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YVES LAURENT

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Microlocal operators with plurisubharmonic growth

YVES LAURENT
Université de Grenoble I, Institut Fourier, Lab. de Maths, B.P. 74, 38402 St. Martin D'Heres, France

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Introduction

The works of Malgrange [13], [14] and Ramis [15] have shown that the classification of the differential equations in the complex plane is closely related to the growth of their solutions. They used the growth of formal power series but if we want to study the case of several variables we have to use microlocal analysis, that is to consider sheaves on the cotangent bundle. In fact the good object seems to be the sheaf of real holomorphic microfunctions $\mathscr{C}^{\mathbb{R}}_{Y|X}$ of [16] and, more precisely, a family of sheaves of the same kind with suitable growth conditions.

The aim of this paper is to define these "real holomorphic microfunctions with growth", prove some cohomological statements and then microlocalize them to get "2-microlocal operators".

We made such a construction in a special case in [11] working with formal microfunctions and obtained a construction of the vanishing cycles of a holonomic $\mathscr{D}$-module. In the next paper [12], we will use the microfunctions and the 2-microlocal operators to construct the irregular vanishing cycles. From this, we will be able to describe precisely the growth of the solutions and get the index theorems which generalize the results of Ramis [15] in the one dimensional case.

In [16], $\mathscr{C}^{\mathbb{R}}_{Y|X}$ was defined as a local cohomology group, that is $\mathscr{C}^{\mathbb{R}}_{Y|X}$ is the microlocalization of the sheaf $\mathscr{O}_X$ of holomorphic functions on the complex manifold X along the submanifold Y of X. In the same way, Andronikof [1] defined a tempered microlocalization and, from it, he deduced the sheaf $\mathscr{C}^{\mathbb{R},f}_{Y|X}$ of real microfunctions with polynomial growth.

We consider here another point of view, which was initiated by Boutet de Monvel [3]: the sheaf is defined locally by a symbolic calculus and explicit formulas are given for the action of a coordinate transformation. Aoki used this method to define some subsheaves of $\mathscr{C}^{\mathbb{R}}_{Y|X}$ in [2].

Boutet de Monvel worked in little neighborhoods of the real domain and Aoki in the germs of the sheaf $\mathscr{C}^{\mathbb{R}}_{Y|X}$. But here we want to define symbols of the new sheaves on global open sets and, moreover, we need results about the vanishing of some cohomology groups to define the sheaves of 2-microlocal operators. That is why we have to consider growths given by plurisubharmonic weights. Then the theorems we need are deduced from the well known results of Hörmander [5] about the cohomology with bounds of holomorphic functions.

Section 1 of this paper is devoted to some results about cohomology with bounds on open conic subsets of $\mathbb{C}^n$.

In the second section, we define the microlocal symbols and show how coordinate transforms act on them. Then we can define the sheaves of real holomorphic microfunctions $\mathscr{C}^{\mathbb{R}}_{Y|X}(r,s)$, $1 \leqslant s \leqslant r \leqslant +\infty$, the microlocal operators and the microdifferential operators. We end the section with cohomological results about these sheaves that are needed in the next section.

In Section 3, we define the 2-microlocal operators as a microlocalization of the real holomorphic microfunctions. Then we establish a symbolic calculus for them and prove that they are invariant under quantized canonical transformations.

Using microlocalization in this definition allows us to consider 2-microlocal operators in the general case of arbitrary lagrangian manifold where no symbolic calculus is possible.

0. Some notations

A lot of different sheaves have been introduced in [16], in [10] and some others will be defined here. This section presents these sheaves and their mutual connections and fixes some notations:

X: a complex analytic manifold of dimension n.
Y: a complex submanifold of X of codimension d.
T^*X: the cotangent bundle to X.
$\dot{T}^*X$: the cotangent bundle without the zero section: $\dot{T}^*X = T^*X - X$.
$\mathbb{P}^*X \simeq \dot{T}^*X/\mathbb{C}^*$ the projective cotangent bundle
$\widehat{\mathbb{P}^*X} = T^*X/\mathbb{C}^* \simeq \mathbb{P}^*X \cup X$.
$\left.\begin{array}{l}\gamma\colon \dot{T}^*X \to \mathbb{P}^*X \\ \hat{\gamma}\colon T^*X \to \widehat{\mathbb{P}^*X}\end{array}\right\}$ the canonical projections
$T^*_Y X$: the conormal bundle to Y that is the kernel of $(T^*X) \times_X Y \to T^*Y$.
$\dot{T}^*_Y X$, $\mathbb{P}^*_Y X$, $\widehat{\mathbb{P}^*_Y X}$, γ_Y, $\hat{\gamma}_Y$ are defined from $T^*_Y X$ as the corresponding object from T^*X.
$\mathscr{O}_X$: the sheaf of holomorphic functions on X.
$\mathscr{D}_X$: the sheaf of differential operators with coefficients in $\mathscr{O}_X$.
$\mathscr{C}^{\mathbb{R}}_{Y|X}$: the sheaf of "real holomorphic microfunctions" it is defined in [16] and [9] as the microlocalization of $\mathscr{O}_X$ along Y:

$$\mathscr{C}^{\mathbb{R}}_{Y|X} = \mu^d_Y(\mathscr{O}_X).$$

(Another definition will be given in Section 2.2). It is a sheaf on T_Y^*X.
$\mathscr{C}_{Y|X}^{\infty} = \hat{\gamma}_Y^{-1}\hat{\gamma}_{Y*}\mathscr{C}_{Y|X}^{\mathbb{R}}$.
$\mathscr{B}_{Y|X}^{\infty} = \mathscr{H}_Y^d(\mathscr{O}_X)$.
By the definitions we have:
$\mathscr{C}_{Y|X}^{\infty}|_{\dot{T}_Y^*X} = \gamma_Y^{-1}\gamma_{Y*}(\mathscr{C}_{Y|X}^{\mathbb{R}}|_{\dot{T}_Y^*X})$ and $\mathscr{B}_{Y|X}^{\infty} = \mathscr{C}_{Y|X}^{\infty}|_Y = \mathscr{C}_{Y|X}^{\mathbb{R}}|_Y$.
If X is identified to the diagonal of $X \times X$, the space $T_X^*(X \times X)$ is isomorphic to T^*X and thus the sheaf $\mathscr{C}_{X|X\times X}^{\mathbb{R}}$ is a sheaf on T^*X.
$\mathscr{E}_X^{\mathbb{R}} = \mathscr{C}_{X|X\times X}^{\mathbb{R}} \otimes_{\mathscr{O}_{X\times X}} \Omega_{X\times X}^{(0,n)}$
where $\Omega_{X\times X}^{(0,n)}$ is the sheaf of holomorphic differential forms on $X \times X$ which are of degree 0 in the first copy of X and of maximum degree n in the second.
$\mathscr{E}_X^{\infty} = \hat{\gamma}^{-1}\hat{\gamma}_*\mathscr{E}_X^{\mathbb{R}} = \mathscr{C}_{X|X\times X}^{\infty} \otimes_{\mathscr{O}_{X\times X}} \Omega_{X\times X}^{(0,n)}$
$\mathscr{D}_X^{\infty} = \mathscr{E}_X^{\mathbb{R}}|_X = \mathscr{E}_X^{\infty}|_X$.
$\mathscr{D}_X^{\infty}$ is the sheaf of differential operators of infinite order.

These sheaves were defined in [16] but $\mathscr{E}_X^{\mathbb{R}}$ and $\mathscr{E}_X^{\infty}$ were denoted respectively by $\mathscr{P}_X^{\mathbb{R}}$ and $\mathscr{P}_X$.

There exists other triples with the same mutual relations as $(\mathscr{C}_{Y|X}^{\mathbb{R}}, \mathscr{C}_{Y|X}^{\infty}, \mathscr{B}_{Y|X}^{\infty})$:

(1) $(\mathscr{C}_{Y|X}^{\mathbb{R},f}, \mathscr{C}_{Y|X}, \mathscr{B}_{Y|X})$: the first sheaf is defined in [1], the two others in [16].
(2) $(\mathscr{E}_X^{\mathbb{R},f}, \mathscr{E}_X, \mathscr{D}_X)$.

In this paper we will define new triples:

(a) $(\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi, r), \mathscr{C}_{Y|X}(\varphi, r), \mathscr{B}_{Y|X}(\varphi, r))$ where φ is a plurisubharmonic function on T_Y^*X and $r \in \mathbb{R}$, $r \geqslant 1$.
(b) $(\mathscr{C}_{Y|X}^{\mathbb{R}}(r, s), \mathscr{C}_{Y|X}(r, s), \mathscr{B}_{Y|X}(r, s))$ is a special case of the preceding with $\varphi(x, \xi) = |\xi|^{1/r}$.

In each case there exists some $\mathscr{E}_X^{\mathbb{R}}(*)\ldots$,

$\mathscr{C}_{Y|X}^{\mathbb{R}}$ is obtained when $\varphi = |\xi|$ and $r = 1$ and $\mathscr{C}_{Y|X}^{\mathbb{R},f}$ when $r = \infty$.

Let us now come to the second microlocalization. We denote by $\Lambda = T_Y^*X$ and Δ the diagonal of $\Lambda \times \Lambda$. Then if $\mathscr{F}$ is a sheaf on $\Lambda \times \Lambda$ we can consider its microlocalization $\mu_\Delta^n(\mathscr{F})$ which is a sheaf on $T^*\Lambda$ and then looking at the canonical map $\hat{\gamma}$: $T^*\Lambda \to \widehat{\mathbb{P}^*\Lambda}$ consider $\hat{\gamma}^{-1}\hat{\gamma}_*\mu_\Delta^n(\mathscr{F})$.

This has been done in [10] with the sheaves $\mathscr{F} = \mathscr{C}_{Y\times Y|X\times X}^{\infty}$ and $\mathscr{F} = \mathscr{C}_{Y\times Y|X\times X}(r, s)$. We got:

$$\mathscr{E}_\Lambda^{2\mathbb{R}} = \mu_\Delta^n(\mathscr{C}_{Y\times Y|X\times X}^{\infty}) \otimes_{\mathscr{O}_{X\times X}} \Omega_{X\times X}^{(0,n)} \quad \text{and} \quad \mathscr{E}_\Lambda^{2\infty} = \hat{\gamma}^{-1}\hat{\gamma}_*\mathscr{E}_\Lambda^{2\mathbb{R}}.$$

In this paper we want to do the same with $\mathscr{C}_{Y\times Y|X\times X}^{\mathbb{R}}$ and the sheaves $\mathscr{C}_{Y\times Y|X\times X}^{\mathbb{R}}(\varphi, r)$, we will get:

$$\mathscr{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(*) = \mu_\Delta^n(\mathscr{C}_{Y\times Y|X\times X}^{\mathbb{R}}(*)) \otimes_{\mathscr{O}_{X\times X}} \Omega_{X\times X}^{(0,n)}$$

and

$$\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*) = \hat{\gamma}^{-1}\hat{\gamma}_*\mathscr{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(*).$$

Here $(*)$ means one of the symbols (φ, r), (r, s) etc. and in fact from these four types of sheaves we will be interested in $\mathscr{E}_\Lambda^{2\infty}$ (cf. [10]) and in $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$ for which we will define a symbolic calculus.

All these definitions remain valid when $\Lambda = T_Y^*X$ is replaced by an arbitrary homogeneous lagrangian submanifold of T^*X as it will be seen in Section 3.

1. Cohomology with bounds on conic open sets

Using Hörmander's methods we extend to cohomology with bounds the classical results on holomorphic functions.

1.1. *Holomorphic functions*

Let U_0 be an open subset of $\mathbb{C}^n$ with coordinates $x=(x_1,\dots,x_n)$ and Γ_0 be an open $\mathbb{R}$-conic subset of $\mathbb{C}^{n'}\backslash\{0\}$ with coordinates $\xi=(\xi_1,\dots,\xi_{n'})$. ($\mathbb{R}$-conic means that Γ_0 is stable under the action of $\mathbb{R}_+^*$ on $\mathbb{C}^{n'}\backslash\{0\}$). We denote by λ the Lebesgue measure on $\mathbb{C}^{n+n'}$.

Throughout this section an open subset V of $U=U_0\times\Gamma_0$ will be $\mathbb{R}$-conic if it is invariant under real positive homotheties in ξ and we will write $V' \Subset V$ if $V'\cap\{|\xi|=1\}$ is relatively compact in V.

Let φ be a plurisubharmonic function on U. For $m\in\mathbb{R}$, V conic open subset of U and $f\in L^2_{\text{loc}}(V)$ we set:

$$\|f\|^2_{V,\varphi,m} = \int_V |f|^2\,e^{-2\varphi}\frac{d\lambda}{(1+|\xi|^2)^m}.$$

Let $W(\varphi, V)$ be the set of the functions $f\in L^2_{\text{loc}}(V)$ such that, for every $V'\Subset V$, the norm $\|f\|_{V',\varphi,m}$ is finite for m large enough and let $S(\varphi, V)$ be the subset of $W(\varphi, V)$ of functions which are holomorphic on V.

When q belongs to $[0,\dots,n+n']$, $W^{(0,q)}(\varphi, V)$ denotes the set of differential forms of type $(0,q)$ with coefficients in $W(\varphi, V)$ and $\tilde{W}^{(0,q)}(\varphi, V)$ is the subset of the forms u such that $\bar\partial u\in W^{(0,q+1)}(\varphi, V)$.

We denote by $\mathscr{S}_U(\varphi)$, $\mathscr{W}_U^{(0,q)}(\varphi)$ and $\tilde{\mathscr{W}}_U^{(0,q)}(\varphi)$ the sheaves on U whose sections on conic open sets V of U are respectively $S(\varphi, V)$, $W^{(0,q)}(\varphi, V)$ and $\tilde{W}^{(0,q)}(\varphi, V)$ (the index U is often omitted).

Theorem 4.4.2 of [5] proves that we have an exact sequence

$$0 \to \mathscr{S}(\varphi) \to \tilde{\mathscr{W}}^{(0,0)}(\varphi) \xrightarrow{\bar\partial} \tilde{\mathscr{W}}^{(0,1)}(\varphi) \to \cdots \xrightarrow{\bar\partial} \tilde{\mathscr{W}}^{(0,n+n')}(\varphi) \to 0$$

and therefore a resolution of $\mathscr{S}(\varphi)$ by soft sheaves (with respect to the conic topology).

LEMMA 1.1.0. *If V is a $\mathbb{R}$-conic open subset of U the following conditions are equivalent:*

(i) *V is a domain of holomorphy.*
(ii) *There exists a continuous plurisubharmonic function ψ on V which is real homogeneous of degree 0 in ξ and such that:*

$$\forall c > 0, \quad \Gamma_c = \{(x, \xi) \in V / \psi(x, \xi) \leqslant c\} \Subset V.$$

In this case, we will say that ψ is a *homogeneous exhaustion* of V.

Proof. If ψ satisfies (ii) $\sup(\psi(x, \xi), |\xi|)$ is an ordinary plurisubharmonic exhaustion of V which proves (i).

Conversely if V is a domain of holomorphy we define the following function on $V \times \mathbb{C}^{n+n'}$:

$$\delta_V(x, \xi; y, \eta) = \sup\{r > 0 / \forall \tau \in \mathbb{C}, |\tau| < r, (x + \tau y, \xi + \tau\eta) \in V\}.$$

It is known that, when V is pseudoconvex, $-\log \delta_V$ is plurisubharmonic on $V \times \mathbb{C}^{n+n'}$ ([4] Theorem 1.7.5).

If $B(r) = \{(y, \eta) \in \mathbb{C}^{n+n'} / |(y, \eta)| < r\}$ then

$$\delta_V(x, \xi) = \inf_{(y,\eta)\in B(1)} \delta_V(x, \xi; y, \eta)$$

is the euclidean distance to the complementary of V.

Now we define, for $j = 1, \ldots, n'$:

$$\delta_V^j(x, \xi) = \inf_{(y,\eta)\in B(1)} \delta_V(x, \xi; y, \eta\xi_j)$$

and

$$\delta_V^h(x, \xi) = \inf_{1 \leqslant j \leqslant n'} \delta_V^j(x, \xi).$$

If V is $\mathbb{R}$-conic $\delta_V(x, \xi; y, \eta\xi_j)$ is real homogeneous of degree 0 in ξ hence the same is true for $\delta_V^h(x, \xi)$.

On the other hand, if $|\xi_j| \leqslant 1$ we have:

$$B(|\xi_j|) \subset \{(y, \eta\xi_j)/(y, \eta \in B(1)\} \subset B(1)$$

hence

$$\delta_V(x, \xi) \leqslant \delta_V^j(x, \xi) \leqslant \frac{1}{|\xi_j|}\delta_V(x, \xi)$$

and if $|\xi| = 1$ we have:

$$\delta_V(x, \xi) \leqslant \delta_V^h(x, \xi) \leqslant \sqrt{n'}\,\delta_V(x, \xi).$$

Therefore the function $\psi(x, \xi) = -\log \delta_V^h(x, \xi)$ is a homogeneous exhaustion of V and is plurisubharmonic because:

$$\psi(x, \xi) = \sup_{1 \leqslant j \leqslant n'} \sup_{(y,\eta) \in B(1)} -\log \delta_V(x, \xi; y, \eta\xi_j).$$

PROPOSITION 1.1.1. *Let V be a $\mathbb{R}$-conic open subset of U. If V is a domain of holomorphy then:*

$$\forall k \geqslant 1, \quad H^k(V, \mathscr{S}(\varphi)) = 0.$$

Proof. Let ψ_0 be a plurisubharmonic homogeneous exhaustion of V, χ be a convex positive increasing C^∞ function on $\mathbb{R}$ and

$$\psi(x, \xi) = \varphi(x, \xi) + \chi(\psi_0(x, \xi))\log(1 + |\xi|^2).$$

As ψ_0 is homogeneous of degree 0 we have

$$(\langle \xi, \partial_\xi \rangle + \langle \bar{\xi}, \partial_{\bar{\xi}} \rangle)\cdot \chi(\psi_0(x, \xi)) = 0$$

and therefore

$$\partial\bar{\partial}\psi = \partial\bar{\partial}\varphi + [\partial\bar{\partial}\chi(\psi_0)]\log(1 + |\xi|^2) + \chi(\psi_0)\partial\bar{\partial}\log(1 + |\xi|^2)$$

which proves that ψ is plurisubharmonic.

From Theorem 4.4.2 of [5], there exists for each $g \in L^2_{(0,q+1)}(V, \psi)$ such that $\bar{\partial}g = 0$ a solution $u \in L^2_{(0,q)}(V, \mathrm{loc})$ of the equation $\bar{\partial}u = g$ which satisfies:

$$\int_V |u|^2 \mathrm{e}^{-2\psi} \frac{\mathrm{d}\lambda}{(1 + |\xi|^2)^2} \leqslant \int_V |g|^2 \mathrm{e}^{-2\psi}\,\mathrm{d}\lambda.$$

Now we remark that for each g in $\tilde{W}^{(0,q+1)}(\varphi, V)$, there exists an increasing convex positive function χ such that

$$\int_V |g|^2 \, e^{-2\psi} \, d\lambda < +\infty.$$

PROPOSITION 1.1.2. *Let V be a $\mathbb{R}$-conic open subset of U, then*

$$\forall k \geqslant n+n', \quad H^k(V, \mathscr{S}(\varphi)) = 0.$$

Proof. From the existence of the resolution $\tilde{\mathscr{W}}^{(0)}(\varphi)$, it follows that $H^k(V, \mathscr{S}(\varphi)) = 0$ for $k > n+n'$. When $k = n+n'$, we can use the proof of Demailly [4], Prop. 9.4.1:

We consider a function ψ_0 which is strongly $(n+n')$-convex on V (i.e. ψ_0 is a C^2-function whose Levi matrix has at least one strictly positive eigenvalue), homogeneous of degree 0 in ξ and such that:

$$\forall c > 0, \quad \Gamma_c = \{(x, \xi) \in V / \psi_0(x, \xi) \leqslant c\} \Subset V.$$

We can now follow the proof of (loc. cit.) using a function

$$\psi(x, \xi) = \varphi(x, \xi) + \chi(\psi_0(x, \xi)) \log(1 + |\xi|^2)$$

with χ convex increasing and positive.

(More generally, this proof gives the Andreotti-Grauert theorem for $\mathscr{S}(\varphi)$, that is: if V is strongly q-complete we have $H^k(V, \mathscr{S}(\varphi)) = 0$ when $k \geqslant q$.)

Let S be a complex analytic manifold and p be the projection $S \times U \to U$. If Ω is a local chart of S the preceding definitions give a sheaf $\mathscr{S}_{\Omega \times U}(\varphi \circ p)$ and glowing these sheaves we get a sheaf $\mathscr{S}_{S \times U}(\varphi)$ on $S \times U$. We denote by $\mathscr{O}_S$ the sheaf of holomorphic functions on S.

PROPOSITION 1.1.3. *Let S be a compact complex analytic manifold and p be the projection $S \times U \to U$. We suppose that φ is a plurisubharmonic function on U. Then*

$$\forall k \geqslant 0, \quad R^k p_* \mathscr{S}_{S \times U}(\varphi) = \mathscr{S}_U(\varphi) \otimes_{\mathbb{C}} H^k(S, \mathscr{O}_S).$$

(If S is compact, $H^k(S, \mathscr{O}_S)$ is a finite dimensional $\mathbb{C}$-vector space by the Cartan-Serre theorem.)

Proof. We will show that when V is an open subset of U satisfying the hypothesis of Proposition 1.1.1, we have:

$$\forall k \geqslant 0, \quad H^k(S \times V, \mathscr{S}_{S \times U}(\varphi)) = \Gamma(V, \mathscr{S}_U(\varphi)) \otimes_{\mathbb{C}} H^k(S, \mathscr{O}_S).$$

We can calculate $H^k(S \times V, \mathscr{S}_{S\times U}(\varphi))$ by the complex $W_\infty^{(0,\cdot)}(\varphi, S \times V)$ which is the subcomplex of $W^{(0,\cdot)}(\varphi, S \times V)$ of the forms whose coefficients are C^∞ in the variables of S (cf. [5], Th. 4.2.5 and Cor. 4.2.6).

The complex $W_\infty^{(0,\cdot)}(\varphi, S \times V)$ is equal to the topological tensor product $W^{(0,\cdot)}(\varphi, V) \hat{\otimes} \mathscr{C}^\infty_{(0,\cdot)}(S)$.

In the complex $\mathscr{C}^\infty_{(0,\cdot)}(S)$, the differential has a close range because its cohomology groups are finite dimensional over $\mathbb{C}$ and in $W^{(0,\cdot)}(\varphi, V)$ the differential is of close range because all cohomology groups are null except one.

So we can apply the topological Künneth theorem and get the result.

COROLLARY 1.1.4. *Let $(\varphi_q)_{q\geqslant 0}$ be an increasing sequence of plurisubharmonic functions on U and let $\mathscr{S}_U(\varphi_\infty) = \varinjlim_q \mathscr{S}_U(\varphi_q)$.*

(i) *If V is a $\mathbb{R}$-conic holomorphy domain in U, then*

$$\forall k \geqslant 2, \quad H^k(V, \mathscr{S}_U(\varphi_\infty)) = 0.$$

(ii) *If V* is a *$\mathbb{R}$-conic open subset of U, then*

$$\forall k \geqslant n+n'+1, \quad H^k(V, \mathscr{S}_U(\varphi_\infty)) = 0.$$

(iii) *If S is a compact complex analytic manifold and $p: S \times U \to U$ the projection we have:*

$$\forall k \geqslant 0, \quad R^k p_* \mathscr{S}_{S\times U}(\varphi_\infty) = \mathscr{S}_U(\varphi_\infty) \otimes_{\mathbb{C}} H^k(S, \mathscr{O}_S).$$

Proof. Let ψ_0 be a plurisubharmonic homogeneous exhaustion of V.

We set

$$V_c = \{(x, \xi) \in V / \psi_0(x, \xi) < c\} \quad \text{and} \quad \Gamma_c = \{(x, \xi) \in V / \psi_0(x, \xi) \leqslant c\}.$$

From Proposition 1.1.1, we have

$$\forall q \geqslant 0, \quad \forall c > 0, \quad \forall k \geqslant 1, \quad H^k(V_c, \mathscr{S}(\varphi_q)) = 0$$

and as $\Gamma_c = \bigcap_{c'>c} V_c$ we have:

$$\forall q \geqslant 0, \quad \forall c > 0, \quad \forall k \geqslant 1, \qquad H^k(\Gamma_c, \mathscr{S}(\varphi_q)) = 0.$$

The sets Γ_c are compacts (for the conic topology!), so as inductive limit commutes with cohomology on compact sets we get:

$$H^k(\Gamma_c, \mathscr{S}(\varphi_\infty)) = \varinjlim_q H^k(\Gamma_c, \mathscr{S}(\varphi_q)) = 0 \quad \text{when } k \geqslant 1.$$

As $V = \bigcup_{c>0} \Gamma_c$, Mittag-Leffler's theorem ([17], Proposition 13.23) gives that if for $c > 0$, $H^{k-1}(\Gamma_c, \mathscr{S}(\varphi_\infty)) = 0$ then

$$H^k(V, \mathscr{S}(\varphi_\infty)) = \varprojlim_{c} H^k(\Gamma_c, \mathscr{S}(\varphi_\infty)) = 0.$$

So we obtain (i) and the part (ii) is proved in the same way applying Proposition 1.1.2 to any increasing family of compact subsets of V.

For (iii), we consider a point α in U and fondamental system $(\Gamma_c)_{c>0}$ of compact holomorphically convex neighborhoods of α. We have:

$$\begin{aligned} R^k p_* \mathscr{S}_{S\times U}(\varphi_\infty)_\alpha &= \varinjlim_{c>0} H^k(S\times\Gamma_c, \mathscr{S}_{S\times U}(\varphi_\infty)) \\ &= \varinjlim_{c>0} \varinjlim_{q} H^k(S\times\Gamma_c, \mathscr{S}_{S\times U}(\varphi_q)) \\ &= \varinjlim_{q} \varinjlim_{c} \Gamma(\Gamma_c, \mathscr{S}_U(\varphi_q)) \otimes_{\mathbb{C}} H^k(S, \mathcal{O}_S) \\ &= \mathscr{S}_U(\varphi_\infty)_\alpha \otimes_{\mathbb{C}} H^k(S, \mathcal{O}_S). \end{aligned}$$

PROPOSITION 1.1.5. *Let φ be a positive plurisubharmonic function on U.*

(i) *If V is an open $\mathbb{R}$-conic subset of U which is a domain of holomorphy then we have:*

$$\forall k \geqslant 1, \qquad H^k\left(V, \varprojlim_{\varepsilon>0} \mathscr{S}(\varepsilon\varphi)\right) = 0.$$

(ii) *If V is an open $\mathbb{R}$-conic subset of U, then*

$$\forall k \geqslant \dim U, \qquad H^k\left(V, \varprojlim_{\varepsilon>0} \mathscr{S}(\varepsilon\varphi)\right) = 0.$$

(iii) *If S is a compact complex analytic manifold and $p: S\times U \to U$ the projection then we have:*

$$\forall k \geqslant 0, \quad R^k p_* \varprojlim_{\varepsilon>0} \mathscr{S}_{S\times U}(\varepsilon\varphi) = \varprojlim_{\varepsilon>0} \mathscr{S}_U(\varepsilon\varphi) \otimes_{\mathbb{C}} H^k(S, \mathcal{O}_S).$$

If moreover φ is of the special form $\varphi(x,\xi) = \varphi_0(|\xi|)$ then the same results are still true for $\varinjlim_{\varepsilon>0} \mathscr{S}(\varepsilon\varphi)$.

Proof. In the proof of Proposition 1.1.1, the function u satisfying $\bar\partial u = g$ may be chosen in $(\ker\bar\partial)^\perp$ and then the solution u is unique ([4], Remark 6.4.6). Therefore if $g \in L^2_{\mathrm{loc}}(V)$ is in $\tilde W^{(0,q+1)}(\varepsilon\varphi, V)$ for each $\varepsilon > 0$ there is a solution u of $\bar\partial u = g$ in $\bigcap_{\varepsilon>0} \tilde W^{(0,q)}(\varepsilon\varphi, V)$. This proves (i) for $\varprojlim_{\varepsilon>0} \mathscr{S}(\varepsilon\varphi)$ and the parts (ii) and (iii) are proved in the same way.

Let us now consider the inductive limit $\varinjlim_{\varepsilon>0} \mathscr{S}(\varepsilon\varphi)$ and come back to the proof of Proposition 1.1.1.

If g is a section on V of $\varinjlim_{\varepsilon>0} \tilde{\mathscr{W}}^{(0,q+1)}(\varepsilon\varphi)$, there exists two functions χ_1 and χ_2 which are positive convex increasing C^∞-functions on $\mathbb{R}$ such that

$$\int_V |g|^2 \,\mathrm{e}^{-2\psi}\,\mathrm{d}\lambda < +\infty$$

where $\psi(x, \xi) = \chi_1(\psi_0(x, \xi))\varphi(x, \xi) + \chi_2(\psi_0(x, \xi)) \log(1 + |\xi|^2)$.

As in the proof of Proposition 1.1.1, ψ is plurisubharmonic because ψ_0 is plurisubharmonic and homogeneous of degree 0 in ξ while φ and $\log(1 + |\xi|^2)$ depend only on $|\xi|$. By this way we can prove (i) using the proof of Proposition 1.1 and (ii) using the proof of Proposition 1.1.2. The third part of the proposition for $\varinjlim_\varepsilon \mathscr{S}(\varepsilon\varphi)$ is a special case of Corollary 1.1.4(iii).

1.2. *Formal completion*

When $m \in \mathbb{R}$, we denote by $S_m(\varphi, V)$ the subset of $S(\varphi, V)$ of the functions f such that $\|f\|_{V',\varphi,m}$ is finite for each conic open set $V' \Subset V$ and by $\mathscr{S}_m(\varphi)$ the corresponding sheaf.

For $\varepsilon > 0$ the canonical imbedding $\mathscr{S}_m(\varphi) \hookrightarrow \mathscr{S}_{m+\varepsilon}(\varphi)$ gives morphisms $H^k(V, \mathscr{S}_m(\varphi)) \to H^k(V, \mathscr{S}_{m+\varepsilon}(\varphi))$.

LEMMA 1.2.1. *Let V be a $\mathbb{R}$-conic open subset of U.*

(a) *If V is a domain of holomorphy, then for all $k \geqslant 1$, all $m > k$ and $\varepsilon > 0$, the morphism $H^k(V, \mathscr{S}_m(\varphi)) \to H^k(V, \mathscr{S}_{m+\varepsilon}(\varphi))$ is zero.*

(b) *If V is any $\mathbb{R}$-conic subset of U the preceding property is still true if $k \geqslant \dim_{\mathbb{C}} U$.*

Proof. We will write $\mathscr{S}_m$ instead of $\mathscr{S}_m(\varphi)$.

Let $\mathscr{W}_m$ be the subsheaf of $\mathscr{W}(\varphi)$ whose sections on an open set V satisfy $\|f\|_{V',\varphi,m} < +\infty$ for all $V' \Subset V$, $\mathscr{W}_m^{(0,p)}$ be the subsheaf of $\mathscr{W}^{(0,p)}(\varphi)$ of forms with coefficients in $\mathscr{W}_m$. Let $\tilde{\mathscr{W}}_m^{(0,p)}$ the subsheaf of $\mathscr{W}_m^{(0,p)}$ of the forms u such that $\bar{\partial}u \in \mathscr{W}_{m-1}^{(0,p+1)}$ and for $\varepsilon > 0$ fixed, $\mathscr{V}_m^{(0,p)}$ the subsheaf of $\mathscr{W}_m^{(0,p)}$ of the forms u such that $\bar{\partial}u \in \mathscr{W}_{m-1-\varepsilon}^{(0,p+1)}$.

We will denote by $\tilde{\mathscr{W}}_m^{(0,\cdot)}$ the complex

$$0 \to \tilde{\mathscr{W}}_m^{(0,0)} \xrightarrow{\bar{\partial}} \tilde{\mathscr{W}}_{m-1}^{(0,1)} \xrightarrow{\bar{\partial}} \tilde{\mathscr{W}}_{m-2}^{(0,2)} \to \cdots \xrightarrow{\bar{\partial}} \tilde{\mathscr{W}}_{m-n-n'}^{(0,n+n')} \to 0$$

and by $\mathscr{V}_m^{(0,\cdot)}$ the complex

$$0 \to \mathscr{V}_m^{(0,0)} \xrightarrow{\bar\partial} \mathscr{V}_{m-1-\varepsilon}^{(0,1)} \xrightarrow{\bar\partial} \mathscr{V}_{m-2-2\varepsilon}^{(0,2)} \to \cdots \xrightarrow{\bar\partial} \mathscr{V}_{m-(n+n')(1+\varepsilon)}^{(0,n+n')} \to 0.$$

Let us first remark that the sheaves $\tilde{\mathscr{W}}_{m-p}^{(0,p)}$ are soft for the conic topology. In fact, the C^∞-functions which are homogeneous of degree 0 in ξ operate by multiplication on these sheaves.

Secondly, if V is a conic domain of holomorphy, the complex $\Gamma(V, \mathscr{V}_m^{(0,\cdot)})$ is exact in degree $k \geqslant 1$ if $m > k$. To prove this, we take the function ψ_0 of the proof of Proposition 1.1.1 and if χ is a convex increasing positive function we consider

$$\psi_m(x, \xi) = \varphi(x, \xi) + m \log(1+|\xi|^2) + \chi(\psi_0(x, \xi)).$$

By [5] Lemma 4.4.1 applied to $\psi_m(x, \xi) + \log(1+(1+|\xi|^2)^\varepsilon)$ we get that for each $g \in L^2_{(0,q+1)}(V, \psi_m)$ such that $\bar\partial g = 0$ there exists $u \in L^2_{(0,q)}(V, \psi_{m+1+\varepsilon})$ solution of $\bar\partial u = g$. Then we conclude as in 1.1.1 choosing χ big enough so that $g \in L^2_{(0,q+1)}(V, \psi_m)$.

This proves that $\mathscr{V}_m^{(0,\cdot)}$ is a resolution of $\mathscr{S}_m$ and so, for each conic open subset V, the cohomology groups $H^k(V, \mathscr{S}_m)$ are equal to the hypercohomology groups $\mathbb{H}^k(V, \mathscr{V}_m^{(0,\cdot)})$.

If $\alpha > (n+n')\varepsilon$, the canonical map $\mathscr{V}_m^{(0,\cdot)} \hookrightarrow \mathscr{V}_{m+\alpha}^{(0,\cdot)}$ factorize in $\mathscr{V}_m^{(0,\cdot)} \hookrightarrow \tilde{\mathscr{W}}_m^{(0,\cdot)} \hookrightarrow \mathscr{V}_{m+\alpha}^{(0,\cdot)}$ and therefore we have the same factorization in hypercohomology:

$$\mathbb{H}^k(V, \mathscr{V}_m^{(0,\cdot)}) \to \mathbb{H}^k(V, \tilde{\mathscr{W}}_m^{(0,\cdot)}) \to \mathbb{H}^k(V, \mathscr{V}_{m+\alpha}^{(0,\cdot)}).$$

As the sheaves $\tilde{\mathscr{W}}_{m-p}^{(0,p)}$ are soft, the hypercohomology groups of $\tilde{\mathscr{W}}_m^{(0,\cdot)}$ are equal to its cohomology groups $H^k(V, \tilde{\mathscr{W}}_m^{(0,\cdot)}) \stackrel{\text{def}}{=} H^k(\Gamma(V, \tilde{\mathscr{W}}_m^{(0,\cdot)}))$ and the morphism of these groups into $\mathbb{H}^k(V, \mathscr{V}_{m+\alpha}^{(0,\cdot)})$ factorize through the cohomology of $\mathscr{V}_{m+\alpha}^{(0,\cdot)}$:

$$\mathbb{H}^k(V, \tilde{\mathscr{W}}_m^{(0,\cdot)}) = H^k(V, \tilde{\mathscr{W}}_m^{(0,\cdot)}) \to H^k(V, \mathscr{V}_{m+\alpha}^{(0,\cdot)}) \to \mathbb{H}^k(V, \mathscr{V}_{m+\alpha}^{(0,\cdot)}).$$

Finally we proved that $H^k(V, \mathscr{S}_m) \to H^k(V, \mathscr{S}_{m+\alpha})$ factorize in $H^k(V, \mathscr{S}_m) \to H^k(V, \mathscr{V}_{m+\alpha}^{(0,\cdot)}) \to H^k(V, \mathscr{S}_{m+\alpha})$ and this morphism is zero if $H^k(V, \mathscr{V}_{m+\alpha}^{(0,\cdot)}) = 0$ which proves the lemma.

LEMMA 1.2.2. *Let S be a compact complex analytic manifold and $p: S \times U \to U$ be the projection. If $k \geqslant 0$, $m \geqslant k$ and $\varepsilon > 0$ the canonical map*

$$\mathscr{S}_{U,m+\varepsilon}(\varphi) \otimes_{\mathbb{C}} H^k(S, \mathscr{O}_S) \to R^k p_* \mathscr{S}_{S\times U,m+\varepsilon}(\varphi)$$

is injective and its image contains the image of $R^k p_* \mathscr{S}_{S\times U,m}(\varphi)$.

Proof. In the commutative square

$$\begin{array}{ccc} \mathcal{S}_{U,m}(\varphi) \otimes_{\mathbb{C}} H^k(S, \mathcal{O}_S) & \overset{\alpha_m}{\to} & R^k p_* \mathcal{S}_{S\times U,m}(\varphi) \\ \downarrow j_m & & \downarrow \\ \mathcal{S}_U(\varphi) \otimes_{\mathbb{C}} H^k(S, \mathcal{O}_S) & \overset{\alpha}{\to} & R^k p_* \mathcal{S}_{S\times U}(\varphi) \end{array}$$

j_m is injective and α bijective (Proposition 1.1.3) therefore α_m is injective.

Let V be a $\mathbb{R}$-conic domain of holomorphy in U denote by $\mathcal{V}^{\infty(0,\cdot)}_{U,m+\varepsilon}$ the subcomplex of $\mathcal{V}^{(0,\cdot)}_{U,m+\varepsilon}$ of the forms which are C^∞ in the S-variables. It is a resolution of $\mathcal{S}_{m+\varepsilon}$, so as in the proof of Proposition 1.1.3 we get by the topological Künneth's theorem:

$$\Gamma(V, \mathcal{S}_{m+\varepsilon}(\varphi)) \otimes_{\mathbb{C}} H^k(S, \mathcal{O}_S) \tilde{\to} H^k(S \times V, \mathcal{V}^{\infty(0,\cdot)}_{S\times U,m+\varepsilon}).$$

From the preceding proof, the map

$$H^k(S \times V, \mathcal{S}_m(\varphi)) \to H^k(S \times V, \mathcal{S}_{m+\varepsilon}(\varphi))$$

is factorized through $H^k(S \times V, \mathcal{V}^{\infty(0,\cdot)}_{m+\varepsilon}) \approx \Gamma(V, \mathcal{S}_{m+\varepsilon}(\varphi)) \otimes_{\mathbb{C}} H^k(S, \mathcal{O}_S)$ and taking the inductive limit on the neighborhoods of a point α in U we have:

$$R^k p_* \mathcal{S}_m(\varphi)_\alpha \to \mathcal{S}_{m+\varepsilon}(\varphi)_\alpha \otimes_{\mathbb{C}} H^k(S, \mathcal{O}_S) \to R^k p_* \mathcal{S}_{m+\varepsilon}(\varphi)_\alpha$$

which gives the result.

Let us now assume that there exists on U a holomorphic function $\mu(x, \xi)$ whose real part is equivalent to $|\xi|$, i.e.:

$$\exists c_1, c_2 > 0, \quad \forall (x, \xi) \in U, \quad c_1|\xi| \leqslant \mathrm{Re}(\mu(x, \xi)) \leqslant c_2|\xi|$$

(this property is satisfied if $U = U_0 \times \Gamma_0$ with Γ_0 strictly contained in a half space).

PROPOSITION 1.2.3. *Let φ be a plurisubharmonic function on U and let us define*

$$\hat{\mathcal{C}}(\varphi) = \varinjlim_{m\in\mathbb{Z}} \varprojlim_{p\in\mathbb{N}} \mathcal{S}_m(\varphi)/\mathcal{S}_{m-p}(\varphi).$$

(i) *If V is a $\mathbb{R}$-conic open subset of U and a domain of holomorphy, then:*

$$\forall k \geqslant 1, \quad H^k(V, \hat{\mathcal{C}}(\varphi)) = 0.$$

(ii) *If V is a $\mathbb{R}$-conic open subset of U, then*

$$\forall k \geqslant \dim_{\mathbb{C}} U, \quad H^k(V, \hat{\mathscr{C}}(\varphi)) = 0.$$

(iii) *If S is a compact complex analytic manifold and $p\colon S \times U \to U$ the projection, then*

$$\forall k \geqslant 0, \quad R^k p_* \hat{\mathscr{C}}_{S \times U}(\varphi) = \hat{\mathscr{C}}_U(\varphi) \otimes_{\mathbb{C}} H^k(S, \mathcal{O}_S).$$

Proof. The multiplication by $(1+\mu(x,\xi))^N$ gives an isomorphism $\mathscr{S}_m(\varphi) \xrightarrow{\sim} \mathscr{S}_{m+N}(\varphi)$, which proves that the results of Lemma 1.2.1 and 1.2.2 are true for all $m \in \mathbb{R}$.

Let V be a $\mathbb{R}$-conic domain of holomorphy in U. Lemma 1.2.1 proves that for $k \geqslant 0$, $m \in \mathbb{R}$ and $p > 1$, there is a commutative diagram

$$\begin{array}{ccccc}
H^k(V, \mathscr{S}_m) & \xrightarrow{\rho_k} & H^k(V, \mathscr{S}_m/\mathscr{S}_{m-p}) & \rightarrow & H^{k+1}(V, \mathscr{S}_{m-p}) \rightarrow 0 \\
\downarrow{\alpha_k} & & \downarrow{\beta_k} & & \downarrow{\gamma_k} \\
H^k(V, \mathscr{S}_m) & \xrightarrow{\rho_k} & H^k(V, \mathscr{S}_m/\mathscr{S}_{m-p+1}) & \xrightarrow{\partial_k} & H^{k+1}(V, \mathscr{S}_{m-p+1}) \rightarrow 0
\end{array}$$

with $\alpha_k = \mathrm{id}$ and $\gamma_k = 0$ so that the image of β_k is equal to the kernel of ∂_k that is to $H^k(V, \mathscr{S}_m)$.

For all domains of holomorphy V and $k \geqslant 0$, the projective system $H^k(V, \mathscr{S}_m/\mathscr{S}_{m-p})_{p>0}$ satisfy the Mittag-Leffler's condition and from [17] Proposition 13.3.1, we have:

$$\forall k \geqslant 1, \quad H^k\left(V, \varprojlim_p \mathscr{S}_m/\mathscr{S}_{m-p}\right) = \varprojlim_p H^k(V, \mathscr{S}_m/\mathscr{S}_{m-p}).$$

Moreover, in the above diagram, the maps ρ_k are injective for $k \geqslant 1$, so as $\alpha_k = \mathrm{id}$ and $\gamma_k = 0$ the morphism

$$H^k(V, \mathscr{S}_m) \rightarrow \varprojlim_p H^k(V, \mathscr{S}_m/\mathscr{S}_{m-p})$$

is an isomorphism.

Let us denote $\hat{\mathscr{C}}_m = \varprojlim_p \mathscr{S}_m/\mathscr{S}_{m-p}$. We have proved that when $k \geqslant 1$ we have:

$$H^k(V, \mathscr{S}_m) \simeq H^k(V, \hat{\mathscr{C}}_m)$$

and therefore that

$$\forall m \in \mathbb{R}, \quad \forall k \geqslant 1, \quad H^k(V, \hat{\mathscr{C}}_m/\mathscr{S}_m) = 0.$$

Now we remark that $\hat{\mathscr{C}}_m/\mathscr{S}_m$ does not depend on m and is therefore equal to $\hat{\mathscr{C}}/\mathscr{S}$. We have $H^k(V, \hat{\mathscr{C}}/\mathscr{S}) = 0$ when $k \geqslant 1$ and, as the same is true for $\mathscr{S}$, we get

$$\forall k \geqslant 1, \quad H^k(V, \hat{\mathscr{C}}) = 0.$$

The second part of the proposition is proved in the same way. To prove the third part we consider the sheaves

$$\tilde{\mathscr{S}}_m = \prod_{p\geqslant 0} \mathscr{S}_{m-p} \quad \text{and} \quad \tilde{\mathscr{S}} = \varinjlim_m \mathscr{S}_m.$$

It is obvious that $\tilde{\mathscr{S}}_m$ satisfy the result of Lemma 1.1.4 so that $\tilde{\mathscr{S}}$ satisfy:

$$R^k p_* \tilde{\mathscr{S}}_{S\times U} = \tilde{\mathscr{S}} \otimes_{\mathbb{C}} H^k(S, \mathscr{O}_S).$$

Now we remark that we have an exact sequence

$$0 \to \tilde{\mathscr{S}}_m \to \tilde{\mathscr{S}}_m \to \hat{\mathscr{C}}_m \to 0$$

where the first map is $(f_p)_{p\geqslant 0} \mapsto (f_p - f_{p-1})_{p\geqslant 0}$ and the second $(f_p)_{p\geqslant 0} \mapsto (\Sigma_{k=0}^{p-1} f_k)_{p\geqslant 1}$ which gives an exact sequence $0 \to \tilde{\mathscr{S}} \to \tilde{\mathscr{S}} \to \hat{\mathscr{C}} \to 0$ and proves the proposition.

2. Microlocal operators

2.1. *Microlocal symbols*

Let U_0 be an open subset of $\mathbb{C}^n$ and Γ_0 an open $\mathbb{R}$-conic subset of $\mathbb{C}^{n'}$. The coordinates on $\mathbb{C}^n$ will be $y=(y_1,\ldots,y_n)$ and on $\mathbb{C}^{n'}$, $\xi=(\xi_1,\ldots,\xi_{n'})$. Let $U=U_0\times\Gamma_0$.

As in Section 1.1, $\mathbb{R}$-conic means invariant under the positive real homotheties in ξ and, for $\mathbb{R}$-conic sets $V' \Subset V$ means relatively compact for the $\mathbb{R}$-conic topology.

Let $r\in\mathbb{R}$, $r \geqslant 1$, and let φ be a continuous function from U to $\mathbb{R}$ such that $\lim_{|\xi|\to\infty} \varphi(y,\xi)/|\xi|^{1/r} = 0$ locally uniformly.

DEFINITION 2.1.1. Let V be a $\mathbb{R}$-conic open subset of U.

(i) $\mathscr{S}_+(\varphi, V)$ is the set of holomorphic functions on V such that $\forall V' \Subset V$, $\exists C > 0$, $\exists m\in\mathbb{R}$, $\forall(y,\xi)\in V'$,

$$|f(y,\xi)| < C|\xi|^m e^{\varphi(y,\xi)}.$$

(ii) $\mathcal{S}_-(r, V)$ is the set of holomorphic functions on V such that $\forall V' \Subset V, \exists \delta > 0, \exists C > 0, \forall (y, \xi) \in V'$,

$$|f(y, \xi)| < C\,\mathrm{e}^{-\delta|\xi|^{1/r}}.$$

(iii) $\mathcal{S}(\varphi, r, V) = \varprojlim_{V' \Subset V} \mathcal{S}_+(\varphi, V')/\mathcal{S}_-(r, V')$.

We denote by $\mathcal{S}(\varphi, r)$ the sheaf generated by the presheaf $V \mapsto \mathcal{S}(\varphi, r, V)$.

(More precisely, the presheaf generates a sheaf on U with the $\mathbb{R}$-conic topology and $\mathcal{S}(\varphi, r)$ is its preimage on U with the usual topology).

PROPOSITION 2.1.2. *If V is a $\mathbb{R}$-conic domain of holomorphy in U and if there exists a holomorphic function $\mu(x, \xi)$ whose real part is equivalent to $|\xi|^{1/r}$ on V, then*

$$\Gamma(V, \mathcal{S}(\varphi, r)) = \mathcal{S}(\varphi, r, V).$$

($\Gamma(V, \mathcal{S}(\varphi, r))$ is the set of global sections of $\mathcal{S}(\varphi, r)$ on V).

Proof. Let $\mathcal{S}_+(\varphi)$ and $\mathcal{S}_-(r)$ be respectively the sheaves generated by $\mathcal{S}_+(\varphi, V)$ and $\mathcal{S}_-(r, V)$, let $\varphi_q(x, \xi) = -\frac{1}{q}\operatorname{Re}\mu(x, \xi)$.

With the notations of Corollary 1.1.4 we have $\mathcal{S}_-(r) = \varinjlim_{q \in \mathbb{N}^*} \mathcal{S}_U(\varphi_q)$.

The functions φ_q are plurisubharmonic, so from the proof of Corollary 1.1.4, there exists a family $(\Gamma_c)_{c>0}$ of compact subsets of V such that $V = \bigcup_{c>0} \Gamma_c$ and $H^1(\Gamma_c, \mathcal{S}_-(r)) = 0$. Thus we have:

$$\Gamma(V, \mathcal{S}(\varphi, r)) = \varprojlim_c \Gamma(\Gamma_c, \mathcal{S}(\varphi, r)) = \varprojlim_c \Gamma(\Gamma_c, \mathcal{S}_+(\varphi))/\Gamma(\Gamma_c, \mathcal{S}_-(r))$$

$$= \varprojlim_{V' \Subset V} \mathcal{S}_+(\varphi, V')/\mathcal{S}_-(r, V') = \mathcal{S}(\varphi, r, V).$$

The action of coordinate transformations on $\mathcal{S}(\varphi, r)$ cannot be explicit on the spaces $\mathcal{S}(\varphi, r, V)$ so we will now introduce a new class of symbols, equivalent to the preceding one, and which following Boutet de Monvel [3] we will call “formal symbols”.

DEFINITION 2.1.3. Let V be a $\mathbb{R}$-conic open subset of U.

(i) $\hat{\mathcal{S}}_+(\varphi, r, V)$ is the set of the formal series $\Sigma_{k \geqslant 0} f_k(y, \xi)$ whose terms are in $\mathcal{S}_+(\varphi, V)$ and satisfy:
$\forall V' \Subset V, \exists m \in \mathbb{R}, \exists A > 0, \exists c > 0, \exists C > 0, \forall (y, \xi) \in V', \ |\xi| > c, \forall k \geqslant 0$,

$$|f_k(y, \xi)| < CA^k (k!)^r |\xi|^{m-k}\,\mathrm{e}^{\varphi(y, \xi)}.$$

(ii) $\hat{\mathcal{S}}_-(\varphi, r, V)$ is the subset of $\hat{\mathcal{S}}_+(\varphi, r, V)$ of the series $f = \Sigma_{k \geqslant 0} f_k(y, \xi)$ such that $S(f) = \Sigma_{k \geqslant 0} g_k$ with $g_k = \Sigma_{0 \leqslant l \leqslant k} f_l$ is still in $\hat{\mathcal{S}}_+(\varphi, r, V)$.

(iii) $\hat{\mathscr{S}}(\varphi, r, V) = \varprojlim_{V' \Subset V} \hat{\mathscr{S}}_+(\varphi, r, V')/\hat{\mathscr{S}}_-(\varphi, r, V')$.

The map which associates to an element f of $\mathscr{S}_+(\varphi, V)$ the series $\Sigma_{k\geqslant 0} f_k$ defined by $f_0 = f$ and $f_k = 0$ if $k > 0$ sends $\mathscr{S}_+(\varphi, V)$ into $\hat{\mathscr{S}}_+(\varphi, r, V)$ and $\mathscr{S}_-(r, V)$ into $\hat{\mathscr{S}}_-(\varphi, r, V)$ hence defines a map from $\mathscr{S}(\varphi, r, V)$ to $\hat{\mathscr{S}}(\varphi, r, V)$.

LEMMA 2.1.4.

(i) *The map* $\mathscr{S}(\varphi, r, V) \to \hat{\mathscr{S}}(\varphi, r, V)$ *is injective.*
(ii) *Let* ψ *be a continuous function from* U *to* $\mathbb{R}$ *such that*

$$\lim_{|\xi|\to\infty} \psi(y, \xi)/|\xi|^{1/r} = 0 \quad \textit{and} \quad \psi \leqslant \varphi \quad \textit{on } U.$$

Let $f \in \mathscr{S}_+(\varphi, V)$ *and* $\Sigma_{k\geqslant 0} f_k \in \hat{\mathscr{S}}_+(\psi, r, V)$ *such that* $(f - \Sigma_{k\geqslant 0} f_k) \in \hat{\mathscr{S}}_-(\varphi, r, V)$, *then* $f \in \mathscr{S}_+(\psi, V)$.

This lemma has been proved in [3], Sections 1.13 and 1.14 when $r = 1$. We may use the same proof when $r > 1$.

THEOREM 2.1.5. *Let* U_0 *be a domain of holomorphy in* $\mathbb{C}^n$, Γ_0 *a convex open* $\mathbb{R}$-*conic subset of* $\mathbb{C}^{n'}$ *and* $V = U_0 \times \Gamma_0$. *The map* $\mathscr{S}(\varphi, r, V) \to \hat{\mathscr{S}}(\varphi, r, V)$ *is bijective.*

REMARK. This theorem has been proved in [3], Theorem 1.2.3 but only for small cones Γ (and for $r = 1$).

Proof. By the hypothesis, Γ is equal to $\mathbb{C}^{n'}$ or is contained in a half space of $\mathbb{C}^{n'}$.

Let us first assume that $\Gamma = \mathbb{C}^{n'}$ and consider an element $\Sigma_{k\geqslant 0} f_k(y, \xi)$ of $\hat{\mathscr{S}}_+(\varphi, r, V')$ with $V' = U_1 \times \mathbb{C}^{n'}$ and U_1 relatively compact open subset of U_0.

As $\lim_{|\xi|\to\infty} \varphi(y, \xi)/|\xi|^{1/r} = 0$, we have from Definition 2.1.3: $\forall U_1' \Subset U_1$, $\exists m \in \mathbb{R}$, $\exists A > 0$, $\exists c > 0$, $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$, $\forall y \in U_1$, $\forall \xi \in \mathbb{C}^{n'}$, $|\xi| > c$, $\forall k \geqslant 0$

$$|f_k(y, \xi)| < C_\varepsilon A^k (k!)^r |\xi|^{m-k} e^{\varepsilon|\xi|^{1/r}}.$$

The Taylor development of the f_k functions is

$$f_k(y, \xi) = \sum_{\alpha \in \mathbb{N}^{n'}} f_k^\alpha(y) \xi^\alpha$$

and Cauchy's inequalities give: $\forall R > c$, $\forall k \geqslant 0$, $\forall \alpha \in \mathbb{N}^{n'}$, $\forall y \in U_1'$,

$$|f_k^\alpha(y)| < C_\varepsilon A^k (k!)^r R^{m-k-|\alpha|} e^{\varepsilon R^{1/r}}.$$

Taking $R = \left[(k+|\alpha|-m)\frac{r}{\varepsilon} \right]^r$ and using Stirling's formula, we get when $k > m$:

$$|f_k^\alpha(y)| < C'_\varepsilon A^k \left(\frac{k!}{(k+|\alpha|-m)!} \right)^r \left(\frac{\varepsilon}{r} \right)^{r(k+|\alpha|-m)}$$

that is, with other constants,
$\forall U'_1 \Subset U$, $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$, $\forall y \in U'_1$, $\forall \alpha \in \mathbb{N}^{n'}$, $\forall k \geqslant 0$,

$$|f_k^\alpha(y)| < C_\varepsilon \varepsilon^{k+|\alpha|} \frac{1}{(|\alpha|!)^r}.$$

This proves that the series $f(y, \xi) = \Sigma_{k \geqslant 0} f_k(y, \xi)$ is convergent and satisfies: $\forall U'_1 \Subset U_1$, $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$, $\forall (y, \xi) \in U'_1 \times \mathbb{C}^{n'}$,

$$|f(y, \xi)| < C_\varepsilon \, e^{\varepsilon |\xi|^{1/r}}.$$

So there exists a function $\varphi_1(y, \xi)$ such that

$$\lim_{|\xi| \to \infty} \varphi_1(y, \xi)/|\xi|^{1/r} = 0, \quad f \in \mathscr{S}_+(\varphi_1, V') \quad \text{and} \quad f = \sum_{k \geqslant 0} f_k \in \hat{\mathscr{S}}(\varphi_1, r, V').$$

From Lemma 2.1.4, it follows that f is an element of $\mathscr{S}(\varphi, r, V')$.

Let us now suppose that $\Gamma \subset \{\operatorname{Re} \xi_1 \geqslant 0\}$. Then if $V' \Subset V$, there exists some $\delta > 0$ such that $V' \subset \{(y, \xi) \in V / \operatorname{Re} \xi_1 \geqslant \delta |\xi|\}$.

For each $\mu > 0$ and $a > 0$, we set

$$g_\mu^a(z) = \frac{z^\mu}{\Gamma(\mu)} \int_0^a e^{-tz} t^{\mu-1} \, dt.$$

The $g_\mu^a(z)$ functions are holomorphic when $\operatorname{Re} z > 0$ and satisty

(i) $1 - g_\mu^a(z) = \dfrac{z^\mu}{\Gamma(\mu)} \displaystyle\int_a^{+\infty} e^{-tz} t^{\mu-1} \, dt$

(ii) $|g_\mu^a(z)| < \dfrac{|z|^\mu}{\Gamma(\mu)} \dfrac{a^\mu}{\mu}$

(iii) $|1 - g_\mu^a(z)| < C^\mu e^{-a\delta'|z|}$ if $\operatorname{Re} z \geqslant \delta |z|$ and $\delta' < \delta$

$$\text{(iv)} \sum_{k>N} \frac{(k-1)!}{|z|^k} |g_k^a(z)| \leqslant \int_0^a \sum_{k>N} t^{k-1} \mathrm{e}^{-t\delta|z|} \mathrm{d}t$$

$$\leqslant \frac{1}{1-a} \int_0^\infty \mathrm{e}^{-t\delta|z|} t^N \mathrm{d}t \leqslant N!|z|^{-N-1} \delta^{-N-1} \frac{1}{1-a}$$

$$\text{if } a < 1 \quad \text{and} \quad \operatorname{Re} z > \delta|z|.$$

Let $\Sigma_{k\geqslant 0} f_k(y, \xi)$ be an element of $\hat{\mathscr{S}}_+(\varphi, r, V')$ with $V' \Subset V$. We fix $V'' \Subset V'$. By definition we have: $\exists m \in \mathbb{R}$, $\exists A > 0$, $\forall c > 0$, $\exists C > 0$, $\forall (y, \xi) \in V''$, $|\xi| > C$, $\forall k \geqslant 0$

$$|f_k(y, \xi)| < CA^k \Gamma(rk) |\xi|^{m-k} \mathrm{e}^{\varphi(y,\xi)}$$

therefore the series $f(y, \xi) = \Sigma_{k\geqslant 0} f_k(y, \xi) g_{rk}^a(\xi_1^{1/r})$ is convergent on $V'' \cap \{|\xi| > c\}$ if $a < A^{-1/r}$ and we have:

$$\left| f(y, \xi) - \sum_{k=0}^{N} f_k(y, \xi) \right| \leqslant \sum_{k=0}^{N} |f_k(y, \xi)| \, |g_{rk}^a(\xi_1^{1/r}) - 1|$$

$$+ \sum_{k>N} |f_k(y, \xi)| \, |g_{rk}^a(\xi^{1/r})|$$

$$\leqslant C_1 (A/\delta)^N (N!)^r |\xi|^{m-N} \mathrm{e}^{\varphi(y,\xi)}.$$

This proves that $f = \Sigma_{k\geqslant 0} f_k$ is in $\hat{\mathscr{S}}(\varphi, r, V'')$ and that f is a holomorphic function on $V'' \cap \{|\xi| > c\}$ which satisfy: $\exists C > 0$, $\forall (x, \xi) \in V''$, $|\xi| > c$,

$$|f(y, \xi)| < C|\xi|^m \mathrm{e}^{\varphi(y,\xi)}.$$

Let C_1 and C_2 such that $c < C_1 < C_2$ and let $\alpha(\xi)$ be a C^∞-function on $\mathbb{C}^{n'}$ such that $\alpha(\xi) = 0$ if $|\xi| < C_1$ and $\alpha(\xi) = 1$ if $|\xi| > C_2$.

The function $\bar{\partial}(\alpha f) = (\bar{\partial}\alpha) f$ has a compact support, thus from Theorem 4.4.2 of [5], there exists a C^∞-function g such that $\bar{\partial} g = \bar{\partial}(\alpha f)$ and $|g(y, \xi)| < C \mathrm{e}^{-\operatorname{Re} \xi_1}$ on V''.

Now, the function $\alpha f - g$ is in $\mathscr{S}_+(\varphi, V'')$ and is equal to $\Sigma_{k\geqslant 0} f_k$ in $\hat{\mathscr{S}}_+(\varphi, r, V'')$.

□

Let us now consider the function $\varphi_r(y, \xi) = |\xi|^{1/r}$. We could define as before the sets $\mathscr{S}(\varphi_r, r, V)$ and $\hat{\mathscr{S}}(\varphi_r, r, V)$ but the map $\mathscr{S}(\varphi_r, r, V) \to \hat{\mathscr{S}}(\varphi_r, r, V)$ is not injective.

Nevertheless, we can define $\mathscr{S}(\varphi_r^-, r, V)$ by replacing the majoration in Definition 2.1.1(i) by the following: $\forall V' \Subset V$, $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$, $\forall (y, \xi) \in V'$,

$$|f(y, \xi)| < C_\varepsilon \mathrm{e}^{\varepsilon|\xi|^{1/r}}$$

and we define $\hat{\mathscr{S}}(\varphi_r^-, r, V)$ as in Definition 2.1.3 replacing the majoration by: $\forall V' \Subset V, \exists A > 0, \exists c > 0, \forall \varepsilon > 0, \exists C_\varepsilon > 0, \forall (y, \xi) \in V', |\xi| > c, \forall k \geqslant 0,$

$$|f_k(y, \xi)| < C_\varepsilon A^k (k!)^r |\xi|^{-k} e^{\varepsilon|\xi|^{1/r}}.$$

By the same proof as in Theorem 2.1.5 we obtain:

PROPOSITION 2.1.6. *If U_0 is a domain of holomorphy and Γ an open $\mathbb{R}$-conic subset of $\mathbb{C}^{n'}$, if $V = U_0 \times \Gamma$ then the map $\mathscr{S}(\varphi_r^-, r, V) \to \hat{\mathscr{S}}(\varphi_r^-, r, V)$ is bijective.*

2.2. *Definition of microlocal operators*

Let X be a complex analytic manifold, Y be a submanifold of X and let T_Y^*X be the conormal bundle to Y in X.

As a fiber bundle, T_Y^*X is provided with a canonical action of $\mathbb{C}$ hence of $\mathbb{R}_+^*$. In the sequel, $\mathbb{R}$-conic will refer to this action.

DEFINITION 2.2.1. Let $r \in \mathbb{R}$, $r \geqslant 1$, and let U be a $\mathbb{R}$-conic open subset of T_Y^*X.

A *r-weight* on U is a continuous plurisubharmonic function φ on U such that $\lim_{t \to +\infty} \varphi(tx^*)/t^{1/r} = 0$ for $t \in \mathbb{R}_+^*$ uniformly for x^* in a compact subset of U.

(The hypothesis "φ plurisubharmonic" will not be used in Section 2.2).

Let $(\tilde{\Omega}, (x_1, \ldots, x_{n'}, y_1, \ldots, y_n))$ be a local coordinate system of X such that if $\Omega = \tilde{\Omega} \cap Y$ we have $\Omega = \{(x, y) \in \tilde{\Omega} / x = 0\}$. Let $(T^*\tilde{\Omega}, (x, y, \xi, \eta))$ be the local coordinate system associated to (x, y). We have $(T_Y^*X) \times_Y \Omega = \{(x, y, \xi, \eta) \in T^*\tilde{\Omega} / x = 0, \eta = 0\}$ and thus an isomorphism between $(T_Y^*X) \times_Y \Omega$ and $U_0 \times \mathbb{C}^{n'}$ where U_0 is an open subset of $\mathbb{C}^n$. This allows us to consider the sheaf $\mathscr{S}(\varphi, r)$ on $U \cap (T_Y^*X \times_Y \Omega)$ and we have now to define the action of a coordinate transformation on $\mathscr{S}(\varphi, r)$. From Theorem 2.1.5, it is enough to define this action on the spaces $\hat{\mathscr{S}}(\varphi, r, V)$.

Let $(\tilde{\Omega}', x', y')$ be another coordinate system, $\chi: \tilde{\Omega} \to \tilde{\Omega}'$ be the coordinate transformation and $\tilde{\chi}: (T_Y^*X) \times_Y \Omega \to (T_Y^*X) \times_Y \Omega'$ the map induced by χ. In fact χ is given by $(x, y) \mapsto (x' = \chi_1(x, y), y' = \chi_2(x, y))$ with $\chi_1(0, y) = 0$ and $\tilde{\chi}$ is given by $\tilde{\chi}(y, \xi) = (y', \xi')$ with $y' = \chi_2(0, y)$ and $\xi' = {}^t\chi_1'(0, y)^{-1}\xi$. Let us define a matrix $M(x, y)$ by $\chi_1(x, y) = M(x, y) \cdot x$ and denote by $J(y)$ the jacobian of $\chi_1(0, y)$.

When V is an open $\mathbb{R}$-conic subset of $(T_Y^*X) \times_Y \Omega \approx U_0 \times \mathbb{C}^{n'}$ and V' its image under $\tilde{\chi}$, we define a map $\chi_*: \hat{\mathscr{S}}(\varphi, r, V) \to \hat{\mathscr{S}}(\varphi, r, V')$ by:

(2.2.1) If $f(y, \xi) = \Sigma_{k \geqslant 0} f_k(y, \xi)$ we set $\chi_* f(y', \xi') = \Sigma_{k \geqslant 0} g_k(y', \xi')$ with $(y', \xi') = \tilde{\chi}(y, \xi)$ and

$$g_k(y', \xi') = J(y)^{-1} \sum_{\substack{\alpha \in \mathbb{N}^{n'} \\ |\alpha| \leqslant k}} \frac{1}{\alpha!} \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial \tilde{\xi}}\right)^\alpha f_{k-|\alpha|}(y, \tilde{\xi} + {}^tM(x, y)\xi') \bigg|_{\substack{\tilde{\xi}=0 \\ x=0}}$$

It is easily verified that χ_* is well defined from $\hat{\mathscr{S}}(\varphi, r, V)$ to $\hat{\mathscr{S}}(\varphi, r, V')$ and that if χ and χ' are two coordinate transforms we have $(\chi\chi')_* = \chi_*\chi'_*$, which proves that χ_* is an isomorphism.

As the sheaf $\mathscr{S}(\varphi, r)$ is generated by the presheaf $V \mapsto \hat{\mathscr{S}}(\varphi, r, V)$, the formula 2.2.1 defines an isomorphism $\chi_*: \mathscr{S}(\varphi, r) \to \tilde{\chi}^{-1}\mathscr{S}(\varphi, r)$.

DEFINITION 2.2.2. Let $r \in \mathbb{R}$, $r \geqslant 1$, and let φ be a r-weight on a $\mathbb{R}$-conic open subset U of T_Y^*X.

The sheaf $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi, r)$ is the sheaf on U which is defined by the gluing of the sheaves $\mathscr{S}(\varphi, r)$ along the isomorphisms χ_*.

If π is the canonical map $\pi: T_Y^*X \to Y$, then $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi, r)$ is a sheaf of $\pi^{-1}(\mathcal{O}_{X|Y})$-modules.

DEFINITION 2.2.3. Under the preceding hypothesis, we set:

$$\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^+, r) = \varinjlim_{c>0} \mathscr{C}_{Y|X}^{\mathbb{R}}(c\varphi, r)$$

$$\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^-, r) = \varprojlim_{c>0} \mathscr{C}_{Y|X}^{\mathbb{R}}(c\varphi, r).$$

In the following text the symbol φ^* in $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^*, r)$ will mean φ, φ^+ or φ^-.

Let us now define the sheaves of microlocal operators.

Let X be a complex analytic manifold. If we identify the diagonal of $X \times X$ to X, the conormal bundle $T_X^*(X \times X)$ is identified to T^*X and thus $\mathscr{C}_{X|X\times X}^{\mathbb{R}}(\varphi^*, r)$ is a sheaf on T^*X. Now φ is a function on an open subset of T^*X.

We denote by $\Omega_{X\times X}^{(0,n)}$ the sheaf of holomorphic differential forms on $X \times X$ of degree 0 in the first variables and of maximum degree $n = \dim X$ in the second.

DEFINITION 2.2.4. The sheaf of microlocal operators of φ^*-type is the sheaf on T^*X defined by:

$$\mathscr{E}_X^{\mathbb{R}}(\varphi^*, r) = \mathscr{C}_{X|X\times X}^{\mathbb{R}}(\varphi^*, r) \otimes_{\mathcal{O}_{X\times X}} \Omega_{X\times X}^{(0,n)}$$

(φ^* means φ, φ^+ or φ^-).

Let $(\Omega, (x_1, \ldots, x_n))$ be a local coordinate system of X, then $T^*\Omega \approx \Omega \times \mathbb{C}^n$. If Ω_0 is a domain of holomorphy in Ω and Γ_0 a convex cone in $\mathbb{C}^n \backslash \{0\}$, then the set of the sections of $\mathscr{E}_X^{\mathbb{R}}(\varphi, r)$ on $V = \Omega_0 \times \Gamma_0$ is $\hat{\mathscr{S}}(\varphi, r, V)$.

Let $\chi: (\Omega, x) \to (\Omega', x')$ be a coordinate transform and $\tilde{\chi}: T^*\Omega \to T^*\Omega'$ be the induced isomorphism. The isomorphism χ_* on $\hat{\mathscr{S}}(\varphi, r, V)$ is given by:

If $f = \Sigma_{k\geqslant 0} f_k(x, \xi)$ and $(x', \xi') = \tilde{\chi}(x, \xi)$ then $\chi_* f(x', \xi') = \Sigma_{k\geqslant 0} g_k(x', \xi')$ with

$$g_k(x', \xi') = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\leqslant k}} \frac{1}{\alpha!}\left(\frac{\partial}{\partial\tilde{x}}\right)^\alpha \left(\frac{\partial}{\partial\tilde{\xi}}\right)^\alpha f_{k-|\alpha|}(x, \tilde{\xi}+{}^tM(x, \tilde{x})\xi')\Big|_{\substack{\tilde{\xi}=0\\ x=\tilde{x}}}$$

In this formula $M(x, \tilde{x})$ is defined by $\chi(x)-\chi(\tilde{x})=M(x, \tilde{x})\cdot(x-\tilde{x})$.

When $P=\Sigma_{k\geqslant 0} P_k(x, \xi)$ and $Q=\Sigma_{k\geqslant 0} Q_k(x, \xi)$ are two formal series of holomorphic functions on $V \subset \mathbb{C}^n \times \mathbb{C}^n$ we set: $P \# Q(x, \xi)=\Sigma_{k\geqslant 0}(P \# Q)_k(x, \xi)$ with

$$(P \# Q)_k(x, \xi) = \sum_{\substack{k=i+j+|\alpha| \\ \alpha\in\mathbb{N}^n}} \frac{1}{\alpha!}\left(\frac{\partial}{\partial\xi}\right)^\alpha P_i(x, \xi)\left(\frac{\partial}{\partial x}\right)^\alpha Q_j(x, \xi). \tag{2.2.2}$$

PROPOSITION 2.2.5. *Let φ_1 and φ_2 be two r-weights on U.*

(i) *If $P\in\hat{\mathscr{P}}(\varphi_1, r, V)$ and $Q\in\hat{\mathscr{P}}(\varphi_2, r, V)$ then*

$$P \# Q\in\hat{\mathscr{P}}(\varphi_1+\varphi_2, r, V).$$

(ii) *If χ is a coordinate transform then*

$$(\chi_* P) \# (\chi_* Q) = \chi_*(P \# Q).$$

These results may be proved by a direct calculation, but it is easier to get them from the fact that the same formulas are true for the usual microdifferential operators.

COROLLARY 2.2.6. *The operation $(P, Q)\mapsto P \# Q$ provides $\mathscr{E}_X^{\mathbb{R}}(\varphi^+, r)$ and $\mathscr{E}_X^{\mathbb{R}}(\varphi^-, r)$ with a structure of sheaves of unitary rings.*

The same formulas provide $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^+, r)$ with a structure of $\mathscr{E}_X^{\mathbb{R}}(\varphi^+, r)$-module and $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^*, r)$ with a structure of $\mathscr{E}_X^{\mathbb{R}}(\varphi^-, r)$-module when φ is a r-weight on a neighborhood of T_Y^*X.

Let $(\tilde{\Omega}, x, y)$ a local coordinate system of X such that $\Omega=\tilde{\Omega}\cap Y=\{(x, y)\in\tilde{\Omega}/x=0\}$ and let $(T_\Omega^*\tilde{\Omega}, (y, \xi))$ be the corresponding coordinate system of T_Y^*X.

The function $\varphi_s(x, \xi)=|\xi|^{1/s}$ is well defined on $T_\Omega^*\tilde{\Omega}$ and is a r-weight if $r<s$. So we may consider the sheaves $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi_s^*, r)$ on $T_\Omega^*\tilde{\Omega}$.

In a coordinate transform the function φ_s is not preserved but if (y, ξ) and (y', ξ') are two coordinate systems, there exist always $C_1>0$ and $C_2>0$ such that $C_1|\xi'|^{1/s} < |\xi|^{1/s} < C_2|\xi'|^{1/s}$, hence the sheaves $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi_s^+, r)$ and $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi_s^-, r)$ do not depend on the local coordinate system.

When $r = s$, it follows from the preceding section that the sheaf $\mathscr{S}(\varphi_r^-, r)$ is well defined in a local chart and is generated by the presheaf $V \mapsto \hat{\mathscr{S}}(\varphi_r^-, r, V)$. Moreover, it is easy to see that the morphisms χ_* act on the spaces $\hat{\mathscr{S}}(\varphi_r^-, r, V)$ and it follows that we can define a sheaf $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi_r^-, r, V)$ by gluing the $\mathscr{S}(\varphi_r^-, r)$ along the χ_*.

DEFINITION 2.2.7. (i) Let r, s in $\mathbb{R}$ such that $s > r \geqslant 1$, the sheaf $\mathscr{C}^{\mathbb{R}}_{Y|X}(s, r)$ is the sheaf which is equal to $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi_s^+, r)$ in each local chart (y, ξ) with $\varphi_s(y, \xi) = |\xi|^{1/s}$.

(ii) The sheaf $\mathscr{C}^{\mathbb{R}}_{Y|X}(r)$ is the sheaf equal to $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi_r^-, r)$ in each local chart with $\varphi_r(y, \xi) = |\xi|^{1/r}$.

(iii) With φ_∞ being the function 1 on T_Y^*X, we set

$$\mathscr{C}^{\mathbb{R}}_{Y|X}(\infty, r) = \mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi_\infty^+, r).$$

If we fix m in Definition 2.1.1, we get the spaces $\mathscr{S}_{+,m}(\varphi, V)$ and, for $\varphi_\infty = 1$, $\mathscr{S}_{+,m}(\varphi_\infty, V)$ is the set of holomorphic functions on V such that

$$\forall V' \Subset V,\ \exists C > 0,\ \forall (y, \xi) \in V',\ |f(y, \xi)| < C|\xi|^m$$

and

$$\mathscr{S}_m(\varphi_\infty, r, V) = \varprojlim_{V' \Subset V} \mathscr{S}_{+,m}(\varphi_\infty, V')/\mathscr{S}_-(r, V').$$

In the same way we can define $\hat{\mathscr{S}}_m(\varphi_\infty, r, V)$ which is clearly invariant under the morphisms χ_*. From this we can define the sheaves $\mathscr{C}^{\mathbb{R}}_{Y|X,m}(\infty, r)$ and we have

$$\mathscr{C}^{\mathbb{R}}_{Y|X}(\infty, r) = \varinjlim_{m} \mathscr{C}^{\mathbb{R}}_{Y|X,m}(\infty, r).$$

DEFINITION 2.2.8 The sheaf $\mathscr{C}^{\mathbb{R}}_{Y|X}(\infty)$ is given by

$$\mathscr{C}^{\mathbb{R}}_{Y|X}(\infty) = \varinjlim_{m \in \mathbb{Z}} \mathscr{C}^{\mathbb{R}}_{Y|X,m}(\infty)$$

with

$$\mathscr{C}^{\mathbb{R}}_{Y|X,m}(\infty) = \varinjlim_{\substack{p \leqslant m \\ p \in \mathbb{Z}}} \mathscr{C}^{\mathbb{R}}_{Y|X,m}(\infty, 1)/\mathscr{C}^{\mathbb{R}}_{Y|X,p}(\infty, 1).$$

DEFINITION 2.2.9 Let X be a complex analytic manifold and let r, s be two real numbers such that $r > s \geqslant 1$ or $r = \infty$.

$$\mathscr{E}_X^{\mathbb{R}}(r, s) = \mathscr{C}_{X|X\times X}^{\mathbb{R}}(r, s) \otimes_{\mathscr{O}_{X\times X}} \Omega_{X\times X}^{(0,\dim X)}$$
$$\mathscr{E}_X^{\mathbb{R}}(r) = \mathscr{C}_{X|X\times X}^{\mathbb{R}}(r) \otimes_{\mathscr{O}_{X\times X}} \Omega_{X\times X}^{(0,\dim X)}$$

$\mathscr{E}_X^{\mathbb{R}}(r, s)$ and $\mathscr{E}_X^{\mathbb{R}}(r)$ are sheaves of rings on T^*X.

For $1 \leqslant r \leqslant +\infty$, we will often write $\mathscr{C}_{Y|X}^{\mathbb{R}}(r, r)$ instead of $\mathscr{C}_{Y|X}^{\mathbb{R}}(r)$ and $\mathscr{E}_X^{\mathbb{R}}(r, r)$ instead of $\mathscr{E}_X^{\mathbb{R}}(r)$ to unify the statements with the case (r, s), $r > s$ but we have to take care because

$$\mathscr{C}_{Y|X}^{\mathbb{R}}(r, s) = \mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi_r^+, s) \quad \text{when } r > s$$

and

$$\mathscr{C}_{Y|X}^{\mathbb{R}}(r, r) = \mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi_r^-, r).$$

PROPOSITION 2.2.10 *The sheaf $\mathscr{E}_X^{\mathbb{R}}(1)$ may be identified with the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of* [16] *and $\mathscr{E}_X^{\mathbb{R}}(\infty, 1)$ with the sheaf $\mathscr{E}_X^{\mathbb{R},f}$ of* [1].

The identity between $\mathscr{E}_X^{\mathbb{R}}(1)$ and $\mathscr{E}_X^{\mathbb{R}}$ is a consequence of Theorem 2.1.1 of Aoki [2], and the other case can be proved by the same method.

REMARK. In [2], Aoki has defined some subsheaves of $\mathscr{E}_X^{\mathbb{R}}$ which he denotes by $\mathscr{E}_X^{\mathbb{R}}(r)$ and which are equal to the sheaves $\mathscr{E}_X^{\mathbb{R}}(r, 1)$ of Definition 2.2.4 here.

On the other hand we defined in [11] a sheaf $\hat{\mathscr{E}}_X^{\mathbb{R}}$ which is denoted here $\mathscr{E}_X^{\mathbb{R}}(\infty)$, it is called the sheaf of formal microlocal operators.

2.3. *Microdifferential operators*

Let X be a complex analytic manifold, Y be a submanifold of X, T_Y^*X be the conormal bundle to Y in X. We identify Y with the null section of T_Y^*X and set $\dot{T}_Y^*X = T_Y^*X - Y$.

Let $\mathbb{P}_Y^*X \approx \dot{T}_Y^*X/\mathbb{C}^*$ be the complex projective conormal bundle and $\gamma: \dot{T}_Y^*X \to \mathbb{P}_Y^*X$ be the canonical projection. Moreover we will consider the set $\widehat{\mathbb{P}_Y^*X} = T_Y^*X/\mathbb{C}^*$ which is isomorphic to the disjointed union of $\mathbb{P}_Y^*X$ and Y and the canonical projection $\hat{\gamma}$: $T_Y^*X \to \widehat{\mathbb{P}_Y^*X}$.

In this section, the function φ will be a r-weight ($r \in \mathbb{R}$, $r \geqslant 1$) which is defined on a $\mathbb{C}^*$-conic open subset U of T_Y^*X. By definition $\mathbb{C}^*$-conic is equivalent to $U = \hat{\gamma}^{-1}(\hat{\gamma}(U))$.

DEFINITION 2.3.1. The sheaves $\mathscr{C}_{Y|X}(\varphi^*, r)$ and $\mathscr{B}_{Y|X}(\varphi^*, r)$ are defined by

$$\mathscr{B}_{Y|X}(\varphi^*, r) = \mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^*, r)|_Y$$
$$\mathscr{C}_{Y|X}(\varphi^*, r) = \hat{\gamma}^{-1}\hat{\gamma}_*\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^*, r)$$

By definition we have

$$\mathscr{C}_{Y|X}(\varphi^*, r)|_{\dot{T}^*_Y X} = \gamma^{-1}\gamma_*(\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^*, r)|_{\dot{T}^*_Y X})$$

and

$$\mathscr{C}_{Y|X}(\varphi^*, r)|_Y = \mathscr{B}_{Y|X}(\varphi^*, r).$$

As before φ^* means φ^+, φ^- or φ and by the same definition we have the sheaves $\mathscr{C}_{Y|X}(r, s)$ and $\mathscr{C}_{Y|X}(r)$ with $1 \leqslant s \leqslant r < +\infty$.

LEMMA 2.3.2 *For each $k \geqslant 1$, we have*

$$R^k\hat{\gamma}_*(\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^*, r)) = 0.$$

Proof. When $T^*_Y X$ is provided with the $\mathbb{R}$-conic topology, $\hat{\gamma}$ is a proper map and thus for each $x \in \widehat{\mathbb{P}^*_Y X}$, each $k \geqslant 0$ and each sheaf $\mathscr{F}$ on $T^*_Y X$ we have

$$R^k\hat{\gamma}_*(\mathscr{F})_x = H^k(\hat{\gamma}^{-1}(x), \mathscr{F}|_{\hat{\gamma}^{-1}(x)}).$$

As $\hat{\gamma}^{-1}(x)$ is isomorphic to $\mathbb{C}$ or to $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, we have $R^k\hat{\gamma}_*(\mathscr{F}) = 0$ if $k \geqslant 2$.

If we take the notations of the proof of Proposition 2.1.2, we have $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi, r) = \mathscr{S}_+(\varphi)/\mathscr{S}_-(r)$ and thus an exact sequence:

$$\begin{array}{ccccc} R^1\hat{\gamma}_*(\mathscr{S}_+(\varphi)) & \to & R^1\hat{\gamma}_*(\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi, r)) & \to & R^2\hat{\gamma}_*(\mathscr{S}_-(r)) \\ & & & & \| \\ & & & & 0. \end{array}$$

From Proposition 1.1.1, we have

$$H^1(\hat{\gamma}^{-1}(x), \mathscr{S}_+(\varphi)|_{\hat{\gamma}^{-1}(x)}) = 0$$

hence $R^1\hat{\gamma}_*(\mathscr{S}_+(\varphi)) = 0$ and $R^1\hat{\gamma}_*(\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi, r)) = 0$. Replacing Proposition 1.1.1 by Proposition 1.1.5, we get the same result for $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^-, r)$.

At last, as $\hat{\gamma}^{-1}(x)$ is compact for the $\mathbb{R}$-conic topology, the cohomology on $\hat{\gamma}^{-1}(x)$ commutes with inductive limits which gives the result for $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^+, r)$.

PROPOSITION 2.3.3. *If V_1 and V_2 are two open sets in $T^*_Y X$ with V_1 not empty,*

V_2 connected and $V_1 \subset V_2$, then the restriction morphism

$$\Gamma(V_2, \mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^*, r)) \to \Gamma(V_1, \mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^*, r))$$

is injective.

Proof. The result is known for the sheaf $\mathscr{C}^{\mathbb{R}}_{Y|X}$ ([16], Chap. III, Th. 2.2.8). By a ramification, we get the same result for $\mathscr{C}^{\mathbb{R}}_{Y|X}(r)$ when $r \geqslant 1$ and now we remark that $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^*, r)$ is always a subsheaf of $\mathscr{C}^{\mathbb{R}}_{Y|X}(r)$.

COROLLARY 2.3.4. *$\mathscr{C}_{Y|X}(\varphi^*, r)$ is a subsheaf of $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^*, r)$.*

Proof. The morphism $\mathscr{C}_{Y|X}(\varphi^*, r) \to \mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^*, r)$ is given by the restriction morphism:

$$\Gamma(\hat{\gamma}^{-1}(\hat{\gamma}(V)), \mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^*, r)) \to \Gamma(V, \mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^*, r))$$

and it is injective by Proposition 2.3.3.

With the notations of Section 2.1 we have:

PROPOSITION 2.3.5. *Let U_0 be open in $\mathbb{C}^n$ and $U = U_0 \times \mathbb{C}^{n'}$. Let $r \in \mathbb{R}$, $r \geqslant 1$, φ a r-weight on U.*

If V is a $\mathbb{R}$-conic open subset of U which is a domain of holomorphy then:

$$\Gamma(V, \mathscr{S}(\varphi, r)) = \hat{\mathscr{S}}(\varphi, r, V) = \hat{\mathscr{S}}_+(\varphi, r, V)/\hat{\mathscr{S}}_-(\varphi, r, V).$$

Proof. When $V = V_0 \times \mathbb{C}^{n'}$, this result is a consequence of Theorem 2.1.5, thus we may suppose that $V \subset U_0 \times \mathbb{C}^{n'} \setminus \{0\}$.

Let $\hat{\mathscr{S}}_+(\varphi, r)$ and $\hat{\mathscr{S}}_-(\varphi, r)$ the sheaves whose sections on $\mathbb{R}$-conic open sets V are respectively $\hat{\mathscr{S}}_+(\varphi, r, V)$ and $\hat{\mathscr{S}}_-(\varphi, r, V)$.

From Theorem 2.1.5, $\mathscr{S}(\varphi, r)$ is the quotient sheaf $\hat{\mathscr{S}}_+(\varphi, r)/\hat{\mathscr{S}}_-(\varphi, r)$ and to prove the proposition we have to show that if V is a domain of holomorphy then

$$H^1(V, \hat{\mathscr{S}}_-(\varphi, r)) = 0.$$

This is proved as Proposition 1.1.1 by Theorem 4.4.2 of [5].

As in the preceding section we consider a local coordinate system $(\tilde{\Omega}, (x_1, \ldots, x_n, y_1, \ldots, y_n))$ of X such that

$$\Omega = \tilde{\Omega} \cap Y = \{(x, y) \in \tilde{\Omega} / x = 0\} \quad \text{and} \quad (T^*_{\Omega}\tilde{\Omega}, (y, \xi))$$

the corresponding coordinate system of $T^*_Y X$.

Let V be a $\mathbb{R}$-conic domain of holomorphy in $T^*_{\Omega}\tilde{\Omega}$. From Proposition 2.1.2, a section of $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^*, r)$ on V has a symbol in $\Gamma(V, \mathscr{S}(\varphi^*, r))$ if V is contained in a half space but from Proposition 2.3.5, it has always a symbol in $\Gamma(V, \hat{\mathscr{S}}(\varphi^*, r))$.

This applies to $\mathbb{C}^*$-conic open sets, hence to sections of the sheaf $\mathscr{C}_{Y|X}(\varphi^*, r)$: if V is an open subset of $\dot{T}_Y^*X$ whose fibers for γ are connected and if $\tilde{V} = \gamma^{-1}\gamma(V)$ then:

$$\Gamma(V, \mathscr{C}_{Y|X}(\varphi^*, r)) = \Gamma(\tilde{V}, \mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^*, r)) = \hat{\mathscr{S}}(\varphi^*, r, \tilde{V}).$$

(But a section of $\mathscr{C}_{Y|X}(\varphi^*, r)$ can never have a symbol in some $\mathscr{S}(\varphi^*, r, V)$).

From this, we can define a better symbolic calculus for $\mathscr{C}_{Y|X}(\varphi^{\pm}, r)$ analogous to the symbolic calculus of $\mathscr{C}_{Y|X}$ in [16].

THEOREM 2.3.6. *Let $(\tilde{\Omega}, (x, y))$ be a local coordinate system of X, $\Omega = Y \cap \tilde{\Omega}$ and $(T_\Omega^*\tilde{\Omega}, (y, \xi))$ be the corresponding coordinate system of T_Y^*X.*

Let $r \geqslant 1$ and let φ be a r-weight on $T_\Omega^*\tilde{\Omega}$. We suppose that φ satisfy:

(a) $\forall K \Subset \Omega, \forall t \in \mathbb{R}, t \geqslant 1, \exists \alpha_1 > 0, \exists \alpha_2 > 0, \forall y \in K, \forall \xi \in \mathbb{C}^n, |\xi| \geqslant 1,$

$$\varphi(y, t\xi) \leqslant \alpha_1 \varphi(y, \xi) \quad \textit{and} \quad t\varphi(y, \xi) \leqslant \varphi(y, \alpha_2 \xi).$$

(b) $\forall \theta \in \mathbb{R} \quad \varphi(y, e^{i\theta}\xi) = \varphi(y, \xi).$

For each $k \geqslant 0$, we define a function φ_k on the unit sphere $\mathbb{S}_\Omega^*\tilde{\Omega} = \{(y, \xi) \in T_\Omega^*\tilde{\Omega} / |\xi| = 1\}$ by

$$\varphi_k(y, \xi) = \inf_{R \geqslant 1} R^{-k} e^{\varphi(y, R\xi)}.$$

Let V be a $\mathbb{C}^*$-conic open subset of $T_\Omega^*\tilde{\Omega}$. The set of sections of $\mathscr{C}_{Y|X}(\varphi^+, r)$ on V is isomorphic to the set of formal series $\Sigma_{k \in \mathbb{Z}} f_k(y, \xi)$ of holomorphic functions on V such that

(i) *For each k in $\mathbb{Z}$, $f_k(y, \xi)$ is homogeneous of degree k in ξ.*

(ii) $\forall K \Subset V, \exists C_0 > 0, \exists C > 0, \forall k \geqslant 0, \forall (y, \xi) \in K, |\xi| = 1,$

$$|f_k(y, \xi)| \leqslant C_0 C^k \varphi_k(y, \xi).$$

(iii) $\forall K \Subset V, \exists C_0 > 0, \exists C > 0, \forall k < 0, \forall (y, \xi) \in K, |\xi| = 1,$

$$|f_k(y, \xi)| \leqslant C_0 C^{-k}((-k)!)^r.$$

The same result is true for $\mathscr{C}_{Y|X}(\varphi^-, r)$ if we replace the condition (ii) *by:*

(iv) $\forall K \Subset V, \forall \varepsilon > 0, \exists C_\varepsilon > 0, \forall k \geqslant 0, \forall (y, \xi) \in K, |\xi| = 1,$

$$|f_k(y, \xi)| \leqslant C_\varepsilon \varepsilon^k \varphi_k(y, \xi).$$

REMARK. If u is a section of $\mathscr{B}_{Y|X}(\varphi^+, r)$ or $\mathscr{B}_{Y|X}(\varphi^-, r)$ on an open subset U of Y, its symbol will be a formal series $\Sigma_{k\geqslant 0} f_k(y, \xi)$ satisfying conditions (i), and (ii) or (iv).

Proof. We will consider $\mathscr{C}_{Y|X}(\varphi^+, r)$, the other case would be the same. Let us denote by $\hat{\hat{\mathscr{S}}}(\varphi^+, r, V)$ the set of the series which are defined on V and satisfy (i), (ii) and (iii). The presheaf $V \mapsto \hat{\hat{\mathscr{S}}}(\varphi^+, r, V)$ is clearly a sheaf, hence the theorem is local in V and we can suppose that V is a domain of holomorphy.

From Proposition 2.3.5, $\Gamma(V, \mathscr{C}_{Y|X}(\varphi^+, r))$ is isomorphic to

$$\hat{\mathscr{S}}(\varphi^+, r, V) = \varinjlim_{V' \Subset V} \varinjlim_{c>0} \hat{\mathscr{S}}_+(c\varphi, r, V')/\hat{\mathscr{S}}_-(c\varphi, r, V').$$

Let $\Sigma_{k\in\mathbb{Z}} f_k(y, \xi)$ be in $\hat{\hat{\mathscr{S}}}(\varphi_+, r, V)$, we have:

$$\sum_{k\geqslant 0} |f_k(y, \xi)| \leqslant C_0 \sum_{k\geqslant 0} C^k |\xi|^k \varphi_k\left(y, \frac{\xi}{|\xi|}\right)$$

and

$$|\xi|^k \varphi_k\left(y, \frac{\xi}{|\xi|}\right) \leqslant (2C)^{-k}\, \mathrm{e}^{\varphi(y, 2C\xi)} \leqslant (2C)^{-k}\, \mathrm{e}^{2\alpha_1 C\varphi(y,\xi)}$$

hence

$$\sum_{k\geqslant 0} |f_k(y, \xi)| \leqslant C_0\, \mathrm{e}^{2\alpha_1 C\varphi(y,\xi)} \sum_{k\geqslant 0} 2^{-k} \leqslant 2C_0\, \mathrm{e}^{2\alpha_1 C\varphi(y,\xi)}.$$

The formal series $\Sigma_{k\geqslant 0}\, g_k(y, \xi)$ defined by $g_0(y, \xi) = \Sigma_{k\geqslant 0} f_k(y, \xi)$ and $g_k(y, \xi) = f_{-k}(y, \xi)$ when $k \geqslant 1$ is thus an element of $\hat{\mathscr{S}}_+(C'\varphi, r, V)$.

We get a map F from $\hat{\hat{\mathscr{S}}}(\varphi^+, r, V)$ to $\hat{\mathscr{S}}(\varphi^+, r, V)$ and it is easy to see using the Cauchy inequalities that this map is injective.

To show that this map is surjective, we will first suppose that

$$V \subset \{(y, \xi)/\xi_1 \neq 0\},$$

then using the coordinates $(y, \xi_2/\xi_1, \ldots, \xi_n/\xi_1; \xi_1)$ we can suppose that $V = V_0 \times \mathbb{C}^*$ with just one variable ξ.

Let $V_0' \Subset V_0$ and $V' = V_0' \times \mathbb{C}^*$, let $f(y, \xi) = \Sigma_{k\geqslant 0} f_k(y, \xi)$ in $\hat{\mathscr{S}}_+(c\varphi, r, V')$.

The functions f_k have a representation in Laurent series:

$$f_k(y, \xi) = \sum_{l\in\mathbb{Z}} f_{kl}(y)\xi^l$$

and from Definition 2.1.3 and the Cauchy inequalities

$$\exists \mu > 0, \exists A > 0, \exists C > 0, \forall R > 1, \forall y \in V_0', |f_{kl}(y)| \leqslant CA^k(k!)^r R^{-k-l} e^{\mu\varphi(y,R)}.$$

As $\lim_{|\xi|\to\infty} \varphi(y, \xi)/|\xi|^{1/r} = 0$ we have:

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0, \forall R > 0, e^{\varphi(y,R)} \leqslant C_\varepsilon e^{\varepsilon R^{1/r}}$$

hence

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0, \forall R \geqslant 1, \forall y \in V_0', |f_{kl}(y)| \leqslant CC_\varepsilon A^k(k!)^r R^{-k-l} e^{\varepsilon R^{1/r}}$$

and with

$$R = \left(\frac{r(k+l)}{\varepsilon}\right)^r \quad \text{if } k \geqslant 1-l \quad \text{and} \quad R = 1 \text{ if } 0 \leqslant k \leqslant -l \text{ we get:}$$

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0, \forall y \in V_0', \forall (k, l) \in \mathbb{N} \times \mathbb{Z}, k+l \geqslant 1,$$

$$|f_{kl}(y)| \leqslant C_\varepsilon A^k \varepsilon^{k+l} \left(\frac{k!}{(k+l)!}\right)^r$$

$$\forall (k, l) \in \mathbb{N} \times \mathbb{Z}, k+l \leqslant 0, \forall y \in V_0', |f_{kl}(y)| \leqslant CA^k(k!)^r.$$

The series $f^l(y) = \Sigma_{k\geqslant 0} f_{kl}(y)$ is therefore convergent and satisfies:

$$\exists C_0, C_1 > 0, \forall l < 0, \forall y \in V_0', |f^l(y)| \leqslant C_0 C_1^{-l}((-l)!)^r.$$

$$\forall \varepsilon > 0, \exists C_\varepsilon, \forall l \geqslant 0, \forall y \in V_0', |f^l(y)| \leqslant C_\varepsilon \varepsilon^l \frac{1}{(l!)^r}.$$

Now we consider the functions

$$f^+(y, \xi) = \sum_{l\geqslant 0} f^l(y)\xi^l \quad \text{and} \quad f_k^+(y, \xi) = \sum_{l\geqslant 0} f_{kl}(y)\xi^l.$$

We have $f^+(y, \xi) = \Sigma_{k\geqslant 0} f_k^+(y, \xi)$ and $\Sigma_{k\geqslant 0} f_k^+(y, \xi)$ is a formal symbol of $\hat{\mathscr{S}}_+(\varphi^+, r, V')$ hence $f^+(y, \xi)$ is in $\mathscr{S}_+(\varphi^+, V')$ (Lemma 2.1.4) that is:

$$\exists \mu > 0, \exists m \geqslant 0, \exists C > 0, \forall y \in V_0', |f^+(y, \xi)| < c|\xi|^m e^{\mu\varphi(y,\xi)}.$$

The Cauchy inequalities applied to f^+ give:

$$\forall y \in V_0', \forall R > 1, |f^l(y)| < CR^{m-l} e^{\mu\varphi(y,R)}$$

hence the inequalities (ii) by definition of φ_k.

The formal series $\Sigma_{l\in\mathbb{Z}}\, f^l(y)\xi^l$ is thus in $\hat{\hat{\mathscr{S}}}(\varphi^+, r, V')$ and its image by F is the series $\Sigma\, f_k(y, \xi)$.

This proves that F is surjective from $\hat{\hat{\mathscr{S}}}(\varphi^+, r, V')$ to $\hat{\mathscr{S}}(\varphi^+, r, V')$ for each $V' \Subset V$ and, as F is injective, from $\hat{\hat{\mathscr{S}}}(\varphi^+, r, V)$ to $\hat{\mathscr{S}}(\varphi^+, r, V)$ when $V \subset \{\xi_1 \neq 0\}$.

In the general case, the problem being local in V, we may suppose that V is of the preceding type or that V contains $\{\xi = 0\}$. In the latter case, we have $V \simeq V_0 \times \mathbb{C}^n$ and the proof is the same, the $f_k(y, \xi)$'s being represented in Taylor series:

$$f_k(y, \xi) = \sum_{\alpha\in\mathbb{N}^n} f_{k\alpha}(y)\xi^\alpha.$$

EXAMPLE 2.3.7. When $s \geqslant 1$ and $\varphi_s(y, \xi) = |\xi|^{1/s}$, φ_k is the constant $\varphi_k = (ks)^{-ks} e^{ks}$ and the Stirling's formula shows that in Theorem 2.3.6, φ_k may be replaced by $1/(k!)^s$. When $\varphi(y, \xi) = 1$, we cannot apply Theorem 2.3.6 but the same calculus gives:

PROPOSITION 2.3.8. *The set of sections of $\mathscr{C}_{Y|X,m}(r)$ on V is isomorphic to the set of formal series $\Sigma_{k\in\mathbb{Z}}\, f_k(y, \xi)$ of holomorphic function on V such that:*

(i) *For each k in $\mathbb{Z}$, $f_k(y, \xi)$ is homogeneous of degree k in ξ.*
(ii) $\forall k > m$, $f_k(y, \xi) \equiv 0$.
(iii) $\forall K \Subset V$, $\exists C_0 > 0$, $\exists C > 0$, $\forall k < 0$, $\forall (y, \xi) \in K$, $|\xi| = 1$,

$$|f_k(y, \xi)| \leqslant C_0 C^{-k}((-k)!)^r.$$

When $r = \infty$ the condition (iii) has to be removed in Theorem 2.3.6 and in Proposition 2.3.8.

Let X be a complex analytic manifold and $\hat{\gamma}$ be the projection $T^*X \to T^*X/\mathbb{C}^* \simeq \mathbb{P}^*X \cup X$. The sheaves of microdifferential operators are defined by

$$\mathscr{E}_X(\varphi^*, r) = \hat{\gamma}^{-1}\hat{\gamma}_* \mathscr{E}_X^{\mathbb{R}}(\varphi^*, r)$$

and the sheaves of differential operators by

$$\mathscr{D}_X(\varphi^*, r) = \mathscr{E}_X^{\mathbb{R}}(\varphi^*, r)|_X = \mathscr{E}_X(\varphi^*, r)|_X.$$

We define in the same way the sheaves $\mathscr{E}_X(r, s)$ for $1 \leqslant s \leqslant r \leqslant +\infty$.

Theorem 2.3.6 proves that the sheaf $\mathscr{E}_X(r, s)$ is exactly the sheaf which has been defined in [10], Section 1.5.

When $r = 1$, we saw that $\mathscr{E}_X^{\mathbb{R}}(1)$ is the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of [8], [16] (it is denoted by $\mathscr{P}_X^{\mathbb{R}}$

in [16]). The sheaf $\mathscr{E}_X(1)$ is thus the sheaf $\mathscr{E}_X^\infty$ of microdifferential operators of infinite order.

The sheaf $\mathscr{E}_X(\infty, 1)$ is the sheaf $\mathscr{E}_X$ of ordinary microdifferential operators and the equality $\mathscr{E}_X = \hat{\gamma}^{-1}\hat{\gamma}_* {}^{\mathbb{R}}_X(\infty, 1)$ was proved in [1].

At last let us remark that $\mathscr{E}_{X,m}(\infty, 1)$ is the sheaf $\mathscr{E}_{X,m}$ of microdifferential operators of order m and that $\mathscr{E}_X(\infty)$ is the sheaf $\hat{\mathscr{E}}_X$ of formal microdifferential operators.

The symbol of an operator of $\mathscr{E}_X(\varphi^\pm, r)$ where φ satisfies the conditions of Theorem 2.3.6 is a formal series $\Sigma_{k\in\mathbb{Z}} f_k(x, \xi)$ of functions on T^*X which satisfy conditions (i), (ii), (iii) or (i), (iv), (iii). The formulas for the product of two operators and the coordinate transforms are the same as in [16] for $\mathscr{E}_X$.

PROPOSITION 2.3.9. *Let r, s be such that* $1 \leqslant s \leqslant r \leqslant +\infty$, *then*

$$\mathscr{C}^{\mathbb{R}}_{Y|X}(r, s) = \mathscr{E}^{\mathbb{R}}_X(r, s) \otimes_{\mathscr{E}_X} \mathscr{C}_{Y|X}.$$

This proposition is easily proved using the symbolic calculus. In fact if $Y = \{(x, y) \in X / x = 0\}$ then it is well known that

$$\mathscr{C}_{Y|X} \simeq \mathscr{E}_X / \mathscr{E}_X x_1 + \cdots + \mathscr{E}_X x_n + \mathscr{E}_X D_{y_1} + \cdots + \mathscr{E}_X D_{Y_n}$$

as a $\mathscr{E}_X$-module and it is easy to see that the same result is true for $\mathscr{C}^{\mathbb{R}}_{Y|X}(r, s)$ as a $\mathscr{E}^{\mathbb{R}}_X(r, s)$-module and $\mathscr{C}_{Y|X}(r, s)$ as a $\mathscr{E}_X(r, s)$-module.

The same is true for $\mathscr{C}^{\mathbb{R}}_{Y|X}(r)$ and in fact if φ is a r-weight on T^*X in a neighborhood of $T^*_Y X$, if $\tilde{\varphi} = \varphi|_{T^*_Y X}$ then we have:

$$\mathscr{C}^{\mathbb{R}}_{Y|X}(\tilde{\varphi}, r) = \mathscr{E}^{\mathbb{R}}_X(\varphi, r) \otimes_{\mathscr{E}_X} \mathscr{C}_{Y|X}.$$

2.4. *Canonical transformations*

PROPOSITION 2.4.1 (Division theorem). *Let* $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ *be a local coordinate system of* T^*X *and let* (x_0, ξ_0) *be a point of* T^*X *with* $\xi_0 = (0, \ldots, 0, 1)$.

Let P be a microdifferential operator of $\mathscr{E}_X$ *defined in a neighborhood of* (x_0, ξ_0) *whose principal symbol satisfies:*

$$\left(\frac{\partial}{\partial \xi_1}\right)^j \sigma(P)(x_0, \xi_0) = 0 \quad \text{if } j < q \quad \text{and} \quad \left(\frac{\partial}{\partial \xi_1}\right)^q \sigma(P)(x_0, \xi_0) \neq 0.$$

There exists a neighborhood V of (x_0, ξ_0) *such that each section S of* $\mathscr{E}^{\mathbb{R}}_X(\varphi^*, r)$ *defined on* $V' \subset V$ *has a unique representation*

$$S = QP + R$$

with Q and R sections of $\mathscr{E}_X^{\mathbb{R}}(\varphi^*, r)$ on V′, R having a symbol of the form

$$R(x, \xi) = \sum_{j=0}^{q-1} R_j(x, \xi_2, \ldots, \xi_n)\xi_1^j.$$

Proof. It is enough to prove the theorem for $\mathscr{E}_X^{\mathbb{R}}(\varphi, r)$, then we get the cases φ^+ and φ^- by unicity.

Let $f = \Sigma_{k\geqslant 0} f_k(x, \xi)$ be a formal series of holomorphic functions on an open $\mathbb{R}$-conic convex subset V of $\mathbb{C}^n \times (\mathbb{C}^{n'}\setminus\{0\})$, the formal norm of Boutet de Monvel [3] is:

$$N_\varphi^{(r)}(f, T) = \sum_{\substack{k\geqslant 0\\ (\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^n}} \frac{2(2n)^{-k}k!}{(k+|\alpha|)!(k+|\beta|)!^r}\,\frac{(1+|\xi|)^{k+|\beta|}}{e^{\varphi(x,\xi)}}\left|\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial \xi}\right)^\beta f_k(x, \xi)\right| T^{2k+|\alpha|+|\beta|}.$$

From [3] we have

$$N_{\varphi+\varphi'}^{(r)}(f \# g, T) \ll N_\varphi^{(r)}(f, T)N_{\varphi'}^{(r)}(g, T)$$

and $\hat{\mathscr{P}}(\varphi, r, V)$ is the set of formal series f such that $N_\varphi^{(r)}(f, T)$ is convergent for $|T| < T_0$ for some $T_0 > 0$.

Now we can prove the proposition following the proof of Theorem 2.2.1, Ch. II of [16] with this formal norm.

In the same way it can be proved that Proposition 2.4.1 is still true if we replace ξ_1 par x_1. (In this case another proof works as in Kashiwara-Schapira [8], Lemma 6.2.1.)

These division theorems allow us to define quantized canonical transforms using the usual proof of $\mathscr{E}_X$ and $\mathscr{E}_X^\infty$ (cf. [16], Ch. II or [8]). We obtain:

THEOREM 2.4.2. *Let f be a canonical transform from an open set U of T^*X to an open set U′ of T^*X' (with X and X′ complex manifolds of the same dimension).*

Any quantized canonical transformation

$$F: \mathscr{E}_X|_U \to f^{-1}(\mathscr{E}_{X'}|_{U'})$$

extends uniquely in an isomorphism of sheaves of rings

$$F: \mathscr{E}_X^{\mathbb{R}}(\varphi^*, r)|_U \to f^{-1}(\mathscr{E}_{X'}^{\mathbb{R}}(\varphi^*, r)|_{U'})$$

for each $r \geqslant 1$ and each r-weight φ.

The proof of Theorem 3.4.1, Ch. II in [16] gives, with the preceding results:

THEOREM 2.4.3. *The sheaves of ring $\mathscr{E}_X^{\mathbb{R}}(\varphi^+, r)$ and $\mathscr{E}_X^{\mathbb{R}}(\varphi^-, r)$ are faithfully flat over $\mathscr{E}_X$.*

2.5. *Cohomological properties*

Let X be a complex analytic manifold and Y be a submanifold of X. Let $r \in \mathbb{R}$, $r \geqslant 1$, and φ be a r-weight on an open subset U of T_Y^*X.

PROPOSITION 2.5.1. *Let V be an open $\mathbb{R}$-conic subset of U which is contained in some local chart domain (y, ξ) and where it exists a holomorphic function μ whose real part is equivalent to $|\xi|^{1/r}$. (For example V is contained in some halfspace* $\operatorname{Re} \xi_1 > 0$).

(i) $\forall k \geqslant \dim_{\mathbb{C}} X,\ H^k(V, \mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi, r)) = 0.$

(ii) *If V is a domain of holomorphy then*

$$\forall k \geqslant 1, \quad H^k(V, \mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi, r)) = 0.$$

The same results are also true for $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^-, r)$ and when φ is of the form $\varphi(y, \xi) = \varphi_0(|\xi|)$ they are still true for $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^+, r)$.

Proof. As V is contained in a chart domain, $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi, r)$ is identified to $\mathscr{S}(\varphi, r)$ and we can write as in the proof of Proposition 2.1.2:

$$\mathscr{S}(\varphi, r) = \mathscr{S}_+(\varphi)/\varinjlim_q \mathscr{S}(\varphi_q)$$

where $\varphi_q(y, \xi) = -1/q \operatorname{Re} \mu(y, \xi)$ is pluriharmonic on V.

The proposition is true for $\mathscr{S}_+(\varphi)$ by Proposition 1.1.1 and 1.1.2 and it is still true for $\varinjlim_q \mathscr{S}(\varphi_q)$ by Corollary 1.1.4 but only for $k > \dim_{\mathbb{C}} X$ in (i) and for $k > 1$ in (ii).

We conclude using the long exact sequence

$$\cdots \to H^k(V, \mathscr{S}_+(\varphi)) \to H^k(V, \mathscr{S}(\varphi, r)) \to H^{k+1}\left(V, \varinjlim_q \mathscr{S}(\varphi_q)\right) \to \cdots$$

The results for $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^+, r)$ and $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^-, r)$ are proved in the same way using Proposition 1.1.5.

In the same way, Propositions 1.1.3, 1.1.5 and 1.1.6 give

PROPOSITION 2.5.2. *Let S be a compact complex analytic manifold and $p: S \times T_Y^*X \to T_Y^*X$ the projection. Then*

$$\forall k \geqslant 0, \quad R^k p_* \mathscr{C}_{S \times Y|S \times X}^{\mathbb{R}}(\varphi^*, r) = \mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^*, r) \otimes_{\mathbb{C}} H^k(S, \mathscr{O}_S).$$

These results are true, as a special case, for the sheaves $\mathscr{C}_{Y|X}^{\mathbb{R}}(r, s)$ and $\mathscr{C}_{Y|X}^{\mathbb{R}}(r)$ if $r < +\infty$.

The sheaf $\mathscr{C}_{Y|X}^{\mathbb{R}}(\infty)$ is, by definition, the formal completion of $\mathscr{C}_{Y|X}^{\mathbb{R}}(\infty, 1)$, so from Proposition 1.2.3 we get:

PROPOSITION 2.5.3. *The results of Propositions* 2.5.1 *and* 2.5.2 *are still true for* $\mathscr{C}^{\mathbb{R}}_{Y|X}(\infty)$.

PROPOSITION 2.5.4. *Let V be an open $\mathbb{R}$-conic subset of $T^*_Y X$ contained in some local chart domain and whose fibers for the projection $\hat{\gamma}$: $T^*_Y X \to \mathbb{P}^*_Y X \cup Y$ are contractible.*

(i) $\forall k \geqslant \dim_{\mathbb{C}} X$, $H^k(V, \mathscr{C}_{Y|X}(\varphi^*, r)) = 0$.

(ii) *If moreover $\gamma(V)$ is a holomorphy domain, then*

$$\forall k \geqslant 1, \quad H^k(V, \mathscr{C}_{Y|X}(\varphi^*, r)) = 0.$$

(iii) *If S is a compact complex analytic manifold*

$$\forall k \geqslant 0, \quad R^k p_* \mathscr{C}_{S\times Y|S\times X}(\varphi^*, r) = \mathscr{C}_{Y|X}(\varphi^*, r) \otimes_{\mathbb{C}} H^k(S, \mathcal{O}_S)$$

(*when $\varphi^* = \varphi^+$ we have to assume that $\varphi(y, \xi) = \varphi_0(|\xi|)$*).

The proof is easily obtained from Lemma 2.3.2 and the corresponding results for $\mathscr{C}^{\mathbb{R}}$.

As we showed in [7] (cf. [10]), if a family of $\mathcal{O}_X$-modules satisfies the Propositions 2.3.3, 2.5.1 and 2.5.2, it satisfies the "Edge of the Wedge" theorem, that is we have the following result:

THEOREM 2.5.5 ("Edge of the Wedge"). *Let $X = \mathbb{C}^n$ and $Y = \mathbb{C}^{n-p} \times \{0\} \subset \mathbb{C}^n$. Let G be a convex closed subset of $T^*_Y X = \mathbb{C}^{n-p} \times \mathbb{C}^p$ which is $\mathbb{R}$-conic.*

Let x be a point of G. Assume that there exists no complex line L such that $L \cap G$ is a neighborhood of x in L. Then

$$\forall k \neq n, \qquad \mathscr{H}^k_G(\mathscr{F})_x = 0$$

when $\mathscr{F}$ is one of the following sheaves:

(i) $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi, r)$, $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^-, r)$, $\mathscr{C}_{Y|X}(\varphi, r)$, $\mathscr{C}_{Y|X}(\varphi^-, r)$ *when $r \in \mathbb{R}$, $r \geqslant 1$ and φ is r-weight defined on $T^*_Y X$.*

(ii) $\mathscr{C}^{\mathbb{R}}_{Y|X}(\varphi^+, r)$ *and* $\mathscr{C}_{Y|X}(\varphi^+, r)$ *if moreover $\varphi = \varphi_0(|\xi|)$ where $(\xi_1, \ldots, \xi_p)$ are the coordinates on $\mathbb{C}^p$.*

(iii) $\mathscr{C}^{\mathbb{R}}_{Y|X}(r, s)$ *and* $\mathscr{C}_{Y|X}(r, s)$ *for each (r, s) such that $1 \leqslant s \leqslant r \leqslant +\infty$.*

3. The sheaf of 2-microlocal operators

Let X be an analytic manifold and Y a submanifold of X, let $T^*_Y X$ be the conormal bundle to Y in X.

If $\mathscr{F}$ is a sheaf of $\mathbb{C}$-vector spaces on X, the microlocalization of $\mathscr{F}$ along Y is a complex of sheaves on T_Y^*X which is denoted by $\mu_Y(\mathscr{F})$ (more exactly it is an object in the derived category of sheaves on T_Y^*X). We refer to [8], [9] for the definition of the microlocalization which was first used in [16]. We denote by $\mu_Y^k(\mathscr{F})$ the kth cohomology group of $\mu_Y(\mathscr{F})$. Let us recall that when Y is identified to the null section of T_Y^*X we have $\mu_Y(\mathscr{F})|_Y \approx \mathbb{R}\Gamma_Y(\mathscr{F})$ and thus $\mu_Y^k(\mathscr{F})|_Y \approx \mathscr{H}_Y^k(\mathscr{F})$.

When $\mathscr{F}$ is the sheaf $\mathscr{O}_X$ of holomorphic functions on a complex manifold X, $\mu_Y^d(\mathscr{O}_X)$ ($d = \operatorname{codim}_X Y$) is precisely the sheaf $\mathscr{C}_{Y|X}^{\mathbb{R}}$ of [16] which is isomorphic to $\mathscr{C}_{Y|X}^{\mathbb{R}}(1)$ as we stated in Section 2. In the same way:

$$\mathscr{E}_X^{\mathbb{R}}(1) \approx \mathscr{E}_X^{\mathbb{R}} = \mu_X^{\dim X}(\Omega_{X\times X}^{(o,n)}).$$

Most properties of $\mathscr{C}_{Y|X}^{\mathbb{R}}$ and $\mathscr{E}_X^{\mathbb{R}}$ in [16] come from the following property:

$$\forall k \neq d, \quad \mu_Y^k(\mathscr{O}_X) = 0.$$

In [10], we defined the 2-microdifferential operators by substituting $\mathscr{C}_{Y|X}(r, s)$ to $\mathscr{O}_X$. Here we will use the sheaves $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^*, r)$ to get a new class of operators.

3.1. *Definition of 2-microlocal operators*

Let X be a complex analytic manifold, Y be a submanifold of X and $\Lambda = T_Y^*X$ be the conormal bundle to Y in X.

From now on, the symbol $\mathscr{C}_{Y|X}^{\mathbb{R}}(*)$ will mean one of the sheaves that satisfy Theorem 2.5.5 that is:

(i) $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi, r)$ or $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^-, r)$ when r is a real number with $r \geqslant 1$ and φ is a r-weight (Definition 2.2.1) on a $\mathbb{R}$-conic open subset of Λ.
(ii) $\mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi^+, r)$ if φ is a r-weight such that there exists a coordinate system (y, ξ) of T_Y^*X where φ is a function of $|\xi|$.
(iii) $\mathscr{C}_{Y|X}^{\mathbb{R}}(r, s)$ when r, s are numbers such that $1 \leqslant s \leqslant r \leqslant +\infty$.

From $\mathscr{C}_{Y|X}^{\mathbb{R}}(*)$ will be defined some sheaves for which we will keep the same notation, for example if we write as in Definition 2.3.1: $\mathscr{B}_{Y|X}(*) = \mathscr{C}_{Y|X}^{\mathbb{R}}(*)|_Y$, this means that $\mathscr{B}_{Y|X}(\varphi, r) = \mathscr{C}_{Y|X}^{\mathbb{R}}(\varphi, r)|_Y$, $\mathscr{B}_{Y|X}(r) = \mathscr{C}_{Y|X}^{\mathbb{R}}(r)|_Y$ and the same formula for all cases (i), (ii), (iii) here above.

The diagonal of $\Lambda \times \Lambda$ will be denoted by Δ and the isomorphism $\Delta \approx \Lambda$ gives an identification $T_\Delta^*(\Lambda \times \Lambda) \simeq T^*\Lambda$.

DEFINITION 3.1.1. The sheaf of 2-microlocal operators on $T^*\Lambda$ is:

$$\mathscr{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(*) = \mu_\Delta^n(\mathscr{C}_{Y\times Y|X\times X}^{\mathbb{R}}(*)) \otimes_{\mathscr{O}_{X\times X}} \Omega_{X\times X}^{(o,n)}$$

where $n = \dim X = \dim \Lambda$ and $\Omega^{(o,n)}_{X\times X}$ is the sheaf of holomorphic differential forms on $X \times X$ with degree 0 in the first variables.

When Σ is an homogeneous submanifold of Λ of codimension d in Λ, we define:

$$\mathscr{C}^{2(\mathbb{R},\mathbb{R})}_{\Sigma||Y|X}(*) = \mu^d_\Sigma(\mathscr{C}^{\mathbb{R}}_{Y|X}(*)).$$

It is a sheaf on $T^*_\Sigma\Lambda$ and by definition we have:

$$\mathscr{E}^{2(\mathbb{R},\mathbb{R})}_\Lambda(*) = \mathscr{C}^{2(\mathbb{R},\mathbb{R})}_{\Delta||Y\times Y|X\times X}(*) \otimes_{\mathscr{O}_{X\times X}} \Omega^{(o,n)}_{X\times X}.$$

As for the microdifferential operators the basic result is:

PROPOSITION 3.1.2. *Let d be the codimension of Σ in Λ*

$$\forall k \neq d, \qquad \mu^k_\Sigma(\mathscr{C}^{\mathbb{R}}_{Y|X}(*)) = 0.$$

Proof. Using Theorem 2.5.5. we can make the same proof as Theorem 1.3.2, Chap. I of [16].

More generally, let us consider a lagrangian conic submanifold of T^*X (here conic means complex conic).

Let $\mathscr{N}$ be a simple holonomic left $\mathscr{E}_X$-module with support Λ, its dual $\mathscr{N}^* = \mathscr{E}xt^n_{\mathscr{E}_X}(\mathscr{N}, \mathscr{E}_X)$ is a right $\mathscr{E}_X$-module and $\mathscr{N}^{*a} = \mathscr{N}^* \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}$ is a simple holonomic left $\mathscr{E}_X$-module (Ω_X is the sheaf of holomorphic differential form of maximum degree on X).

It was proved in [6] and [10], that the simple holonomic $\mathscr{E}_{X\times X}$-module $\mathscr{M}_\Lambda = \mathscr{N} \hat{\otimes} \mathscr{N}^{*a}$ is independent of the choice of $\mathscr{N}$ and thus is globally defined in a neighborhood of the diagonal Δ of $\Lambda \times \Lambda$ in $T^*X \times X$.

DEFINITION 3.1.3. We set:

$$\mathscr{M}^{\mathbb{R}}_\Lambda(*) = \mathscr{E}^{\mathbb{R}}_{X\times X}(*) \otimes_{\mathscr{E}_{X\times X}} \mathscr{M}_\Lambda$$

and we define the sheaf of 2-microlocal operators on $T^*\Lambda$ of type $(*)$ as:

$$\mathscr{E}^{2(\mathbb{R},\mathbb{R})}_\Lambda(*) = \mu^n_\Delta(\mathscr{M}^{\mathbb{R}}_\Lambda(*)) \otimes_{\mathscr{O}_{X\times X}} \Omega^{(o,n)}_{X\times X}.$$

When $\Lambda = T^*_Y X$ we have $\mathscr{M}_\Lambda = \mathscr{C}_{Y\times Y|X\times X}$ and the definition is the same as Definition 2.1.1 (by Proposition 2.3.9).

From now on we will exclude the case (φ, r) in $(*)$ to get the following:

PROPOSITION 3.1.4. *$\mathscr{E}^{2(\mathbb{R},\mathbb{R})}_\Lambda(*)$ is a sheaf of rings.*

The proof of this proposition is the same as the proof of Theorem 2.1.5 in [10]. It is based on the isomorphism:

$$\begin{aligned}&\mathbb{R}\,\mathrm{Hom}_{\mathscr{E}_X^{\mathbb{R}}(*)}(\mathscr{E}_X^{\mathbb{R}}(*)\otimes_{\mathscr{E}_X}\mathscr{N},\ \mathscr{E}_X^{\mathbb{R}}(*)\otimes_{\mathscr{E}_X}\mathscr{N})\\&\quad=\mathbb{R}\,\mathrm{Hom}_{\mathscr{E}_X}(\mathscr{N},\mathscr{N})\otimes_{\mathscr{E}_X}\mathscr{E}_X^{\mathbb{R}}(*)\approx\mathbb{C}_\Lambda.\end{aligned}$$

For $\mathscr{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(\varphi, r)$ we have no ring structure but an external operation:

$$\mathscr{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(\varphi_1, r)\times\mathscr{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(\varphi_2, r)\to\mathscr{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(\varphi_1+\varphi_2, r).$$

Let us denote by $\hat{\gamma}_\Lambda$ the canonical projection from $T^*\Lambda$ to $T^*\Lambda/\mathbb{C}^*\approx(\mathbb{P}^*\Lambda)\sqcup\Lambda$.

DEFINITION 3.1.5. We define a sheaf of rings on $T^*\Lambda$ by:

$$\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)=\hat{\gamma}_\Lambda^{-1}\hat{\gamma}_{\Lambda *}\mathscr{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(*)$$

and we denote by $\mathscr{D}_\Lambda^{2(\mathbb{R},\infty)}(*)$ the sheaf $\mathscr{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(*)|_\Lambda$.

REMARK. By the definition we have:

$$\mathscr{D}_\Lambda^{2(\mathbb{R},\infty)}(*)=\mathscr{H}_\Delta^n(\mathscr{M}_\Lambda^{\mathbb{R}}(*))\otimes_{\mathscr{O}_{X\times X}}\Omega_{X\times X}^{(o,n)}.$$

Let Λ_1 and Λ_2 be two lagrangian conic submanifolds of T^*X_1 and T^*X_2 respectively and let $\varphi\colon T^*X_1\to T^*X_2$ a canonical transformation which exchanges Λ_1 and Λ_2. Let $\tilde{\varphi}\colon T^*\Lambda_1\to T^*\Lambda_2$ be the map induced by φ.

Each quantized canonical transformation $\Phi\colon\mathscr{E}_{X_1}\to\varphi^{-1}\mathscr{E}_{X_2}$ induces an isomorphism $\mathscr{M}_{\Lambda_1}(*)\to(\varphi\otimes\varphi)^{-1}\mathscr{M}_{\Lambda_2}(*)$ and thus isomorphism of sheaves of rings:

$$\Phi\colon\mathscr{E}_{\Lambda_1}^{2(\mathbb{R},\mathbb{R})}(*)\to\tilde{\varphi}^{-1}\mathscr{E}_{\Lambda_2}^{2(\mathbb{R},\mathbb{R})}(*)$$

and

$$\Phi\colon\mathscr{E}_{\Lambda_1}^{2(\mathbb{R},\infty)}(*)\to\tilde{\varphi}^{-1}\mathscr{E}_{\Lambda_2}^{2(\mathbb{R},\infty)}(*).$$

3.2. *Symbols of 2-microlocal operators*

We show in this section that when Λ is of the special form T_Y^*X, we can define symbols for the operators of $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$ to make explicit calculations on these operators.

The method we use here is the method of [10] that is of [16]. Using Aoki's method [2] we could in the same way define a symbolic calculus for the sheaf $\mathscr{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(*)$.

Let Y be a submanifold of X and let $(x_1, \dots, x_p, t_1, \dots, t_q)$ a local coordinate system of X such that $Y = \{(x, t) \in X/t = 0\}$. If (x, t, ξ, τ) is the corresponding coordinate system of T^*X we have

$$\Lambda = T_Y^*X = \{(x, t, \xi, \tau) \in T^*X/t = 0, \xi = 0\}$$

and we denote by (x, τ, x^*, τ^*) the local coordinate system of $T^*\Lambda$ associated to (x, τ).

DEFINITION 3.2.1. Let (r, s) be two real numbers with $1 \leqslant s < r < +\infty$. Let $\alpha = (x_0, \tau_0, x_0^*, \tau_0^*)$ be a point of $T^*\Lambda$ with $\tau_0 \neq 0$.

(i) $\hat{\mathscr{P}}^2_{\alpha,+}(r, s)$ is the set of formal series

$$\sum_{\substack{k \geqslant 0 \\ i \in \mathbb{Z}}} u_{ik}(x, \tau, x^*, \tau^*)$$

such that:

(3.2.1) There exists an open neighborhood V of α in $T^*\Lambda$, $\mathbb{R}$-conic in (τ, x^*) and $\mathbb{C}^*$-conic in (x^*, τ^*), such that for each $(i, k) \in \mathbb{Z} \times \mathbb{N}$, u_{ik} is holomorphic on V and homogeneous of degree i in (x^*, τ^*).

(3.2.2) $\forall K \Subset V, \exists c > 0, \exists C > 0, \exists A > 0, \forall \varepsilon > 0, \exists C_\varepsilon > 0,$
$\forall (x, \tau, x^*, \tau^*) \in K, \forall \lambda > 0,$

(a) $\forall i \geqslant 0, \forall k \geqslant 0, |u_{ik}(x, \lambda\tau, x^*, \lambda^{-1}\tau^*)| < C_\varepsilon \dfrac{\varepsilon^i}{i!} A^k (k!)^s \lambda^{-k} e^{c\lambda^{1/r}}.$

(b) $\forall i < 0, \forall k \geqslant 0, |u_{ik}(x, \lambda\tau, x^*, \lambda^{-1}\tau^*)| < C^{-i}(-i)! A^k (k!)^s \lambda^{-k} e^{c\lambda^{1/r}}.$

(ii) $\hat{\mathscr{P}}^2_{\alpha,-}(r, s)$ is the subset of $\hat{\mathscr{P}}^2_{\alpha,+}(\infty, r)$ of series $\sum_{k \geqslant 0} u_{ik}$

such that the series $S(u) = \Sigma\, v_{ik}$ given by $v_{ik} = \Sigma_{0 \leqslant l < k}\, u_{ik}$ is an element of $\hat{\mathscr{P}}^2_{\alpha,+}(r, s)$.

(iii) $\hat{\mathscr{P}}^2_\alpha(r, s) = \hat{\mathscr{P}}^2_{\alpha,+}(r, s) / \hat{\mathscr{P}}^2_{\alpha,-}(r, s)$.

(iv) When $1 \leqslant s = r < +\infty$, $\hat{\mathscr{P}}^2_\alpha(r, r)$ has the same definition except that (3.2.2) is modified in:

$$\forall K \Subset V, \forall c > 0, \exists C \dots \tag{3.2.3}$$

(v) When $1 \leqslant s < r = +\infty$, we have to replace the function:

$$e^{c\lambda^{1/r}} \text{ by } (1 + \lambda)^c.$$

(vi) When $r = s = +\infty$, the condition 3.2.2 has to be replaced by:

(3.2.4) $\forall K \Subset V, \exists m \in \mathbb{R}, \exists C > 0, \forall \varepsilon > 0, \exists C_\varepsilon > 0, \forall k \geqslant 0, \exists A_k > 0,$
$\forall (x, \tau, x^*, \tau^*) \in K, \forall \lambda > 0.$

(a) $\forall i \geqslant 0, \forall k \geqslant 0 \ |u_{ik}(x, \lambda\tau, x^*, \lambda^{-1}\tau^*)| < C_\varepsilon \frac{\varepsilon^i}{i!} A_k (1+\lambda)^{m-k}.$

(b) $\forall i < 0, \forall k \geqslant 0 \ |u_{ik}(x, \lambda\tau, x^*, \lambda^{-1}\tau^*)| < C^{-i}(-i)! A_k (1+\lambda)^{m-k}.$

THEOREM 3.2.2. *Let r, s be such that $1 \leqslant s \leqslant r \leqslant +\infty$ and let $\alpha = (x_0, \tau_0, x_0^*, \tau_0^*)$ be a point of $T^*\Lambda$ with $\tau_0 \neq 0$.*

The set $\hat{\mathscr{S}}_\alpha^2(r, s)$ is equal to the set of germs of the sheaf $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(r, s)$ at the point α.

If $P \in \mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(r, s)_\alpha$, a symbol of P is an element of $\hat{\mathscr{S}}_{\alpha,+}^2(r, s)$ whose image in $\hat{\mathscr{S}}_\alpha^2(r, s)$ corresponds to P.

THEOREM 3.2.3. *Let P and Q be two elements of $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(r, s)_\alpha$ with respective symbols*

$$p = \sum_{i\in\mathbb{Z}, k\geqslant 0} p_{ik}(x, \tau, x^*, \tau^*) \quad \text{and} \quad q = \sum_{i\in\mathbb{Z}, k\geqslant 0} q_{ik}(x, \tau, x^*, \tau^*).$$

Then the product $R = PQ$ of P and Q in the ring $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(r, s)$ has a symbol $r = p \# q$ defined by:

$$r_{\lambda\mu}(x, \tau, x^*, \tau^*)$$
$$= \sum_{\substack{\lambda = i+j-|\alpha|-|\beta| \\ \mu = k+l}} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \left(\frac{\partial}{\partial x^*}\right)^\alpha \left(\frac{\partial}{\partial \tau}\right)^\beta p_{ik}(x, \tau, x^*, \tau^*) \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial \tau^*}\right)^\beta q_{kl}(x, \tau, x^*, \tau^*). \tag{3.2.4}$$

The proof of these two theorems is exactly the same as the proof of Theorem 2.3.1 and 2.3.3 in [10], replacing the symbols of $\mathscr{C}_{Y|X}^\infty$ by those of $\mathscr{C}_{Y|X}^{\mathbb{R}}(r, s)$, that is by the elements of $\hat{\mathscr{S}}(r, s)$ of Section 2.1.

We have supposed that $\tau_0 \neq 0$, because when $\tau_0 = 0$, we are in the case of 2-microdifferential operators, that is in the case of [10] as we will see in the next section.

In fact, the same calculations give symbols for each $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(\varphi^*, r)$ with due modification in the definitions. We can define $\hat{\mathscr{S}}_\alpha^2(\varphi, r)$ in the following way:

DEFINITION 3.2.4. Let $r \in \mathbb{R}$, $r \geqslant 1$ and let $\varphi(x, t, \xi, \tau)$ be a r-weight on T^*X defined in a neighborhood of $\Lambda = T_Y^*X$.

For ε sufficiently small we define:

$$\varphi_\varepsilon(x, \tau) = \sup_{\substack{|t_1| = \cdots = |t_q| = \varepsilon \\ |\xi_1| = \cdots = |\xi_p| = \varepsilon|\tau|}} \varphi(x, t, \xi, \tau)$$

then $\hat{\mathscr{S}}^2_{\alpha,+}(\varphi, r)$ is the set of series

$$\sum_{\substack{k \geq 0 \\ i \in \mathbb{Z}}} u_{ik}(x, \tau, x^*, \tau^*)$$

which satisfy (3.2.1) and:

(3.2.5) $\forall K \Subset V, \exists C > 0, \exists c > 0, \exists A > 0, \forall \varepsilon > 0, \exists C_\varepsilon > 0,$
$\forall (x, \tau, x^*, \tau^*) \in K, \forall \lambda > 0,$

(a) $\forall i \geq 0, \forall k \geq 0, |u_{ik}(x, \lambda\tau, x^*, \lambda^{-1}\tau^*)| < C_\varepsilon \dfrac{\varepsilon^i}{i!} A^k (k!)^r \lambda^{-k} e^{\varphi_\varepsilon(x, \lambda\tau)}$.

(b) $\forall i < 0, \forall k \geq 0, |u_{ik}(x, \lambda\tau, x^*, \lambda^{-1}\tau^*)| < C^{-i}(-i)! A^k (k!)^r \lambda^{-k} e^{\varphi_c(x, \lambda\tau)}$.

$\hat{\mathscr{S}}^2_{\alpha,-}(\varphi, r)$ is the set of series u such that $S(u) \in \hat{\mathscr{S}}^2_{\alpha,+}(\varphi, r)$ and

$$\hat{\mathscr{S}}^2_\alpha(\varphi, r) = \hat{\mathscr{S}}^2_{\alpha,+}(\varphi, r) / \hat{\mathscr{S}}^2_{\alpha,-}(\varphi, r).$$

Then we can define:

$$\hat{\mathscr{S}}^2_\alpha(\varphi^+, r) = \varinjlim_{c > 0} \hat{\mathscr{S}}^2_\alpha(c\varphi, r)$$

and

$$\hat{\mathscr{S}}^2_\alpha(\varphi^-, r) = \varinjlim_{c > 0} \hat{\mathscr{S}}^2_\alpha(c\varphi, r).$$

We now have:

THEOREM 3.2.5. *The set* $\hat{\mathscr{S}}^2_\alpha(\varphi^*, r)$ *is equal to* $\mathscr{E}^{2(\mathbb{R},\infty)}_\Lambda(\varphi^*, s)_\alpha$.

(As before φ^* means $\varphi^+, \varphi, \varphi^-$.)

The formulas of Theorem 3.2.3 give the ring structure of $\mathscr{E}^{2(\mathbb{R},\infty)}_\Lambda(\varphi^+, r)$ and $\mathscr{E}^{2(\mathbb{R},\infty)}_\Lambda(\varphi^-, r)$.

When α is a point of the null section Λ of $T^*\Lambda$, that is when we consider sections of $\mathscr{D}^2_\Lambda(r, s)$ or $\mathscr{D}^2_\Lambda(\varphi^*, r)$, Theorems 3.2.2 and 3.2.5 are still true but the symbols are of the following special type:

The functions $p_{ik}(x, \tau, x^*, \tau^*)$ are equal to 0 when $i < 0$ and are homogeneous polynomials of degree i in (x^*, τ^*) when $i \geq 0$.

Instead of the formal symbols of $\mathscr{C}^{\mathbb{R}}_{Y|X}$ we could use the "convergent symbols", that is the sets $\mathscr{S}(\varphi, r, V)$, and define "convergent symbols" of $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$:

DEFINITION 3.2.6. Let (r,s) be two real numbers with $1 \leqslant s < r < +\infty$ and $\alpha = (x_0, \tau_0, x_0^*, \tau_0^*)$ be a point of $T^*\Lambda$ with $\tau_0 \neq 0$.

Let V be an open neighborhood of α in $T^*\Lambda$, $\mathbb{R}$-conic in (τ, x^*) and $\mathbb{C}^*$-conic in (x^*, τ^*) and $a > 0$.

(i) $\mathscr{S}^2_{\alpha,+}(r,s)(a,V)$ is the set of formal series $\Sigma_{i\in\mathbb{Z}} u_i(x,\tau,x^*,\tau^*)$ such that for each $i \in \mathbb{Z}$, u_i is a holomorphic function on $V \cap \{|\tau| > a\}$ homogeneous of degree i in (x^*, τ^*) and such that:
$\forall K \Subset V \cap \{|\tau| > a\}$, $\exists c > 0$, $\exists C > 0$, $\exists C > 0$, $\forall (x,\tau,x^*,\tau^*) \in K$,

(a) $\forall i \geqslant 0$, $|u_i(x,\tau,x^*,\tau^*)| < C_a \dfrac{a^i}{i!} \mathrm{e}^{c|\tau|^{1/r}}$.

(b) $\forall i < 0$, $|u_i(x,\tau,x^*,\tau^*)| < C^{-i}(-i)!\, \mathrm{e}^{c|\tau|^{1/r}}$.

(ii) $\mathscr{S}^2_{\alpha,-}(r,s)(a,V)$ is the subset of $\mathscr{S}^2_{\alpha,+}(r,s)(a,V)$ of formal series $\Sigma_{i\in\mathbb{Z}} u_i(x,\tau,x^*,\tau^*)$ such that:
$\forall K \Subset V \cap \{|\tau| > a\}$, $\exists c > 0$, $\exists C > 0$, $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$, $\forall (x,\tau,x^*,\tau^*) \in K$,

(a) $\forall i \geqslant 0$, $|u_i(x,\tau,x^*,\tau^*)| < C_a \dfrac{a^i}{i!} \mathrm{e}^{-c|\tau|^{1/s}}$.

(b) $\forall i < 0$, $|u_i(x,\tau,x^*,\tau^*)| < C^{-i}(-i)!\, \mathrm{e}^{-c|\tau|^{1/s}}$.

(iii) $\mathscr{S}^2_\alpha(r,s) = \varinjlim_{a>0} \varinjlim_{V\ni\alpha} \mathscr{S}^2_{\alpha,+}(r,s)(a,V)/\mathscr{S}^2_{\alpha,-}(r,s)(a,V)$.

We may extend this definition to each (r,s) such that $1 \leqslant s \leqslant r \leqslant +\infty$ as in Definition 3.2.1 and we have:

THEOREM 3.2.7. *The canonical map* $\mathscr{S}^2_\alpha(r,s) \to \hat{\mathscr{S}}^2_\alpha(r,s)$ *is bijective.*

This theorem may be proved in the same way as Theorem 2.1.5 or, by proving directly that $\mathscr{S}^2_\alpha(r,s) = \mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(r,s)_\alpha$ (replacing $\hat{\mathscr{S}}$ by $\mathscr{S}$ in the proof of Theorem 3.2.2).

3.3. *Bicanonical transformations*

If we replace $\mathscr{C}^{\mathbb{R}}_{Y|X}(*)$ by $\mathscr{C}_{Y|X}(*)$ in the definition of $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$ we get the sheaf $\mathscr{E}_\Lambda^{2\infty}(*)$ of 2-microdifferential operators. In the case of $\mathscr{C}_{Y|X}(r,s)$ we get the sheaf $\mathscr{E}_\Lambda^{2\infty}(r,s)$ which has been studied in [10].

The injective morphism $\mathscr{C}_{Y|X}(*) \hookrightarrow \mathscr{C}^{\mathbb{R}}_{Y|X}(*)$ of Corollary 2.3.4 gives a morphism $\mathscr{E}_\Lambda^{2\infty}(*) \to \mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$ which is injective from Lemma 3.1.2.

Let us recall ([10], Theorem 2.3.1) that the sections of $\mathscr{E}_\Lambda^{2\infty}(r,s)$ are bijectively mapped to the symbols of the type $\Sigma_{(i,j)\in\mathbb{Z}^2} P_{ij}(x,\tau,x^*,\tau^*)$ where P_{ij} is a holomorphic function which is homogeneous of degree i in (x^*,τ^*) and j in (τ, x^*) and where the family (P_{ij}) satisfy suitable estimates (cf. Remark (ii) hereafter). Of course the same is true for $\mathscr{E}_\Lambda^{2\infty}(\varphi^*, r)$ with due modifications on the estimates.

As the symbols of $\mathscr{E}_\Lambda^{2\infty}(*)$ and $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$ are calculated in the same way from the symbols of $\mathscr{C}_{Y|X}(*)$ and $\mathscr{C}_{Y|X}^{\mathbb{R}}(*)$ respectively, the morphism of Theorem 2.3.6 gives:

PROPOSITION 3.3.1 *If P is an operator of $\mathscr{E}_\Lambda^{2\infty}(*)$ with symbol $\Sigma_{(i,j)\in\mathbb{Z}^2} P_{ij}(x, \tau, x^*, \tau^*)$, its image in $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$ is given by the following symbol of $\hat{\mathscr{S}}_{\alpha,+}^2(*)$:*

$$\begin{cases} \forall i\in\mathbb{Z}, & u_{i,0}(x, \tau, x^*, \tau^*)=\Sigma_{l\geqslant 0} P_{i,i+l}(x, \tau, x^*, \tau^*) \\ \forall i\in\mathbb{Z},\ \forall k\geqslant 1, & u_{ik}(x, \tau, x^*, \tau^*)=P_{i,i-k}(x, \tau, x^*, \tau^*). \end{cases} \tag{3.3.1}$$

REMARKS. (i) The Cauchy inequalities show immediately that if the symbol u defined by (3.3.1) is in $\hat{\mathscr{S}}_{\alpha,-}^2(*)$ then the functions P_{ij} are all identically zero which corresponds to the injectivity of $\mathscr{E}_\Lambda^{2\infty}(*) \hookrightarrow \mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$.

(ii) The estimates of Theorem 2.3.1 in [10] which define the symbols of $\mathscr{E}_\Lambda^{2\infty}(r, s)$ are exactly those which make the symbol defined by (3.3.1) belong to $\hat{\mathscr{S}}_{\alpha,+}^2(r, s)$.

(iii) If $\alpha = (x_0, \tau_0, x_0^*, \tau_0^*)$ is a point of $T^*\Lambda$ such that $\tau_0=0$, that is if $\alpha\in(T^*\Lambda)\times_\Lambda Y$, the morphism $\mathscr{E}_{\Lambda,\alpha}^{2\infty}(*)\to\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$, α is an isomorphism for on the null section Y of $\Lambda = T_Y^*X$ we have $\mathscr{C}_{Y|X}^{\mathbb{R}}(*)|_Y = \mathscr{C}_{Y|X}^{\infty}(*)|_Y$.

This gives a symbol to the elements of $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}$ in the case which was excluded in Theorem 3.2.2.

For $1\leqslant s\leqslant r\leqslant \infty$, we defined in [10] the sheaf $\mathscr{E}_\Lambda^2(r, s)$ of 2-microdifferential operators of finite order which is a subring of $\mathscr{E}_\Lambda^{2\infty}(r, s)$ and we associated to each section P of $\mathscr{E}_\Lambda^2(r, s)$ a principal symbol $\sigma_\Lambda^{(r,s)}(P)$ which is a holomorphic function on $T^*\Lambda$ independent of the local coordinate system.

When $r = s$, $\sigma_\Lambda^{(r,s)}(P)$ is always a non-zero function.

THEOREM 3.3.2. (Division theorem.) *Let α be a point of $T^*\Lambda$, and ω a neighborhood of α with local coordinates (x, τ, x^*, τ^*).*

Let u be one of the coordinates x_i, τ_j, x_i^* or τ_j^* which vanish at α and let U be the operator whose symbol is u (e.g. if $u=x_1$ then U is the multiplication by x_1 and if $u=x_1^*$, then U is $\partial/\partial x_1$).

Let P in $\mathscr{E}_\Lambda^2(r, s)$ and let m be the first integer such that

$$\frac{\partial^m}{\partial u^m}\sigma_\Lambda^{(r,s)}(P)(\alpha) \neq 0,$$

we assume that $m < +\infty$.

Let $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$ be some of the sheaves of 2-microlocal operators such that $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(r, s) \subset \mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$.

Each S in $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$ *has one and only one representation:*

$$S = QP + \sum_{\nu=0}^{m-1} U^\nu R^{(\nu)}$$

where $Q, R^{(1)}, \ldots, R^{(m-1)}$ *are in* $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$ *and the operators* $R^{(1)}, \ldots, R^{(m-1)}$ *have symbols in* $\hat{\mathscr{S}}^2_{\alpha,+}(*)$ *independent of u.*

Proof. Let us first assume that $u = x_1$. As in the proof of Lemma 6.2.1 of [8], we consider the operator

$$\frac{1}{2i\pi}\frac{1}{s-x_1}$$

which is well defined when $|s| > |x_1|$ and we apply to it Theorem 2.7.1 of [10]:

$$\frac{1}{2i\pi}\frac{1}{s-x_1} = P(x, \tau, x^*, \tau^*)G(s, x, \tau, x^*, \tau^*) + K(s, x, \tau, x^*, \tau^*)$$

with

$$K = \sum_{\nu=0}^{m-1} x_1^\nu K^{(\nu)}(s, x'', \tau, x^*, \tau^*) \quad (x'' = (x_2, \ldots, x_p)).$$

Let $S \in \hat{\mathscr{S}}^2_{\alpha,+}(*)$, it is a formal series of holomorphic functions so it satisfies the Cauchy formula:

$$\text{if } |x_1| < \varepsilon \quad S(x, \tau, x^*, \tau^*) = \int_{|s|=\varepsilon} \frac{1}{2i\pi}\frac{1}{s-x_1} S(s, x'', \tau, x^*, \tau^*)\,ds.$$

We define $Q, R^{(1)}, \ldots, R^{(m-1)}$ by setting:

$$Q(x, \tau, x^*, \tau^*) = \int_{|s|=\varepsilon} G(s, x, \tau, x^*, \tau^*)S(s, x'', \tau, x^*, \tau^*)\,ds,$$

$$R^{(\nu)}(x, \tau, x^*, \tau^*) = \int_{|s|=\varepsilon} K^{(\nu)}(s, x'', \tau, x^*, \tau^*)S(s, x'', \tau, x^*, \tau^*)\,ds.$$

In these formulas, the products GS and $K^{(\nu)}S$ are given by (3.2.4) and therefore Q and $R^{(\nu)}$ are belonging to $\hat{\mathscr{S}}^2_{\alpha,+}(*)$ and we have:

$$S = PQ + \sum_{\nu=0}^{m-1} x_1^\nu R^{(\nu)}.$$

Let us prove that this representation is unique in $\hat{\mathscr{P}}^2_{\alpha,+}(*)$:

From Theorem 2.7.2 of [10], we may assume that P is of the form

$$x_1^m + \sum_{\nu=0}^{m-1} x_1^\nu P^{(\nu)}(x'', \tau, x^*, \tau^*)$$

and that there exists some $C > 0$ such that $P = x_1^m \tilde{P}$ with $\tilde{P}$ invertible when $|x_1| > C$.

If there exists some Q and $R^{(\nu)}$ such that:

$$P(x, \tau, x^*, \tau^*)Q(x, \tau, x^*, \tau^*) = -\sum_{\nu=0}^{m-1} x_1^\nu R^{(\nu)}(x'', \tau, x^*, \tau^*)$$

we will obtain when $|x_1| > C$:

$$Q(x, \tau, x^*, \tau^*) = -\tilde{P}^{-1} \sum_{\nu=0}^{m-1} x_1^{\nu-m} R^{(\nu)}(x'', \tau, x^*, \tau^*)$$

Q is thus a formal series of holomorphic functions in x_1 which vanish in $x_1 = +\infty$ hence $Q = 0$.

The same proof will work in $\hat{\mathscr{P}}^2_{\alpha,-}(*)$ and thus the representation is unique in $\hat{\mathscr{P}}^2_\alpha(*) = \hat{\mathscr{P}}^2_{\alpha,+}(*)/\hat{\mathscr{P}}^2_{\alpha,-}(*)$ that is in $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$.

Let us now consider the case where u is one of the variables τ_j, then we can make a quantized canonical transformation which transforms τ_j into x_i and use the first case. (Quantized canonical transformations operate on $\mathscr{C}^{\mathbb{R}}_{Y|X}(*)$ by Theorem 2.4.2 and Proposition 2.3.9. and thus on $\mathscr{E}_\Lambda^{2(\mathbb{R},\infty)}(*)$.)

When u is one of the x_i^*'s or τ_j^*'s, we may suppose that α is not on the zero section (otherwise the theorem would be a special case of Theorem 2.7.1 in [10]) and then, after a quantized canonical transformation, suppose that $u = x_1^*$ and that $x_p^* \neq 0$ at α. Then the proof is the same as before replacing

$$\frac{1}{s - x_1} \quad \text{by} \quad \frac{1}{x_p^* s - x_1^*}.$$

Let X and X' be two complex analytic manifolds, Λ and Λ' be two lagrangian homogeneous submanifolds of T^*X and T^*X' respectively.

The manifold Λ is provided with the action of $\mathbb{C}^*$ induced by the structure of vector bundle on T^*X, this action induces an action of $\mathbb{C}^*$ on $T^*\Lambda$ which is not equal to the action of $\mathbb{C}^*$ given by the vector bundle structure $T^*\Lambda \to \Lambda$.

Let us recall ([10] §2.9) that a bihomogeneous canonical transformation from $T^*\Lambda$ to $T^*\Lambda'$ is an analytic isomorphism from an open subset Ω of $T^*\Lambda$ to an open subset Ω' of $T^*\Lambda'$ which preserves the canonical 2-form and the two actions of $\mathbb{C}^*$.

We proved in [10] Theorem 2.9.11 that if $\varphi:\Omega\to\Omega'$ is a bihomogeneous canonical transformation, then there exists a (non unique) isomorphism of sheaves of rings:

$$\Phi: \mathscr{E}^2_\Lambda(r, s) \to \varphi^{-1}\mathscr{E}^2_\Lambda(r, s)$$

such that $\sigma^{(r,s)}_\Lambda(\Phi(P)) = \sigma^{(r,s)}_\Lambda(P)\circ\varphi$ for each (r, s) such that $1 \leqslant s \leqslant r \leqslant +\infty$.

Using Theorem 3.3.2 when $r=\infty$, $s=1$ and the same proof as [10] we get:

THEOREM 3.3.3. *The isomorphism* Φ *extends uniquely to an isomorphism:*

$$\Phi: \mathscr{E}^{2(\mathbb{R},\infty)}_\Lambda(*) \to \varphi^{-1}\mathscr{E}^{2(\mathbb{R},\infty)}_\Lambda(*).$$

Using Theorems 3.3.2 and 3.3.3 and the method of the proof of Theorem 2.9.12 of [10] we get:

THEOREM 3.3.4. $\mathscr{E}^{2(\mathbb{R},\infty)}_\Lambda(r,r)$ *is faithfully flat on* $\mathscr{E}^2_\Lambda(r,r)$ *for each* $r\in[1,+\infty]$.

COROLLARY 3.3.5. *Let* $\pi: T^*\Lambda\to\Lambda$ *be the canonical map, then* $\mathscr{E}^{2(\mathbb{R},\infty)}_\Lambda(r,r)$ *is flat on* $\pi^{-1}(\mathscr{E}_X|_\Lambda)$ *while* $\mathscr{D}^{2(\mathbb{R},\infty)}_\Lambda(r,r)$ *is flat on* $\mathscr{E}_X|_\Lambda$.

PROPOSITION 3.3.6. *The sheaf* $\mathscr{M}^{\mathbb{R}}_\Lambda(*)$ *is a* $\mathscr{D}^{2(\mathbb{R},\infty)}_\Lambda(*)$*-module and we have canonical isomorphisms:*

$$\mathscr{D}^{2(\mathbb{R},\infty)}_\Lambda(*)\otimes_{\mathscr{E}_{X|\Lambda}}\mathscr{M}_\Lambda \approx \mathscr{D}^{2(\mathbb{R},f)}_\Lambda(*)\otimes_{\mathscr{E}_{X|\Lambda}}\mathscr{M}_\Lambda \approx \mathscr{M}^{\mathbb{R}}_\Lambda(*).$$

Proof. The fact that $\mathscr{M}^{\mathbb{R}}_\Lambda(*)$ is a $\mathscr{D}^{2(\mathbb{R},\infty)}_\Lambda(*)$-module is proved in the same way as Proposition 3.1.4 (in fact it is much more simple).

From this structure we get canonical morphisms:

$$\mathscr{D}^{2(\mathbb{R},\infty)}_\Lambda(*)\otimes_{\mathscr{E}_X}\mathscr{M}_\Lambda\to\mathscr{M}^{\mathbb{R}}_\Lambda(*) \quad\text{and}\quad \mathscr{D}^{2(\mathbb{R},f)}_\Lambda(*)\otimes_{\mathscr{E}_X}\mathscr{M}_\Lambda\to\mathscr{M}^{\mathbb{R}}_\Lambda(*)$$

and to see that they are isomorphism we can make a quantized canonical transformation and then we may suppose that Λ is of the form T^*_YX. The proposition becomes:

$$\mathscr{D}^{2(\mathbb{R},\infty)}_\Lambda(*)\otimes_{\mathscr{E}_X}\mathscr{C}_{Y|X}\xrightarrow{\sim}\mathscr{C}^{\mathbb{R}}_{Y|X}(*)$$

and the morphism is easily calculated with the symbols. Then it is clear that it is an isomorphism.

References

[1] Andronikof, E.: Microlocalisation Tempérée des Distributions et des Fonctions Holomorphes, Thèse à l'université Paris-Nord, 1987.
[2] Aoki, T.: Invertibility for microdifferential operators of infinite order, *Publ. R.I.M.S. Kyoto* 18 (1982), 1–29.
[3] Boutet de Monvel, L.: Opérateurs pseudo-différentiels d'ordre infini, *Ann. Inst. Fourier (Grenoble)*, 22 (1972), 229–268.
[4] Demailly, J. P.: Analytic Geometry Vol. I, to appear.
[5] Hörmander, L.: An Introduction to Complex Analysis in Several Variables, Von Nostrand-Princeton, 1966.
[6] Kashiwara, M. and Kawaï, T.: Second microlocalization and asymptotic expansions, *Lecture Notes in Physics*, Springer 126 (1980), 21–76.
[7] Kashiwara, M. and Laurent, Y.: Théorèmes d'annulation et deuxième microlocalisation, *Preprint*, Université Orsay, 1983.
[8] Kashiwara, M. and Schapira, P.: Microhyperbolic systems, *Acta Math.* 142 (1979), 1–55.
[9] Kashiwara, M. and Schapira, P.: Microlocal study of sheaves, *Astérisque* 128 (1985).
[10] Laurent, Y.: Théorie de la deuxième microlocalisation dans le domaine complexe, *Progress in Math.*, Birkhäuser 53 (1985).
[11] Laurent, Y.: Vanishing cycles and second microlocalization, in *Algebraic Analysis* Vol. I, Academic Press (1988), 381–391.
[12] Laurent, Y.: Irregular vanishing cycles of D-modules, in preparation.
[13] Malgrange, B.: Sur les points singuliers des équations différentielles, *L'enseignement Mathématique* XX(1–2) (1974), 147–176.
[14] Malgrange, B.: Sur la réduction formelle des équations différentielles à singularités irrégulières, Prépublication, Grenoble I, 1979.
[15] Ramis, J. P.: Théorèmes d'indices Gevrey pour les équations différentielles ordinaires, *Memoirs of A.M.S.* 48 (1984).
[16] Sato, M., Kawaï, T. and Kashiwara, M., Hyperfunctions and pseudo-differential equations, *Lecture Notes in Math.*, Springer 287 (1983).
[17] Grothendieck, A.: E.G.A. III, *Publ. Math. IHES* 11 (1961).