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Hyperhopfian groups and approximate fibrations

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Continuing an investigation launched in [D1], this paper presents a setting in which a proper map defined on an arbitrary manifold of a given dimension can be quickly recognized as an approximate fibration, by virtue of having all point preimages of a certain fixed homotopy type. More precisely, the aim is to describe closed n-manifolds N which force maps $p: M \to B$ to be approximate fibrations, when M is an $(n+2)$-manifold and each $p^{-1}b$ has the homotopy type (or, shape) of N. The results center upon an algebraic property pertaining to $\pi_1(N)$, the hyperhopfian condition appearing in the title, which typically is sufficient (but not necessary) for achieving the desired conclusion. We call a finitely presented group G *hyperhopfian* if every homomorphism $\psi: G \to G$ with $\psi(G)$ normal and $G/\psi(G)$ cyclic is necessarily an automorphism.

The standard setting involves several notational items employed throughout: a specific closed n-manifold N; an $(n+k)$-manifold M, with $k = 2$ the nearly universal rule; a usc (i.e., upper semicontinuous) decomposition $\mathscr{G}$ of M into copies of N up to (shape) homotopy equivalence; the associated decomposition space $B = M/\mathscr{G}$, and the usual decomposition map $p: M \to B$. Equivalently, for such N and M, $p: M \to B$ is a proper, closed, surjective mapping and each $p^{-1}b$ is (shape) homotopy equivalent to N. When $k = 2$, B is known to be a 2-dimensional manifold [D-W]; in any event, B is taken to be finite-dimensional. The issue to be addressed is the following:

MAIN QUESTION. Under what conditions on N is $p: M \to B$ an approximate fibration?

If it is, as a payoff Coram and Duvall [C-D1, Cor. 3.5] have an exact sequence relating the various homotopy groups of N, M and B, analogous to the one for genuine fibrations. The sequence provides the most efficient means available for extracting structural information about M from that of N and B.

Strictly speaking, one says a closed n-manifold N is a *codimension k fibrator* (respectively, a *codimension k orientable fibrator*) if whenever $\mathscr{G}$ is a usc decomposition of an arbitrary (respectively, orientable) $(n+k)$-manifold M such that each $g \in \mathscr{G}$ is shape equivalent to N, then $p: M \to B$ is an approximate fibration. However, we limit the focus here to the doubly orientable situation, requiring it of both M and N, and drop the word "orientable" from the terminology.

There are two prominent topics here: information concerning hyperhopfian groups in section 4 and applications to codimension 2 fibrators in section 5. The information developed about finite groups includes specific data and, more generally, lists certain collections in which hyperhopficity is recognized by the absence of an obvious obstruction: cyclic direct factors. Among infinite groups, except for the anomalous $Z_2 * Z_2$, a hopfian group is hyperhopfian if it is a nontrivial free product of finitely presented, residually finite groups (Theorem 4.11) or if it has a presentation with at least two more generators than relations (Theorem 4.8). Proofs of these algebraic statements entail topological considerations. The main result, Theorem 5.4, assures that a closed n-manifold N with hyperhopfian fundamental group is a codimension 2 fibrator, provided every degree one map $N \to N$ inducing an automorphism of $\pi_1(N)$ is a homotopy equivalence. In dimension 3 this means that by and large all nontrivial connected sums of 3-manifolds, excluding those homotopy equivalent to $RP^3 \# RP^3$, are codimension 2 fibrators. At the other extreme, a 3-manifold with finite fundamental group Γ is a codimension 2 fibrator if Γ has no cyclic direct factor. In higher dimensions the hyperhopfian property holds a less exalted position, for ordinarily a closed manifold is a codimension 2 fibrator if it has non-zero Euler characteristic (Theorem 5.10), where the implicit restriction is again to those manifolds N such that degree one mappings $N \to N$ are homotopy equivalences. For instance, a closed 4-manifold N is such a fibrator if $\pi_1(N)$ is hopfian and $H_1(N)$ is finite (Corollary 5.11).

Basic connections between the algebra and the geometry of the setting appear in Lemmas 5.1 and 5.2. If the manifold N generating the decomposition $\mathscr{G}$ has hyperhopfian fundamental group, then any retraction $R: U \to g_0 \in \mathscr{G}$ defined on a neighborhood of g_0 in M restricts to a degree one map $g \to g_0$ and induces an isomorphism $\pi_1(g) \to \pi_1(g_0)$, for all $g \in \mathscr{G}$ sufficiently close to g_0.

By way of preliminaries, section 2 contrasts older hopfian properties with the hyperhopfian one; section 3 sets forth a reminder of why manifolds that cover themselves in regular, cyclic fashion cannot be codimension 2 fibrators and outlines a new method, due to F. C. Tinsley, for constructing such things.

Originally the plan for this paper was to provide a classification of codimension 2 fibrators among aspherical 3-manifolds, in terms of geometric structures. Among surfaces geometry neatly regulates the fibrators; among 3-manifolds geometry's governance is somewhat fuzzier but the effects still interesting. However, as work progressed, new directions evolved and the results became increasingly independent of dimension. Consequently, we treat aspherical 3-manifolds, which demand markedly different techniques, in another place [D3].

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1. Definitions

All manifolds are understood to be connected, metric and boundaryless. Whenever the presence of boundary is tolerated, the object will be called a manifold with boundary.

Homology and cohomology groups are computed with integer coefficients. The *degree* of a map $f: N \to N$, where N is a closed, connected (orientable) n-manifold, sometimes called the absolute degree, for emphasis, is the nonnegative integer d such that the induced endomorphism of $H_n(N) \cong Z$ amounts to multiplication by d, up to sign.

The symbol χ is used to denote Euler characteristic.

Let N be a closed n-manifold. A usc decomposition $\mathscr{E}$ of a manifold M is N-*like* if $\dim M/\mathscr{E} < \infty$ and each $g \in \mathscr{E}$ is shape equivalent to N. For simplicity or familiarity, we shall assume each $g \in \mathscr{E}$ in an N-like decomposition to be an ANR having the homotopy type of N; experts can easily adapt the proofs to the more general situation.

A proper map $p: M \to B$ between locally compact ANR's is called an *approximate fibration* if it has the following approximate homotopy lifting property: given an open cover Ω of B, an arbitrary space X, and two maps $f: X \to M$ and $F: X \times I \to B$ such that $pf = F_0$, there exists a map $F': X \times I \to M$ such that $F'_0 = f$ and pF' is Ω-close to F. The latter means: for each $z \in X \times I$ there exists $U_z \in \Omega$ such that $\{F(z), pF'(z)\} \subset U_z$.

The *continuity set* of $p: M \to B$, usually denoted as C, consists of all points $x \in B$ such that, under any retraction $R: p^{-1}U \to p^{-1}x$ defined over a neighborhood $U \subset B$ of x, x has another neighborhood $V_x \subset U$ such that $R | p^{-1}b: p^{-1}b \to p^{-1}x$ is a degree one map for all $b \in V_x$. By way of explanation for the terminology, such neighborhoods V_x form the domains of local winding functions α_x to the nonnegative integers, where $\alpha_x(b)$ is defined to be the degree of $R | p^{-1}b$, and C then equals the set of points x for which α_x is continuous at x.

2. Consequences of hopfian properties in fundamental groups

Recall that a group G is *hopfian* if every epimorphism $\Psi: G \to G$ is an isomorphism, while it is *cohopfian* if every injection $\Phi: G \to G$ is an isomorphism. One significant aspect of cohopficity is: no path-connected and locally simply connected space with cohopfian fundamental group can be properly covered by itself.

Repeating the new term for contrast, we say a finitely presented group G is *hyperhopfian* if every homomorphism $\psi: G \to G$ with $\psi(G)$ normal and $G/\psi(G)$ cyclic is an isomorphism (onto). Hyperhopfian groups G submit to a key restraint: if $\mu: G \to \Gamma$ is an injection having normal image with $\Gamma/\mu(G)$ cyclic and if $\theta: \Gamma \to G$ is an epimorphism, then $\theta\mu: G \to G$ is an automorphism. Finitely presented simple groups are hyperhopfian, obviously, as are the fundamental groups of all compact surfaces with negative Euler characteristic (a class which includes all finitely generated nonabelian free groups). On the other hand, no group which splits off a cyclic direct factor has this property. Also, hyperhopfian groups are hopfian, by definition, but they are only partially cohopfian. The related topological feature clearly exposes the limited cohopfian phenomenon: no path-connected and locally simply connected space with hyperhopfian fundamental groups can be a proper, regular, cyclic cover of itself.

Call a closed manifold N *hopfian* if it is orientable and every degree one map $N \to N$ is a homotopy equivalence. The notion aids in efficiently identifying approximate fibrations. Whether $\pi_1(N)$ a hopfian group necessarily makes N a hopfian manifold is part of a significant, old unsolved problem, due to Hopf and recently reexamined by Hausmann [Ha].

THEOREM 2.1. *If N is a closed hopfian manifold and $\mathscr{E}$ is an N-like decomposition of an orientable* $(n+k)$-manifold M, then $p: M \to B$ *is an approximate fibration over its continuity set.*

Proof. This follows immediately from the definition of hopfian manifold and from Coram and Duvall's characterization [C-D2] of approximate fibrations in terms of movability properties.

THEOREM 2.2. *A closed, orientable n-manifold N is a hopfian manifold if any one of the following conditions holds*:

(1) $n \leqslant 4$ *and* $\pi_1(N)$ *is hopfian*;
(2) $\pi_1(N)$ *is hopfian and its integral group ring,* $Z\pi_1(N)$, *is Noetherian*;
(3) $\pi_1(N)$ *contains a nilpotent subgroup of finite index*; *or*
(4) $\pi_i(N)$ *is trivial,* $1 < i < n-1$, *and* $\pi_1(N)$ *is hopfian.*

ADDENDUM. *A* 3-*manifold N is hopfian if its prime factors* (*in a connected sum decomposition*) *are either virtually Haken or have finite or cyclic fundamental group.*

Proof. That each of the first three conditions implies N hopfian was shown by Hausmann [Ha]. That the fourth has the same effect was derived by Swarup [S2, Lemma 1.1], who also took care of the 3-dimensional version of (1) in an earlier paper [S1]. The addendum follows because in this class, which conceivably includes all closed 3-manifolds, fundamental groups are residually finite and therefore are hopfian [He, Theorem 1.1] (also see [He] for terminology).

The work of both Hausmann and Swarup greatly influenced the development of the results here.

3. Manifolds that regularly, cyclically cover themselves.

Manifolds that regularly, cyclically cover themselves include those with circle factors, those that fiber over S^1 with periodic monodromy, doubles of twisted I-bundles, and things arising from a construction related to me by F. C. Tinsley.

The key Orbit Space Construction of [D1], outlined in this paragraph, reveals why such manifolds fail to be codimension 2 fibrators. Let Λ be a finite cyclic group of order $k > 1$ acting freely on a given closed (connected, orientable) n-manifold N^n. Determine a semifree Λ-action of rotations on the plane, E^2, fixing the origin. Let M^{n+2} denote the orbit space of $N^n \times E^2$ resulting from the free diagonal Λ-action, and let $p: M^{n+2} \to B$ denote the quotient map with point inverses equal to the various images of $N^n \times \{\text{point}\}$ in the orbit space. In any case $p: M^{n+2} \to B$ fails to be an approximate fibration, as decomposition elements near the image N_0 of $N^n \times 0$ in M^{n+2} retract to N_0 via a degree k map (factoring through the covering $N^n \to N^n/\Lambda$). When N^n/Λ is homeomorphic to N^n, this construction yields an N^n-like decomposition of M^{n+2}.

The Tinsley Construction works for any finitely presented non-cohopfian group G with trivial Whitehead group such that G is the fundamental group of an aspherical finite complex K. Let $K' \to K$ denote a covering determined by a proper subgroup of $\pi_1(K)$ isomorphic to G. By asphericity, K, K' are homotopy equivalent, and the Whitehead group hypothesis translates to the existence of a new complex L collapsing to both K' and K. Embed L in a high-dimensional Euclidean space, and consider the boundary M of a regular neighborhood. Then $\pi_1(M) \cong G$ (provided $\dim M - \dim L > 1$) and M bounds a regular neighborhood of both K' and K, from which it follows that, in general, M nontrivially covers itself. A relevant example is

$$G = \langle x, y \mid y^{-1}x^{-1}yx = x^m \rangle \qquad (m \geqslant 2);$$

the subgroup generated by x, y^m has a presentation identical to the above, and Waldhausen's result [W] about torsion-free 1-relator groups gives the triviality of $Wh(G)$. This specific example G fails to be hyperhopfian, and the manifold M determined through the Tinsley Construction covers itself regularly and cyclically.

4. Conditions implying a group is hyperhopfian

The results of this section afford easy detection of hyperhopfian groups amid certain classes, initially among the finite groups, and later among the infinite ones. While much of the first part is a bit specialized, it fits together neatly with the conclusion (Theorem 4.7) that a finite fundamental group of a closed 3-manifold is hyperhopfian if and only if it has no cyclic direct factor. In the infinite case, we show that a nontrivial free product of finitely presented, residually finite groups is hyperhopfian unless two groups are involved, both cyclic of order 2 (Theorem 4.11), and that every hopfian group with $s > t + 1$ generators and t relations is hyperhopfian (Theorem 4.8).

Since finitely generated (nontrivial) Abelian groups are never hyperhopfian, commutators play a role. We denote the commutator subgroup of a given group G as G' or, occasionally, as $[G, G]$. Stated below are relevant elementary features.

LEMMA 4.1. *If $\psi: G \to G$ is a homomorphism with normal image such that $G/\psi(G)$ is cyclic, then $\psi(G) \supset G'$ and the projection $G \to G/\psi(G)$ induces a epimorphism $G/G' \to G/\psi(G)$.*

REMARK 4.2. Every hopfian perfect group is hyperhopfian. (Recall that a group G is *perfect* if $G = G'$.) So are the generalized dihedral groups $D'(2^k, 2n+1)$ of order $2^k \cdot (2n+1)$ $(k > 0)$, where

$$D'(2^k, 2n+1) = \langle x, y \mid x^{2^k} = y^{2n+1} = 1, x^{-1}yx = y^{-1} \rangle.$$

Referring to such a group just as Γ for the moment, we see that Γ' is generated by y, any proper subgroup $H \supset \Gamma'$ of Γ is Abelian, and $|H|$ is a multiple of $2n + 1$. There can be no epimorphism $\psi: \Gamma \to H$, as it would induce an epimorphism of Γ/Γ', the cyclic group of order 2^k, to H.

The quaternionic group $Q = \langle c, d \mid c^2 = (cd)^2 = d^2 \rangle$ of order 8 has $Q' = \{1, c^2\}$ and $Q/Q' \cong Z_2 \oplus Z_2$; since the proper subgroups of Q containing Q' with cyclic quotient (generated by c, d, cd, respectively) are all cyclic of order 4, Lemma 4.1 implies Q is hyperhopfian. More generally, the same is true of the groups $T(8, 3^k) = T$ of order $8 \cdot 3^k$, $k > 0$, where

$$T = \langle x, y, z \mid x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^k} = 1 \rangle.$$

Here T' is the quaternionic group Q generated by x and y, and its abelianization is cyclic of order 3^k. There can be no epimorphism $T \to T' = Q$, because there is none between their abelianizations. As z^3 belongs to the center of T, any proper subgroup H of T containing T' has a direct product representation $H = Q \times \Lambda$, for some subgroup Λ. Consequently, there is no epimorphism $T \to H$, for it would give rise to one from T to Q. In particular, $T(8, 3)$ happens to be the binary tetrahedral group

$$T^* = \langle x, y \mid x^2 = (xy)^3 = y^3, x^4 = 1 \rangle,$$

although exhibition of an isomorphism is probably harder than direct verification that T^* is hyperhopfian.

THEOREM 4.3. *Suppose G is a finite group such that G/G' has square-free order relatively prime to that of G'. Then G is hyperhopfian if and only if G has no cyclic direct factor.*

Proof. One direction is obvious. For the other, suppose $\psi: G \to G$ is a homomorphism with $\psi(G)$ normal and $G/\psi(G)$ cyclic of order k. Then $\psi(G) \supset G'$ and k divides $|G/G'|$. Moreover, the representation of G as an extension

$$1 \to \ker \psi \to G \to \psi(G) \to 1$$

shows $|G| = |\ker \psi| \cdot |\psi(G)|$. Since $|\ker \psi| = k$ is relatively prime to $|\psi(G)|$, $\psi(G) \cap \ker \psi = \{1\}$. This shows G to be the direct product $\psi(G) \times \ker \psi$. Finally, the projection $G \to G/\psi(G)$ restricts to an isomorphism $\ker \psi \to G/\psi(G)$, indicating G has $\ker \psi$ as a cyclic direct factor.

COROLLARY 4.4. *A finite group G of square-free order is hyperhopfian if and only if G has no cyclic direct factor. In particular, every nonabelian group of order pq, where p and q are distinct primes, is hyperhopfian.*

EXAMPLES. For p prime, the solvable group of order p^4

$$\langle x, y \mid x^{p^2} = y^{p^2} = 1, y^{-1}xy = x^{1+p} \rangle$$

fails to be hyperhopfian but has no nontrivial factorization as a direct product. The same statement holds for the group of order $4n^2$, $n > 1$, having presentation

$$\langle a, b \mid a^n = b^n = 1, b^{-1}ab = a^{-1} \rangle.$$

LEMMA 4.5. *Let G be a group with G' hyperhopfian and $G/G' \cong Z_2$. Then either $G \cong G' \times Z_2$ and G' is perfect or G is hyperhopfian.*

Proof. If there exists an epimorphism $\psi: G \to G'$, then $[G':\psi(G')] \leqslant 2$ and G' hyperhopfian imply $\psi \mid G': G' \to G'$ is an automorphism. This gives $G = G' \times \ker \psi$ and G' perfect. In light of Lemma 4.1, the only other possibility is for G to be hyperhopfian.

REMARK 4.6. The binary octahedral group

$$O^* = \langle x, y \mid x^2 = (yx)^3 = y^4, x^4 = 1 \rangle$$

of order 48 is hyperhopfian, since $O^*/[O^*, O^*] \cong Z_2$ and $[O^*, O^*]$ is the nonperfect, hyperhopfian group

$$\langle a, b \mid a^2 = (ab)^3 = b^3, a^4 = 1\rangle \cong T^*,$$

formed by $a = y^2$ and $b = y^{-1}x$.

In a similar but more complicated vein, one can verify hyperhopficity of the dihedral group

$$D^*_{4m} = \langle x, y \mid x^2 = (xy)^2 = y^m\rangle.$$

When $m = 2n + 1$, this group coincides with $D'(4, m)$; when $m = 2k$ the argument that $D = D^*_{4m}$ is hyperhopfian proceeds by induction on k, with the initial step $k = 1$ completed previously, since D then equals the group Q of Remark 4.2. Generally, the abelianization of D is $Z_2 \oplus Z_2$, with $[D, D]$ generated by $\{x^2, y^2\}$. The three subgroups of D determining cyclic quotients are generated by $\{x^2, y\}$, $\{x, y^2\}$, $\{xy, y^2\}$, respectively, the first of which is isomorphic to $Z_2 \oplus Z_m$, while the other two have presentation:

$$D^*_{2m} = \langle c, y \mid c^2 = (cy)^2 = y^{m/2}\rangle,$$

where c stands for either x or xy. There is no epimorphism of D onto the former, by application of Lemma 4.1, nor to the latter, by induction and the argument of Lemma 4.5.

A similar analysis can be used to check that for pairwise relatively prime integers $8n$, k, l, the groups

$$Q(8n, k, l) = \langle x, y, z \mid x^2 = (xy)^2 = y^{2n}, z^{kl} = 1, xzx^{-1} = z^{-1}, yzy^{-1} = z^{-1}\rangle,$$

where $r \equiv -1 \pmod{k}$ and $r \equiv +1 \pmod{l}$, are hyperhopfian.

THEOREM 4.7. *A finite group Γ isomorphic to the fundamental group of a closed 3-manifold is hyperhopfian if and only if Γ has no cyclic direct factor.*

Proof. Milnor [Mil] provides a list of all finite groups potentially the fundamental group of a closed 3-manifold, and Lee [L] updated the list to exclude two of Milnor's classes. Remarks 4.2 and 4.6 point out that all remaining groups of this type with no cyclic direct factor are hyperhopfian.

At this point the subject shifts to infinite groups.

THEOREM 4.8. *Suppose the group G has a presentation consisting of s generators and t relations, $s > t + 1$. Then G is hyperhopfian if and only if G is hopfian.*

Proof. To address the nontrivial implication, construct a finite CW complex

K having 1 vertex, s edges, t 2-cells and $\pi_1(K) \cong G$. Given a subgroup H with $[G:H] = k < \infty$, form the covering $p: K' \to K$ corresponding to H. Then

$$\chi(K') = 1 - \beta_1(K') + \beta_2(K') = k(1 - s + t)$$

so

$$\begin{aligned} \beta_1(K') &= k(s - t - 1) + 1 + \beta_2(K') \\ &\geqslant k(s - t - 1) + 1. \end{aligned}$$

When $k \geqslant s$ we have $\beta_1(K') > s \geqslant \beta_1(K)$, which precludes the existence of any epimorphism $G \to H$, as the induced homomorphism on abelianizations would yield $\beta_1(K) \geqslant \beta_1(K')$. We reach the same end for arbitrary $k > 1$ by showing next that $\beta_2(K') \geqslant \beta_2(K)$, for then

$$\beta_1(K') = k(s - t - 1) + 1 + \beta_2(K') > (s - t - 1) + 1 + \beta_2(K) = \beta_1(K).$$

Order the 2-cells $D_1, \ldots, D_t$ of K. Generators of $H_2(K)$ are determined, consecutively, as some multiple of ∂D_i is null-homologous in $P_i = L \cup \bigcup\{D_j : j < i\}$, where L is the 1-skeleton of K. Let $\mu: S_i \to P_i$ denote a map of a compact, orientable, 2-manifold with boundary S_i realizing this null-homology. The pull-back

$$\begin{array}{ccc} S_i^* & \overset{\mu'}{\cdots\cdots>} & p^{-1}(P_i) \\ \vdots & & \big\downarrow p| \\ S_i & \xrightarrow{\ \mu\ } & P_i \end{array}$$

shows the existence of a nontrivial 2-cycle α_i carried by $\mu'(S_i^*) \cup p^{-1}(D_i) \subset K'$. Since the carrier of α_i nontrivially involves 2-cells in $p^{-1}(D_i)$ while that of α_j, $j < i$, does not, the cycles $\{\alpha_i\}$ are linearly independent.

Now consider a would-be homomorphism $\psi: G \to G$ with $\psi(G)$ normal and $G/\psi(G)$ infinite cyclic, and let $\nu: G \to G/\psi(G)$ denote projection. The kernel κ of the composition

$$G \to G/\psi(G) \cong Z \to Z/(s + 1)Z$$

is generated by $\psi(G)$ and any $m \in \nu^{-1}(s + 1)$, and we see, as before, that the covering $K_\kappa \to K$ determined by κ has $\beta_1(K_\kappa) > s + 1$, an impossibility. The only alternative is $\psi(G) = G$, whence ψ is an isomorphism for hopfian groups G.

REMARK. The direct product of the integers with a free group on t generators illustrates the sharpness of 4.8.

COROLLARY 4.9. A free group on s generators is hyperhopfian whenever $1 < s < \infty$.

COROLLARY 4.10. *The fundamental group of any closed surface S with $\chi(S) < 0$ is hyperhopfian.*

It is well known that $\pi_1(S)$ is hopfian.

THEOREM 4.11. *If G_1, G_2 are nontrivial, finitely generated residually finite groups and $G_2 \neq Z_2$, then $G_1 * G_2$ is hyperhopfian.*

Proof. Consider $G = G_1 * G_2$, where G_i $(i = 1, 2)$ has no further nontrivial decomposition as a free product. The general case $G = G_1 * \cdots * G_m$, $m > 2$, is proved by the same means as this special one. Let $\psi: G \to G$ be a homomorphism with $H = \psi(G)$ a normal subgroup such that G/H is cyclic. Here $[G:H] = k < \infty$ [S-W].

Construct connected 2-complexes X_i with $\pi_1(X_i) \cong G_i$ $(i = 1, 2)$, join them with an edge e to form another complex $X \supset X_1 \cup X_2$, and examine the covering $q: X^* \to X$ corresponding to the subgroup H. Regularity of q ensures that the components of $q^{-1}(X_i)$ are pairwise homeomorphic. Let K_i denote a component of $q^{-1}(X_i)$. Obviously $q \mid K_i$ gives a regular cyclic cover of X_i having order k_i dividing k, with $k_1 \neq 1$ or $k_2 \neq 1$. For definiteness assume $k_2 \neq 1$. Moreover, $\pi_1(X^*)$ is a free product of k/k_1 copies of $\pi_1(K_1)$, k/k_2 copies of $\pi_1(K_2)$, and a free group F.

Examination of certain first homology groups shows F to be trivial. The free part of $H_1(X^*)$ has rank $\beta_1(X^*)$ equal to the sum of $(k/k_1) \cdot \beta_1(K_1)$, $(k/k_2) \cdot \beta_1(K_2)$, and the rank of the abelianized free group, while similarly $\beta_1(X) = \beta_1(X_1) + \beta_1(X_2)$. Since ψ induces an epimorphism from the Abelianization of G to that of H, $\beta_1(X) \geqslant \beta_1(X^*)$. Being of finite index in $H_1(X_i)$, $q_*(H_1(K_i))$ has free part isomorphic to $H_1(X_i)$, so $\beta_1(K_i) \geqslant \beta_1(X_i)$ for $i = 1, 2$. Hence, $F = 1$.

Geometrically, this implies that $q \mid K_1$ is 1-1, for otherwise one could produce a loop in X^* as a composition of paths $\alpha_1\gamma_1\alpha_2\gamma_2 \cdots \alpha_m\gamma_m$, $m > 1$, where $q(\alpha_i)$ is contained in one of X_1, X_2, $q(\alpha_{i+1})$ is contained in the other, $q(\gamma_i) \subset e$, and the various γ_j are pairwise disjoint. Such a loop would necessarily be carried by the free part of the graph of groups used to describe $\pi_1(X^*)$.

Clearly $\pi_1(X_2)$ is generated by $q_{\#}(\pi_1(K_2))$ and one additional element sent to a generator of G/H. Denoting the rank of a group G (the minimum cardinality required for a set of generators) as $\rho(G)$, we have

$$\rho(\pi_1(K_2)) \geqslant \rho(\pi_1(X_2)) - 1.$$

In what follows we frequently use the consequence of Grushko's theorem that $\rho(A * B) = \rho(A) + \rho(B)$.

Case 1. $k \geqslant 3$. This can be ruled out immediately, for it leads to the impossibility

$$\begin{aligned}\rho(H) &= k \cdot \rho(\pi_1(K_1)) + (k/k_2) \cdot \rho(\pi_1(K_2)) \\ &\geqslant k \cdot \rho(\pi_1(X_1)) + \rho(\pi_1(X_2)) - 1 \\ &> \rho(\pi_1(X_1)) + \rho(\pi_1(X_2)) \\ &= \rho(G).\end{aligned}$$

Case 2. $k = 2$. The same impossibility as in Case 1 occurs if $\rho(\pi_1(K_1)) > 1$. We can identify $\pi_1(K_1)$ with the cyclic group Z_p, for the reasoning used previously to show F trivial indicates $\pi_1(K_1)$ is not infinite cyclic. Accordingly,

$$H_1(X) \cong Z_p \oplus H_1(X_2) \quad \text{and} \quad H_1(X^*) \cong Z_p \oplus Z_p \oplus H_1(K_2). \tag{ζ}$$

Hence $\beta_1(K_2) = \beta_1(X^*) = \beta_1(X) = \beta_1(X_2)$. Since $H_1(X)$ surjects to $H_1(X^*)$ and $H_1(K_2)$ surjects to a subgroup of index 2 in $H_1(X_2)$,

$$|\text{torsion } H_1(X_2)| \leqslant 2 \cdot |\text{torsion } H_1(K_2)|.$$

Similarly,

$$|\text{torsion } H_1(X)| \leqslant 2 \cdot |\text{torsion } H_1(X^*)|.$$

Comparisons with (ζ) reveal $p = 2$.

Keep in mind that ψ lifts to an epimorphism

$$\psi' : G_1 * G_2 \cong Z_2 * G_2 \to Z_2 * Z_2 * \pi_1(K_2) \cong \pi_1(X^*),$$

and the composite $\psi' \circ q_\#$ provides a homomorphism

$$\psi' \circ q_\# : Z_2 * Z_2 * \pi_1(K_2) \to Z_2 * Z_2 * \pi_1(K_2)$$

whose image has index $\leqslant 2$. Repeating the preceding sort of rank arguments (regard G_1 as $Z_2 * Z_2$), we see the index cannot equal 2. This means that $\psi' \circ q_\#$ is an epimorphism. By Gruenberg's work [G] on root properties, the free product of residually finite groups is residually finite, which implies that $Z_2 * Z_2 * \pi_1(K_2)$

is hopfian and $\psi' \circ q_{\#}$ is an automorphism. As a result, the sequence

$$\ker \psi' \to Z_2 * G_2 \xrightarrow{\psi'} Z_2 * Z_2 * \pi_1(K_2)$$

has a direct product splitting. However, this is impossible, since free products are never direct products.

Consequently, $k = 1$. As above, $G_1 * G_2$ is hopfian, implying $\psi: G_1 * G_2 \to H \cong G_1 * G_2$ is an isomorphism, as required.

The argument establishes variations such as the following.

COROLLARY 4.12. *If G_1, G_2 are nontrivial, finitely generated groups such that $|G_2| \neq 2$ and $G_1 * G_2$ is hopfian, then $G_1 * G_2$ is hyperhopfian.*

COROLLARY 4.13. *If G_1, G_2 are nontrivial, finitely generated hyperhopfian groups with $G_1 * G_2$ hopfian, then $G_1 * G_2$ is hyperhopfian.*

Thus, in principle the hyperhopfian property is closed with respect to free products. It is closed neither under HNN extensions nor under free products with amalgamation (recall that the double of a twisted I-bundle over a nonorientable surface 2-1 covers itself). The matter of closure with respect to direct products is still open. On this front, Im [I] has shown that all Cartesian products of surfaces of negative Euler characteristic have hyperhopfian groups.

5. Codimension 2 fibrators

To streamline the envisioned topological applications, we say that a decomposition $\mathscr{G}$ of a manifold into ANRs has Property $R\cong$ if, for each $g_0 \in \mathscr{G}$, a retraction $R: U \to g_0$ defined on some open set $U \supset g_0$ induces π_1-isomorphisms $(R \mid g)_{\#}: \pi_1(g) \to \pi_1(g_0)$ for all g sufficiently close to g_0. If this property holds for one retraction $R: U \to g_0$, then it holds for every retraction $R': U' \to g_0$ defined on another open set $U' \supset g_0$. Similarly, we say that $\mathscr{G}$ has Property $R_*\cong$ if $R: U \to g_0$ as above restricts to H_1-isomorphisms $(R \mid g)_*: H_1(g) \to H_1(g_0)$ for all g sufficiently close to g_0 (and all $g_0 \in \mathscr{G}$). Property $R\cong$ is the fundamental effect of hyperhopficity, whereas the weaker Property $R_*\cong$, in context, causes $R \mid g: g \to g_0$ to be degree one.

LEMMA 5.1. *If N^n is a closed n-manifold with hyperhopfian fundamental group and $\mathscr{G}$ is an N^n-like decomposition of an $(n+2)$-manifold M, then $\mathscr{G}$ has Property $R\cong$.*

Proof. There is no difficulty at $g_0 \in \mathscr{G}$ in the preimage of the continuity set, C; any neighborhood retraction $R: U \to g_0$ restricts to a degree one map $R \mid g: g \to g_0$, which induces a π_1-epimorphism; the hopfian hypothesis gives Property $R\cong$.

Just as in [D1, Theorem 3.1], points of $B\backslash C$ are isolated in B, and we can reduce immediately to the situation where the decomposition space B is identical to E^2 and the decomposition map p is an approximate fibration over the complement of the origin, 0.

Use g_0 to represent $p^{-1}(0)$. First we claim g_0 is a strong deformation retract of M. Properties of ANRs ensure that g_0 is a strong deformation retract in M of some ($\mathscr{E}$-saturated) neighborhood V. Specify a homotopy pulling E^2 into $p(V)$ and fixing a smaller neighborhood of 0. By [C-D1, Prop. 1.5], there exists an approximately lifted homotopy pulling M into V while fixing a neighborhood of g_0 throughout (first restrict to $M\backslash g_0$; after obtaining the desired lift on the deleted space, fill in across g_0 with the inclusion).

Name the retraction $R: M \to g_0$ promised above.

Fix an arbitrary g from $\mathscr{E}\backslash\{g_0\}$. From the exact homotopy sequence for approximate fibrations [C-D1, Cor. 3.5], we have

$$\pi_2(E^2\backslash\{0\}) \cong 1 \to \pi_1(g) \xrightarrow{\lambda} \pi_1(M\backslash g_0) \to \pi_1(E^2\backslash\{0\}) \to 1 \qquad (*)$$

showing $\pi_1(M\backslash g_0)/\lambda(\pi_1(g)) \cong Z$.

Because g_0 has the homotopy type of a codimension 2 compactum from M, the inclusion $M\backslash g_0 \to M$ induces an epimorphism Ψ of fundamental groups (readily verifiable by passing to universal covers and exploiting homology properties). It follows directly that $R_{\#}\Psi\lambda(\pi_1(g))$ is a normal subgroup of $\pi_1(g_0) \cong \pi_1(N^n)$ having cyclic cokernel. Due to the hyperhopfian hypothesis, $R_{\#}\Psi\lambda \mid \pi_1(g)$ is a (surjective) isomorphism. In other words, the restricted retraction $R \mid g: g \to g_0$ induces a fundamental group isomorphism, and Property $R\cong$ holds.

The next result is basically due to Im [I]. Although results similar to 5.2 appear in the literature, nothing matched it, and Im seems to be the first to have noticed this extremely useful fact.

LEMMA 5.2. *Suppose $\mathscr{E}$ is an N^n-like decomposition of an $(n + 2)$-manifold M with Property $R_*\cong$, and $g_0 \in \mathscr{E}$. Then, for all $g \in \mathscr{E}$ sufficiently close to g_0, a given neighborhood retraction $R: U \to g_0$ restricts to a degree one map $g \to g_0$.*

Proof. Using the reduction and notation of 5.1, we take $B = E^2$, $g_0 = p^{-1}(0 = B\backslash C)$, and $R: M = p^{-1}(E^2) \to g_0$, a strong deformation retraction. For any other $g \in \mathscr{E}$, we verify that $R \mid g$ is a degree one map by checking it gives a cohomology isomorphism between $H^n(g_0) \cong H_2(M, M\backslash g_0)$ and $H^n(g) \cong H_2(M, M\backslash g)$ and applying the universal coefficient theorem to obtain the same for homology. The larger step depends on showing p provides isomorphisms of pairs $H_2(M, M\backslash g_0) \to H_2(B, B\backslash 0)$ and $H_2(M, M\backslash g) \to H_2(B, B\backslash p(g))$, and then spotting an isomorphism between the images. To find the isomorphism of pairs, examine the homology ladder

$$\begin{array}{ccccc}
Z \cong H_2(M, M\backslash g_0) & \xrightarrow{\partial} & H_1(M\backslash g_0) & \longrightarrow & H_1(M) \cong H_1(g_0) \to 0 \\
\downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle p'_*} & & \\
Z \cong H_2(B, B\backslash 0) & \xrightarrow{\cong} & H_1(B\backslash 0) & &
\end{array}$$

Property $R_* \cong$ leads to a splitting $H_1(M\backslash g_0) = \lambda_* H_1(g) \oplus \operatorname{im} \partial$, where $\lambda: g \to M\backslash g_0$ denotes inclusion. As point inverses of p are connected, p' induces epimorphisms of both π_1 and H_1. With $\lambda_*(H_1(g)) \subset \ker p'_*$, diagram chasing yields that p'_* carries $\operatorname{im} \partial$ isomorphically onto $H_1(B\backslash 0)$.

By a similar argument, p restricts to an isomorphism $H_2(M, M\backslash g) \to H_2(B, B\backslash c)$, for $c = p(g)$. Specify a closed disk $D \subset B$ containing 0, c in its interior. One sees what is needed, the desired isomorphism between $H^n(g_0)$ and $H^n(g)$, in the diagram below.

$$\begin{array}{ccccc}
H^n(g_0) & \longleftarrow & H^n(p^{-1}D) & \longrightarrow & H^n(g) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
H_2(M, M\backslash g_0) & \longleftarrow & H_2(M, M\backslash p^{-1}(D)) & \longrightarrow & H_2(M, M\backslash g) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
H_2(B, B\backslash 0) & \xleftarrow{=} & H_2(B, B\backslash D)) & \xrightarrow{\cong} & H_2(B, B\backslash c)
\end{array}$$

Before going on, we give a variation to Lemma 5.2 with other uses.

LEMMA 5.2′. *Suppose $\mathscr{E}$ is an N^n-like decomposition of an $(n + 2)$-manifold M, $g_0 \in \mathscr{E}$, and for all $g \in \mathscr{E}$ sufficiently close to g_0, a given neighborhood retraction $R: U \to g_0$ sends $H_1(g)$ to a finite index subgroup of $H_1(g_0)$. Then $R \mid g: g \to g_0$ has positive degree.*

Proof. By the Universal Coefficient Theorem, a map $f: N_1 \to N_2$ between closed (orientable) n-manifolds has positive degree (integer coefficients implicitly understood) if, with rational coefficients (Q), the induced homomorphism $H^n(N_2: Q) \to H^n(N_1; Q)$ is an isomorphism, which follows by the argument of 5.2.

LEMMA 5.3. *Suppose N^n is a hopfian manifold and $\mathscr{E}$ is an N^n-like decomposition of an $(n + 2)$-manifold M with Property $R\cong$. Then $p: M \to B$ is an approximate fibration.*

Proof. The hypothesis concerning Property $R\cong$ routinely yields that $\mathscr{E}$ has Property $R_* \cong$, and Lemma 5.2 ensures that all $g_0 \in \mathscr{E}$ live above the continuity set of $p: M \to B$. For hopfian manifolds N^n, Theorem 2.1 certifies p is an approximate fibration.

Lemmas 5.1 and 5.3 combine to form the main theorem. Its consequences all rely on Theorem 2.2.

THEOREM 5.4. *All closed, hopfian manifolds with hyperhopfian fundamental group are codimension 2 fibrators.*

COROLLARY 5.5. *If N is a closed, orientable n-manifold such that $\pi_1(N)$ is hyperhopfian and $\pi_i(N)$ is trivial, for $1 < i < n - 1$, then N is a codimension 2 fibrator. In particular, this conclusion holds for aspherical closed manifolds with hyperhopfian fundamental groups.*

COROLLARY 5.6. *Every closed, orientable manifold with finite, hyperhopfian fundamental group is a codimension 2 fibrator.*

COROLLARY 5.7. *Let N^3 be a closed 3-manifold such that $\pi_1(N^3)$ is finite and has no cyclic direct factor. Then N^3 is a codimension 2 fibrator.*

COROLLARY 5.8. *Suppose the hopfian group G is a nontrivial free product of residually finite groups, $G \not\cong Z_2 * Z_2$, and suppose N^3 is a closed orientable 3-manifold with $\pi_1(N^3) \cong G$. Then N^3 is a codimension 2 fibrator.*

COROLLARY 5.9. *Suppose the group G is a nontrivial free product, $G \not\cong Z_2 * Z_2$, and suppose N^3 is a closed orientable 3-manifold with $\pi_1(N^3) \cong G$, where N^3 has a connected sum decomposition consisting of virtually Haken manifolds and/or those with finite or cyclic fundamental groups. Then N^3 is a codimension 2 fibrator.*

Next comes a result leading to less restrictive analogs of the two last corollaries for 4-manifolds. This provides unequivocal evidence that codimension 2 fibrators neither are characterized in terms of hyperhopfian fundamental groups nor determined solely by their groups. For instance, real projective space RP^{2n+1} is a codimension 2 fibrator [D1] but $\pi_1(RP^{2n+1}) \cong Z_2$ certainly is not hyperhopfian; the ensuing results furnish a large collection of additional examples. Moreover, it is useful to compare any 4-manifold N having $\pi_1(N) \cong Z_2 * Z_2$ to $RP^3 \# RP^3$, a known nonfibrator with the same group, for by Corollary 5.11 N is a fibrator.

THEOREM 5.10. *Every closed, hopfian n-manifold N with $\pi_1(N)$ hopfian and $\chi(N) \neq 0$ is a codimension 2 fibrator.*

Proof. Let $p: M \to B$ be a closed map defined on an $(n + 2)$-manifold M such that $p^{-1}b \approx N$ for all $b \in B$. Just as in proof of Lemma 5.1, localize to the situation in which $B = E^2$ and p is an approximate fibration over $C = E^2 \backslash 0$. (Remark: each point preimage above C has a neighborhood retraction transporting nearby preimages in degree one fashion, and the hypothesis on N being a hopfian manifold with hopfian group ensures that the restricted retractions are homotopy equivalences.)

Use g_0 to represent $p^{-1}(0)$, which again is a strong deformation retract of M under a retraction $R: M \to g_0$. In light of Lemma 5.3, it suffices to show that the restriction of R induces an epimorphism $H_1(g) \to H_1(g_0)$ for $p^{-1}b = g \neq g_0$. Suppose that is not the case.

From the exact homotopy sequence for approximate fibrations [CD, Cor. 3.5], we have

$$\pi_2(E^2\backslash 0) \cong 1 \to \pi_1(N) \to \pi_1(M\backslash g_0) \to \pi_1(E^2\backslash 0) \to 1 \tag{$*$}$$

showing that $\pi_1(N)$ injects onto a normal subgroup of $\pi_1(M\backslash g_0)$ having infinite cyclic quotient.

Fix an arbitrary $g = p^{-1}b$ in $M\backslash g_0$. The fiber fundamental group $\pi_1(N)$ in formula ($*$) above is carried by $\pi_1(g)$. Hence, there exists an epimorphism of $\pi_1(M\backslash g_0)$ to Z with kernel determined by the image of $\pi_1(g)$. Because g_0 has the homology type of a codimension 2 manifold in M, the inclusion $M\backslash g_0 \to M$ induces an epimorphism at the first homology level. For $j: g \to M$ the inclusion, this implies $H_1(M)/j_*(H_1(g)) \cong H_1(g_0)/R_*(H_1(g))$ is a cyclic group T (nontrivial, by an earlier supposition), so there exists an induced epimorphism $\pi_1(M) \to H_1(M) \to T$ whose kernel K contains $j_\#(\pi_1(g)) \subset \pi_1(M)$.

Form the (cyclic) covering $\theta: M' \to M$ corresponding to K. Since the deformation retraction R lifts to another deformation retraction $M' \to \gamma_0 = \theta^{-1}(g_0)$, we can regard θ as providing a cyclic covering $\theta': N' \approx \gamma_0 \to N$ such that order θ' = order θ.

Case 1. $[\pi_1(M):K] = \infty$. We will verify that $N' \approx \gamma_0$ has finitely generated homology. The contradiction that $\chi(N) = 0$ will follow immediately from work of Milnor (Assertion 6 [Mi2]).

Here γ_0 has the homotopy type of M' (and N'), and $M'\backslash\gamma_0$ is partitioned into copies of N (namely, the components of the various sets $(p \circ \theta)^{-1}(z)$, $z \neq 0$) and the associated decomposition map (which is a closed mapping) makes the following diagram commutative:

$$\begin{array}{ccc} M'\backslash\gamma_0 & \longrightarrow & B' \\ \Big\downarrow{\scriptstyle\theta} & & \Big\downarrow{\scriptstyle v} \\ M\backslash g_0 & \xrightarrow{\;p\;} & E^2\backslash 0 \end{array}$$

Now $v: B' \to E^2\backslash 0$ is an infinite cyclic covering and $M\backslash\gamma_0 \to B' \approx E^2$ is an approximate fibration, as before, from which it follows that $M'\backslash\gamma_0$ has the homotopy type of N.

We give an inductive argument showing each $H_k(\gamma_0)$ to be finitely generated. Obviously true for $k = 0$, 1, suppose it is also true for all $j < k$, where $k \geqslant 2$. Then $H_{k-2}(\gamma_0)$ finitely generated implies the same for each of the following, in turn: $H_c^{n+k-2}(\gamma_0)$; $H_k(M', M'\backslash\gamma_0)$; and $H_k(M') \cong H_k(\gamma_0)$. The first implication results from Poincaré duality in γ_0; the second, from Poincaré-Lefshetz duality in M'; and the last, from inspection of the long exact sequence for the pair $(M', M'\backslash\gamma_0)$.

Case 2. $[\pi_1(M): K] = k > 1$. By Lemma 5.2′, R restricts to a map $g \to g_0$ of positive degree. Thus, $R|g$ lifts to a positive degree map $R': g \to \gamma_0$ with $R|g = \theta R'|g$. But the existence of a positive degree map $N_1 \to N_2$ between closed, orientable manifolds yields the relationship $\beta_i(N_1) \geqslant \beta_i(N_2)$ for arbitrary Betti numbers β_i (see [Mu, p. 399]; applying this with both $\theta|\gamma_0$ and $R'|g$, we find

$$\beta_i(N) = \beta_i(g) \geqslant \beta_i(N') \geqslant \beta_i(g_0) = \beta_i(N)$$

for all i, thereby determining

$$\chi(N) = \chi(N') = k \cdot \chi(N),$$

which gives the contradiction $\chi(N) = 0$.

COROLLARY 5.11. *If N^4 is a closed orientable 4-manifold such that $\pi_1(N^4)$ is hopfian and $H_1(N^4)$ is finite, then N^4 is a codimension 2 fibrator.*

Proof. Here $\beta_1(N^4) = \beta_3(N^4) = 0$, so $\chi(N^4) \geqslant 2$.

COROLLARY 5.12. *Every closed orientable 4-manifold N^4 with finite fundamental group is a codimension 2 fibrator.*

No counterexamples to 5.12 are known in other dimensions.

COROLLARY 5.13. *Let N be a hopfian 2n-manifold with $\pi_1(N)$ a hopfian group and $H_{2i-1}(N)$ finite for $0 < i < n$. Then N is a codimension 2 fibrator.*

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