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1. Introduction

The concept of *dilation* was introduced and investigated by several important mathematicians [2]. Given probability measures P, Q on the σ-field of Borel subsets of a topological space S, we say that *Q is a dilation of P relatively to a set K of functions $S \to R$, and write $P \prec_K Q$, iff $\int f \, \mathrm{d}P \leqslant \int f \, \mathrm{d}Q$ for all $f \in K$*. The set of functions K is usually a cone. It is possible that, although Q does not dilate P relatively to K, it nearly does so in some sense, giving rise to an *approximate dilation* of P. A natural approach is to employ a 'distance' of type

$$\delta(P, Q) := \inf \left\{ \varepsilon \geqslant 0 \,\middle|\, \int f \, \mathrm{d}P \leqslant \int f \, \mathrm{d}Q + \varepsilon L(f), f \in K \right\}$$

where $L(f) \geqslant 0$ measures the 'size' of f.

We allow any cone of bounded functions which is *admissible*, i.e., *a convex cone of continuous functions containing the constants and being invariant under the operation* $\vee$. The latter means that $\max\{f, g\} \in K$ whenever $f, g \in K$. Initially $L(f)$ will be taken as the oscillation of f. Afterwards, other approximate dilations will also be discussed. Theorem 10, summarized in Fig. 1, is our main result.

2. Notations

In this paper A^c denotes the complement of the set A; $\mathscr{B} = \mathscr{B}(S)$ the σ-field of Borel subsets of a topological space S; $C(S)$ the set of all continuous functions $S \to R$; $C_b(S)$ the set of all functions in $C(S)$ which are bounded; 'distribution function' is abbreviated as df; K' is the set of all $f \in K$ (K is a cone of functions) with $\inf f = 0$ and $\sup f = 1$; $\mathscr{M}(S)$ the set of all probability measures on the σ-field of Borel subsets of S; osc f stands for 'oscillation of the function f', i.e., $\operatorname{osc} f := \sup f - \inf f$; δ_s represents the Dirac measure at the point s; the symbols $\vee$, $\wedge$ have the usual meaning, i.e., they denote the maximum and the minimum operation, respectively; lsc abbreviates 'lower semicontinuous'; and, finally, iff stands for 'if and only if'.

We begin with a lemma, essential for the fundamental Theorem 7. It was suggested by Lemma 4 in [2], to which it reduces when $\varepsilon = 0$.

3. LEMMA. *Let S be a compact Hausdorff space and $K \subset C(S)$ an admissible cone. Let $P, Q \in \mathcal{M}(S)$ be such that $\int f \,\mathrm{d}P \leqslant \int f \,\mathrm{d}Q + \varepsilon \operatorname{osc} f$ for all $f \in K$. Let us fix bounded functions $\alpha, \beta, \phi_i\colon S \to R$, where α and β are Borel measurable and $\phi_i \geqslant 0$, $i = 1, \ldots, n$. Further let us fix $f_i \in K$, $i = 1, \ldots, n$. Then*

$$\inf_{s,t \in S} \left[\alpha(s) + \beta(t) + \sum_{i=1}^{n} (f_i(s) - f_i(t))\phi_i(s) \right] \geqslant 0 \tag{1}$$

implies

$$\int \alpha \,\mathrm{d}P + \int \beta \,\mathrm{d}Q + \varepsilon \operatorname{osc} \beta \geqslant 0. \tag{2}$$

Proof. The proof is patterned after that of Lemma 3 in [2]. As in that lemma, the crucial step consists of defining an auxiliary function $\hat{\beta}\colon R^n \to \bar{R} := [-\infty, +\infty]$ having convenient properties. The Euclidean space R^n will be equipped with the usual coordinatewise partial ordering. Throughout the rest of the proof we will use the notation $f := (f_1, \ldots, f_n)$. Also $\boldsymbol{\beta}\colon S \to R$ will be the lsc regularization of β. It is given by $\boldsymbol{\beta}(t) := \varliminf_{s \to t} \beta(s)$. Of course (1) holds true with $\boldsymbol{\beta}$ in place of β.

Let $x \in R^n$ and consider the sequences

$$(p_1, p_2, \ldots) \in [0, 1]^\infty \quad \text{with } p_1 + p_2 + \cdots = 1, \tag{3}$$

$$(t_1, t_2, \ldots) \in S^\infty \quad \text{with } x \leqslant \sum_j p_j f(t_j). \tag{4}$$

Set

$$T_x := \left\{ \sum_j p_j \boldsymbol{\beta}(t_j) \,|\, (3) \text{ and } (4) \text{ hold} \right\}$$

and define

$$\hat{\beta}(x) := \inf T_x \quad \text{if } T_x \neq \varnothing, \qquad \text{and} \qquad \hat{\beta}(x) := +\infty \quad \text{if } T_x = \varnothing.$$

It is easy to see that $\hat{\beta}(x)$ is finite on and only on the set $U := \{x \in R^n \,|\, x \leqslant y$ for some $y \in \operatorname{conv} f(S)\}$. Here the notation $\operatorname{conv} f(S)$ indicates the convex hull of $f(S)$. The properties of $\hat{\beta}$ that we are interested in are: (i) $-\alpha \leqslant \hat{\beta} \circ f \leqslant \beta$, (ii) $\hat{\beta}$ is increasing, (iii) $\hat{\beta}$ is convex, and (iv) $\hat{\beta}$ is lsc. The last one is the more important; it is the Lemma 4 in [2], where we need the lower semicontinuity of $\boldsymbol{\beta}$.

Let us prove the property (i). Taking $(p_n) := (1, 0, \ldots)$ and $(t_n) := (t, t, \ldots) \in S^\infty$, we see that $\hat{\beta}(t) \in T_{f(t)}$, hence $\hat{\beta}(f(t)) \leqslant \boldsymbol{\beta}(t)$, that is,

$$\hat{\beta} \circ f \leqslant \boldsymbol{\beta} \leqslant \beta \quad \text{on } S. \tag{5}$$

For the first inequality in (i), fix $s \in S$, set $x := f(s)$ and take sequences (p_j), (t_j) verifying (3) and (4), respectively. In particular

$$f(s) \leqslant \sum_j p_j f(t_j). \tag{6}$$

Let us apply (1) with $t := t_j$; afterwards, we multiply by p_j and sum over j obtaining

$$\alpha(s) + \sum_j p_j \boldsymbol{\beta}(t_j) + \sum_{i=1}^{n} \left[f_i(s) - \sum_j p_j f_i(t_j) \right] \phi_i(s) \geqslant 0,$$

which gives, using (6), $\alpha(s) + \Sigma_j p_j \boldsymbol{\beta}(t_j) \geqslant 0$. This together with the definition of $\hat{\beta}$ yield $\alpha(s) + \hat{\beta} \circ f(s) \geqslant 0$ so that, by (5),

$$-\alpha \leqslant \hat{\beta} \circ f \leqslant \beta \quad \text{on } S. \tag{7}$$

That $\hat{\beta}$ is increasing is immediate.

The convexity is easy: let p, $q \in [0, 1]$ with $p + q = 1$, x, $y \in R^n$ and

$$\sum_j p_j \boldsymbol{\beta}(t_j) \in T_x, \qquad \sum_j q_j \boldsymbol{\beta}(t_j) \in T_y.$$

Therefore it is readily seen that

$$\left[\sum_j p p_j \boldsymbol{\beta}(t_j) + \sum_j q q_j \boldsymbol{\beta}(t_j) \right] \in T_{px+qy},$$

hence

$$\hat{\beta}(px + qy) \leqslant p \sum_j p_j \boldsymbol{\beta}(t_j) + q \sum_j q_j \boldsymbol{\beta}(t_j),$$

which produces

$$\hat{\beta}(px + qy) \leqslant p \inf T_x + q \inf T_y = p\hat{\beta}(x) + q\hat{\beta}(y),$$

so $\hat{\beta}$ is convex indeed.

It is known that a convex lsc function like $\hat{\beta}$ restricted to U, which is a convex set with non-empty interior, is the limit of an increasing sequence $(h_{(\nu)})$ of functions $h_{(\nu)} := h_1 \vee \cdots \vee h_\nu$, where, for $i = 1, \ldots, \nu$, h_i is the restriction to U of an affine function on R^n given by $h_i(x) := \langle A_i, x \rangle + a_i$, $A_i \in R^n$, $a_i \in R$. Here $\langle \cdot, \cdot \rangle$ is the usual inner product. Since $\hat{\beta}$ is increasing, we can suppose that all the h_i's are increasing, equivalently, that $A_i \geqslant 0$. As K contains the constants, the linear combinations $h_i \circ f \in K$, thus also $h_{(\nu)} \circ f \in K$ for all $\nu \in N$, because K is invariant under the operation $\vee$, so that

$$\int h_{(\nu)} \circ f \, \mathrm{d}P \leqslant \int h_{(\nu)} \circ f \, \mathrm{d}Q + \varepsilon \operatorname{osc}(h_{(\nu)} \circ f) \quad \text{for all } \nu \in N.$$

Therefore by the Monotone Convergence Theorem

$$\int \hat{\beta} \circ f \, \mathrm{d}P \leqslant \int \hat{\beta} \circ f \, \mathrm{d}Q + \varepsilon \, \overline{\lim} \operatorname{osc}(h_{(\nu)} \circ f).$$

It is obvious that $\sup h_{(\nu)} \circ f \leqslant \sup \hat{\beta} \circ f$. Further $\lim_\nu (\inf h_{(\nu)} \circ f) = \inf \hat{\beta} \circ f$ by Dini's lemma. Thus $\overline{\lim} \operatorname{osc}(h_{(\nu)} \circ f) \leqslant \operatorname{osc}(\hat{\beta} \circ f) \leqslant \operatorname{osc} \beta$. Putting all together, one arrives at the inequality

$$\int \hat{\beta} \circ f \, \mathrm{d}P \leqslant \int \hat{\beta} \circ f \, \mathrm{d}Q + \varepsilon \operatorname{osc} \beta. \tag{8}$$

Finally, using (7) and (8), we conclude that

$$\begin{aligned} \int \alpha \, \mathrm{d}P + \int \beta \, \mathrm{d}Q &\geqslant \int \alpha \, \mathrm{d}P + \int \hat{\beta} \circ f \, \mathrm{d}Q \\ &\geqslant \int (\alpha + \hat{\beta} \circ f) \, \mathrm{d}P - \varepsilon \operatorname{osc} \beta \geqslant -\varepsilon \operatorname{osc} \beta. \end{aligned} \qquad \square$$

Let $P, Q \in \mathscr{M}(S)$. We will describe the property $\int f \, \mathrm{d}P \leqslant \int f \, \mathrm{d}Q + \varepsilon \operatorname{osc} f$, for all f in a subset L of $C_b(S)$ also by saying that Q is an *approximate dilation* or an *ε-dilation* of P relatively to L.

The following theorem supplies an equivalent definition of 'ε-dilation' relatively to an admissible cone $K \subset C(S)$ for the case that S is a compact metric space. It says that a necessary and sufficient condition for Q to be an ε-dilation of P relatively to K is that one can find a probability measure $\lambda \in \mathscr{M}(S^2)$ that satisfies

$$\int (f(s) - f(t)) \phi(s) \lambda(\mathrm{d}s, \mathrm{d}t) \leqslant 0 \quad \text{for all } f \in K, \ \phi \in C^+(S) \tag{9}$$

and whose first marginal is P and second marginal is 'ε-close' to Q.

4. THEOREM. *Let S be a compact metric space, $K \subset C(S)$ an admissible cone, $\varepsilon \geqslant 0$, and $P, Q \in \mathcal{M}(S)$. Then $\int f \, \mathrm{d}P \leqslant \int f \, \mathrm{d}Q + \varepsilon \operatorname{osc} f$ for all $f \in K$ iff there exists $\lambda \in \mathcal{M}(S^2)$ satisfying (9) and, in addition,*

$$\int \alpha(s)\lambda(\mathrm{d}s, \mathrm{d}t) \leqslant \int \alpha \, \mathrm{d}P \quad \textit{for all } \alpha \in C(S), \tag{10}$$

$$\int \beta(t)\lambda(\mathrm{d}s, \mathrm{d}t) \leqslant \int \beta \, \mathrm{d}Q + \varepsilon \operatorname{osc} \beta \quad \textit{for all } \beta \in C(S). \tag{11}$$

Proof. 'If': Fix $f \in K$. Applying (10) with $\alpha = -f$, (9) with $\phi = 1$, (11) with $\beta = f$, one finds that $\int f \, \mathrm{d}P \leqslant \int f(s)\lambda(\mathrm{d}s, \mathrm{d}t) \leqslant \int f(t)\lambda(\mathrm{d}s, \mathrm{d}t) \leqslant \int f \, \mathrm{d}Q + \varepsilon \operatorname{osc} f$.

'Only if': By Theorem 7 in [4], the existence of a measure $\lambda \in M(S^2)$ satisfying (9), (10) and (11) is equivalent to the implication (1) ⇒ (2). Thus the 'only if' part follows from Lemma 3. □

In the following lemma the equivalence (b) ⇔ (c) is known. See for example [3].

5. LEMMA. *Assume S is a metric space, $\varepsilon \geqslant 0$ and P, $Q \in \mathcal{M}(S)$. Then the following are equivalent:*

(a) $\int \alpha \, \mathrm{d}P \leqslant \int \alpha \, \mathrm{d}Q + \varepsilon \operatorname{osc} \alpha$ *for all* $\alpha \in C_b(S)$;
(b) $|P(B) - Q(B)| \leqslant \varepsilon$ *for all* $B \in \mathcal{B}(S)$;
(c) $\|P - Q\| \leqslant 2\varepsilon$.

Proof. We will show that (a) ⇒ (b) ⇒ (c) ⇒ (a).

(a) ⇒ (b): Since the indicator function 1_A of an open set $A \subset S$ is lsc, it is the pointwise limit of an increasing sequence of non-negative functions in $C_b(S)$. So (a) implies through the Monotone Convergence Theorem that $P(A) \leqslant Q(A) + \varepsilon$ for all open sets $A \subset S$. Now (b) follows by regularity of P.

(b) ⇒ (c): Let $\mu := (P + Q)/2$ and consider $f := \mathrm{d}P/\mathrm{d}\mu$, $g := \mathrm{d}Q$, the Radon-Nikodym derivatives. We have, using (b), $\|P - Q\| = \int |f - g| \, \mathrm{d}\mu \leqslant 2\varepsilon$.

(c) ⇒ (a): Let μ, f and g be as in the proof of (b) ⇒ (c), $\alpha \in C_b(S)$ and $c := -(\sup \alpha + \inf \alpha)/2$. Therefore $2\|\alpha + c\| = \operatorname{osc} \alpha$ and

$$\begin{aligned}\int \alpha \, \mathrm{d}P - \int \alpha \, \mathrm{d}Q &= \int (\alpha + c)(f - g) \, \mathrm{d}\mu \leqslant \|\alpha + c\| \int |f - g| \, \mathrm{d}\mu \\ &= \|\alpha + c\| \cdot \|P - Q\| \leqslant \varepsilon \operatorname{osc} \alpha.\end{aligned}$$

□

6. DEFINITIONS. In view of Theorem 4 and Lemma 5 it becomes natural to study the five quantities $\varepsilon_i(P, Q)$, $i = 1, \ldots, 5$, defined as follows.

Let S be a topological space, $K \subset C_b(S)$ an admissible cone and P, $Q \in \mathcal{M}(S)$.

Here the dilations will be relative to K. Let us define

$$E_1 := \left\{\varepsilon \geqslant 0 \mid \int f\,\mathrm{d}P \leqslant \int f\,\mathrm{d}Q + \varepsilon \operatorname{osc} f \text{ for all } f \in K\right\},$$

$$E_2 := \left\{\varepsilon \geqslant 0 \mid \text{there exists } Q' \in \mathscr{M}(S) \text{ with } P \prec Q' \text{ and } \|Q' - Q\| \leqslant 2\varepsilon\right\},$$

$$E_3 := \left\{\varepsilon \geqslant 0 \mid \text{there exists } P' \in \mathscr{M}(S) \text{ with } P' \prec Q \text{ and } \|P' - P\| \leqslant 2\varepsilon\right\},$$

$$E_4 := \left\{\varepsilon \geqslant 0 \mid \text{there exists } P', Q' \in \mathscr{M}(S) \text{ with } P' \prec Q', \right.$$

$$\left. \|P' - P\| \leqslant 2\varepsilon \text{ and } \|Q' - Q\| \leqslant 2\varepsilon\right\},$$

$$E_5 := \left\{\varepsilon \geqslant 0 \mid \text{there exists } P', Q' \in \mathscr{M}(S) \text{ with } P' \prec Q' \right.$$

$$\left. \text{and } \|P' - P\| + \|Q' - Q\| \leqslant 2\varepsilon\right\}.$$

Now we define

$$\varepsilon_i(P, Q) := \inf E_i, \quad i = 1, \ldots, 5. \tag{12}$$

It is trivial to see that $E_2 \subset E_1$, $E_3 \subset E_1$ and $E_2 \subset E_5 \subset E_4$. Now, if S is a compact metric space, taking Q' as the second marginal of the measure λ, it follows from Theorem 4 that $E_1 \subset E_2$. To summarize, whenever S is compact metric space $E_3 \subset E_1 = E_2 \subset E_5 \subset E_4$, thus we have proved the important

7. THEOREM. *If S is a compact metric space, $\varepsilon_4 \leqslant \varepsilon_5 \leqslant \varepsilon_2 = \varepsilon_1 \leqslant \varepsilon_3$.*

8. REMARKS. (i) Later on it will be seen that $\varepsilon_5 = \varepsilon_1$ and that the first and last inequalities in Theorem 7 are frequently strict.

(ii) If $P \prec Q$, then $\varepsilon_i(P, Q) = 0$, $i = 1, \ldots, 5$.

(iii) We always have $0 \leqslant \varepsilon_i(P, Q) \leqslant 1$, $i = 1, \ldots, 5$.

(iv) Obviously

$$\varepsilon_1(P, Q) = \sup_{\substack{\operatorname{osc} f \leqslant 1 \\ f \in K}} \left[\int f\,\mathrm{d}P - \int f\,\mathrm{d}Q\right]. \tag{13}$$

(v) Theorem 4 is false for non-compact spaces. For such spaces the condition

$\int f\,\mathrm{d}P \leqslant \int f\,\mathrm{d}Q + \varepsilon \operatorname{osc} f$ for all $f \in K$ is obviously necessary but no longer sufficient for (9), (10) and (11). To see that the named condition fails to be sufficient, consider $S := [0, 1)$, take $P := \delta_{1/2}$ and $Q := \delta_0$ and let K consist of all increasing convex functions on S. One can show that $\varepsilon_1(P, Q) = 1/2$ and that there is no $Q' \in \mathscr{M}(S)$ dilating P with $\|Q' - Q\| \leqslant 2\varepsilon$. This contradicts the inclusion $E_1 \subset E_2$, thus Theorem 4. □

From (13) it follows immediately that ε_1 satisfies the triangle inequality. But ε_1 is not symmetric. The mapping $(P, Q) \mapsto \delta_1(P, Q) := \varepsilon_1(P, Q) + \varepsilon_1(Q, P)$ is a pseudo-metric on $\mathscr{M}(S)$, in fact a metric when K is a determining class for $\mathscr{M}(S)$ (for instance, S a convex compact metrizable subset of a topological vector space and $K \subset C(S)$ the cone of convex functions). It is not difficult to prove that a sequence (P_n) in $\mathscr{M}(S)$ converges with respect to δ_1, i.e., $\delta_1(P_n, P) \to 0$ for some $P \in \mathscr{M}(S)$ iff the sequence of linear functionals $f \mapsto \int f\,\mathrm{d}P_n$ converges uniformly on $K \cap \{f \in C(S) \mid \|f\| = 1\}$. As a consequence, if K is a determining class for $\mathscr{M}(S)$, the δ_1-topology on $\mathscr{M}(S)$ is finer than the weak topology.

Neither ε_3 nor ε_4 satisfy the triangle inequality as Examples 9 and 13 will show. On the other hand it is easy to see that $\varepsilon_4(P, R) \leqslant 2[\varepsilon_4(P, Q) + \varepsilon_4(Q, R)]$.

9. EXAMPLE. A case where $\varepsilon_3(P, R) > \varepsilon_3(P, Q) + \varepsilon_3(Q, R)$. Let $S := [0, 1]$, $K \subset C(S)$ be the cone of all convex functions, $P := \delta_{1/2}$, $Q := (1/2)(\delta_0 + \delta_1)$ and $R := \delta_0$. For each $f \in K$, $f(1/2) \leqslant (1/2)f(0) + (1/2)f(1)$, so that $P \prec Q$, hence $\delta_3(P, Q) = 0$. Also $\delta_3(Q, R) \leqslant \|Q - R\|/2 = 1/2$. Since $P' \prec R$ requires $P' = \delta_0$, it follows that $\delta_3(P, R) = \|\delta_0 - \delta_{1/2}\|/2 = 1$. □

Probably there is no easy formula for computing the value ε_i, $i = 1, \ldots, 5$, but the next theorem and corollary are an important step in this direction.

10. THEOREM. *Let S be a compact space, $K \subset C(S)$ an admissible cone, P, $Q \in \mathscr{M}(S)$ and u, $v \geqslant 0$ constants. Then that there exist P', $Q' \in \mathscr{M}(S)$ such that*

$$\|P' - P\| \leqslant 2u,\ \|Q' - Q\| \leqslant 2v,\ P' \prec_K Q' \tag{14}$$

if and only if, for all $f \in K$ with $\inf f = 0$ *and all $c \in R$ with* $0 < c \leqslant \sup f$, we have

$$\int f \wedge c\,\mathrm{d}P \leqslant \int f\,\mathrm{d}Q + uc + v \sup f. \tag{15}$$

Proof. By the very definition of ε_2, (14) is equivalent to the existence of $P' \in \mathscr{M}(S)$ such that

$$\|P' - P\| \leqslant 2u,\ \varepsilon_2(P', Q) \leqslant v. \tag{16}$$

By Lemma 5 and the equality $\varepsilon_2 = \varepsilon_1$, condition (16) on P' is equivalent to

$$\int \alpha \, dP' \leqslant \int \alpha \, dP + u \operatorname{osc} \alpha, \quad \text{for all } \alpha \in C(S)$$

$$\int f \, dP' \leqslant \int f \, dQ + v \operatorname{osc} f, \quad \text{for all } f \in K. \tag{17}$$

Since $C(S)$ and K are cones, Theorem 5 in [4] tells us that a $P' \in \mathcal{M}(S)$ satisfying (17) exists iff, for all $f_j \in K$, $\alpha_i \in C(S)$, and m, $n \in N$, we have that

$$\inf\left(\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{n} f_j\right) \geqslant 0 \tag{18}$$

implies

$$\sum_{i=1}^{m}\left(\int \alpha_i \, dP + u \operatorname{osc} \alpha_i\right) + \sum_{j=1}^{n}\left(\int f_j \, dQ + v \operatorname{osc} f_j\right) \geqslant 0. \tag{19}$$

Letting $\alpha := \Sigma\, \alpha_i$ and $f := \Sigma\, f_j$, then $\alpha \in C(S)$ and $f \in K$, since the cones $C(S)$ and K are convex. As $\operatorname{osc} \alpha \leqslant \Sigma \operatorname{osc} \alpha_i$ and $\operatorname{osc} f \leqslant \Sigma \operatorname{osc} f_j$, it suffices to establish the implication

$$\alpha \in C(S),\ f \in K,\ \inf(\alpha + f) \geqslant 0$$
$$\Rightarrow \int \alpha \, dP + \int f \, dQ + u \operatorname{osc} \alpha + v \operatorname{osc} f \geqslant 0. \tag{20}$$

Introducing $h := \alpha + f$, this is equivalent to the requirement that

$$\int f \, dP - \int f \, dQ \leqslant \int h \, dP + u \operatorname{osc}(f - h) + v \operatorname{osc} f,$$
$$\text{if } f \in K,\ h \in K,\ h \in C^{+}(S). \tag{21}$$

Given $f \in K$, we want to choose $h \in C^{+}(S)$ so as to minimize the right-hand side of (21). Put $a := \inf(f - h)$ and $c := \sup(f - h)$ so that $\operatorname{osc}(f - h) = c - a$ and $a \leqslant f - h \leqslant c$, or $f - c \leqslant h \leqslant f - a$. As $h \geqslant 0$, setting $h_0 := (f - c)^{+} := (f - c) \vee 0$, we have $f - c \leqslant h_0 \leqslant h \leqslant f - a$. Further $f - c \leqslant h_0 \leqslant f - a$, or $a \leqslant f - h_0 \leqslant c$, which shows that $\operatorname{osc}(f - h_0) \leqslant c - a = \operatorname{osc}(f - h)$. Since $0 \leqslant h_0 = (f - c)^{+} \leqslant h$ and $\operatorname{osc}(f - h_0) \leqslant \operatorname{osc}(f - h)$, it is clear from (21) that it suffices to consider only functions of the form $h := (f - c)^{+}$, where c is a constant. Observing that $f - (f - c)^{+} = f \wedge c$, (21) is equivalent to

$$\int f \wedge c\, dP - \int f\, dQ \leqslant u \operatorname{osc}(f \wedge c) + v \operatorname{osc} f, \quad \text{for all } f \in K,\ c \in R. \tag{22}$$

Let us show that in (22) we only need

$$\inf f < c \leqslant \sup f. \tag{23}$$

For, the choice $c > \sup f$ is the same as the choice $c = \sup f$, because in both cases $f \wedge c = f$. If $c \leqslant \inf f$, then $\int f \wedge c\, dP = c$ and $\int f\, dQ \geqslant \inf f \geqslant c$ so that (23) is always true.

Since K contains the constants we can always take $\inf f = 0$, in which case $\operatorname{osc} f = \sup f$. Thus the proof will be complete if we show that $\operatorname{osc}(f \wedge c) = c$. Indeed, by (23) $\inf(f \wedge c) = \inf f = 0$ and $\sup(f \wedge c) = 0$. □

Besides using only functions $f \in K$ with $\inf f = 0$ in (15) one may also assume without loss of generality that $\sup f = 1$. Hence (15), thus also (14), is equivalent to

$$tu + v \geqslant \phi(t), \quad \text{for all } 0 \leqslant t \leqslant 1. \tag{24}$$

Here $\phi(t) := \sup\{\int f \wedge t\, dP - \int f\, dQ \mid f \in K,\ \inf f = 0,\ \sup f = 1\}$.

The set of relations (24) represents a family $(H_t)_{t\in[0,1]}$ of closed half planes. The intersection

$$A := A(P, Q, K) := \left(\bigcap_{t\in[0,1]} H_t \right) \cap \{(u, v) \in R^2 \mid u \geqslant 0,\ v \geqslant 0\}$$

is a closed convex subset of R^2. The pairs $(u, v) \in A$ are precisely the pairs for which there exist $P', Q' \in \mathscr{M}(S)$ satisfying (14).

Considering the definitions of $\varepsilon_i(P, Q)$ it is clear that

$$\varepsilon_2(P, Q) = \inf\{v \mid (0, v) \in A\},$$

$$\varepsilon_3(P, Q) = \inf\{u \mid (u, 0) \in A\},$$

$$\varepsilon_4(P, Q) = \inf\{u \mid (u, u) \in A\},$$

$$\varepsilon_5(P, Q) = \inf\{u + v \mid (u, v) \in A\}.$$

The geometric meaning of $\varepsilon_1 = \varepsilon_2$, ε_3, ε_4 and ε_5 is clear. So putting all together we have the situation described in Fig. 1.

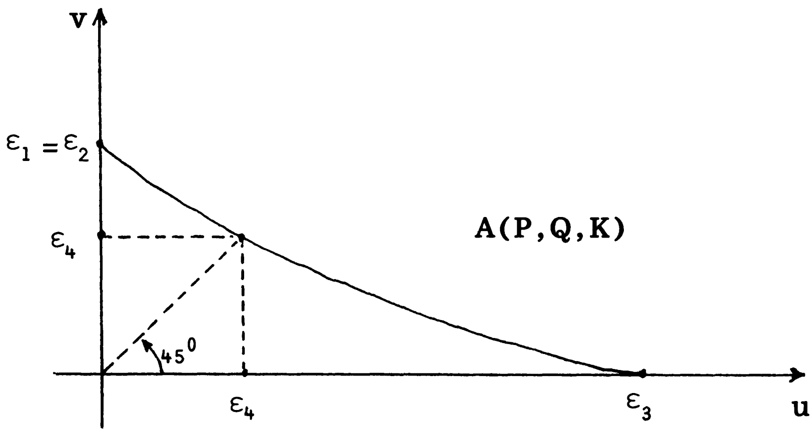

Fig. 1

The only thing that is not clear is how ε_5 fits into the picture. In fact one has:

11. COROLLARY. $\varepsilon_5 = \varepsilon_2$.

Proof. The function $t \mapsto \phi(t)$ in (24) is increasing. Hence $\varepsilon_2(P, Q) = \phi(1)$. Therefore, taking $t = 1$ in (24), all points $(u, v) \in A$ satisfy $u + v \geqslant \varepsilon_2(P, Q)$. The equality sign is attained at $(0, \varepsilon_2(P, Q))$. This proves that $\varepsilon_5 = \varepsilon_2$. □

Let S be a compact metric space with a partial order, and K the cone of all continuous increasing functions that assume their minimum at every point of $U := \operatorname{supp} Q$, the support of Q. Note that such a cone K is not only invariant under the operation $\vee$ but also under $\wedge$. Letting $P \in \mathscr{M}(S)$ be arbitrary, we have as $\phi(t)$ in (24)

$$\phi(t) = \int t \wedge 1_{U^c}(s) P(\mathrm{d}s) = tP(U^c),$$

which leads to $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 2\varepsilon_4 = \varepsilon_5 = P(U^c)$ (see Fig. 2).

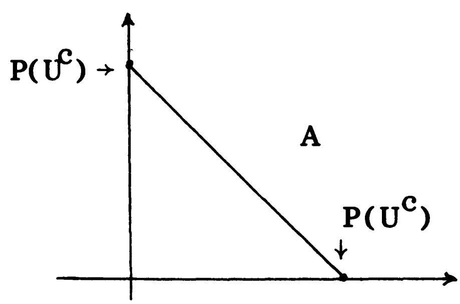

Fig. 2

The above expression for $\phi(t)$ was possible because K' (see Notations) is filtering from the right (see [1], p. 145), i.e., given $f, g \in K'$, there exists $h \in K'$ with $f, g \leqslant h$. In general, if S is a compact space with a partial ordering, $K \subset C(S)$ an admissible cone such that K' is filtering from the right, and $Q \in \mathscr{M}(S)$ is such that each $f \in K'$ assumes its minimum at every point of supp Q, then (24) takes the form

$$tu + v \geqslant \phi(t) = t - \int_0^t F(s)\, ds, \quad t \in [0, 1], \tag{25}$$

where F is the P-distribution functions of $s \mapsto \sup_{f \in K'} f(s) := f^*(s)$.

It is true in general that the right slope r of the lower boundary of a region $A(P, Q)$ at $(0, \varepsilon_1)$ is given by the formula $r = -\inf\{t \in [0, 1] \mid \phi$ is constant on $[t, 1]\}$, where ϕ is as in (24). Now, if $\phi(t)$ is the right-hand side in (25), then, as it is easy to see, the formula for r specializes to

$$r = -\inf\{t \in [0, 1] \mid F(t) = 1\}. \tag{26}$$

Similarly, it is true in general that the left slope l of the lower boundary of $A(P, Q)$ at $(\varepsilon_3, 0)$ is obtained by the formula $l = -\sup\{t_1 \in [0, 1] \mid \phi(t)/t$ is constant on $(0, t_1]\}$, which, in the situation of (25), becomes

$$l = -\sup\{t_1 \in [0, 1] \mid F \text{ is constant on } [0, t_1]\}. \tag{27}$$

12. EXAMPLE. Let S be the interval $[0, 1]$, $K \subset C(S)$ the cone of convex increasing functions, $P \in \mathscr{M}(S)$ the measure with density $[1/(b-a)]1_{[a,b]}(s)\, ds$ where $0 \leqslant a < b \leqslant 1$, and $Q := \delta_0$. The corresponding df F is given by $F(s) := (s-a)/(b-a)$ if $s \in [a, b]$. Hence here $r = -b$ and $l = -a$, which show that the right slope of the lower boundary of A at $(0, \varepsilon_1)$ can be any number in $[-1, 0)$ and its left slope at $(\varepsilon_3, 0)$ any number in $(-1, 0]$. We observe also that here $\varepsilon_1(P, \delta_0) = 1 - \int_0^1 F(s)\, ds = (a+b)/2$, so that ε_1 *can be close to* 0 *or* 1. □

13. Measures P'_t, Q'_t realizing the boundary of $A(P, Q)$

As it was already observed, $A(P, Q)$ is a closed subset of R^2. This means that, for each point (u, v) on the boundary of $A(P, Q)$, one can attain both equality signs in (14) by a suitable choice of P' and Q'. Let us now give an example where P', Q' can be explicitly described.

Let S be a compact metric space and $K \subset C(S)$ an admissible cone. Suppose K' possesses a largest element f^*. Choose $P \in \mathscr{M}(S)$ and let F be the P-distribution function of f^*. Suppose there is a unique point y in S with $f^*(y) = 0$

and a unique point y' in S with $f^*(y') = 1$. (Example: let S be a compact space with a partial ordering, a least element y and a greatest element y', and let $K \subset C(S)$ be the cone of all convex increasing functions.) Choose $Q = \delta_y$. A little calculation readily shows that here ϕ in (24) is given by $\phi(t) = t - \int_0^t F(s)\,ds$ so that (24) reads $tu + v \geqslant t - \int_0^t F(s)\,ds$ for all $t \in [0, 1]$. Hence we obtain that the part of the lower boundary of $A(P, \delta_y, K)$ not contained in the coordinate axes is a smooth curve (envelope) with parametric equations $u(t) = 1 - F(t)$, $v(t) = tF(t) - \int_0^t F(s)\,ds$, $t \in [0, 1]$. Here we are assuming that P has no atom.

Define P'_t, $Q'_t \in \mathcal{M}(S)$ by

$$P'_t(E) := P[E \cap (f^* \leqslant t)] + u(t)\delta_y(E),$$

$$Q'_t(E) := v(t)\delta_{y'}(E) + (1 - v(t))\delta_y(E).$$

Certainly $\|P'_t - P\| = 2u(t)$ and $\|Q'_t - Q\| = 2v(t)$. Moreover, given $f \in K'$,

$$\int f\,dP'_t \leqslant \int f^*\,dP'_t = \int_{[0,t]} s\,dF(s) + uf^*(y)$$

$$= \int_{[0,t]} s\,dF(s) = tF(t) - \int_0^t F(s)\,ds = v(t),$$

and

$$\int f\,dQ'_t = v(t)f(y') + [1 - v(t)]f(y) = v(t).$$

Thus $\int f\,dP'_t \leqslant \int f\,dQ'_t$. This proves that $P'_t \prec Q'_t$.

14. The triangle inequality fails for ε_4

Let $S := [0, 1]$, $K \subset C(S)$ be the cone of decreasing convex functions and $Q := (1/2)(\delta_0 + \delta_1)$. We want to show that,

$$\varepsilon_4(\delta_{1/2}, \delta_1) > \varepsilon_4(\delta_{1/2}, Q) + \varepsilon_4(Q, \delta_1). \tag{28}$$

Let us first compute $\varepsilon_4(\delta_{1/2}, \delta_1)$. Here (25) applies. The function $s \mapsto -s + 1$ is the largest element in K and its $\delta_{1/2}$-distribution function is $F = 1_{[1/2,\infty)}$. Using (25) we obtain the following family of half planes

$$tu + v \geqslant \begin{cases} t, & \text{if } t \leqslant 1/2 \\ 1/2, & \text{if } t \geqslant 1/2. \end{cases}$$

Thus $u + 2v = 1$ is the equation of the lower boundary of $A(\delta_{1/2}, \delta_1)$. Letting

$v = u$ in that equation, we conclude that $\varepsilon_4(\delta_{1/2}, \delta_1) = 1/3$.

Next consider $\varepsilon_4(\delta_{1/2}, Q)$. Here it is easier going back to (24). We have

$$\phi(t) = \sup_{f \in K'} \left[\int (f \wedge t)\, \mathrm{d}\delta_{1/2} - \int f\, \mathrm{d}Q \right] = \begin{cases} t - 1/2, & t \leqslant 1/2 \\ 0, & t \geqslant 1/2. \end{cases}$$

The equation of the important part of the lower boundary of $A(\delta_{1/2}, Q)$ is $u + 2v = 0$, from which, letting $v = u$, we obtain $\varepsilon_4(\delta_{1/2}, Q) = 0$.

As to $\varepsilon_4(Q, \delta_1)$, here again (25) applies. The Q-distribution function F of $s \mapsto -s+1$ has values $F(s) = 0$ if $s < 0$, $F(s) = 1/2$ if $0 \leqslant s < 1$ and $F(s) = 1$ if $s \geqslant 1$. By (25)

$$tu + v \geqslant t - \int_0^t F(s)\, \mathrm{d}s = t - \tfrac{1}{2}t = \tfrac{1}{2}t, \; t \in [0, 1].$$

So the part of the lower boundary of $A(Q, \delta_1)$ we are interested in is given by $u + v = 1/2$, $u \in [0, 1/2]$, showing that $\varepsilon_4(Q, \delta_1) = 1/4$. Therefore $\varepsilon_4(\delta_{1/2}, Q) + \varepsilon_4(Q, \delta_1) = 1/4$. Thus (28) is proved. □

When we dealt with cones both invariant under max and min operation, the corresponding picture, Fig. 2, was very peculiar. In particular $\varepsilon_2 = \varepsilon_3 = 2\varepsilon_4$ in that situation. Let us show that this is always so whenever the cone has the mentioned property through the following proposition.

15. PROPOSITION. *Let S be a compact metric space, $K \subset C(S)$ an admissible cone which is invariant under the operation $\wedge$ and let P, $Q \in \mathcal{M}(S)$. Then the portion of the boundary of $A(P, Q, K)$ not contained in the u-axis is a line segment with slope -1. In particular $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 2\varepsilon_4 = \varepsilon_5$ at (P, Q).*

Proof. The lower boundary of $A(P, Q, K)$ has slope $\leqslant 1$ (in absolute value). But so has the corresponding set $A(P, Q, -K)$, where $-K := \{f \mid -f \in K\}$. Since $A(P, Q, -K)$ is simply the reflexion $\{(v, u) \mid (u, v) \in A(P, Q, K)\}$ of $A(P, Q, K)$, the lower boundary of the latter is a straight line of slope -1. □

Before ending this article it is worthwhile to make the following

16. REMARK. Let S be a compact metric space, $K \subset C(S)$ an admissible cone and P, $Q \in \mathcal{M}(S)$. Using the definition of $\varepsilon_1(P, Q)$, Theorems 7 and 10 and Corollary 11, we have

$$\varepsilon_i(P, Q) = \sup_f \left[\int f\, \mathrm{d}P - \int f\, \mathrm{d}Q \right], \quad i = 1, 2, 5;$$

$$\varepsilon_3(P, Q) = \sup_{f,t} \left[\frac{1}{t} \int f \wedge t\, \mathrm{d}P - \frac{1}{t} \int f\, \mathrm{d}Q \right];$$

$$\varepsilon_4(P, Q) = \sup_{f,t} \left[\frac{1}{1+t} \int f \wedge t\, \mathrm{d}P - \frac{1}{1+t} \int f\, \mathrm{d}Q \right];$$

where f runs over K' and t over $(0, 1)$. It follows that, endowing $\mathscr{M}(S)$ with the weak topology, the function $(P, Q) \mapsto \varepsilon_1(P, Q)$, $i = 1, \ldots, 5$, is lsc and convex. It is easy to produce examples showing that those functions are not (weakly) continuous. $\square$

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