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On the unitary dual of the classical Lie groups II. Representations of $SO(n, m)$ inside the dominant Weyl Chamber

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Abstract. In this paper I prove that if $G = SO_0(n, m)$ and X is an irreducible unitary Harish-Chandra module of G whose infinitesimal character minus half the sum of the positive roots is dominant, then X is isomorphic to a Zuckerman derived functor module induced via cohomological parabolic induction from a one dimensional unitary character of a subgroup. The proof is by reduction to a subgroup of G of smaller dimension: arguing by contradiction we find two K-types where the Hermitian form is indefinite.

1. Introduction

In addition to being the sequel of an earlier publication, this paper generalizes the argument used in part I (Salamanca-Riba, [2]), to prove the main theorem for the case of $SL(n, \mathbb{R})$. It therefore illustrates how the general case for the proof of that main theorem (Theorem 2.1 in this paper) should be approached. In other words, if G is a simple real Lie group with Cartan involution θ and complexified Lie algebra $\mathfrak{g}$, the general result I am alluding to is the following. Fix a Cartan subalgebra $\mathfrak{h}$ and a set of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{h})$.

CONJECTURE 1.1. Suppose X is an irreducible unitary Harish-Chandra module of G with infinitesimal character corresponding to a dominant weight $\gamma \in \mathfrak{h}^*$. Assume further that

$$(\gamma - \rho, \alpha^\vee) \quad \text{for all } \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}) \quad \text{with } \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha. \tag{1.1}$$

Then there are a θ-stable complex parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$ and a one-dimensional unitary character λ of the Levi-subgroup L of $\mathfrak{q}$ such that X is isomorphic to the Zuckerman module (see Section 2)

$$A_{\mathfrak{q}}(\lambda) = \mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda).$$

(See part I for undefined terms. There, $\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}}(Y)$ for any Harish-Chandra module Y, is also denoted $\mathscr{R}_{\mathfrak{q}}^{S}(Y)$.)

I outlined an algorithm for a case-by-case proof of this conjecture in part I and I mentioned that the case of $SO_0(n, m)$ was analogous to the one of $Sp(n, \mathbb{R})$. I also summarized the algorithm as a reduction to a special case of a proper subgroup of the same type in Cartan's classification and a real form of $GL(m, \mathbb{C})$.

However, if we look closely at the argument for $Sp(n, \mathbb{R})$ in part I, it is obvious that the choice of the appropriate subgroups is a very ad hoc one. It can be done for $SO(n, m)$ following the same ideas, but this paper contains a new idea which arose from the search for a more canonical subgroup. This idea (mainly Lemma 3.4, its proof and Lemma 3.5) simplifies the argument and lends itself to a non case-by-case proof of the same result for every classical Lie group. It also makes the argument run more parallel to the one used for $SL(n, \mathbb{R})$.

Throughout this paper I will refer the reader to part I for a few of the results that were also used there, especially if the original sources (which the reader may find in the bibliography of part I) state these results differently. I include the main theorems in Section 2.

The theorem that I actually prove is Theorem 2.2 which states that if X is not an $A_{\mathfrak{q}}(\lambda)$ the reason is because it contains two K-types where the restriction of the Hermitian form on X is indefinite. Lemma 3.4 and 3.5 mimic Lemma 7.5 in part I and the discussion following it, and Lemma 3.3 and 3.6 are analogous to the concluding argument of chapter 7 in the same paper.

The argument goes as follows. One of the main tools in dealing with unitary representations is the Dirac operator inequality (Lemma 6.1, part I), which states that if a representation is unitary then the highest weight of any of its K-types must satisfy a certain inequality.

Hence, if we argue by contradiction, we would like to show that if X is not a module of the desired form then Dirac inequality must fail for some K-type of X. However, to carry out this idea is impractical. Lemmas 3.3–3.6 give then a reduction to a subgroup of G for which proving that this inequality fails is very easy. Computing this inequality involves a choice of non-compact positive roots and the new element in this proof consists of choosing one set of non-compact positive roots given by the highest weight of the lowest K-type of X plus the sum of the compact positive roots (this choice is not unique but it does not matter).

This choice of positive roots essentially determines the subgroup L. It contains the subgroup L_μ of G, which Vogan ([2], Chap. 6) attaches to a lowest K-type μ of a Harish-Chandra module X. This in turn implies that (a) of Theorem 2.2 holds. Then, with the help of Lemma 6.3(a) in part I and using Theorem 2.3, one can show that (b) and (c) of Theorem 2.2 hold.

In Section 2 we set up notation, state the main theorems and give a few lemmas which will be needed in the proof. Section 3 is devoted to the proof of Theorem 2.2. At times it is necessary to specify which type of group is $SO(n, m)$ in terms of the parity of n and m. On a first reading, the reader may choose to fix one in order to maintain some continuity in the argument.

2. Notation and main theorems

Let $G = SO_0(n, m)$ and $\mathfrak{g}_0$ its Lie algebra. The maximal compact subgroup K of G is

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| A \in SO(n),\ D \in SO(m) \right\}.$$

If θ is the Cartan involution defined by

$$\theta(X) = -{}^tX,$$

then

$$\mathfrak{p}_0 = \{X \in \mathfrak{g}_0 \mid X = {}^tX\}.$$

The maximal compact Cartan subgroup of G is $H = TA$ where

$$T = \begin{cases} \left[\begin{array}{ccc|ccc} g(t_1) & & & & & \\ & \ddots & & & 0 & \\ & & g(t_p) & & & \\ \hline & & & g(s_1) & & \\ & 0 & & & \ddots & \\ & & & & & g(s_q) \end{array}\right] & n = 2p,\ m = 2q \\[2em] \left[\begin{array}{cccc|ccc} g(t_1) & & & & & & \\ & \ddots & & & & & \\ & & g(t_p) & & & & \\ & & & 1 & & & \\ \hline & & & & g(s_1) & & \\ & & & & & \ddots & \\ & & & & & & g(s_q) \end{array}\right] & n = 2p+1,\ m = 2q \\[2em] \left[\begin{array}{cccc|cccc} g(t_1) & & & & & & & \\ & \ddots & & & & & & \\ & & g(t_p) & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & g(s_1) & & \\ & & & & & & \ddots & \\ & & & & & & & g(s_q) \end{array}\right] & n = 2p + 1,\ m = 2q + 1. \end{cases}$$

Here, t_i, s_j are in $\mathbb{R}$ and

$$g(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \quad \text{for all } t \in \mathbb{R}.$$

Let $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ be the Lie algebra of H and write $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ for its complexification. Then $\mathfrak{a} = 0$ except when m, n are both odd. The roots of $\mathfrak{t}$ in $\mathfrak{k}$ and $\mathfrak{p}$

are

$$\Delta(\mathfrak{k},\mathfrak{t})=\begin{cases} B=\{\pm(e_i\pm e_j)\mid 1\leqslant i<j\leqslant p \text{ or } p<i<j\leqslant p+q\}; & \text{for } n=2p,\ m=2q\\ B\cup\{\pm e_k\mid 1\leqslant k\leqslant p\}; & \text{for } n=2p+1,\ m=2q\\ B\cup\{e_k\mid 1\leqslant k\leqslant p+q\}; & \text{for } n=2p+1,\\ & \quad m=2q+1,\end{cases}$$

$$\Delta(\mathfrak{p},\mathfrak{t})=\begin{cases} D=\{\pm(e_i\pm e_j)\mid 1\leqslant i\leqslant p \text{ and } p<j\leqslant p+q\}; & \text{for } n=2p,\ m=2q\\ D\cup\{\pm e_k\mid p<k\leqslant p+q\}; & \text{for } n=2p+1,\ m=2q\\ D\cup\{\pm e_k\mid 1\leqslant k\leqslant p+q\}; & \text{for } n=2p+1,\ m=2q+1.\end{cases}$$

The construction of the modules that we will use to parametrize the unitary Harish-Chandra modules of $SO_0(n,m)$ was given by Zuckerman [6] (see also part I, Sections 3–4). Given a θ-stable parabolic subalgebra $\mathfrak{q}\subseteq\mathfrak{g}$ with Levi subgroup $L\subset G$ and a Harish-Chandra module (π^L, X_L) of L, Zuckerman constructs a $(\mathfrak{g},K)$ module $\mathscr{R}^{\mathfrak{g}}_{\mathfrak{q}}(X_L)$. Under certain conditions, $\mathscr{R}^{\mathfrak{g}}_{\mathfrak{q}}(X_L)$ is unitary and irreducible. Set $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$, where $\mathfrak{l}=\mathrm{Lie}(L)_{\mathbb{C}}$, the conditions we are interested in are, if $\Pi^L\colon L\to\mathrm{Aut}(\mathbb{C})$ is one-dimensional, $\lambda\colon\mathfrak{l}\to\mathrm{End}(\mathbb{C})$ is the corresponding representation of $\mathfrak{l}=\mathrm{Lie}(L)_{\mathbb{C}}$ and $(\lambda,\alpha^{\vee})\geqslant 0$ for all $\alpha\in\Delta(\mathfrak{u})$. Let $A_{\mathfrak{q}}(\lambda)=\mathscr{R}^{\mathfrak{g}}_{\mathfrak{q}}(\mathbb{C}_\lambda)$. Then $A_{\mathfrak{q}}(\lambda)$ is irreducible and unitary (see Speh-Vogan [3] and Vogan [5]).

The main theorem that we want to prove is

THEOREM 2.1. *Let $G=SO_0(n,m)$. Suppose X is an irreducible Harish-Chandra module with infinitesimal character γ satisfying* (1.1) *and a positive definite Hermitian form $\langle\,,\rangle$. Then X is isomorphic to some module $A_{\mathfrak{q}}(\lambda)$.*

We will argue by contradiction. Assuming that X is not isomorphic to any one of these modules we will prove the following

THEOREM 2.2. *Let $G=SO_0(n,m)$ and $\mathfrak{g}=\mathfrak{g}_0\otimes\mathbb{C}$ and X an irreducible Harish-Chandra module of G endowed with a non-zero Hermitian form $\langle\,,\,\rangle$ and infinitesimal character as in* (1.1).

If $X\not\cong A_{\mathfrak{q}'}(\lambda')$ *for any* $\mathfrak{q}',\lambda'$, *then there are, a θ-stable parabolic $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$, a Harish-Chandra module X_L of L, the Levi factor of $\mathfrak{q}$ and $(L\cap K)$-types δ_1^L, δ_2^L such that*

(a) *X is the unique irreducible subquotient of $\mathscr{R}_{\mathfrak{q}}(X_L)$ and X occurs only once as a composition factor of $\mathscr{R}^{\mathfrak{g}}_{\mathfrak{q}}(X_L)$,*
(b) *the Hermitian dual of X_L is endowed with a Hermitian form $\langle\,,\rangle^L\neq 0$ and $\langle\,,\,\rangle^L|_{(V\delta_1^L+V\delta_2^L)}$ is indefinite,*
(c) *choose $\Delta^+(\mathfrak{k},\mathfrak{t})=\Delta^+(\mathfrak{l}\cap\mathfrak{k})\cup\Delta(\mathfrak{u}\cap\mathfrak{k})$. Then if δ_i^L has highest weight μ_i^L, then $\mu_i=\mu_i^L+2\rho(\mathfrak{u}\cap\mathfrak{p})$ is $\Delta^+(\mathfrak{k})$-dominant.*

Assume this is true. The proof is given in Section 3. We will use the following result to finish the proof of Theorem 2.1.

THEOREM 2.3 (Vogan) (see part I, Theorem 5.8). *Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$ be a θ-stable parabolic with Levi factor L and X_L a Harish-Chandra module of L. Suppose X is an irreducible Harish-Chandra module of G with a non-zero Hermitian form $\langle\,,\rangle$ and X is the unique irreducible submodule of $\mathscr{R}^{\mathfrak{g}}_{\mathfrak{q}}(X_L)$ and that it occurs there only once as a composition factor.*

*Suppose further that X^h_L, the Hermitian dual of X_L has a Hermitian form $\langle\,,\rangle^L$. If $\delta^L \in (L \cap K)\hat{}$ is a K-*type of X_L *(i.e. $X_L(\delta^L) \neq 0$), δ^L has highest weight μ^L and $\mu = \mu^L + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ is dominant for $\Delta(\mathfrak{u} \cap \mathfrak{k})$, then if $\delta \in \hat{K}$ has highest weight μ, $X(\delta) \neq 0$ and* $\operatorname{sign}(\langle\,,\rangle|_{X(\delta)}) = \operatorname{sign}(\langle\,,\rangle^L|_{X_L(\delta^L)})$.

Using these two theorems we can show that the form $\langle\,,\rangle$ on X is indefinite as follows. Since there are δ^L_1, $\delta^L_2 \in (L \cap K)\hat{}$ such that $\langle\,,\rangle^L|_{V_{\delta^L_1} + V_{\delta^L_2}}$ is indefinite, then by Theorem 2.3, $X(\delta_j) \neq 0$ for $j = 1, 2$ and $\langle\,,\rangle|_{(V_{\delta_1} + V_{\delta_2})}$ is indefinite. This proves Theorem 2.1.

To prove Theorem 2.2 we will need three lemmas. Suppose G is a quasisplit Lie group, $K \subseteq G$ a maximal compact subgroup and $\mathfrak{k}_0 = \operatorname{Lie}(K)$. Vogan in [4] gives the definition of a fine (for G) K-type $\mu \in \hat{K}$. (Definition 4.3.9). Using this definition the following lemma can be proved, by reducing to $SU(2, 1)$, $SL(2, \mathbb{C})$ and $Sp(2, \mathbb{R})$.

LEMMA 2.4 (Vogan, unpublished). *Let G be a quasisplit Lie group and $\mu \in \hat{K}$, then μ is a fine K-type for G if and only if the following three conditions hold. Let $\mathfrak{h}$ be the fundamental Cartan subalgebra of $\mathfrak{g}$.*

(i) *Suppose $\beta \in \Delta(\mathfrak{g}, \mathfrak{h})$ is an element of a strongly orthogonal set $\{\beta_i\}$ of non-compact imaginary roots, of maximal order. Then*

$$|(\mu, \beta^\vee)| \leqslant 1.$$

(ii) *Suppose $\beta, \alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ are orthogonal imaginary roots, with α compact and β noncompact such that β and α are not strongly orthogonal. Then*

$$|(\mu, \alpha^\vee)| \leqslant 1.$$

(iii) *μ is trivial on the identity component of the compact part of any quasisplit Cartan subgroup $H^S \subseteq G$.*

LEMMA 2.5. *Let μ be a fine K-type. Then for any maximal strongly orthogonal set B of imaginary non-compact roots*

$$\mu = \sum_{\substack{a_i = \pm\frac{1}{2}, 0 \\ \beta_i \in B}} a_i \beta_i.$$

Proof. By condition (iii) of Lemma 2.4. If μ is fine then μ lives in the span of any maximal strongly orthogonal set $B=\{\beta_1, \beta_2 \cdots \beta_t\}$ of imaginary non-compact roots. Hence

$$\mu = \sum_{\substack{a_i \in \mathbb{R} \\ \beta_i \in B}} a_i \beta_i,$$

but by condition (i) of Lemma 2.4

$$1 \geqslant |(\mu, \check{\beta_j})| = \left| \frac{2(\Sigma_{\beta_i \in B} a_i \beta_i, \beta_j)}{(\beta_j, \beta_j)} \right| = \sum_{\beta_i \in B} \frac{2|a_i(\beta_i, \beta_j)|}{(\beta_j, \beta_j)} = 2|a_j|.$$

Hence $|a_j| \leqslant \frac{1}{2}$ and since μ is integral this means that $a_j = 0, \pm\frac{1}{2}$ $(j = 1, 2 \cdots t)$.

LEMMA 2.6. *Let μ be a non-trivial fine K-type and choose a maximal strongly orthogonal subset B of positive imaginary non-compact roots and a positive root system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t}) \supset B$ so that μ is Δ^+-dominant and*

$$\mu = \sum_{\substack{\beta_i \in B \\ c_i = 0, \frac{1}{2}}} c_i \beta_i.$$

Set $\rho_n = \rho(\Delta^+(\mathfrak{p}))$, $\rho_c = \rho(\Delta^+(\mathfrak{k}))$, $\rho = \rho(\Delta^+)$ and $\delta = \mu - \rho_n + w\rho_c$ where $w \in W_k$ makes $\mu - \rho_m w \Delta_{\mathfrak{k}}^+$-dominant. Then $(\delta, \delta)_{\mathfrak{k}} \leqslant (\rho, \rho)_{\mathfrak{k}}$, and equality holds only if $c_i = 0$, $\forall \beta_i \in B$.

Proof. By Lemma 2.5, we can choose B as desired. Then

$$\delta = \sum_{\substack{\beta_i \in B \\ c_i = 0, \frac{1}{2}}} c_i \beta_i - \frac{1}{2} \sum_{\sigma \in \Delta_n^+} \sigma + \sum_{\substack{\alpha \in \Delta_{\mathfrak{k}}^+ \\ b_\alpha = \pm\frac{1}{2}}} b_\alpha \alpha$$

$$= -\rho_n + \sum c_i \beta_i + \sum_{\substack{\alpha \in \Delta_{\mathfrak{k}}^+ \\ b_\alpha = \pm\frac{1}{2}}} \left(-\frac{1}{2} + b_\alpha + \frac{1}{2} \right) \alpha$$

$$= -\left[\rho - \sum_{c_i = 0, \frac{1}{2}} c_i \beta_i - \sum_{b'_\alpha = 0, 1} b'_\alpha \alpha \right].$$

Now $\rho - \frac{1}{2}\alpha = \frac{1}{2}(\rho - \alpha) + \frac{1}{2}\rho = \frac{1}{2}(\sigma_\alpha \rho) + \frac{1}{2}\rho$. Hence $\forall \alpha \in \Delta^+$, $\rho - \frac{1}{2}\alpha \in H[W \cdot \rho] =$ convex hull of $W \cdot \rho$. By induction on $\#\{\beta_i, \alpha \mid c_i, b'_\alpha \neq 0\}$ we can show that $\delta \in H[W \cdot \rho]$. Therefore, $(\delta, \delta) \leqslant (\rho, \rho)$ and equality holds only if $\delta = w_1 \cdot \rho$ for some $w_1 \in W$. Note that $\delta = -[\rho - 2\rho(B_0) - \rho(B_1)]$, where $B_0 \subset \Delta_c^+$ and $B_1 \subset \Delta_n^+$. Then $\delta = w_1 \cdot \rho$ implies that $B_1 = \phi$ and hence $c_i = 0$, $\forall \beta_i \in B$. This proves Lemma 2.6.

3. Proof of Theorem 2.2

In what follows let X be an irreducible Harish-Chandra module with a non-zero Hermitian form $\langle\,,\rangle$. Let μ=highest weight of a lowest K-type (LKT) V_μ of X. After conjugating by an outer automorphism of K we may assume

$$\mu=(a_1,a_2\cdots a_p\,|\,b_1,b_2\cdots b_q) \text{ where } a_1\geqslant a_2\geqslant\cdots\geqslant a_p\geqslant 0$$
$$\text{and } b_1\geqslant b_2\geqslant\cdots\geqslant b_q\geqslant 0. \tag{3.1}$$

Fix $\Delta^+(\mathfrak{k})$ such that μ is dominant. Then

$$2\rho_c=\begin{cases}(2p-2,2p-4\cdots 2,0\,|\,2q-2,2q-4\cdots 2,0) & \text{if } G=SO_0(2p,2q)\\ (2p-1,2p-3\cdots 3,1\,|\,2q-2,2q-4\cdots 2,0) & \text{if } G=SO_0(2p+1,2q)\\ (2p-1,2p-3\cdots 3,1\,|\,2q-1,2q-3\cdots 3,1) & \text{if } G=SO_0(2p+1,2q+1).\end{cases} \tag{3.2}$$

Since we want to argue by contradiction we will give some necessary and sufficient conditions for V_μ to be the LKT of a module $A_{\mathfrak{q}}(\lambda)$.

Recall from Vogan [4] that to construct a θ-stable parabolic subalgebra we need a weight $\chi\in i(\mathfrak{t}_0^c)^*$. Suppose

$$\chi=(\underbrace{\chi_1\cdots\chi_1}_{p_1\text{ times}},\underbrace{\chi_2\cdots\chi_2}_{p_2\text{ times}}\cdots\underbrace{\chi_k\cdots\chi_k}_{p_k},\underbrace{0\cdots 0}_{r}\,|\,\underbrace{\chi_1\cdots\chi_1}_{q_1}\cdots\underbrace{\chi_k\cdots\chi_k}_{q_k},\underbrace{0\cdots 0}_{s}) \tag{3.3}$$

where $\chi_1>\chi_2>\cdots>\chi_k>0$, $p_i, q_i\geqslant 0$.

Set $r=p-\Sigma p_i$, $s=q-\Sigma q_i$, $n=p+q$, $n_i=p_i+q_i$.

Set

$$\Delta(\mathfrak{u}(\chi))=\{\alpha\in\Delta(\mathfrak{g},\mathfrak{t}^c)\,|\,(\chi,\alpha)>0\},$$
$$\Delta(\mathfrak{l}(\chi))=\{\alpha\in\Delta(\mathfrak{g},\mathfrak{t}^c)\,|\,(\chi,\alpha)=0\}.$$

Denote by $\mathfrak{q}(\chi)$, $\mathfrak{u}(\chi)$, and $\mathfrak{l}(\chi)$ the subalgebras spanned by these subsets, where $\mathfrak{q}(\chi)=\mathfrak{u}(\chi)+\mathfrak{l}(\chi)$. Consider first the case $G=SO_0(2p,2q)$, an easy calculation shows that

$$2\rho(\mathfrak{u}\cap\mathfrak{k})=(\underbrace{c_1\cdots c_1}_{p_1},\underbrace{c_2\cdots c_2}_{p_2}\cdots\underbrace{c_k\cdots c_k}_{p_k},\underbrace{0\cdots 0}_{r}\,|\,\underbrace{d_1\cdots d_1}_{q_1}\cdots\underbrace{d_k\cdots d_k}_{q_k},\underbrace{0\cdots 0}_{s})$$

where, for $j = 1, 2\cdots k$

$$c_j = 2\left(p-\sum_{i=1}^{j-1} p_i\right)-p_j-1, \qquad d_j = 2\left(q-\sum_{i=1}^{j-1} q_i\right)-q_j-1$$

$$2\rho(\mathfrak{u}\cap\mathfrak{p}) = (d_1+1\cdots d_1+1\cdots d_k+1\cdots d_k+1,\ 0\cdots 0\,|\,c_1+1\cdots c_1 + 1\cdots c_k+1\cdots c_k+1,\ 0\cdots 0)$$

$$2\rho(\mathfrak{u}) = (u_1\cdots u_1\cdots u_k\cdots u_k,\ 0\cdots 0\,|\,u_1\cdots u_1\cdots u_k\cdots u_k,\ 0\cdots 0)$$

$$\text{where } u_j = c_j+d_j+1 = 2\left(n-\sum_{i=1}^{j-1} n_i\right)-n_j-1. \tag{3.4}$$

Now assume $G=SO_0(2p+1,2q)$. Then if χ is as in (3.3).

$$2\rho(\mathfrak{u}\cap\mathfrak{k})=(\underbrace{c_1'\cdots c_1'}_{p_1\text{ times}},\underbrace{c_2'\cdots c_2'}_{p_2}\cdots\underbrace{c_k'\cdots c_k'}_{p_k},\underbrace{0\cdots 0}_{r}\,|\,\underbrace{d_1\cdots d_1}_{q_1},\underbrace{d_2\cdots d_2}_{q_2}\cdots\underbrace{d_k\cdots d_k}_{q_k},\underbrace{0\cdots 0}_{s})$$

where

$$c_j' = 2\left(p-\sum_{i=1}^{j-1} p_i\right)-p_j=c_j+1 \quad\text{and}\quad d_j$$

as before

$$2\rho(\mathfrak{u}\cap\mathfrak{p}) = (\underbrace{d_1+1,\ d_1+1\cdots d_1+1}_{p_1}\cdots\underbrace{d_k+1\cdots d_k+1}_{p_k},\underbrace{0\cdots 0}_{r}\,|$$
$$\underbrace{c_1'+1\cdots c_1'+1}_{q_1}\cdots\underbrace{c_k'+1\cdots c_k'+1}_{q_k},\underbrace{0\cdots 0}_{s})$$

$$2\rho(\mathfrak{u}) = (\underbrace{u_1'\cdots u_1'}_{p_1},\underbrace{u_2'\cdots u_2'}_{p_2}\cdots\underbrace{u_k'\cdots u_k'}_{p_k},\underbrace{0\cdots 0}_{r}\,|\,\underbrace{u_1'\cdots u_1'}_{q_1},\underbrace{u_2'\cdots u_2'}_{q_2}\cdots\underbrace{u_k'\cdots u_k'}_{q_k},\underbrace{0\cdots 0}_{s})$$

$$\begin{aligned} u_j' = c_j'+d_j+1 &= 2\left(p-\sum_{i=1}^{j-1} p_i\right)-p_j+2\left(q-\sum_{i=1}^{j-1} q_i\right)-q_j-1+1 \\ &= 2\left(n-\sum_{i=1}^{j-1} n_i\right)-n_j = u_j+1. \end{aligned}$$

Now, if $G = SO_0(2p+1, 2q+1)$, then

$$2\rho(\mathfrak{u}\cap\mathfrak{k}) =$$

$$(\underbrace{c'_1\cdots c'_1}_{p_1}, \underbrace{c'_2\cdots c'_2}_{p_2}\cdots \underbrace{c'_k\cdots c'_k}_{p_k}, \underbrace{0\cdots 0}_{r} \mid \underbrace{d'_1\cdots d'_1}_{q_1}, \underbrace{d'_2\cdots d'_2}_{q_2}\cdots \underbrace{d'_k\cdots d'_k}_{q_k}, \underbrace{0\cdots 0}_{s})$$

where

$$c'_j \text{ is as before } d'_j = d_j+1 = 2\left(q-\sum_{i=1}^{j-1} q_i\right)-q_j$$

$$2\rho(\mathfrak{u}\cap\mathfrak{p}) =$$

$$(\underbrace{d'_1+1\cdots d'_1+1}_{p_1}\cdots \underbrace{d'_k+1\cdots d'_k+1}_{p_k}, \underbrace{0\cdots 0}_{r} \mid \underbrace{c'_1+1\cdots c'_1+1}_{q_1}\cdots \underbrace{c'_k+1\cdots c'_k+1}_{q_k}, \underbrace{0\cdots 0}_{s})$$

$$2\rho(\mathfrak{u}) =$$

$$(\underbrace{u'_1+1\cdots u'_1+1}_{p_1}\cdots \underbrace{u'_k+1\cdots u'_k+1}_{p_k}, \underbrace{0\cdots 0}_{r} \mid \underbrace{u'_1+1\cdots u'_1+1}_{q_1}\cdots \underbrace{u'_k\cdots u'_k+1}_{q_k}, \underbrace{0\cdots 0}_{s})$$

$$u'_j+1 = c'_j+d'_j+1 = 2\left(n-\sum_{i=1}^{j-1} n_i\right)-n_j+1.$$

We can now give the conditions for our lowest K-type μ. Since $\mu\in i(\mathfrak{t}_0^c)^*$, μ determines a θ-stable parabolic subalgebra $\mathfrak{q}(\mu)$. However, this is not quite the parabolic $\mathfrak{q}$ of the $A_{\mathfrak{q}}(\lambda)$ module that μ should determine. By Proposition 4.4 and Lemmas 4.6 and 4.8 of Salamanca-Riba [2] we may assume that our weight μ determines the compact part of the parabolic. Then write

$$\begin{cases} \mathfrak{q}\cap\mathfrak{k} = \mathfrak{q}(\mu)\cap\mathfrak{k} = \mathfrak{u}(\mu)\cap\mathfrak{k}+\mathfrak{l}(\mu)\cap\mathfrak{k} \\ \mu' = \mu+2\rho(\mathfrak{u}\cap\mathfrak{k}) \\ \mathfrak{q}' = \mathfrak{q}(\mu'). \end{cases} \tag{3.5}$$

Note that $\mathfrak{q}' \neq \mathfrak{q}(\mu)$ but their compact parts coincide. An easy argument shows that

PROPOSITION 3.1. *In the above setting write*

$$\mu+2\rho(\mathfrak{u}\cap\mathfrak{k}) = (\underbrace{z_1\cdots z_1}_{p_1}\cdots \underbrace{z_k\cdots z_k}_{p_k}, \underbrace{0\cdots 0}_{r} \mid \underbrace{z_1\cdots z_1}_{q_1}\cdots \underbrace{z_k\cdots z_k}_{q_k}, \underbrace{0\cdots 0}_{s})$$

then V_μ is the LKT of an $A_{\mathfrak{q}}(\lambda)$ iff

$$\begin{cases} \text{If } G=SO_0(2p,2q),\ z_i-z_{i+1}\geqslant n_i+n_{i+1}\ (i=1,2\cdots k-1 \text{ and} \\ z_k\geqslant n_k+2(r+s)-1 \\ \text{If } G=SO_0(2p+1,2q),\ z_i-z_{i+1}\geqslant n_i+n_{i+1} \\ \qquad z_k\geqslant n_k+2(r+s) \\ \text{If } G=SO_0(2p+1,2q+1),\ z_i-z_{i+1}\geqslant n_i+n_{i+1} \\ \qquad z_k\geqslant n_k+2(r+s)+1. \end{cases} \tag{3.6}$$

The proof follows from the conditions on λ and μ given in Salamanca-Riba [2] 4.1 and 4.3. Assume now that μ is not the LKT of an $A_{\mathfrak{q}}(\lambda)$. Rewrite the coordinates of μ as

$$\mu=(\underbrace{a_1\cdots a_1}_{p_1}\cdots\underbrace{a_k\cdots a_k}_{p_k},\underbrace{0\cdots 0}_{r}\,|\,\underbrace{b_1\cdots b_1}_{q_1}\cdots\underbrace{b_k\cdots b_k}_{q_k},\underbrace{0\cdots 0}_{s})$$

where $a_k>0$ and $b_k>0$.

If μ were to satisfy the conditions of Proposition 3.1 then

$$a_k\geqslant\begin{cases} q_k+2s & \text{if } G=SO_0(2p,2q) \text{ or } SO_0(2p+1,2q) \\ q_k+2s+1 & \text{if } G=SO_0(2p+1,2q+1) \end{cases}$$
$$b_k\geqslant\begin{cases} p_k+2r & \text{if } G=SO_0(2p,2q) \\ p_k+2r+1 & \text{if } G=SO_0(2p+1,2q) \text{ or } SO_0(2p+1,2q+1). \end{cases} \tag{3.7}$$

Let $\Delta^+_\mu=\Delta^+(\mathfrak{g},\mathfrak{h})$ be a θ-stable positive root system making $\mu+2\rho_c$ dominant. The roots of $\mathfrak{t}$ in $\mathfrak{g}$, $\Delta(\mathfrak{g},\mathfrak{t})$ are the restriction of $\Delta(\mathfrak{g},\mathfrak{h})$ to $\mathfrak{t}$.

Write $\Pi_\mu=\Pi_\mu(\mathfrak{g},\mathfrak{t})$ for the set of simple roots restricted to $\mathfrak{t}$. Vogan attaches to μ a quasisplit subgroup L_μ and a weight η^μ in $L_\mu\cap K$ which is the highest weight of a fine $(L_\mu\cap K)$-type (see Vogan [4], Proposition 5.3.3 and Definition 5.3.22).

Having in mind this construction we obtain the highest weights of the fine K-types for the quasisplit group $G=SO_0(m,n)$.

G is quasisplit if $|m-n|\leqslant 2$ which gives the following results.

1. If $G=SO_0(2p,2p)$ the fine K-types are

$$\{(\underbrace{0\cdots 0}_{p\text{ times}}\,|\,\underbrace{0\cdots 0}_{p\text{ times}});\ (\underbrace{1\cdots 1}_{r\text{ times}},\underbrace{0\cdots 0}_{p-r}\,|\,\underbrace{0\cdots 0}_{p});\ (\underbrace{0\cdots 0}_{p}\,|\,\underbrace{1\cdots 1}_{r},\underbrace{0\cdots 0}_{p-r})\};$$

2. If $G=SO_0(2p, 2p-2)$,

$$\{\underbrace{0\cdots0}_{p}\,|\,\underbrace{0\cdots0}_{p-1});(\underbrace{0\cdots0}_{p}\,|\,\underbrace{1\cdots1}_{r},\ \underbrace{0\cdots0}_{p-1-r})\};$$

3. If $G=SO_0(2q+1, 2q)$,

$$\{(\underbrace{0\cdots0}_{q}\,|\,\underbrace{0\cdots0}_{q});\ (\underbrace{0\cdots0}_{q}\,|\,\underbrace{1\cdots1}_{r},\underbrace{0\cdots0}_{q-r})\};$$

4. $G=SO_0(2q-1, 2q)$, $\{(\underbrace{0\cdots0}_{q-1}\,|\,\underbrace{0\cdots0}_{q})\}$;

5. $SO_0(2p+1, 2p+1)$,

$$\{(\underbrace{0\cdots0}_{p}\,|\,\underbrace{0\cdots0}_{p});\ (\underbrace{1\cdots1}_{r},\underbrace{0\cdots0}_{p-r}\,|\,\underbrace{0\cdots0}_{p});\ (\underbrace{0\cdots0}_{p}\,|\,\underbrace{1\cdots1}_{r},\underbrace{0\cdots0}_{p-r})\}$$

6. $SO_0(2p+1, 2p-1)$,

$$\{(\underbrace{0\cdots0}_{p}\,|\,\underbrace{0\cdots0}_{p-1});\ (\underbrace{0\cdots0}_{p}\,|\,\underbrace{1\cdots1}_{r},\underbrace{0\cdots0}_{p-r-1})\}$$

This computation follows from the definition of fine K-type.

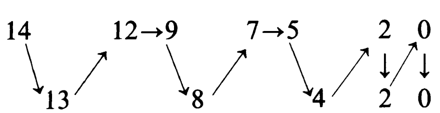
Recall that fine K-types have the property that the quasisplit subgroup attached to them is all of G. With this information we can obtain the subgroup L_μ attached to μ: L_μ will be a product of quasisplit subgroups and η^μ will be fine on each quasisplit factor of L_μ.

Hence, we follow Vogan's algorithm in the fine case and we can conclude that the general picture will be as follows. Write $\mu+2\rho_c=(x_1, x_2\cdots x_p\,|\,y_1, y_2\cdots y_q)$.

We can form an array of 2 rows with the coordinates of $\mu+2\rho_c$ so that they are aligned in decreasing order from left to right, having the first p coordinates in the first row and the last q in the second row. For example if

$$\mu+2\rho_c=(14, 12, 9, 7, 5, 2, 0\,|\,13, 8, 4, 2, 0),$$

the array would look like:



This array gives a choice of Δ_μ^+, compatible with $\Delta^+(\mathfrak{k})$: the simple roots are given by the arrows. (This choice is not unique.) Because the terms in each row decrease by at least 2, the entire array is a union of maximal blocks of the following types. If $G = SO(2p, 2q)$:

1. $$\begin{array}{|ll|} \hline r & r-2\cdots r-2k \\ r & r-2\cdots r-2k \\ \hline \end{array}$$

with $r-2k > 0$

2. $$\begin{array}{ll} r & \cdots r-2k \\ & r-1\cdots \quad r-2k-1 \end{array}, \qquad \begin{array}{ll} & r-1\cdots \quad r-2k-1 \\ r & \cdots r-2k \end{array}$$

with $r-2k > 1$

3. $$\begin{array}{l} r \quad \cdots \quad r-2k \\ r-1\cdots r-2k+1 \end{array}, \qquad \begin{array}{l} r-1\cdots r-2k+1 \\ r \quad \cdots \quad r-2k \end{array}$$

with $r-2k > 1$

4. $$\begin{array}{|ll|} \hline 2m-2\cdots 2 & 0 \\ 2m-2\cdots 2 & 0 \\ \hline \end{array}$$

5. $$\begin{array}{llll} 2m-1 \quad \cdots \quad 2m-2r+1 & & 2m-2r-2\cdots 2 & 0 \\ & 2m-2 \quad \cdots \quad 2m-2r & 2m-2r-2\cdots 2 & 0 \end{array}$$

or

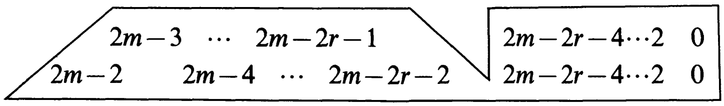
$$\begin{array}{llll} & 2m-2 \quad \cdots \quad 2m-2r & 2m-2r-2\cdots 2 & 0 \\ 2m-1 \quad \cdots \quad 2m-2r+1 & & 2m-2r-2\cdots 2 & 0 \end{array}$$

with $r \geqslant 1$

6. $$\begin{array}{lll} 2m-2 & 2m-4\cdots 2 & 0 \\ & 2m-4\cdots 2 & 0 \end{array}, \qquad \begin{array}{lll} & 2m-4\cdots 2 & 0 \\ 2m-2 & 2m-4\cdots 2 & 0 \end{array}$$

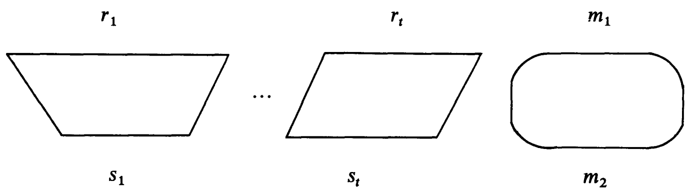
7. $$\begin{array}{llll} 2m-2 & 2m-4\cdots 2m-2r-2 & 2m-2r-4\cdots 2 & 0 \\ & 2m-3 \quad \cdots \quad 2m-2r-1 & 2m-2r-4\cdots 2 & 0 \end{array}$$

or



with $r \geqslant 1$ (3.8)

1, 2 and 3 will give $U(p, q)$ factors of L_μ, 4 and 5, an $SO_0(2m, 2m)$ factor and 6 and 7, an $SO_0(2m, 2m-2)$ factor.
The coordinates of μ can then be grouped by the blocks that $\mu+2\rho_c$ determines as follows. All blocks except the last one on the right will be of type 1 to 3. The last one will be of any of the types 4 to 7.



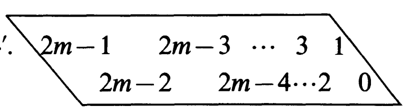
(3.9)

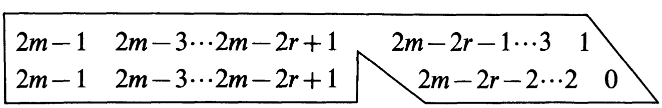
with $|p_i - q_i| \leqslant 1$ and $|m_1 - m_2| \leqslant 1$.
If $G = SO(2p+1, 2q)$, $\mu + 2\rho_c$ will give the following types of pictures.

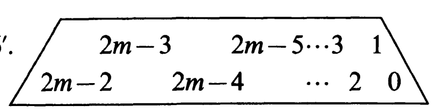
1′. Like type 1 but with $r - 2k > 1$

2′. Same as type 2

3′. Like type 3, with $r - 2k > 0$

4′. 

5′. 

6′. 

(3.8′)

Again, as in the case of $SO_0(2p, 2q)$, 1′, 2′, 3′ give also $U(p, q)$ factors of L_μ; 4′, 5′ give an $SO_0(2m+1, 2m)$ factor and 6′ gives an $SO_0(2m-1, 2m)$ factor. Also, the coordinates of μ can be grouped to give a picture of the form (3.9).

If $G = SO(2p+1, 2q+1)$, we will have the following pictures.

1″. As type 1, with $r-2k > 1$

2″. As type 2, with $r-2k-1 > 1$

3″. Same as type 3

4″.

$$\begin{array}{ccc} 2m-1 & 2m-3\cdots 3 & 1 \\ 2m-1 & 2m-3\cdots 3 & 1 \end{array}$$

5″.

$$\begin{array}{cccc|cc} 2m & 2m-2 & \cdots & 2m-2r+2 & 2m-2r-1\cdots 3 & 1 \\ & 2m-1 & 2m-3\cdots 2m-2r+1 & & 2m-2r-1\cdots 3 & 1 \end{array}$$

or

$$\begin{array}{ccc|cc} 2m-1 & 2m-3\cdots 2m-2r+1 & & 2m-2r-1\cdots 3 & 1 \\ 2m & 2m-2\cdots 2m-2r+2 & & 2m-2r-1\cdots 3 & 1 \end{array}$$

with $r \geqslant 1$

6″.

$$\begin{array}{ccc} 2m-1 & 2m-3\cdots 3 & 1 \\ & 2m-3\cdots 3 & 1 \end{array} \quad \text{or} \quad \begin{array}{ccc} & 2m-3\cdots 3 & 1 \\ 2m-1 & 2m-3\cdots 3 & 1 \end{array}$$

7″.

$$\begin{array}{ccccc|cc} 2m-1 & 2m-3\cdots & 2m-2r+1 & & 2m-2r-1 & 2m-2r-3\cdots 3 & 1 \\ & 2m-2 & 2m-4\cdots & 2m-2r & & 2m-2r-3\cdots 3 & 1 \end{array}$$

or

$$\begin{array}{cccc|cc} 2m-2 & 2m-4\cdots & 2m-2r & & 2m-2r-3\cdots 3 & 1 \\ 2m-1 & 2m-3\cdots 2m-2r+1 & & 2m-2r-1 & 2m-2r-3\cdots 3 & 1 \end{array}$$

with $r \geqslant 1$. (3.8″)

Once again 1″, 2″, 3″ correspond to $U(p, q)$ factors; 4″, 5″, to $SO_0(2m+1, 2m)$ and 6″, 7″ to $SO_0(2m-1, 2m)$ and we get a picture like (3.9).

The above discussion proves the following

PROPOSITION 3.2. *Let $G = SO_0(n, m)$ and X a Harish-Chandra module of G with a lowest K-type of highest weight μ and L_μ, the quasisplit subgroup attached to μ by Vogan. If μ gives picture* (3.9), *then*

$$L_\mu \cong U(r_1, s_1) \times U(r_2, s_2) \times \cdots \times U(r_t, s_t) \times SO_0(N, M), \tag{3.10}$$

where $N \equiv n(\text{mod}\, 2)$, $M \equiv m(\text{mod}\, 2)$, $|N-M| \leqslant 2$ and $|r_i - s_i| \leqslant 1$.

Set

$$L_1 = U(R,S), \quad R = \sum_{i=1}^{t} r_i, \quad S = \sum_{i=1}^{t} s_i; \qquad L_2 = SO_0(N,M) \tag{3.11}$$

then $L = L_1 \times L_2 \supseteq L_\mu$ and if $\mathfrak{l} = \mathfrak{l}_0 \otimes \mathbb{C}$, $\mathfrak{l} \supseteq \mathfrak{l}_\mu$ and we can define a subalgebra $\mathfrak{u} \subset \mathfrak{u}_\mu$ by $\mathfrak{u}_\mu = \mathfrak{u} + (\mathfrak{u}_\mu \cap \mathfrak{l})$ then $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \supset \mathfrak{q}_\mu$ and by induction by stages (a) of Theorem 2.2 holds, i.e., there is an $(\mathfrak{l}, L\cap K)$-module X_L, such that X occurs only once as a composition factor of $R_{\mathfrak{q}}^{\dim(\mathfrak{u}\cap\mathfrak{l})}(X_L)$. We can see X_L as the exterior tensor product $X_L = X_{L_1} \otimes X_{L_2}$ with X_{L_i} an $(\mathfrak{l}_i, L_i \cap K)$-module. By Corollary 5.3 in Salamanca-Riba [2], (b) of Theorem 2.2 holds also. Set $\mu^L = \mu - 2\rho(\mathfrak{u}\cap\mathfrak{p})$ and $\mu_i = \mu^L|_{L_i}$, then μ^L(resp. μ_i) is the highest weight of a lowest $(L\cap K)$-type of X_L (resp. X_{L_i}).

LEMMA 3.3. *With notation as in the preceding paragraph, if X is unitary, with infinitesimal character satisfying* (1.1), *then there is a parabolic subalgebra* $\mathfrak{q}_1 \subseteq \mathfrak{l}_1$ *and a character* $\lambda_1 : L_1 \to \mathbb{C}$ *such that*

$$X_{L_1} \cong A_{\mathfrak{q}_1}(\lambda_1).$$

Proof. By Theorem 5.7 of Salamanca-Riba [2], if $X_{L_1} \not\cong A_{\mathfrak{q}_1}(\lambda_1)$ for any $\mathfrak{q}_1, \lambda_1$, then there is $\beta \in \Delta(\mathfrak{l}_1 \cap \mathfrak{p})$ such that $\langle\, , \rangle^L$ is indefinite on the sum $V_{\mu_1} \oplus V_{\mu_1+\beta}$.

By Theorem 2.3 we need to check that for this $\beta \in \Delta(\mathfrak{l}_1 \cap \mathfrak{p})$, $\mu + \beta$ is dominant for $\Delta^+(\mathfrak{k})$. This will show that X is not unitary.

If $\mu = (a_1, a_2, \ldots, a_R, a_{R+1} \cdots a_p \,|\, b_1, b_2 \cdots b_S, b_{S+1} \cdots b_q)$ with R and S as in (3.11), then clearly, since

$$\Delta(\mathfrak{l}_1 \cap \mathfrak{p}) = \begin{cases} B = \{\pm(e_i \pm e_j) \,|\, 1 \leqslant i \leqslant R;\ p < j \leqslant S\} & \text{if } n, m, \text{ even} \\ BU\{\pm e_k \,|\, p < k \leqslant S\} & \text{if } n, \text{ odd} \\ BU\{\pm e_k \,|\, 1 \leqslant k \leqslant R \text{ or } p < k \leqslant S\} & \text{if } n, m, \text{ odd} \end{cases}$$

$\mu + \beta$ is dominant unless $a_R = a_{R+1}$ or $b_S = b_{S+1}$. Note that a_{R+1} and b_{S+1} are either 0 or 1. If $R = S = 0$, then we have a fine K-type for $SO_0(n, m)$ and by Lemma 2.6 X is not unitary. Then if either R or S are non-zero, this means that μ does not satisfy (3.7) and we can prove that X will not be unitary using the following 2 lemmas.

LEMMA 3.4. *Let* $\mu \in i\mathfrak{t}_0^*$ *dominant for* $\Delta^+(\mathfrak{k}, \mathfrak{t})$ *and* Δ_μ^+ *as defined above. Suppose that*

$$(\mu + 2\rho_c, \delta^\vee) \leqslant 2, \forall \delta \in \Delta_\mu^+ \quad \textit{simple}, \tag{3.12}$$

then Dirac inequality fails on μ.

LEMMA 3.5. *We keep the notation of the above lemmas. Suppose that either $a_R = a_{R+1}$ or $b_S = b_{S+1}$. Let $\{\delta_1, \delta_2, \ldots, \delta_d\}$ be an ordering of Π_μ given by the blocks that $\mu + 2\rho_c$ determines and let t be the smallest integer, such that $1 \leqslant t \leqslant d$ and $(\mu + 2\rho_c, \check{\delta_i}) \leqslant 2$ for all $i \geqslant t$. Then X is not unitary.*

Proof of Lemma 3.4. We first prove the following

Claim. $(\mu - \rho_{n,\mu}, \check{\alpha}) \leqslant 0$ for all simple $\alpha \in \Delta^+(\mathfrak{k})$ where $\rho_{n,\mu} = \rho_\mu - \rho_c$ and $\rho_\mu = \rho(\Delta^+_\mu)$.

(a) If α is also simple for Δ^+_μ, then $(\mu + 2\rho_c, \check{\alpha}) = (\mu, \check{\alpha}) + 2(\rho_c, \check{\alpha}) = (\mu, \check{\alpha}) + 2 \Rightarrow (\mu, \check{\alpha}) = 0$ and $(\mu - \rho_{n,\mu}, \check{\alpha}) = (-\rho_{n,\mu}, \check{\alpha}) < 0$, since $\rho_{n,\mu}$ is $\Delta^+(\mathfrak{k})$-dominant.

(b) If α is not simple for Δ^+_μ and $(\mu, \check{\alpha}) = 0$ we still can argue as in (i).

(c) If α is not simple for Δ^+_μ and $(\mu, \check{\alpha}) > 0$ then we need to look at the blocks of simple roots that $\mu + 2\rho_c$ determines:

As before write Π_μ for the simple roots of Δ^+_μ. From Figures 3.8 (resp. 3.8″) and 3.9 we may deduce that if $(\mu, \check{\alpha}) > 0$ for some compact simple root α, not in Π_μ, then either

$$\begin{cases} \text{(i)} \ \alpha = \beta_1 + \beta_2, \\ \text{(ii)} \ \alpha = \beta_1 + \alpha_1 + \beta_2, \quad \text{or} \\ \text{(iii)} \ \alpha = \beta_1 + \alpha_1 + \alpha_2 + \beta_2 \end{cases} \tag{3.13}$$

where the β_j's are non compact and the α_i's, compact roots in Π_μ. We have (again from Figures 3.8, 3.8″ and 3.9), in case (i),

$$(\mu + 2\rho_c, \check{\alpha}) \leqslant (\mu + 2\rho_c, \check{\beta_1}) + (\mu + 2\rho_c, \check{\beta_2}) \leqslant 3,$$

in case (ii),

$$(\mu + 2\rho_c, \check{\alpha}) \leqslant (\mu + 2\rho_c, \check{\beta_1} + \check{\alpha_1} + \check{\beta_2}) \leqslant 4,$$

in case (iii),

$$(\mu + 2\rho_c, \check{\alpha}) \leqslant (\mu + 2\rho_c, \check{\beta_1} + \check{\alpha_1} + \check{\alpha_2} + \check{\beta_2}) \leqslant 5$$

i.e., in these cases, $\alpha = \Sigma_{i=1}^k \gamma_i$ $(k = 2, 3, 4)$, and

$$(\mu + 2\rho_c, \check{\alpha}) \leqslant k + 1, \tag{3.14}$$

moreover, $(\rho_\mu, \alpha^\vee) = k$, since ρ_μ decreases by one on each simple root in Π_μ. Then

$$(\mu - \rho_{n,\mu}\alpha^\vee) = (\mu - (\rho_\mu - \rho_c), \alpha^\vee) = (\mu + 2\rho_c - \rho_\mu - \rho_c, \alpha^\vee)$$
$$= (\mu + 2\rho_c, \alpha^\vee) - (\rho_\mu, \alpha^\vee) - (\rho_c, \alpha^\vee) \leqslant k + 1 - k - 1 = 0.$$

Hence $\mu - \rho_{n,\mu}$ is dominant for $-\rho_c$ and the claim follows for $SO_0(2p, 2q)$ and $SO_0(2p+1, 2q+1)$.

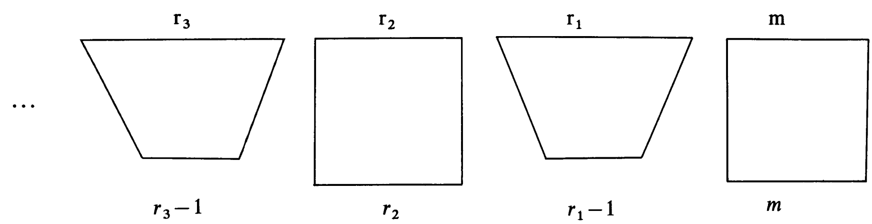
From Figures 3.8′ and 3.9, when $G = SO_0(2p+1, 2q)$, if $(\mu, \alpha^\vee) > 0$ for α compact simple not in Π_μ, then only cases (i) and (ii) in 3.13 hold, i.e., either

(i) $\alpha = \beta_1 + \beta_2$,
(ii) $\alpha = \beta_1 + \alpha_1 + \beta_2$

and again (3.14) holds by inspection of Figures 3.8′ and 3.9. Hence, since $(\rho_\mu, \alpha^\vee) = k$ in this case also, the claim follows for $SO_0(2p+1, 2q)$. Now, since $\mu - \rho_{n,\mu}$ is dominant for $-\Delta^+(\mathfrak{k})$ then the Dirac expression becomes

$$\mu - \rho_{n,\mu} - \rho_c = \mu - (\rho_\mu - \rho_c) - \rho_c = \mu - \rho_\mu.$$

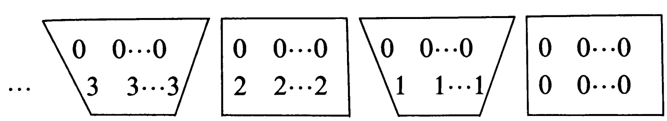
Then, we will prove Lemma 3.4 if we can show that $(\mu - \rho_\mu, \mu - \rho_\mu) < (\rho_\mu, \rho_\mu)$, whenever μ satisfies 3.12. Again, from Figures 3.8 and 3.9, we may conclude that if μ satisfies 3.12 then, at worst we may have



(3.15)

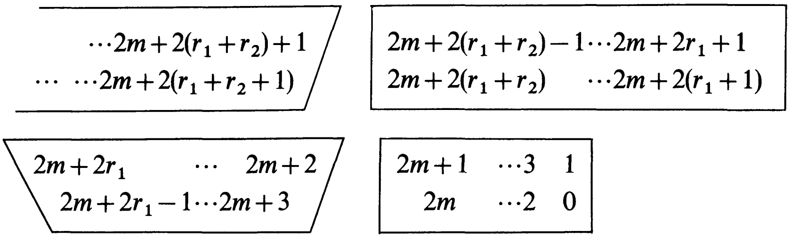
where the jumps from one block to the other are exactly 2 on the top row.

If we write the coordinates of μ in two rows, respecting the blocks to which they correspond, we get



(3.16)

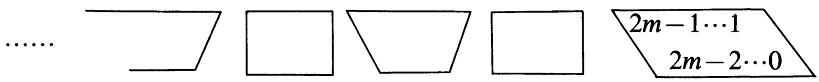
and writing the coordinates of ρ_μ in these blocks we have



(3.17)

Clearly $(\mu-\rho_\mu, \mu-\rho_\mu) < (\rho_\mu, \rho_\mu)$ unless $r_1 = r_2 = \cdots = r_k = 0$. This is the worst case since any other configuration of blocks will make $(\mu-\rho_\mu, \mu-\rho_\mu)$ at least as bad as this one.

From Figure 3.8′ the worst situation is



and the K-type μ will have the same coordinates as in 3.16. So clearly if we arrange ρ_μ as in 3.17 we will have

$$(\mu-\rho_\mu, \mu-\rho_\mu) < (\rho_\mu, \rho_\mu).$$

Similarly, the worst case from 3.8″ and 3.9 is when the last block of the sequence in 3.15 is of type 4″ (in 3.8″) and Lemma 3.4 follows then for $SO_0(2p+1, 2q+1)$.

Proof of Lemma 3.5. The set of roots $\Pi_1 = \{\gamma_1, \ldots, \gamma_{t-1}\}$ forms a group $L_1' = U(R', S')$ and $\Pi_2 = \{\gamma_t, \ldots, \gamma_d\}$ form a group $L_2' = SO(N', M')$. Simply take in Figure 3.9 all the blocks, starting from right to left up to the first time that the jump to the left from one block to the next is greater than 2. These blocks will give L_2, then take all the roots of the leftover blocks to form L_1'.

If $L' = L_1' \times L_2'$, then $L' \supset L$ and there is $\mathfrak{q}' \supset \mathfrak{q}$ such that (a) of Theorem 2.2 holds. (By Vogan [4], 6.5.9(g), 6.5.12(b) and 6.3.10). Corollary 5.3 in part I implies then that $X_{L'}^h$ has a Hermitian form $\langle\, , \rangle^{L'}$ i.e. (b) of Theorem 2.2 holds.

Also $\mu^{L'} = \mu - 2\rho(\mathfrak{u}' \cap \mathfrak{p})$ is the LKT of $X_{L'}$, and by Lemma 3.4, Dirac inequality fails on $\mu_2' = \mu^{L'}|_{L_2'}$.

By Lemma 6.3(a) (part I), there is a K-type $V_{\eta_2'}$ in $V_{\mu_2'} \otimes (\mathfrak{l}_2' \cap \mathfrak{p})$ that makes $\langle\, , \rangle^{L'}|_{V_{\eta_2'} \otimes V_{\mu_2'}}$ indefinite. Since $(\mu + 2\rho_c, \check{\gamma}_{t-1}) > 2$ and $\alpha \in \Delta(\mathfrak{u}' \cap \mathfrak{k})$ is expressed as a sum

$$\alpha = \sum_{\gamma \in B} \gamma$$

with $B \subseteq \Pi_\mu$ and $B \cap \Pi_i \neq \phi$, $i = 1, 2$ then, since the ordering of Π_μ is given by the blocks determined by $\mu+2\rho_c$, $\gamma_{t-1}\in B$ and $(\mu+2\rho_c,\alpha^{\vee})>2$. Hence $(\mu,\alpha^{\vee})>0$ and therefore, since $\eta=\eta_2+2\rho(\mathfrak{u}'\cap\mathfrak{p})=\mu+\beta'$ for some $\beta'\in\Delta(\mathfrak{l}_2'\cap\mathfrak{p})$ we have that η is dominant for $\Delta(\mathfrak{u}'\cap\mathfrak{k})$ and (c) of Theorem 2.2 holds. By Theorem 2.3, X is not unitary.

This proves Lemma 3.5. Now, to conclude the proof of Lemma 3.3, we just argue as in the proof of Lemma 3.5, with $L=L_1\times L_2$ and $\beta\in\Delta(\mathfrak{l}_1\cap\mathfrak{p})$. Since $\mu+\beta$ is dominant, if $X_{L_1}\ncong A_{\mathfrak{q}_1}(\lambda_1)$ then X is not unitary.

We will conclude the proof of Theorem 2.2 if we prove the following

LEMMA 3.6. *We use the notation of* 3.11 *and Lemma* 3.3 *and its proof. Assume* $X_{L_1}\cong A_{\mathfrak{q}_1}(\lambda_1)$, *with* $a_R>a_{R+1}$ *and* $b_S>b_{S+1}$ *then Theorem* 2.2 *is true.*

Proof. If $a_R>a_{R+1}$ and $b_S>b_{S+1}$ we will consider two cases.

(1) Suppose first that $\mu_2=\mu^L|_{L_2}$ is trivial. Then Dirac inequality fails on μ_2 unless the infinitesimal character γ_2, of X_{L_2} is $\rho_{\mathfrak{l}_2}$. If $\gamma_2\neq\rho_{\mathfrak{l}_2}$ then by Lemma 6.3 in Salamanca-Riba [2] we can find $\beta\in\Delta(\mathfrak{l}_2\cap\mathfrak{p})$ such that the signature of the form of X_{L_2} on V_{μ_2} and $V_{\mu_2+\beta}$ is indefinite. Clearly if $\mu_2+\beta$ is dominant for $\Delta^+(\mathfrak{l}_2\cap\mathfrak{k})$ then, by the hypothesis of the lemma, $\mu+\beta$ is dominant for $\Delta^+(\mathfrak{k})$, and by Theorem 2.3, X is not unitary.

Now, if $\gamma_2=\rho_{\mathfrak{l}_2}$ then the Langland's subquotient of X_{L_2} is the trival rep. Hence the Langlands subquotient of $\mathscr{R}_{\mathfrak{q}}(X_{L_1}\otimes X_{L_2})=\mathscr{R}_{\mathfrak{q}}(X_{L_1})\otimes\mathscr{R}_{\mathfrak{q}}(X_{L_2})$ is $X\cong\mathscr{R}_{\mathfrak{q}}(A_{\mathfrak{q}_1}(\lambda_1))\otimes R_{\mathfrak{q}}(\text{trivial})$. By induction by stages, X is an $A_{\mathfrak{q}}(\lambda)$ module.

(2) Now suppose that μ^2 is not trivial. Since μ^2 is fine, by Lemma 2.6, Dirac inequality fails on μ_2 and again, by the above argument in the case μ^2 is trivial, we can show that $\langle\,,\rangle$ will be indefinite. This concludes the proof of Theorem 2.2.

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