Because $j^*A = 0$, A is a complex whose cohomology is $(f_1, \dots, f_n)$-torsion. $\mathrm{RHom}(B, A) = B \otimes A$ because B is self-dual. Because $B = \otimes_{i=1}^n (R \xrightarrow{f_i} R)$, it suffices to prove by induction the case where $n = 1$; if A is an object in $D(R)$ with f-torsion homology, and $(R \xrightarrow{f} R) \otimes A = 0$, then $A = 0$. But if $(R \xrightarrow{f} R) \otimes A = 0$, then multiplication by f is an isomorphism on the cohomology of A, which is f-torsion. Thus A has vanishing cohomology, hence $A = 0$.

STEP 3. (end of proof of Proposition 6.1). Let $i_*i^!(A) \to A \to j_*j^*(A) \to \Sigma i_*i^!(A)$ be the localization triangle. Applying $\mathrm{RHom}(B, -)$ to this triangle, we deduce by Step 1 an isomorphism

$$\mathrm{RHom}(B, i_*i^!(A)) \xrightarrow{\sim} \mathrm{RHom}(B, A).$$

By Step 2, $i_*i^!(A)$ is zero if and only if $\mathrm{RHom}(B, i_*i^!(A)) = 0$. But $i_*i^!(A) = 0$ iff A is local; thus the proof is complete. □

Next we define:

DEFINITION 6.2. Let $\mathscr{T}$ be a triangulated category. The *bounded part* $\mathscr{T}^b$ of $\mathscr{T}$ is defined to be the full subcategory of objects $X \in \mathscr{T}$ such that for all families $\{Y_\lambda, \lambda \in \wedge\}$ of objects in $\mathscr{T}$, the morphism

$$\bigoplus_\lambda \mathrm{Hom}(X, Y_\lambda) \to \mathrm{Hom}(X, \oplus Y_\lambda)$$

is an isomorphism.

PROPERTY 6.3. $\mathscr{T}^b$ is an épaisse subcategory of $\mathscr{T}$.

Proof. Clear.

PROPOSITION 6.4. *If* $= D(R)$, *then* $\mathscr{T}^b = D^b(R)$.

Proof. Every object $X \in D(R)$ is a direct limit of finite subcomplexes. Using Section 4, we see that if $X \in \mathscr{T}^b$, then $\mathrm{Hom}(X, X) = \mathrm{Hom}(X, \underrightarrow{\mathrm{holim}}\, Y_\alpha)$ must factor through Y_α for some α. Thus X is a retract of a finite chain complex Y_α. By Proposition 3.4, $X \in D^b(R)$.

Now we are ready for the smashing conjecture:

CONJECTURE 6.5. Let $\mathscr{T}$ be a triangulated category with direct sums. Let

$L \subset \mathcal{T}$ be a localizing category. Then L is smashing if and only if L is generated by $L \cap \mathcal{T}^b$.

REMARK 6.6. This conjecture seems to be due to Ravenel (where $\mathcal{T}$ is the category of spectra). What we tried to show here is that it is also natural and interesting in the algebraic case; Proposition 6.1 should be viewed as evidence for the conjecture.

The conjecture, as stated, is probably false. One probably needs to make added hypotheses on $\mathcal{T}$; for instance, that every object of $\mathcal{T}$ is a direct limit of objects of $\mathcal{T}^b$.

Before we end the section, we owe the reader a proof of Corollary 5.5. We remind the reader:

6.7 (Corollary 5.5). Let X be a quasi-compact, separated scheme. The natural forgetful functor

$$F: D(qc/_X) \to D_{qc}(X)$$

is an equivalence of categories.

Proof. Suppose $X = \bigcup_{i=1}^n U_i$, where the U_i's are open affines. The proof is by induction on n. Theorem 5.1 is the statement for $n = 1$, and we will assume we know it for $n - 1$.

Then $X = U \cup V$ where $U = U_1$ and $V = \bigcup_{i \geqslant 2} U_i$. Let $j_1: U \to X, j_2: V \to X$ be the inclusions. Let $Y = X - U$, and $i: Y \to X$ be the inclusion.

For any object $\mathcal{S} \in D_{qc}(X)$, there is a triangle

$$i_* i^! \mathcal{S} \to \to (j_{1*}) j_1^* \mathcal{S} \to \Sigma i_* i^! \mathcal{S}.$$

But $j_1^* \mathcal{S}$ is a sheaf on U, hence in $D(qc/_U)$. Because X is separated, $(j_1)_* j_1^* \mathcal{S}$ is in $D(qc/_X)$. Similarly, $i_* i^! \mathcal{S}$ is $(j_2)_* \mathcal{S}$ of an object in $Dqc(V) = D(qc/_V)$ the equality being by the induction hypothesis. Finally, the map $(j_1)_* j_1^* \mathcal{S} \to \Sigma i_* i^! \mathcal{S}$ is also really $(j_2)_*$ of a morphism $j_2^*(j_1)_* j_1^* \mathcal{S} \to j_2^* \Sigma i_* i^! \mathcal{S}$. It immediately follows that it is the image of a map in $D(qc/_X)$. Therefore $\mathcal{S}$, being the third edge of a triangle on a morphism in $D(qc/_X)$, is in $D(qc/_X)$.

This shows that the functor F is surjective on objects. The proof that it is fully faithful is very similar, and is left to the reader. $\square$

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