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## On the cohomology of a general fiber of a polynomial map

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### Introduction

Let  $X = \mathbf{C}^n$ ,  $S = \mathbf{C}$ , and  $f: X \rightarrow S$  a map defined by a polynomial which is also denoted by  $f$ . Then  $f$  induces a topological fibration over a Zariski-open subset of  $S$ . It is interesting whether we can compute algebraically the cohomology of a generic fiber  $F = f^{-1}(t)$  using the polynomial ring  $\mathcal{O} := \mathbf{C}[x_1, \dots, x_n]$  and the polynomial  $f$ . Of course, we can calculate the cohomology using the de Rham cohomology of the (scheme theoretic) generic fiber of  $f$ , but it is not quite computable.

In the weighted homogeneous case, the answer was given by [3]. Let  $\Omega$  denote the complex of global algebraic differential forms on  $X$  (i.e.,  $\Omega^p$  is a free  $\mathcal{O}$ -module with a basis  $dx_{i_1} \wedge \dots \wedge dx_{i_p}$  ( $i_1 < \dots < i_p$ )). Define a differential  $D_f$  on  $\Omega$  by

$$D_f(\omega) = d\omega - df \wedge \omega \quad \text{for } \omega \in \Omega^p. \quad (0.1)$$

Then we have an isomorphism

$$H^{k+1}(\Omega, D_f) \cong \tilde{H}^k(F, \mathbf{C}) \quad \text{for any } k, \quad (0.2)$$

if  $f$  is weighted homogeneous, where  $\tilde{H}$  denotes reduced cohomology. (In [loc. cit.],  $D_f$  was denoted by  $\bar{D}_f$ . See also (2.10) below.) In this paper, we prove

(0.3) **THEOREM.** *The isomorphism (0.2) holds for any polynomial  $f$ .*

The proof uses the theory of algebraic Gauss–Manin system which is a generalization of [4] (see, for example, [1]), and also the theory of monodromic algebraic  $\mathcal{D}$ -modules. Let  $\int_f \mathcal{O}_X$  denote the algebraic Gauss–Manin system, which is defined by the direct image of  $\mathcal{O}_X$  by  $f$  as algebraic  $\mathcal{D}$ -module. Let  $t$  be the coordinate of  $S$ , and  $\partial_t = \partial/\partial t$ . Then we have a natural quasi-isomorphism

(see (2.7)):

$$R\Gamma\left(S, \text{Cone}\left(\partial_t - \text{id}: \int_f \mathcal{O}_X \rightarrow \int_f \mathcal{O}_X\right)\right) \xrightarrow{\sim} (\Omega^*, D_f)[n]. \tag{0.4}$$

Here  $\partial_t - \text{id}$  is *analytically* equivalent to  $\partial_t$  (because  $\partial_t - 1 = e^t \partial_t e^{-t}$  in  $\mathcal{D}_{S^{\text{an}}}$ ). Let  $\int_f^p \mathcal{O}_X$  denote the  $p$ th cohomology of  $\int_f \mathcal{O}_X$ . Its restriction to a Zariski open subset of  $S$  is a vector bundle (i.e., a locally free sheaf) whose fiber is isomorphic to the cohomology of the fiber of  $f$  (see (2.3)). We take the direct image of  $\int_f^p \mathcal{O}_X$  by the compactification  $S \rightarrow \mathbf{P}^1$ , and compute its *analytic* local cohomology at infinity (see (2.8)). Then we get the assertion using the theory of monodromical  $\mathcal{D}$ -modules (see (2.9)).

It should be noted that Theorem (0.3) is essentially of *algebraic nature*, and the local analytic version of (0.3) does not hold. For example,  $(\Omega_{X^{\text{an}}}, D_f)[1]$  is not quasi-isomorphic to Deligne’s vanishing cycle sheaf, because  $D_f$  is *analytically* equivalent to the natural differential  $d$  using  $e^f$ .

### 1. Monodromical $D$ -modules of one variable

In this section, we gather some elementary facts from the theory of monodromical algebraic  $\mathcal{D}$ -modules of one variable, which should be well known to specialists.

(1.1) Let  $S$  denote the affine line  $\mathbf{C}$  with coordinate  $t$  (i.e.,  $S = \text{Spec } \mathbf{C}[t]$ ). Let  $S^* = S \setminus \{0\}$  with a natural inclusion  $j: S^* \rightarrow S$ . Let  $\mathcal{D}_S$  be the sheaf of algebraic differential operators on  $S$  [1], [5]. We denote by  $R$  the global sections of  $\mathcal{D}_S$ , which is the Weyl algebra  $\mathbf{C}[t, \partial_t]$ . Let  $M_{\text{coh}}(\mathcal{D}_S)$  be the category of coherent  $\mathcal{D}_S$ -modules, and  $M_{\text{fin}}(R)$  the category of finite  $R$ -modules. We have an equivalence of categories

$$M_{\text{coh}}(\mathcal{D}_S) = M_{\text{fin}}(R) \tag{1.1.1}$$

by the global section functor  $\Gamma(S, *)$ .

Let  $S^{\text{an}}$  denote the underlying complex analytic space of  $S$ . We have a functor

$$\text{An}: M_{\text{coh}}(\mathcal{D}_S) \rightarrow M_{\text{coh}}(\mathcal{D}_{S^{\text{an}}}) \tag{1.1.2}$$

by  $M \rightarrow M^{\text{an}} := \mathcal{O}_{S^{\text{an}}} \otimes_{\mathcal{O}_S} M$ , where the pull-back by the natural morphism  $S^{\text{an}} \rightarrow S$  is omitted. Then the de Rham functor  $\text{DR}_S$  is given by

$$\text{DR}_S(M) = \text{Cone}(\partial_t: M^{\text{an}} \rightarrow M^{\text{an}}) \tag{1.1.3}$$

using the coordinate  $t$  to trivialize  $\Omega_S^1$  (see (2.1.2) below).

(1.2) For  $M \in M_{\text{coh}}(\mathcal{D}_S)$ , let  $M(S) = \Gamma(S, M)$ , and

$$M(S)^\alpha = \bigcup_{i \geq 0} \text{Ker}((t\partial_t - \alpha)^i: M(S) \rightarrow M(S)) \quad \text{for } \alpha \in \mathbf{C}. \tag{1.2.1}$$

Then

$$tM(S)^\alpha \subset M(S)^{\alpha+1}, \quad \partial_t M(S)^\alpha \subset M(S)^{\alpha-1}, \tag{1.2.2}$$

and we have isomorphisms

$$t: M(S)^{\alpha-1} \xrightarrow{\sim} M(S)^\alpha, \quad \partial_t: M(S)^\alpha \xrightarrow{\sim} M(S)^{\alpha-1} \quad \text{for } \alpha \neq 0. \tag{1.2.3}$$

In fact,  $t\partial_t$  is bijective on  $M(S)^\alpha$  for  $\alpha \neq 0$ , because  $t\partial_t = \alpha$  on  $\text{Gr}_i^K M(S)^\alpha$  with  $K_i M(S)^\alpha = \text{Ker}(t\partial_t - \alpha)^{i+1}$  (similarly for  $\partial_t t$ ).

(1.3) DEFINITION. We say that  $M \in M_{\text{coh}}(\mathcal{D}_S)$  is *monodromical* if  $M$  is generated by  $M(S)^\alpha$  ( $\alpha \in \mathbf{C}$ ) over  $\mathcal{D}_S$ . Let  $M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$  denote the full subcategory of  $M_{\text{coh}}(\mathcal{D}_S)$  consisting of monodromical  $\mathcal{D}_S$ -modules. Then  $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$  is called *meromorphic* (resp. *microlocal*) *type* if the action of  $t$  (resp.  $\partial_t$ ) on  $M(S)$  is bijective.

REMARK. The condition of monodromical  $\mathcal{D}_S$ -module is equivalent to that any element of  $M(S)$  is annihilated by a polynomial of  $t\partial_t$ . So it is stable by extensions in  $M_{\text{coh}}(\mathcal{D}_S)$ .

(1.4) LEMMA. For  $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$ , we have a natural isomorphism

$$\bigoplus_{\alpha \in \mathbf{C}} M(S)^\alpha \xrightarrow{\sim} M(S), \tag{1.4.1}$$

and  $M(S)^\alpha$  is finite dimensional over  $\mathbf{C}$ . In particular, the functor  $M \rightarrow M(S)^\alpha$  is exact.

*Proof.* The injectivity of (1.4.1) is clear using the action of  $t\partial_t$  on  $M(S)$ . Since the condition of monodromical  $\mathcal{D}_S$ -module is equivalent to the surjectivity of

$$\bigoplus_{\alpha \in \mathbf{C}} \mathcal{D}_S \otimes_{\mathbf{C}} M(S)^\alpha \rightarrow M, \tag{1.4.2}$$

the surjectivity of (1.4.1) follows from (1.2.3), taking the global section of (1.4.2). We have  $\dim_{\mathbf{C}} M(S)^\alpha < \infty$ , because  $M(S)^\alpha$  is finitely generated over  $\mathbf{C}[N]$  with  $N = -(t\partial_t - \alpha)$ .

REMARK. By (1.2.3) and (1.4.1),  $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$  is meromorphic (resp.

microlocal) type if and only if

$$t: M(S)^{-1} \rightarrow M(S)^0 \text{ (resp. } \partial_t: M(S)^0 \rightarrow M(S)^{-1} \text{)} \tag{1.4.3}$$

is bijective.

(1.5) LEMMA. *Let  $M \in M_{\text{coh}}(\mathcal{D}_S)$  such that  $\text{supp } M \subset \{0\}$ . Then  $M$  is monodromical, and  $M(S)^\alpha = 0$  except for negative integers  $\alpha$ .*

*Proof.* The assumption is equivalent to that any element of  $M(S)$  is annihilated by a sufficiently high power of  $t$ . Then we can check the assertion using  $\partial_t^i t^i = \prod_{0 < j \leq i} (t\partial_t + j)$ .

REMARK. For  $M$  as above,  $M$  is a finite direct sum of  $\mathcal{B}$  in the proof of (1.8) by (1.2.3). This is a special case of Kashiwara’s equivalence (see [1]).

(1.6) LEMMA. *Let  $\Lambda$  be a subset of  $\mathbf{C}$  such that  $0 \in \Lambda$  and the natural morphism  $\Lambda \rightarrow \mathbf{C}/\mathbf{Z}$  is bijective. Let  $\Lambda' = \Lambda \cup \{-1\}$ . Let  $\mathcal{C}$  be the category whose object is a family of  $\mathbf{C}$ -vector spaces  $V^\alpha (\alpha \in \Lambda')$  with morphisms  $u: V^0 \rightarrow V^{-1}$ ,  $v: V^{-1} \rightarrow V^0$ , and  $N: V^\alpha \rightarrow V^\alpha (\alpha \in \Lambda \setminus \{0\})$  such that  $\bigoplus_{\alpha \in \Lambda'} V^\alpha$  is finite dimensional, and  $vu$ ,  $uv$  and  $N$  are nilpotent. Then we have an equivalence of categories*

$$M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}} = \mathcal{C} \tag{1.6.1}$$

by associating  $M(S)^\alpha$ ,  $\partial_t$ ,  $t$  and  $t\partial_t - \alpha$  to  $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$ .

*Proof.* This follows from (1.2.2–3) and (1.4.1).

(1.7) COROLLARY. *We have an equivalence of categories*

$$M_{\text{coh}}(\mathcal{D}_S)_{\text{mon},*} \text{ (resp. } M_{\text{coh}}(\mathcal{D}_S)_{\text{mon},!} \text{)} = V(\mathbf{C}, T), \tag{1.7.1}$$

where the left-hand side is the category of monodromical  $\mathcal{D}_S$ -modules of meromorphic (resp. microlocal) type, the right-hand side is the category of finite dimensional  $\mathbf{C}$ -vector spaces with an automorphism  $T$ , and the functor is defined by  $M \rightarrow \bigoplus_{\alpha \in \Lambda} M(S)^\alpha$  with  $T = \exp(-2\pi i t \partial_t)$ .

REMARK. Using (1.6), we can show that the category  $M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$  is equivalent to the category of regular holonomic  $\mathcal{D}_{S^{\text{an}},0}$ -modules (for which an equivalence of categories similar to (1.6.1) holds). The terms ‘meromorphic’ and ‘microlocal’ are originally used in this case (see [8]).

(1.8) PROPOSITION. *Let  $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$ . Then  $M$  is regular holonomic [1].*

*Proof.* Since the action of  $t\partial_t - \alpha$  on  $M(S)^\alpha$  is nilpotent, we may assume  $\sum_{\alpha \in \Lambda} \dim M(S)^\alpha = 1$  by (1.6), taking the graduation of a finite filtration on  $M$  (because regular holonomic  $\mathcal{D}$ -modules are stable by extensions [loc. cit.]). Then we can check that  $M$  is isomorphic to one of the following:

- (i)  $\mathcal{O}_S = \mathcal{D}_S/\mathcal{D}_S\partial_t$ ,
- (ii)  $\mathcal{B} := \mathcal{D}_S/\mathcal{D}_St$ ,
- (iii)  $M(\alpha) := \mathcal{D}_S/\mathcal{D}_S(t\partial_t - \alpha)$  ( $\alpha \in \Lambda \setminus \{0\}$ ),

depending on the  $\alpha$  such that  $M(S)^\alpha \neq 0$ . So we get the assertion.

REMARK. We can show that a regular holonomic  $\mathcal{D}_S$ -module is monodromical, if and only if its restriction to  $S^*$  is finite over  $\mathcal{O}_{S^*}$  (i.e., a vector bundle with connection [2]). In fact, we may assume that  $M|_{S^*}$  is a vector bundle by (1.10) below. Then the assertion is reduced to case where the action of  $t$  on  $M$  is bijective using the localization of  $M$  by  $t$  (because  $M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$  is stable by extensions in  $M_{\text{coh}}(\mathcal{D}_S)$ , see Remark after (1.3)). Then the assertion follows [2] (see also (1.11) below).

(1.9) PROPOSITION. For  $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$ , there exists uniquely  $M' \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$  of meromorphic (resp. microlocal) type with a morphism  $M \rightarrow M'$  (resp.  $M' \rightarrow M$ ) inducing an isomorphism on  $S^*$ .

Proof. By (1.6) there exists uniquely  $M' \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$  with a morphism  $M \rightarrow M'$  (resp.  $M' \rightarrow M$ ), such that

$$\begin{aligned} M'(S)^\alpha &= M(S)^\alpha \quad \text{for } -\alpha \notin \mathbf{N} \setminus \{0\}, \\ \partial_t^i: M(S)^0 &\xrightarrow{\sim} M(S)^{-i} \quad (\text{resp. } t^i: M(S)^{-i} \xrightarrow{\sim} M(S)^0) \end{aligned} \tag{1.9.1}$$

for  $i > 0$ . Then the morphism induces an isomorphism on  $S^*$  by (1.5).

REMARK. In the standard notation (see [1]),  $M'$  is denoted by  $j_*j^*M$  (resp.  $j_!j^*M$ ). Here  $j_*$  is really the direct image as Zariski sheaf (because  $M'$  is the localization of  $M$  by  $t$ ), but  $j_!$  is not. In fact,  $j_!$  is defined by  $\mathbf{D}j_*\mathbf{D}$  with  $\mathbf{D}$  the dual functor (see [loc. cit.]). We have

$$\mathbf{D}R_S(j_*j^*M) = Rj_*j^*\mathbf{D}R_S(M) \quad (\text{cf. [2]}), \tag{1.9.2}$$

$$\mathbf{D}R_S(j_!j^*M) = j_!j^*\mathbf{D}R_S(M) \quad (\text{cf. [1]}). \tag{1.9.3}$$

See also (1.12) below for (1.9.3).

(1.10) COROLLARY. For  $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$ , the restriction of  $M$  to  $S^*$  is a free  $\mathcal{O}_{S^*}$ -module of rank  $\sum_{\alpha \in \Lambda} \dim M(S)^\alpha$ . In particular,  $M^{\text{an}}|_{S^*}$  is a vector bundle with connection [2], and  $\mathbf{D}R_S(M)[-1]|_{S^*}$  is a local system.

Proof. It is enough to show the first assertion. We may assume  $M$  meromorphic type by (1.9). Then  $M(S)$  is a free  $\mathbf{C}[t, t^{-1}]$ -module of rank  $\sum_{\alpha \in \Lambda} \dim M(S)^\alpha$  by (1.2.3) and (1.4.3), and the assertion follows.

(1.11) PROPOSITION. Let  $L$  be a local system on  $S^*$  with complex coefficients,  $L_\infty$  the group of multivalued sections of  $L$  with the monodromy  $T$ , and  $L_\infty^\alpha$  the

$\exp(-2\pi i\alpha)$ -eigenspace of  $L_\infty$  with respect to  $T_s$ , where  $T = T_s T_u$  is the Jordan decomposition. Then there exists uniquely  $M \in M_{\text{coh}(\mathcal{D}_S)_{\text{mon}}}$  of meromorphic (resp. microlocal) type with an isomorphism

$$L = \text{DR}_S(M)[-1]_{|_{S^*}}, \tag{1.11.1}$$

where  $\text{DR}_S$  is as in (1.1.3). Furthermore, we have a canonical isomorphism

$$M(S)^\alpha = L_\infty^\alpha \tag{1.11.2}$$

for  $\alpha \in \Lambda$ , such that  $-(t\partial_t - \alpha)$  corresponds to  $N := (\log T_u)/2\pi i$ .

*Proof.* By (1.9) it is enough to show the assertion for  $M$  meromorphic type. By (1.7), there exists uniquely  $M \in M_{\text{coh}(\mathcal{D}_S)_{\text{mon}}}$  of meromorphic type with the isomorphism (1.11.2). Then  $M^{\text{an}}$  is identified with a  $\mathcal{O}_S[t^{-1}]$ -submodule of  $j_{\star}(\mathcal{O}_{S^*} \otimes_{\mathbb{C}} L)$  generated by

$$t^\alpha \exp(-(\log t)N)u \tag{1.11.3}$$

for  $u \in L_\infty^\alpha$  with  $\alpha \in \Lambda$  (see [2]), because (1.11.3) satisfies the same relation as the element of  $M(S)^\alpha$  corresponding to  $u \in L_\infty^\alpha$  by definition of  $M$  (and  $M^{\text{an}}|_{S^*}$  and  $\mathcal{O}_{S^*} \otimes_{\mathbb{C}} L$  have the same rank). In particular,  $M^{\text{an}}|_{S^*} = \mathcal{O}_{S^*} \otimes_{\mathbb{C}} L$ , and the assertion follows.

**REMARK.** This proposition shows that we have an equivalence of categories between the category of monodromical  $\mathcal{D}_S$ -modules of meromorphic type and the category of local systems on  $S^*$ , in a compatible way with (1.7.1). Note that the isomorphism (1.11.2) depends on the choice of the branch of  $\log t$  (i.e., the choice of a lift of 1 to a universal covering of  $S^*$ ).

(1.12) **PROPOSITION.** *If  $M \in M_{\text{coh}(\mathcal{D}_S)_{\text{mon}}}$  is microlocal type, we have*

$$(\text{DR}_S(M))_0 = 0, \tag{1.12.1}$$

$$R\Gamma(S^{\text{an}}, \text{DR}_S(M)) = 0. \tag{1.12.2}$$

*Proof.* By (1.10),  $\text{DR}_S(M)[-1]_{|_{S^*}}$  is a local system, and it is enough to show (1.12.1). Using a filtration defined in the category of monodromical  $\mathcal{D}_S$ -modules of microlocal type, we may assume  $\sum_{\alpha \in \Lambda} \dim M(S)^\alpha = 1$ . Then it is isomorphic to  $M(\alpha) = \mathcal{D}_S/\mathcal{D}_S(t\partial_t - \alpha)$  if  $M(S)^\alpha = \mathbb{C}$  for  $\alpha \in \Lambda \setminus \{0\}$  (see the proof of (1.8)). In the other case, we can check that  $M$  is isomorphic to  $\mathcal{D}_S/\mathcal{D}_S(t\partial_t)$ . Then we can check the assertion (see also [8]).

## 2. Algebraic Gauss-Manin system

(2.1) Let  $f: X \rightarrow Y$  be a morphism of smooth complex algebraic varieties. The direct image  $\int_f M$  of a  $\mathcal{D}_X$ -module  $M$  is defined by

$$\int_f M = Rf_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^L M) \quad \text{with} \quad \mathcal{D}_{Y \leftarrow X} = \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} (\omega_Y)^\vee), \tag{2.1.1}$$

where  $\omega_X$  is the dualizing sheaf, and  $\vee$  denotes the dual line bundle. See [1], [6], [10], etc. Note that, if  $f$  is an open embedding,  $\int_f M$  is defined by the sheaf theoretic direct image. In the case  $Y$  is the affine line  $S$ , the direct image  $\int_f M$  will be more explicitly expressed later (see (2.6)).

We denote the cohomological direct image  $\mathcal{H}^p \int_f M$  by  $\int_f^p M$ . For  $M = \mathcal{O}_X$ , the direct image  $\int_f \mathcal{O}_X$  (or  $\int_f^p \mathcal{O}_X$ ) is called the *Gauss-Manin system* of  $f$  [7].

For a  $\mathcal{D}_X$ -module  $M$ , let

$$DR_X(M) = (\Omega_X^*(M))^{\text{an}}[\dim X], \tag{2.1.2}$$

where  $\Omega_X^*(M)$  denotes the de Rham complex as in [1], [2], and  $\text{an}$  is defined as in (1.1.2). By [1] we have:

(2.2) **PROPOSITION.** *If  $M$  is regular holonomic, the cohomological direct images  $\int_f^p M$  are regular holonomic, and we have a natural isomorphism*

$$DR_Y \left( \int_f M \right) = Rf_*(DR_X(M)). \tag{2.2.1}$$

(2.3) **COROLLARY.** *Assume  $f$  smooth with relative dimension  $r$ , and  $\int_f^p \mathcal{O}_X$  is a locally free  $\mathcal{O}_Y$ -module of finite rank (i.e., a vector bundle with integrable connection [2]). Let  $L^p$  be the local system defined by the horizontal sections of  $(\int_f^p \mathcal{O}_X)^{\text{an}}$ . Then we have natural isomorphisms*

$$DR_Y \left( \int_f^p \mathcal{O}_X \right) = L^p[\dim Y], \tag{2.3.1}$$

$$L^p = R^{p+r}f_* \mathbf{C}_{X^{\text{an}}}. \tag{2.3.2}$$

*Proof.* The first assertion follows from the Poincaré lemma. Then the second follows from (2.2.1) using  $DR_X(\mathcal{O}_X) = \mathbf{C}_{X^{\text{an}}}[\dim X]$ , because  $\mathcal{O}_X$  is regular holonomic by definition [1].



REMARK. If we assume that  $R^{p+r}f_*\mathbf{C}_{X^{\text{an}}}$  is a local system, the corollary follows also from the relative version of [4] using a desingularization of the divisor at infinity of a compactification of  $f$ , because we may replace  $Y$  by its Zariski-open subset.

(2.4) LEMMA. *Let  $f: X \rightarrow Y$  be as in (2.1), and assume  $Y$  is the affine line  $S$  with coordinate  $t$  as in Section 1 so that  $f$  is identified with a function on  $X$ . Let  $\theta = \partial_t - \text{id}$ , and*

$$K_f = \text{Cone} \left( \theta: \int_f \mathcal{O}_X \rightarrow \int_f \mathcal{O}_X \right). \tag{2.4.1}$$

Then we have a natural isomorphism

$$(K_f)^{\text{an}} = \text{DR}_S \left( \int_f \mathcal{O}_X \right), \tag{2.4.2}$$

where  $\text{an}$  is defined as in (1.1.2).

*Proof.* This follows from  $\partial_t - 1 = e^t \partial_t e^{-t}$  in  $\mathcal{D}_S^{\text{an}}$ .

(2.5) PROPOSITION. *For  $f: X \rightarrow S$  as above, assume  $X^{\text{an}}$  contractible and purely  $n$ -dimensional. Then we have a natural isomorphism*

$$R\Gamma(S^{\text{an}}, (K_f)^{\text{an}}) = \mathbf{C}[n]. \tag{2.5.1}$$

*Proof.* This follows from (2.2) and (2.4).

REMARK. We can apply (2.2) also to the direct image of  $\mathcal{O}_X$  by  $X \rightarrow pt$  and the direct image of  $\int_f^p \mathcal{O}_X$  by  $S \rightarrow pt$ . In this case, (2.2) means the commutativity of the direct image with the functor  $\text{An}$  in (1.1.2), and follows also from [4] and [2] (see also (1.9.2) above) respectively.

(2.6) Let  $f: X \rightarrow S$  be as in (2.4), and  $M$  a  $\mathcal{D}_X$ -module. We define a structure of  $\mathcal{D}_X$ -module on  $M \otimes_{\mathbf{C}} \mathbf{C}[\partial_t]$  by

$$g(u \otimes \partial_t^i) = gu \otimes \partial_t^i, \quad \zeta(u \otimes \partial_t^i) = \zeta u \otimes \partial_t^i - (\zeta f)u \otimes \partial_t^{i+1} \tag{2.6.1}$$

for  $g \in \mathcal{O}_X$ ,  $\zeta \in \Theta_X$  and  $u \in M$ . It has also the action of  $R = \mathbf{C}[t, \partial_t]$  (see (1.1)) by

$$\partial_t(u \otimes \partial_t^i) = u \otimes \partial_t^{i+1}, \quad t(u \otimes \partial_t^i) = fu \otimes \partial_t^i - iu \otimes \partial_t^{i-1}, \tag{2.6.2}$$

which commutes with the action of  $\mathcal{D}_X$ . Then  $M \otimes_{\mathbf{C}} \mathbf{C}[\partial_t]$  is identified with the direct image of  $M$  by the embedding  $i_f$  by the graph of  $f$ , and  $u \otimes \partial_t^i$  is identified

with  $\partial_t^i \delta(t - f) \otimes u$ . Here  $\delta(t - f)$  is the delta function with support  $\{f = t\}$ , and satisfies the relation

$$t\delta(t - f) = f\delta(t - f), \quad \zeta\delta(t - f) = -(\zeta f)\partial_t \delta(t - f), \tag{2.6.3}$$

which gives (2.6.1–2).

Since the direct image of a  $\mathcal{D}$ -module by a smooth projection with fiber  $X$  is given by the sheaf theoretic direct image of the relative de Rham complex shifted by  $\dim X$  (see for example [1]), the direct image  $\int_f M$  is expressed as

$$\int_f M = Rf_* (\Omega_X(M \otimes_{\mathbb{C}} \mathbb{C}[\partial_t])[\dim X]), \tag{2.6.4}$$

factorizing  $f$  into the closed embedding  $i_f$  and the projection. This can be also obtained by using induced  $\mathcal{D}$ -modules [9]. Note that, if  $f$  is an affine morphism, the derived direct image  $Rf_*$  can be replaced by  $f_*$ .

(2.7) PROPOSITION. For  $f: X \rightarrow S$  as above, assume  $X = \mathbb{C}^n$ . Then we have a natural quasi-isomorphism (0.4) in the introduction.

Proof. By (2.6.4),  $\int_f \mathcal{O}_X$  is expressed by  $f_*(\Omega_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t])[n]$ , where the differential of  $\Omega_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$  is given by

$$\omega \otimes \partial_t^i \rightarrow d\omega \otimes \partial_t^i - df \wedge \omega \otimes \partial_t^{i+1}. \tag{2.7.1}$$

See (2.6.1). Then we have a short exact sequence of complexes

$$0 \rightarrow \Omega_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t] \xrightarrow{d} \Omega_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t] \rightarrow \Omega_X \rightarrow 0, \tag{2.7.2}$$

where the differential of  $\Omega_X$  is  $D_f$ . Then we get the assertion taking the exact functors  $f_*$  and  $\Gamma(S, *)$ .

(2.8) PROPOSITION. Let  $\bar{S} = \mathbb{P}^1$  with a natural inclusion  $j: S \rightarrow \bar{S}$ . Let  $M$  be a regular holonomic  $\mathcal{D}_S$ -module, and  $K = \text{Cone}(\theta: M \rightarrow M)$  for  $\theta$  as above. Let  $U$  be a Zariski-open subset of  $S$  on which  $M$  is locally free over  $\mathcal{O}_U$ , and denote the local system  $\text{DR}_S(M)[-1]|_{U^{\text{an}}}$  by  $L$ . Then we have a canonical isomorphism

$$H_{\{t>0\}}^0((j_* K)^{\text{an}}) = L_t \tag{2.8.1}$$

for  $t \in U^{\text{an}} (\subset \mathbb{C})$  such that  $\text{Im } t = 0$  and  $\text{Re } t \gg 0$ . Furthermore,

$$H_{\{t>0\}}^i((j_* K)^{\text{an}}) = 0 \quad \text{for } i \neq 0. \tag{2.8.2}$$

Proof. Let  $V = \{t \in \mathbb{C} : |t| > R\} \subset U^{\text{an}}$  for  $R$  sufficiently large, and  $V' =$

$V \cup \{\infty\}$ . Then  $(j'_*M)^{\text{an}}|_V$  is an extension of  $M^{\text{an}}|_V$  as regular holonomic  $\mathcal{D}_V$ -module such that the action of the local coordinate  $s (= t^{-1})$  is bijective, and such an extension is unique by [2]. So  $H^i_{\{\infty\}}((j'_*K)^{\text{an}})$  is uniquely determined by  $M^{\text{an}}|_V$ , or equivalently, by  $L|_V$  (see [loc. cit.]). Since  $V$  is homotopy equivalent to  $(S^*)^{\text{an}}$ , we may assume  $U = S^*$ , and  $M$  is a monodromical  $\mathcal{D}_S$ -module of microlocal type by (1.11). Then

$$H^i(S^{\text{an}}, K^{\text{an}}) = H^i(S^{\text{an}}, \text{DR}_S(M)) \tag{2.8.3}$$

by the same argument as the proof of (2.4). So it is zero for any  $i$  by (1.12.2), and we may replace  $H^i_{\{\infty\}}((j'_*K)^{\text{an}})$  in (2.8.1–2) by  $H^i(\bar{S}^{\text{an}}, (j'_*K)^{\text{an}})$  using the long exact sequence:

$$\rightarrow H^i_{\{\infty\}}((j'_*K)^{\text{an}}) \rightarrow H^i(\bar{S}^{\text{an}}, (j'_*K)^{\text{an}}) \rightarrow H^i(S^{\text{an}}, K^{\text{an}}) \rightarrow \tag{2.8.4}$$

By GAGA, we have

$$H^i(\bar{S}^{\text{an}}, (j'_*K)^{\text{an}}) = H^i(\bar{S}, j'_*K) = H^i(S, K), \tag{2.8.5}$$

where the last isomorphism follows from the exactness of  $j'_*$ . Moreover,

$$R\Gamma(S, K) = \text{Cone}(\theta: M(S) \rightarrow M(S)) = \bigoplus_{\alpha \in \Lambda} M(S)^\alpha \tag{2.8.6}$$

using (1.2.3) and (1.4.3). So the assertion follows from the isomorphism (1.11.2). Here we identify  $L_t$  with  $L_\infty$  by taking a lift of  $t$  to a universal covering of  $S^*$ , at which  $\log t$  is real valued.

(2.9) *Proof of Theorem (0.3).* Let  $K_f$  be as in (2.4), and  $j': S \rightarrow \bar{S}$  as above. By GAGA and (2.7), we have

$$H^i(\bar{S}^{\text{an}}, (j'_*K_f)^{\text{an}}) = H^i(\bar{S}, j'_*K_f) = H^i(S, K_f) = H^{i+n}(\Omega, D_f). \tag{2.9.1}$$

We have the long exact sequence (2.8.4) with  $K$  replaced by  $K_f$ . Let  $F = f^{-1}(t)$  for  $t$  as in (2.8.1). Then it is enough to show a canonical isomorphism

$$H^i_{\{\infty\}}((j'_*K_f)^{\text{an}}) = H^{i+n-1}(F, \mathbb{C}) \tag{2.9.2}$$

by (2.5.1), because we can check  $H^0(\Omega, D_f) = 0$  so that the morphism

$$\mathbb{C} = H^{-n}(S^{\text{an}}, (K_f)^{\text{an}}) \rightarrow H^1_{\{\infty\}}((j'_*K_f)^{\text{an}}) \tag{2.9.3}$$

is injective. Then, applying (2.8) to  $M = \int_f^p \mathcal{O}_X$ , the assertion (2.9.2) follows from (2.3).

(2.10) REMARKS. (i) If  $f$  is weighted homogeneous,  $\int_f^p \mathcal{O}_X$  ( $p \neq 1 - n$ ) and  $\int_f^{1-n} \mathcal{O}_X/\mathcal{O}_S$  are monodromical  $\mathcal{D}_S$ -modules of microlocal type, and the decomposition (1.4.1) by the action of  $t\partial_t$  is induced by the grading of  $\Omega'$  compatible with  $f$  as in [3]. This implies that the isomorphism (0.2) is compatible with the action of monodromy as in [loc. cit.].

(ii) In Theorem A of [3],  $\delta$  does not induce an isomorphism for  $k=0$ . The definition of  $D_f$  should be replaced by (0.1) in this paper, which is denoted by  $\bar{D}_f$  in [loc. cit.].

(iii) In the proof of (1.8) of [3], it is better to use Coim  $\Delta$  instead of  $\bar{\Omega} = \text{Ker } \Delta$ .

(2.11) EXAMPLE:  $f = x^2y + x$ . This is a generalized weighted homogeneous polynomial admitting *negative* weights, and the spectral sequence as in [3] does not converge. In fact, the  $E_0$ -complex is isomorphic to the Koszul complex  $(\Omega', df \wedge)$ , and is acyclic, but the general fiber  $F = f^{-1}(t)$  is isomorphic to  $\mathbf{C}^*$  so that  $\tilde{H}^0(F, \mathbf{C}) = 0$ ,  $\tilde{H}^1(F, \mathbf{C}) = \mathbf{C}$ . We can check  $H^2(\Omega', D_f) = \mathbf{C}$  as follows.

We have  $f_x = 2xy + 1$ ,  $f_y = x^2$ , and

$$D_f(-x^i y^j dx) = jx^i y^{j-1} dx dy - x^{i+2} y^j dx dy$$

$$D_f(x^i y^j dy) = ix^{i-1} y^j dx dy - 2x^{i+1} y^{j+1} dx dy - x^i y^j dx dy.$$

Let  $\phi: \Omega^2 \rightarrow \mathbf{C}$  be a map defined by  $\phi(x^i y^j dx dy) = (-1)^i j! / (2j - i + 1)!$  for  $2j + 1 \geq i$ , and 0 otherwise. Then  $\phi$  induces the isomorphism  $H^2(\Omega', D_f) = \mathbf{C}$ .

**Added in the proof.** We are informed that a similar result is obtained by B. Malgrange and P. Deligne independently using the theory of Fourier transformation.

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