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KĘSTUTIS IVINSKIS

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A variational Torelli theorem for cyclic coverings of high degree

KĘSTUTIS IVINSKIS

Universität - GH Essen, Fachbereich 6, Mathematik, Universitätsstr. 3, D-4300 Essen 1, Germany

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Introduction

The theory of the Infinitesimal Variation of Hodge Structure (abbreviated as IVHS) developed by Carlson and Griffiths in [CG] together with the method of symmetrizers introduced by Donagi [Do] and refined by Green [Gre] gives a powerful tool for attacking the generic Torelli problem. By a result of Cox, Donagi and Tu [CDT] the generic Torelli problem can be reduced to the variational Torelli problem. In this article we show using the methods of Donagi and Green a variational Torelli theorem for smooth cyclic coverings of sufficiently high degree with ample branchlocus. We arrive at this result by using a new way of recovering the variety in question from its IVHS.

This paper contains parts of the author's 1990 dissertation [Iv] at the University of Bonn. My advisers were F. Hirzebruch and E. Viehweg, who also posed me the problem treated here. It is a pleasure to thank both of them for their invaluable advices.

1. Notations and discussion of the main results

All varieties and schemes are supposed to be defined over the field $\mathbb{C}$ of complex numbers. The notation used by us will essentially be the same as in [Ha]. As a standard reference for the theory of IVHS we refer to [CGGH] and [PS].

DEFINITION 1.1. A pair $(Y, \mathscr{L})$ is called a polarized variety of dimension n, if Y is a smooth, projective variety of dimension n and $\mathscr{L}$ is an ample, invertible sheaf on Y. Two polarized varieties $(Y, \mathscr{L})$ and $(Y', \mathscr{L}')$ are isomorphic, if there is an isomorphism $\tau: Y \to Y'$ such that the equivalence class $[\mathscr{L}]$ of $\mathscr{L}$ in the Néron-Severi group $NS(Y) = DIV(Y)/(\text{numeric equivalence})$ is the same as the class $[\tau^*\mathscr{L}']$.

DEFINITION 1.2. A morphism $g: \mathscr{Y} \to S$ is called a smooth family of polarized varieties of dimension n with respect to a relatively ample sheaf $\mathscr{L}_{\mathscr{Y}}$ on $\mathscr{Y}$, if $\mathscr{Y}$ and S are connected varieties and the morphism g is smooth, proper and has connected smooth fibres $\mathcal{Y}_s = g^{-1}(s)$ of dimension n. We regard the fibres $\mathcal{Y}_s$, $s \in S$, as varieties polarized by the restriction of $\mathcal{L}_{\mathcal{Y}}$ to $\mathcal{Y}_s$.

Let $g: \mathcal{Y} \to S$ be a smooth family of polarized varieties of dimension n with respect to the relatively ample sheaf $\mathcal{L}_{\mathcal{Y}}$ on $\mathcal{Y}$. We choose a basepoint $s \in S$. Let $Y = g^{-1}(s)$ be the fibre over s and $(H_{\mathbb{Z}}, F^{\cdot}, Q)$ the polarized Hodge structure induced by the kth primitive cohomology of $(Y, \mathcal{L}_{\mathcal{Y}|Y})$ with $H_{\mathbb{Z}} = H^k(Y, \mathbb{Z})/(\text{Torsion})$. The polarized variation of Hodge structure defined by the kth primitive cohomology of the fibres of this family induces a holomorphic period map $\varphi_\Gamma: S \to \Gamma\backslash D$, where D is the appropriate period domain and Γ is a discrete subgroup of the isometry group $G_{\mathbb{Z}} = \text{Aut}(H_{\mathbb{Z}}, Q)$ containing the monodromy group $\text{im}\{\pi_1(S, s_0) \to G_{\mathbb{Z}}\}$.

We denote by $\approx$ the equivalence relation on S given by isomorphy of polarized varieties. For $\varphi_\Gamma(s_1) = \varphi_\Gamma(s_2)$ with $s_1, s_2 \in S$ one would like to conclude that the fibres $\mathcal{Y}_{s_1}$ and $\mathcal{Y}_{s_2}$ of g are isomorphic as polarized varieties. Hence the aim is the injectivity of the map

$$\tilde{\varphi}_\Gamma: S/\approx\ \to \Gamma\backslash D$$

induced by φ_Γ on the set theoretic quotient $S/\approx$. This is the global Torelli statement one would like to get for the family $g: \mathcal{Y} \to S$.

By using the differential of the period map this problem can be attacked with the technique of IVHS as proposed by Carlson and Griffiths in [CG]. But the application of this method has a weaker result: one gets the global Torelli statement just for an open dense subset of S.

By definition generic Torelli for the family $g: \mathcal{Y} \to S$ and the group Γ asserts that on the complement $S_0 \subset S$ of a proper analytic subvariety the map $S_0/\approx\ \to \Gamma\backslash D$ induced by $\tilde{\varphi}_\Gamma$ is injective. The open set $S_0 \subset S$ can be chosen in such a way that S_0 and $S_0/\approx$ are smooth varieties, the natural map $\pi_0: S_0 \to S_0/\approx$ is a smooth morphism and the map $S_0/\approx\ \to \Gamma\backslash D$ inherits the structure of a holomorphic map ([CDT, Thm. in §2 and §3]).

DEFINITION 1.3. Variational Torelli holds for the family $g: \mathcal{Y} \to S$ with respect to the equivalence relation $\approx$, if the following condition is true:

> On the complement $S_0 \subset S$ of a proper analytic subvariety of S for all $s \in S_0$ the isomorphy class of the polarized variety $(\mathcal{Y}_s, \mathcal{L}_{\mathcal{Y}|\mathcal{Y}_s})$ is uniquely determined by the algebraic part of the IVHS induced by $(\mathcal{Y}_s, \mathcal{L}_{\mathcal{Y}|\mathcal{Y}_s})$.

By a result of Cox, Donagi and Tu [CDT, Thm. in §3] it suffices to check the variational Torelli condition for concluding the generic Torelli for the family $g: \mathcal{Y} \to S$. Hence the following implication holds

Variational Torelli $\Rightarrow$ Generic Torelli.

Now we fix a smooth projective variety Y of dimension n with an ample invertible sheaf $\mathscr{L}$. Let $N \geqslant 2$ be a natural number and $s \in H^0(Y, \mathscr{L}^N)$ a section with smooth, reduced zero divisor $X = \{s = 0\}$.

The section s defines by the injection $\mathscr{L}^{-N} \xrightarrow{\cdot s} \mathcal{O}_Y$ an $\mathcal{O}_Y$-algebra structure on $\bigoplus_{i=0}^{N-1} \mathscr{L}^{-i}$. The smooth variety

$$Z = \mathrm{Spec}_{\mathcal{O}_Y}\left(\bigoplus_{i=0}^{N-1} \mathscr{L}^{-i}\right)$$

together with the natural morphism $f: Z \to Y$ is called a cyclic covering of Y with respect to $\mathscr{L}^N \simeq \mathcal{O}_Y(X)$.

The variety Z has two natural polarizations: on one hand there is the canonical polarization and on the other hand there is the polarization given by $f^*\mathscr{L}$. We will work with the second one. In the case of multicanonical coverings, i.e. $\mathscr{L} = \omega_Y$, both polarizations coincide by the adjunction formula $\omega_Z = f^*(\omega_Y \otimes \mathscr{L}^{N-1})$.

In Theorem 6.1 we state the following variational Torelli result.

MAIN RESULT. Let $(Y, \mathscr{L})$ be a polarized variety and N be a sufficiently large positive number. Then for each cyclic covering $f: Z \to Y$ with respect to $\mathscr{L}^N = \mathcal{O}_Y(X)$, where X is a smooth, reduced divisor on Y, the polarized variety $(Z, f^*\mathscr{L})$ is uniquely determined by the algebraic part of its IVHS.

Beside the technical assumption that the degree of the covering has to be large enough, the main requirement is that we take into consideration IVHS'es together with a fixed operation of the Galois group, where the isomorphisms of the IVHS are compatible with this operation.

An important ingredient, which we use for recovering the variety Z from its IVHS in the proof of Theorem 6.1, is the $I_{(1,1)}$-criterion (see Proposition 5.1).

2. Prolongation bundles and primitive De Rham cohomology

It is well known that prolongation bundles play an important role in the theory of IVHS (see e.g. [Gre]). The usefulness of prolongation bundles in this context relies on a relation between primitive cohomology and prolongation bundles proved by Ogus [Og]. Fix a smooth projective variety Y. The interpretation of the primitive cohomology group $H^p_0(Y, \Omega^q_Y)$ by a cohomology group depending on a prolongation bundle (see (2.3.1)) gives a way of handling the primitive cohomology groups of Y in the framework of IVHS.

Since the relation with this result in [Og] seems not to be documented in the existing literature on IVHS, we will recall in this chapter Ogus' way of looking at primitive cohomology in the setup needed by us.

Let $\mathscr{I}_\Delta$ be the ideal sheaf of the diagonal Δ of $Y \times Y$ and $Y_{(2)}$ the subscheme of $Y \times Y$ defined by the ideal sheaf $\mathscr{I}_\Delta^2$. For a locally free invertible sheaf $\mathscr{L}$ on Y one defines the sheaf $P(\mathscr{L})$ by

$$P(\mathscr{L}) = p_{1*}(p_2^*\mathscr{L}_{|Y_{(2)}}).$$

Here p_1 and p_2 denote the projections p_i: $Y \times Y \to Y$. By [At] there is a natural short exact sequence of $\mathscr{O}_Y$-modules

$$0 \to \Omega_Y^1 \otimes \mathscr{L} \to P(\mathscr{L}) \to \mathscr{L} \to 0. \tag{2.1.1}$$

Hence $P(\mathscr{L})$ is a locally free sheaf of rank $n+1$ on Y. The previous sequence induces by dualizing and tensoring with $\mathscr{L}$ the short exact sequence

$$0 \to \mathscr{O}_Y \to \Sigma_Y(\mathscr{L}) := P(\mathscr{L})^\vee \otimes_{\mathscr{O}_Y} \mathscr{L} \to T_Y \to 0. \tag{2.1.2}$$

The $\mathscr{O}_Y$-module $\Sigma_Y(\mathscr{L})$ is called the prolongation bundle of $\mathscr{L}$. For simplicity we will denote $\Sigma_Y(\mathscr{L})$ also by Σ_Y. By [At] the extension class to the sequence (2.1.2) for $\mathscr{L}$ is given by the element $-2\pi i \cdot c_1(\mathscr{L}) \in H^1(Y, \Omega_Y^1)$ and hence for $k \in \mathbb{Z} - \{0\}$ the $\mathscr{O}_Y$-modules $\Sigma_Y(\mathscr{L})$ and $\Sigma_Y(\mathscr{L}^k)$ are canonically isomorphic.

Let $s \in H^0(Y, \mathscr{L})$ be a section with smooth and reduced zero divisor $X = \{s = 0\}$ and $U = Y - X$ the open complement of X with the natural embedding $j: U \to Y$. By defining $f \mapsto f \otimes 1 \otimes s$ for a local section f in $\mathscr{O}_Y$ one gets an $\mathscr{O}_Y$-linear short exact sequence

$$0 \to \mathscr{O}_Y \xrightarrow{\cdot 1 \otimes 1 \otimes s} P(\mathscr{L}) \to \Omega_Y^1\langle X\rangle \otimes \mathscr{L} \to 0. \tag{2.1.3}$$

The sequence (2.1.3) induces by dualizing and tensoring with $\mathscr{L}$ the short exact sequence

$$0 \to T_Y\langle -X\rangle \to \Sigma_Y \to \mathscr{L} \to 0. \tag{2.1.4}$$

DEFINITION 2.2 ([Gre]). For a coherent sheaf $\mathscr{F}$ on Y and a global section s of $\mathscr{L}$ the Jacobi system $J_{Y,\mathscr{F}}$ of $\mathscr{F}$ is defined by

$$J_{Y,\mathscr{F}} = \operatorname{im}\{H^0(Y, \Sigma_Y \otimes \mathscr{L}^{-1} \otimes \mathscr{F}) \to H^0(Y, \mathscr{F})\}.$$

The map $H^0(Y, \Sigma_Y \otimes \mathscr{L}^{-1} \otimes \mathscr{F}) \to H^0(Y, \mathscr{F})$ is induced by the sequence (2.1.4). For simplicity we will denote $J_{Y,\mathscr{F}}$ also by $J_{\mathscr{F}}$.

We define $P_X(\mathscr{L})$ to be locally free $\mathscr{O}_Y$-submodule of the sheaf $j_*P(\mathscr{L})$, which is generated by $P(\mathscr{L})$ and elements of the form $\dfrac{ds_\alpha}{s_\alpha} \otimes T_\alpha$, where s_α is a local equation for X and T_α is the fibre coordinate of $\mathscr{L}$. The $\mathcal{O}_Y$-module $(P_X(\mathscr{L}) \otimes \mathscr{L}^{-1})^\vee$ is denoted by $\Sigma_Y\langle X\rangle$. By [EV2, Appendix B, Prop. B.1] the natural exact sequence

$$0 \to \Omega_Y^1\langle X\rangle \to \Sigma_Y^\vee\langle X\rangle \to \mathcal{O}_Y \to 0 \tag{2.2.1}$$

splits. This splitting will be used in Lemma 3.2.

We recall now Ogus' interpretation of the primitive cohomology of a smooth, projective variety by means of the primitive De Rham complex. Let $\mathscr{L}$ be an invertible ample sheaf on Y. As the Hodge filtration on the cohomology groups $H^i(Y, \mathbb{C})$ can be given by the *filtration bête* of the De Rham complex $(\Omega_Y^\bullet, d)$ one gets by [Og] an analogous statement for the primitive cohomology with respect to $\mathscr{L}$. Let

$$L_0 = \operatorname{Spec}_{\mathcal{O}_Y}\left(\bigoplus_{l=-\infty}^{\infty} \mathscr{L}^{-1}\right)$$

with the natural morphism $\pi: L_0 \to Y$ be the $\mathbf{A}^1$-bundle one gets by deleting the zero section from the total space of $\mathscr{L}$. Let $(\Omega_{L_0}^\bullet, d_{L_0})$ be the De Rham complex on L_0. The subcomplex of terms of degree 0 of the complex $(\pi_*\Omega_{L_0}^\bullet, \pi_* d_{L_0})$ is called the *primitive De Rham complex* of Y with respect to the sheaf $\mathscr{L}$. It can be identified with the complex $(\wedge^\bullet \Sigma_Y^\vee, d)$, where the differential d is induced by the differential of the primitive De Rham complex.

For a complex of sheaves $\mathscr{K}^\bullet$ on Y let

$$\mathbb{H}^i(Y, \mathscr{K}^\bullet)$$

be the ith hypercohomology group of $\mathscr{K}^\bullet$ on Y and $\mathscr{K}^\bullet[r]$ the complex of sheaves shifted r places to the left, i.e. at the ith place of the complex $\mathscr{K}^\bullet[r]$ one puts $\mathscr{K}^{i+r}$. One denotes by $\{F^p\mathscr{K}^\bullet\}$ the *filtration bête* of the complex $\mathscr{K}^\bullet$, i.e. at the qth place of the complex $F^p\mathscr{K}^\bullet$ one puts $\mathscr{K}^q$ for $q \geqslant p$ and elsewhere 0.

The following sequence of complexes is exact

$$0 \to \Omega_Y^\bullet \xrightarrow{\sigma} \wedge^\bullet \Sigma_Y^\vee \xrightarrow{\rho} \Omega_Y^\bullet[-1] \to 0. \tag{2.2.2}$$

Here $\Omega_Y^\bullet$ is the De Rham complex on Y with differential d, the complex $\Omega_Y^\bullet[-1]$ is the De Rham complex shifted to the right with differential $-d$ and $\wedge^\bullet \Sigma_Y^\vee$ is the primitive De Rham complex. The maps σ and ρ are given by the maps σ_p and ρ_p in the following exact sequence

$$0 \to \Omega_Y^p \xrightarrow{\sigma_p} \wedge^p \Sigma_Y^\vee \xrightarrow{\rho_p} \Omega_Y^{p-1} \to 0 \tag{2.2.3}$$

of wedge products induced by the prolongation bundle sequence (2.1.2).

For the fibre bundle $\pi: L_0 \to Y$ the exact sequence of complexes (2.2.2) induces the Gysin sequence

$$\cdots \to \mathbb{H}^{i-2}(Y, \Omega_Y^\cdot) \to \mathbb{H}^i(Y, \Omega_Y^\cdot) \to \mathbb{H}^i(Y, \wedge^\cdot \Sigma_Y^\vee) \to \mathbb{H}^{i-1}(Y, \Omega_Y^\cdot) \to \cdots,$$

where the connecting homomorphism $\mathbb{H}^{i-2}(Y, \Omega_Y^\cdot) \to \mathbb{H}^i(Y, \Omega_Y^\cdot)$ is given by cup-product with a multiple of the class $c_1(\mathscr{L}) \in H^1(Y, \Omega_Y^1)$. By the Hard Lefschetz theorem one gets for $0 \leqslant i \leqslant n$ the following short exact sequence

$$0 \to \mathbb{H}^{i-2}(Y, \Omega_Y^\cdot) \to \mathbb{H}^i(Y, \Omega_Y^\cdot) \to \mathbb{H}^i(Y, \wedge^\cdot \Sigma_Y^\vee) \to 0. \tag{2.2.4}$$

For $0 \leqslant k \leqslant n$ the primitive cohomology group

$$\ker \left\{ H^k(Y, \mathbb{C}) \xrightarrow{\cup c_1(\mathscr{L})^{n-k+1}} H^{2n-k+2}(Y, \mathbb{C}) \right\}$$

of $H^k(Y, \mathbb{C})$ is denoted by $H_0^k(Y, \mathbb{C})$. Furthermore one defines for $0 \leqslant p + q \leqslant n$ the primitive cohomology group $H_0^p(Y, \Omega_Y^q)$ by

$$H_0^p(Y, \Omega_Y^q) = H_0^k(Y, \mathbb{C}) \cap H^p(Y, \Omega_Y^q).$$

By the Poincaré lemma there is a natural quasi-isomorphism of complexes $\mathbb{C}_Y \to \Omega_Y^\cdot$ and therefore we identify the cohomology groups $H^i(Y, \mathbb{C})$ and $\mathbb{H}^i(Y, \Omega_Y^\cdot)$.

PROPOSITION 2.3. *For $0 \leqslant k \leqslant n$ the sequence* (2.2.4) *induces a natural isomorphism*

$$H_0^k(Y, \mathbb{C}) \xrightarrow{\simeq} \mathbb{H}^k(Y, \wedge^\cdot \Sigma_Y^\vee).$$

This isomorphism is compatible with the Hodge decomposition of $H_0^k(Y, \mathbb{C})$ and hence induces for p, q with $p + q = k$ the following isomorphisms

$$H_0^p(Y, \Omega_Y^q) \xrightarrow{\simeq} H^p(Y, \wedge^q \Sigma_Y^\vee). \tag{2.3.1}$$

The spectral sequence

$$E_1^{p,q} = H^q(Y, \wedge^p \Sigma_Y^\vee) \Rightarrow H^{p+q}(Y, \wedge^\cdot \Sigma_Y^\vee) \xrightarrow{\simeq} H_0^{p+q}(Y, \mathbb{C})$$

induced by the filtration bête *degenerates in E_1.*

Proof. For a proof see [Og, Theorem 1.9]. □

3. Hodge theory of cyclic coverings

Let Y be a smooth, projective variety of dimension n and $\mathscr{L}$ an invertible sheaf on Y. Let $f: Z \to Y$ be a cyclic covering with respect to $\mathscr{L}^N \simeq \mathcal{O}_Y(X)$, where X is a smooth reduced divisor on Y. We denote the divisor $(f^*X)_{\mathrm{red}}$ by X'. The Galois group of Z over Y is a cyclic group of degree N. We may choose a generator σ of G and a primitive root of unity ζ, such that σ is operating on the direct summand $\mathscr{L}^{-i}$ of $f_*\mathcal{O}_Z$ as multiplication by ζ^i. If $f_*\mathscr{F}$ is locally free for a sheaf $\mathscr{F}$ on Z and the group G operates on $f_*\mathscr{F}$, we denote by $(f_*\mathscr{F})_i$ the eigenspace for ζ^i.

Pushdown of sheaves

A standard way of interpreting a cohomology group $H^i(Z, \mathscr{F})$ for a sheaf $\mathscr{F}$ on Z is by application of the Leray spectral sequence. Since for finite morphisms we have $R^if_*\mathscr{F} = 0$ for $i > 0$, one gets natural isomorphisms

$$H^i(Z, \mathscr{F}) \simeq H^i(Y, f_*\mathscr{F})$$

for $i > 0$. For later use we apply the functor f_* to some standard sheaves on Z. By [EV1] one gets the following lemma.

LEMMA 3.1. *The following formulas hold for $p \geqslant 1$:*

1. $f_*\mathcal{O}_Z = \bigoplus_{i=0}^{N-1} \mathscr{L}^{-i}$

2. $f_*\Omega_Z^p = \Omega_Y^p \oplus \left(\bigoplus_{i=1}^{N-1} \Omega_Y^p\langle X\rangle \otimes \mathscr{L}^{-i} \right)$

3. $f_*\Omega_Z^p\langle X'\rangle = \bigoplus_{i=0}^{N-1} \Omega_Y^p\langle X\rangle \otimes \mathscr{L}^{-i}$

4. $f_* \wedge^p T_Z = \wedge^p T_Y \otimes \mathscr{L}^{-N+1} \oplus \left(\bigoplus_{i=1}^{N-1} \wedge^p T_Y\langle -X\rangle \otimes \mathscr{L}^{-N+1+i} \right)$

5. $f_* \wedge^p T_Z\langle -X'\rangle = \bigoplus_{i=0}^{N-1} \wedge^p T_Y\langle -X\rangle \otimes \mathscr{L}^{-N+1+i}$

Since we are interested in the primitive cohomology of Z with respect to $f^*\mathscr{L}$, we examine the application of the functor f_* to the sheaves $\wedge^p\Sigma_Z$ and $\wedge^p\Sigma_Z^\vee$ with $\Sigma_Z = P(f^*\mathscr{L})^\vee \otimes f^*\mathscr{L}$. The sheaf $\Sigma_Y = P(\mathscr{L})^\vee \otimes \mathscr{L}$ on Y is denoted by Σ_Y. The following lemma shows that the application of the functor f_* to the sheaves $\wedge^p\Sigma_Z$ and $\wedge^p\Sigma_Z^\vee$ behaves very well.

LEMMA 3.2. *The following formulas hold for* $p \geqslant 1$:

1. $f_* \wedge^p \Sigma_Z^\vee = \wedge^p \Sigma_Y^\vee \otimes \left(\bigoplus_{i=1}^{N-1} (\Omega_Y^p\langle X \rangle \otimes \mathscr{L}^{-i} \oplus \Omega_Y^{p-1}\langle X \rangle \otimes \mathscr{L}^{-i}) \right)$

2. $f_* \wedge^p \Sigma_Z = \wedge^p \Sigma_Y \otimes \mathscr{L}^{-N+1}$

$$\oplus \left(\bigoplus_{i=1}^{N-1} (\wedge^p T_Y\langle -X \rangle \otimes \mathscr{L}^{-N+1+i} \oplus \wedge^{p-1} T_Y\langle -X \rangle \otimes \mathscr{L}^{-N+1+i}) \right)$$

Proof. By application of wedge products to the exact splitting sequence (2.2.1) we get for $p \geqslant 1$ the exact splitting sequence

$$0 \to \Omega_Y^p\langle X \rangle \to \wedge^p \Sigma_Y^\vee \langle X \rangle \to \Omega_Y^{p-1}\langle X \rangle \to 0. \tag{3.2.2}$$

Since by [Viel, Lemma 1.6] the relation $f^*\Omega_Y^p\langle X \rangle = \Omega_Z^p\langle X' \rangle$ holds, one gets the equality $f^* \wedge^p \Sigma_Y^\vee\langle X \rangle = \wedge^p \Sigma_Z^\vee\langle X' \rangle$ by application of the functor f^* to the sequence (3.2.2). For $p \geqslant 1$ this implies

$$f_* \wedge^p \Sigma_Z^\vee\langle X' \rangle = (\wedge^p \Sigma_Y^\vee\langle X \rangle) \otimes \left(\bigoplus_{i=0}^{N-1} \mathscr{L}^{-i} \right).$$

It can be checked simply that for $p = 1, \ldots, n+1$ the cokernel of the natural embedding $\wedge^p \Sigma_Z^\vee \to \wedge p \Sigma_Z^\vee\langle X' \rangle$ is isomorphic to the $\mathcal{O}_{X'}$-module

$$\wedge^{p-1} \Sigma_{X'}^\vee = \wedge^{p-1}(P(f^*\mathscr{L}_{|X'}) \otimes_{\mathcal{O}_{X'}} f^*\mathscr{L}_{|X'}^{-1}).$$

This gives the exact sequence

$$0 \to \wedge^p \Sigma_Z^\vee \to \wedge^p \Sigma_Z^\vee\langle X' \rangle \to \wedge^{p-1} \Sigma_{X'}^\vee \to 0. \tag{3.2.3}$$

By applying f_* to (3.2.3) one gets the exact sequence

$$0 \to f_* \wedge^p \Sigma_Z^\vee \xrightarrow{\alpha} (\wedge^p \Sigma_Y^\vee\langle X \rangle) \otimes \left(\bigoplus_{i=0}^{N-1} \mathscr{L}^{-i} \right) \xrightarrow{\beta} \wedge^{p-1} \Sigma_X^\vee \to 0 \tag{3.2.4}$$

where $\Sigma_X^\vee = P(\mathscr{L}_{|X}) \otimes \mathscr{L}_{|X}^{-1}$. For a affine covering $\{U_\alpha\}$ of Y the sheaf $\Sigma_Z^\vee$ can be represented on $V_\alpha = f^{-1}(U_\alpha)$ as

$$\Sigma_{Z|V_\alpha}^\vee = \left(\bigoplus_{i=1}^{n} \mathcal{O}_{V_\alpha} dx_i \right) \oplus \mathcal{O}_{V_\alpha} \frac{dT_\alpha}{T_\alpha},$$

where T_α is a fibre coordinate of $\mathscr{L}$ on U_α and $x_1, \ldots, x_n$ are local coordinates of

Z on V_α. The natural operation of G on $\Sigma_Z^\vee$ is given by

$$\sigma\left(f\cdot dx_1\wedge\cdots\wedge dx_n + g\cdot\frac{dT_\alpha}{T_\alpha}\right) = \sigma f\cdot d(\sigma x_1)\wedge\cdots\wedge d(\sigma x_n) + \sigma g\cdot\frac{dT_\alpha}{T_\alpha}.$$

Therefore the maps in the sequence (3.2.4) are G-equivariant. Since $\wedge^{p-1}\Sigma_X^\vee$ is a G-invariant sheaf, we get

$$(\ker\beta)^G = \wedge^p\Sigma_Y^\vee.$$

The map α in (3.2.4) induces an isomorphism on the part, which is not G-invariant. Hence the first assertion follows. Now we prove the second assertion. An application of the duality theory for finite flat morphisms (vgl. [Ha, Kap. III, Ex. 6.10]) to the morphism f gives

$$f_*\wedge^p\Sigma_Z = \mathscr{HOM}_{\mathcal{O}_Y}(\mathscr{L}^{N-1}\otimes f_*\wedge^p\Sigma_Z^\vee, \mathcal{O}_Y).$$

By putting the first result into this equation the second assertion follows. □

IVHS of cyclic coverings

For the abstract definition of an IVHS of weight k we refer to [CGGH] and [PS].

Let $\mathscr{L}$ be an ample invertible sheaf on Y. To avoid different cases we suppose from now on for the sake of simplicity that the dimension n of Y is bigger than 1 (Case of curves is analogous). Now we examine the IVHS of weight k induced by the polarized variety $(Z, f^*\mathscr{L})$. The vectorspace $H_0^1(Z, T_Z)$ is defined to be the set of $v\in H^1(Z, T_Z)$ with the property $v\cup c_1(\mathscr{L}) = 0$ in $H^2(Z, \mathcal{O}_Z)$. Another interpretation is the following

$$H_0^1(Z, T_Z) = \operatorname{im}\{H^1(Z, \Sigma_Z)\xrightarrow{(*)} H^1(Z, T_Z)\}, \tag{3.2.5}$$

where the map $(*)$ is induced by the prolongation bundle sequence of the sheaf $f^*\mathscr{L}$. For $p = 1,\ldots,k$ the canonical map

$$H_0^1(Z, T_Z)\xrightarrow{\delta_p}\mathrm{HOM}_{\mathbb{C}}(H_0^{k-p}(Z, \Omega_Z^p), H_0^{k-p+1}(Z, \Omega_Z^{p-1}))$$

decomposes as a result of the eigenspace decompositions of the vectorspaces $H_0^1(Z, T_Z)$, $H_0^{k-p}(Z, \Omega_Z^p)$ and $H_0^{k-p+1}(Z, \Omega_Z^{p-1})$ into a direct sum

$$\delta_p = \bigoplus_{i=0}^{N-1}\delta_{p,i}$$

with

$$\delta_{p,i} = (\delta_{p,i,0}, \ldots, \delta_{p,i,N-1}),$$

where for $0 \leqslant i, j \leqslant N - 1$ the map $\delta_{p,i,j}$ is given by

$$T^{(i)} \xrightarrow{\delta_{p,i,j}} \mathrm{HOM}_{\mathbb{C}}(U_p^{(j)}, U_{p-1}^{(i+j-[i+j/N]\cdot N)}).$$

Here we use the following notation

$$T^{(i)} = (H_0^1(Z, T_Z))_i \quad \text{for } 0 \leqslant i \leqslant N - 1 \text{ and}$$

$$U_p^{(j)} = (H_0^{k-p}(Z, \Omega_Z^p))_j \quad \text{for } 0 \leqslant j \leqslant N - 1.$$

Now we define the notion of an IVHS-isomorphism of cyclic coverings. Let $(Y, \mathscr{L})$ and $(Y', \mathscr{L}')$ be two polarized varieties of dimension n. Let $f: Z \to Y$ and $f: Z' \to Y'$ be cyclic coverings with respect to $\mathscr{L}^N = \mathcal{O}_Y(X)$ and $\mathscr{L}'^N = \mathcal{O}_{Y'}(X')$. Here X is a smooth reduced divisor on Y and X' is a smooth reduced divisor on Y'. The IVHS of weight k induced by $(Z, f^*\mathscr{L})$ respectively $(Z', f^*\mathscr{L}')$ is denoted by

$$(F^{\cdot}, Q, T, \delta)$$

and

$$(F'^{\cdot}, Q', T', \delta'),$$

with

$$F^p = \bigoplus_{r \geqslant p} H_0^{k-r}(Z, \Omega_Z^r), \quad F'^p = \bigoplus_{r \geqslant p} H_0^{k-r}(Z', \Omega_{Z'}^r),$$

$$T = H_0^1(Z, T_Z) \quad \text{and} \quad T' = H_0^1(Z', T_{Z'}).$$

REMARK 3.3. To simplify notation we here just use the algebraic part of the IVHS and call this the IVHS. Recall that in some geometric situations a polarized variety can be reconstructed from the algebraic part of its IVHS. For such a situation Donagi introduced in [Do2] the notion of *Variational Torelli.*

DEFINITION 3.4. An IVHS-isomorphism

$$(F^{\cdot}, Q, T, \delta) \xrightarrow[\simeq]{\psi} (F'^{\cdot}, Q', T', \delta')$$

is called an IVHS-isomorphism of cyclic coverings, if for $p = 1, \ldots, k$ and $0 \leqslant i$,

$j \leqslant N-1$ the following diagram

$$\begin{array}{ccc} T^{(i)} & \xrightarrow{\delta_{p,i,j}} & \mathrm{HOM}_{\mathbb{C}}(U_p^{(j)}, U_{p-1}^{(i+j-[i+j/N]\cdot N)}) \\ \wr\Vert\downarrow & & \wr\Vert\downarrow \\ T'^{(i)} & \xrightarrow{\delta'_{p,i,j}} & \mathrm{HOM}_{\mathbb{C}}(U_p'^{(j)}, U_{p-1}'^{(i+j-[i+j/N]\cdot N)}) \end{array}$$

is commutative. In this diagram the vertical isomorphisms are induced by ψ.

Now we take a closer look to the maps $\delta_{p,i,j}$. The ampleness of the invertible sheaf $\mathscr{L}$ gives by [EV2, (2.8)]

$$H^{k-p}(Y, \Omega_Y^{p-1}\langle X\rangle \otimes \mathscr{L}^{-j}) = 0 \quad \text{for } j = 1, \ldots, N-1 \quad \text{and} \quad p = 0, \ldots, k.$$

By Lemma 3.1 and Lemma 3.2 one gets

$$T^{(i)} \simeq \begin{cases} H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{L}^{-i}) & \text{for } i = 0, \ldots, N-2 \\ H^1(Y, \Sigma_Y \otimes \mathscr{L}^{-N+1}) & \text{for } i = N-1 \end{cases} \tag{3.4.1}$$

and

$$(H_0^{k-p}(Z, \Omega_Z^p))_j \simeq \begin{cases} H^{k-p}(Y, \wedge^p \Sigma_Y^\vee) & \text{for } j = 0 \\ H^{k-p}(Y, \Omega^p\langle X\rangle \otimes \mathscr{L}^{-j}) & \text{for } j = 1, \ldots, N-1 \end{cases} \tag{3.4.2}$$

For $i = 0, \ldots, N-2$ the isomorphism (3.4.1) is induced by composition with the isomorphism

$$T^{(i)} \simeq (H^1(Y, f_*\Sigma_Z))_i \xrightarrow{\simeq} (H^1(Y, f_*T_Z))_i \simeq H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{L}^{-i})$$

and isomorphism (3.4.2) is induced by the isomorphism in Proposition 2.3.

For $p = 1, \ldots, k$, $0 \leqslant i \leqslant N-2$, $1 \leqslant j \leqslant N-1$ and $i+j \leqslant N-1$ we denote by $\tilde{\delta}_{p,i,j}$ the following map of cohomology groups on Y

$$H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{L}^{-i})$$
$$\xrightarrow{\tilde{\delta}_{p,i,j}} \mathrm{HOM}_{\mathbb{C}}(H^{k-p}(Y, \Omega_Y^p\langle X\rangle \otimes \mathscr{L}^{-j}), H^{k-p+1}(Y, \Omega_Y^{p-1}\langle X\rangle \otimes \mathscr{L}^{-i-j}))$$

induced by cupproduct and contraction of sheaves. The corresponding maps $\delta_{p,i,j}$ play an important role, because for $i+j < N-1$ the $\mathcal{O}_Y$-module structure of the pushdowns under f_* of the respective sheaves is easier to control (compare

[Kn, 3.2]). The compatibility of the Leray spectral sequence with cupproduct and contraction gives the following statement.

LEMMA 3.5. *For* $p=1,\dots,k$, $0\leqslant i\leqslant N-2$, $1\leqslant j\leqslant N-1$ *and* $i+j\leqslant N-1$ *the following diagram is commutative*

$$\begin{array}{ccc} T^{(i)} & \xrightarrow{\delta_{p,i,j}} & HOM_{\mathbb{C}}(U_p^{(j)}, U_{p-1}^{(i+j)}) \\ \wr\Vert\downarrow & & \wr\Vert\downarrow \\ H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{L}^{-i}) & \xrightarrow{\tilde{\delta}_{p,i,j}} & HOM_{\mathbb{C}}(H^{k-p}(Y, \Omega_Y^p\langle X\rangle \otimes \mathscr{L}^{-j}), \\ & & H^{k-p+1}(Y, \Omega_Y^{p-1}\langle X\rangle \otimes \mathscr{L}^{-i-j})) \end{array}$$

The vertical isomorphisms are induced by the isomorphisms in (3.4.1) *and* (3.4.2).

REMARK 3.6. The cohomology group $H^1(Y, \Sigma_Y \otimes \mathscr{L}^{-N+1})$ vanishes, if N is a sufficiently large number. Hence we get in this case the following equalities

$$H^1(Z, T_Z) = H^1_0(Z, T_Z) = \bigoplus_{i=0}^{N-2} H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{L}^{-N+1-i}).$$

Jacobisystems

By the ampleness of $\mathscr{L}$ the global vanishing theorem for integral parts of $\mathbb{Q}$-divisors [EV2, (2.8)] gives here $H^q(Y, \Omega_Y^p\langle X\rangle \otimes \mathscr{L}^{-i}) = 0$ for $1 \leqslant i \leqslant N-1$ and $p+q \neq n$. Hence we get an isomorphism $H^q(Y, \Omega_Y^p) \xrightarrow{\sim} H^q(Z, \Omega_Z^p)$ for $p+q \neq n$. The interesting part of the cohomology $H^*(Z, \mathbb{C})$ of Z is therefore in the middle part $H^n(Z, \mathbb{C})$. Now we are going to interpret the summand

$$H^p(Y, \Omega_Y^{n-p}\langle X\rangle \otimes \mathscr{L}^{-i})$$

of the Hodge structure $H^n(Z, \mathbb{C})/\mathrm{im}\, H^n(Y, \mathbb{C})$ as well as the summand

$$H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{L}^{-i})$$

of the cohomology group $H^1(Y, T_Z)$ with the help of Jacobi systems (see Definition 2.2). This allows at least for big N and certain i and j a good control of the maps $\tilde{\delta}_{p,i,j}$ and hence $\delta_{p,i,j}$ in Lemma 3.5.

PROPOSITION 3.7. *Let* $\mathscr{F}$ *and* $\mathscr{G}$ *be locally free sheaves on* Y. *There exist maps induced by the prolongation bundle sequence* (2.1.4)

$$\frac{H^0(Y, \omega_Y \otimes \mathscr{L}^{N(p+1)} \otimes \mathscr{G})}{J_{\omega_Y \otimes \mathscr{L}^{N(p+1)} \otimes \mathscr{G}}} \xrightarrow{\psi_p(\mathscr{G})} H^p(Y, \Omega_Y^{n-p}\langle X\rangle \otimes \mathscr{G}) \text{ for } p = 0,\dots,n$$

and

$$\frac{H^0(Y, \mathscr{L}^N \otimes \mathscr{F})}{J_{\mathscr{L}^N \otimes \mathscr{F}}} \xrightarrow{\phi(\mathscr{F})} H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{F}),$$

such that the following diagram is commutative for $p = 0, \ldots, n-1$:

$$\begin{tikzcd}
H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{F}) \otimes_{\mathbb{C}} H^p(Y, \Omega_Y^{n-p}\langle X\rangle \otimes \mathscr{G}) \arrow[r, "(*)"] & H^{p+1}(Y, \Omega_Y^{n-p-1}\langle X\rangle \otimes \mathscr{F} \otimes \mathscr{G}) \\
\dfrac{H^0(Y, \mathscr{L}^N \otimes \mathscr{F})}{J_{\mathscr{L}^N \otimes \mathscr{F}}} \otimes_{\mathbb{C}} \dfrac{H^0(Y, \omega_Y \otimes \mathscr{L}^{N(p+1)} \otimes \mathscr{G})}{J_{\omega_Y \otimes \mathscr{L}^{N(p+1)} \otimes \mathscr{G}}} \arrow[u, "\phi(\mathscr{F}) \otimes \psi_p(\mathscr{G})"] \arrow[r, "(**)"] & \dfrac{H^0(Y, \omega_Y \otimes \mathscr{L}^{N(p+2)} \otimes \mathscr{F} \otimes \mathscr{G})}{J_{\omega_Y \otimes \mathscr{L}^{N(p+2)} \otimes \mathscr{F} \otimes \mathscr{G}}} \arrow[u, "\psi_{p+1}(\mathscr{F} \otimes \mathscr{G})"']
\end{tikzcd}$$

The map $(*)$ *is induced by composition of cupproduct and the contraction map*

$$T_Y\langle -X\rangle \otimes \Omega_Y^{n-p}\langle X\rangle \to \Omega_Y^{n-p-1}\langle X\rangle.$$

The map $(**)$ *is induced by multiplication of global sections. The map* $\phi(\mathscr{F})$ *is injective.*

By supposing additional assumptions we get the following statements:

(1) *The condition* $H^s(Y, \wedge^{n-s}\Sigma_Y^\vee \otimes \mathscr{L}^{N(p-s)} \otimes \mathscr{G}) = 0$ *for* $s = 1, \ldots, p-1$ *implies the injectivity of* $\psi_p(\mathscr{G})$.

(2) *The condition* $H^{s+1}(Y, \wedge^{n-s}\Sigma_Y^\vee \otimes \mathscr{L}^{N(p-s)} \otimes \mathscr{G}) = 0$ *for* $s = 0, \ldots, p-1$ *implies the surjectivity of* $\psi_p(\mathscr{G})$.

(3) *If* $H^1(Y, \Sigma_Y \otimes \mathscr{F})$ *vanishes, then* $\phi(\mathscr{F})$ *is an isomorphism.*

Proof. For a locally free sheaf $\mathscr{G}$ on Y and $p \in \mathbb{N}$ with $0 \leqslant p \leqslant n$ we denote by

$$(K^\cdot, d_\cdot) \text{ with } K^i := (\wedge^{n-p+i+1}\Sigma_Y^\vee) \otimes \mathscr{L}^{N(i+1)} \otimes \mathscr{G} \text{ for } i \geqslant 0$$

the Koszul complex, which is induced by the surjection $\Sigma_Y \otimes \mathscr{L}^{-N} \twoheadrightarrow \mathcal{O}_Y$. The complex $K^\cdot$ is exact except at the first place, where we have $\ker d_0 = \Omega_Y^{n-p}\langle X\rangle \otimes \mathscr{G}$. Hence $\Omega_Y^{n-p}\langle X\rangle \otimes \mathscr{G}$ and $K^\cdot$ are quasiisomorphic as complexes. In particular we get $K^p = \omega_Y \otimes \mathscr{L}^{N(p+1)} \otimes \mathscr{G}$. We denote by α_p the composition of the following natural maps:

$$\begin{tikzcd}
H^0(Y, \omega_Y \otimes \mathscr{L}^{N(p+1)} \otimes \mathscr{G}) \arrow[r, "\alpha_p"] \arrow[d, "\wr\|"'] & H^p(Y, \Omega_Y^{n-p}\langle X\rangle \otimes \mathscr{G}) \\
\mathbb{H}^p(Y, K^p[-p]) \arrow[r, "\alpha"] & \mathbb{H}^p(Y, K^\cdot) \arrow[u, "\wr\|"']
\end{tikzcd}$$

The map α is induced by the natural inclusion of complexes $K^p[-p] = F^pK^\bullet \hookrightarrow K^\bullet$. For the kernel of α we have

$$\ker\alpha = \mathrm{im}\{\mathbb{H}^{p-1}(Y, K^\bullet/F^pK^\bullet) \xrightarrow{\beta} \mathbb{H}^p(Y, F^pK^\bullet)\}.$$

For the composition of β and the map

$$\mathbb{H}^{p-1}(Y, K^{p-1}[-p+1]) \xrightarrow{\ \gamma\ } \mathbb{H}^{p-1}(Y, K^\bullet/F^pK^\bullet)$$

induced by $K^{p-1}[-p+1] \hookrightarrow K^\bullet/F^pK^\bullet$ the relation $\mathrm{im}(\beta\cdot\gamma) \subset \ker\alpha$ holds. The map induced by α_p on

$$\frac{H^0(Y, \omega_Y \otimes \mathscr{L}^{N(p+1)} \otimes \mathscr{G})}{J_{\omega_Y \otimes \mathscr{L}^{N(p+1)} \otimes \mathscr{G}}} \xrightarrow{\ \simeq\ } \frac{\mathbb{H}^p(Y, K^p[-p])}{\mathrm{im}(\beta\cdot\gamma)}$$

is denoted $\psi_p(\mathscr{G})$.

Since the group $H^s(Y, K^{p-s-1})$ is zero for $s = 1, \ldots, p-1$, we get by the second spectral sequence for the hypercohomology an isomorphism between $H^0(Y, K^{p-1})$ and $\mathbb{H}^{p-1}(Y, K^\bullet/F^pK^\bullet)$. This implies the first assertion. Since the group $H^{s+1}(Y, K^{p-s-1})$ is zero for $s = 0, \ldots, p-1$, we get $\mathbb{H}^p(Y, K^\bullet/F^pK^\bullet) = 0$ and hence the second assertion follows. The map $\phi(\mathscr{F})$ is induced by the exact sequence (2.1.4), which immediately gives the third assertion. The commutativity of the diagram follows by a simple argument using the second spectral sequence of the hypercohomology as in [Iv, Proposition 5.7] (compare [Fl, §2]). $\square$

We remark that the vanishing conditions in Proposition 3.7 can be controlled with the help of the prolongation bundle sequence (2.1.2). The maps $\phi(\mathscr{F})$ and $\psi_p(\mathscr{G})$ where $\mathscr{F}$ and $\mathscr{G}$ are certain negative powers of the invertible sheaf $\mathscr{L}$ are of special interest.

DEFINITION 3.8. For $i = 0, \ldots, N-1$ the map $\psi_p(\mathscr{L}^{-i})$ is denoted by $\psi_{p,i}$ and for $k = 0, \ldots, N-2$ the map $\phi(\mathscr{L}^{-k})$ is denoted by ϕ_k.

To simplify in the situation of Definition 3.8 the control of the vanishing conditions in Proposition 3.7 we introduce the numbers $r_{Y,\mathscr{L}}$ and $s_{Y,\mathscr{L}}$.

DEFINITION 3.9. The natural numbers $r_{Y,\mathscr{L}}$ and $s_{Y,\mathscr{L}}$ are defined by

$$r_{Y,\mathscr{L}} = \min\{l \mid H^p(Y, \wedge^{n-p}\Sigma_Y^\vee \otimes \mathscr{L}^{l'}) = 0 \quad \text{and}$$
$$H^p(Y, \wedge^{n-p+1}\Sigma_Y^\vee \otimes \mathscr{L}^{l'}) = 0 \quad \text{for } l' \geqslant l > 0 \text{ and } p > 0\}$$

and

$$s_{Y,\mathscr{L}} = \min\{l \mid H^1(Y, T_Y \otimes \mathscr{L}^{-l'}) = 0 \text{ for } l' \geqslant l > 0\}.$$

The numbers $r_{Y,\mathscr{L}}$ and $s_{Y,\mathscr{L}}$ depend just on the polarized variety $(Y, \mathscr{L})$. As an immediate corollary of Proposition 3.7 one gets the following statement.

COROLLARY 3.10. *For $N \in \mathbb{N}$ and $i \in \{0, \dots, N-1\}$ it holds*

(1) *If $i \leqslant N - r_{Y,\mathscr{L}}$ holds, then $\psi_{p,i}$ is an isomorphism for $p = 0, \dots, n$.*
(2) *If $k \geqslant s_{Y,\mathscr{L}}$ holds, then ϕ_k is an isomorphism.*

DEFINITION 3.11. We denote by ω the invertible sheaf $\omega_Y^2 \otimes \mathscr{L}^{N(n+1)}$.

Now we state Green's generalized Macaulay-Duality [Gre, Thm. 2.15] in the form we are going to apply it. The sheaf ω has a similar role as the canonical sheaf in the Serre duality.

PROPOSITION 3.12 (Macaulay-Duality). *Let $\mathscr{E}$ be a locally free sheaf on Y and $a \in \mathbb{Z}$. If the following vanishing assumptions hold*

(1) $H^q(Y, \wedge^{q+1}\Sigma_Y \otimes \mathscr{E} \otimes \mathscr{L}^{a-N(q+1)}) = 0$ *for* $q = 1, \dots, n-1$,
(2) $H^q(Y, \wedge^{q}\Sigma_Y \otimes \mathscr{E} \otimes \mathscr{L}^{a-Nq}) = 0$ *for* $q = 1, \dots, n-1$,
(3) $H^q(Y, \wedge^{q+1}\Sigma_Y \otimes \mathscr{L}^{-N(q+1)}) = H^q(Y, \wedge^{q}\Sigma_Y \otimes \mathscr{L}^{-Nq}) = 0$
for $q = 1, \dots, n-1$,
(4) $H^0(Y, T_Y \otimes \mathscr{L}^{-N}) = 0$,

then

$$\frac{H^0(Y, \omega)}{J_\omega} \simeq \mathbb{C}$$

and the pairing

$$\frac{H^0(Y, \mathscr{E} \otimes \mathscr{L}^a)}{J_{\mathscr{E} \otimes \mathscr{L}^a}} \otimes_{\mathbb{C}} \frac{H^0(Y, \mathscr{E}^\vee \otimes \mathscr{L}^{-a} \otimes \omega)}{J_{\mathscr{E}^\vee \otimes \mathscr{L}^{-a}\omega}} \to \frac{H^0(Y, \omega)}{J_\omega} \simeq \mathbb{C}$$

induced by cupproduct is nondegenerate.

Proof. [Gre, Thm. 2.15]. □

REMARK 3.13. The fourth assumption in Proposition 3.12 can be skipped, because by [Wa, Thm. 1] it follows from $H^0(Y, T_Y \otimes \mathscr{L}^{-N}) \neq 0$ that $(Y, \mathscr{L}^N)$ must be isomorphic $(\mathbb{P}^n, \mathcal{O}(1))$ or to $(\mathbb{P}^1, \mathcal{O}(2))$. Hence in this case we must have $N = 1$ respectively $n = 1$ and this are uninteresting cases for us.

REMARK 3.14. Let $\mathscr{E}$ be a fixed sheaf on Y. If N is sufficiently large, then the vanishing conditions in Proposition 3.12 are fulfilled.

PROPOSITION 3.15. *Let N be a sufficiently large natural number. For $k \geqslant 0$ the following map is an isomorphism:*

$$\left(\frac{H^0(Y, \omega_Y^2 \otimes \mathscr{L}^{Nn+k})}{J_{\omega_Y^2 \otimes \mathscr{L}^{Nn+k}}}\right)^\vee \xrightarrow{\simeq} H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{L}^{-k}).$$

Proof. The proof is similar to the proof of Proposition 3.12. Here we take $\mathscr{E} = \omega_Y^2$ and $a = Nn + k$. □

4. Symmetrizer

For $\mathbb{C}$-vectorspaces U, V, W and a linear map $q: U \otimes_{\mathbb{C}} V \to W$ the $\mathbb{C}$-vectorspace

$$B(q) = \{T \in \mathrm{HOM}(U, V) \mid q(u_1 \otimes T(u_2)) = q(u_2 \otimes T(u_1)) \text{ for all } u_1, u_2 \in U\}$$

is called the symmetrizer space of the map q. The natural linear map

$$\begin{aligned} s(q): U \otimes_{\mathbb{C}} B(q) &\to V \\ \sum_i u_i \otimes T_i &\mapsto \sum_i T(u_i) \end{aligned}$$

is called the symmetrizer map of q or just the symmetrizer of q (see [Do]). Let Y be a smooth projective variety of dimension n and $\mathscr{L}$ an ample invertible sheaf on Y. We now state a theorem by Green [Gre, Thm. 2.21], which we apply to our situation in Proposition 4.2.

THEOREM 4.1 (Green's generalized symmetrizer lemma). *Let $\mathscr{E}$ be a locally free sheaf on Y and $\mathscr{M}$ a locally free invertible sheaf on Y generated by global sections. If N is a sufficiently large number then the complex*

$$\begin{aligned} &\wedge^2 H^0(Y, \mathscr{M}) \otimes_{\mathbb{C}} \frac{H^0(Y, \mathscr{M}^{-2} \otimes \mathscr{E}^\vee \otimes \mathscr{L}^{-N} \otimes \omega)}{J_{\mathscr{M}^{-2} \otimes \mathscr{E}^\vee \otimes \mathscr{L}^{-N} \otimes \omega}} \\ &\quad \to H^0(Y, \mathscr{M}) \otimes_{\mathbb{C}} \frac{H^0(Y, \mathscr{M}^{-1} \otimes \mathscr{E}^\vee \otimes \mathscr{L}^{-N} \otimes \omega)}{J_{\mathscr{M}^{-1} \otimes \mathscr{E}^\vee \otimes \mathscr{L}^{-N} \otimes \omega}} \\ &\quad \to \frac{H^0(Y, \mathscr{E}^\vee \otimes \mathscr{L}^{-N} \otimes \omega)}{J_{\mathscr{E}^\vee \otimes \mathscr{L}^{-N} \otimes \omega}} \to 0 \end{aligned}$$

induced by the Koszul complex for the evaluation map

$$H^0(Y, \mathscr{M}) \otimes \mathscr{O}_Y \to \mathscr{M}$$

is exact in the middle and right place.

Proof. This is the statement of [Gre, Thm. 2.21] in the dual form. □

Now we prove the symmetrizer result we need for the proof of our variational Torelli result in Theorem 6.1. For the definition of the number $r_{Y,\mathscr{L}}$ see Definition 3.9.

PROPOSITION 4.2. *Let k be a natural number with $k \geqslant r_{Y,\mathscr{L}}$ such that the following properties are fulfilled for $j \in \{0, 1\}$:*

(1) *The sheaf $\omega_Y \otimes \mathscr{L}^k$ is generated by global sections.*
(2) $H^q(Y, \wedge^q \Sigma_Y \otimes \omega_Y^{-1} \otimes \mathscr{L}^{-k-j}) = 0$ *for* $q = 1, \dots, n-1$.
(3) $H^0(Y, \Sigma_Y \otimes \omega_Y^{-1} \otimes \mathscr{L}^{-k-j}) = 0$.

Let N be a sufficiently large natural number and $s \in H^0(Y, \mathscr{L}^N)$ a section with smooth reduced zero divisor $X = \{s = 0\}$. Put $i = N - k$. For the natural map

$$H^0(Y, \Omega_Y^n\langle X\rangle \otimes \mathscr{L}^{-i}) \otimes_{\mathbb{C}} H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{L}^{-j}) \xrightarrow{\delta} H^1(Y, \Omega_Y^{n-1}\langle X\rangle \otimes \mathscr{L}^{-i-j})$$

induced by cupproduct and contraction the following statements hold

1. *The symmetrizer space $B(\delta)$ is canonically isomorphic to $H^0(Y, \omega_Y^{-1} \otimes \mathscr{L}^{i-j})$ and the symmetrizer map $\delta_1 = s(\delta)$ is*

$$H^0(Y, \omega_Y \otimes \mathscr{L}^{N-i}) \otimes_{\mathbb{C}} H^0(Y, \omega_Y^{-1} \otimes \mathscr{L}^{i-j}) \xrightarrow{\delta_1} H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{L}^{-j}).$$

2. *The symmetrizer space $B(\delta_1)$ is canonically isomorphic to $H^0(Y, \omega_Y^{-2} \otimes \mathscr{L}^{2i-N-j})$ and the symmetrizer map $\delta_2 = s(\delta_1)$ is*

$$H^0(Y, \omega_Y \otimes \mathscr{L}^{N-i}) \otimes_{\mathbb{C}} H^0(Y, \omega_Y^{-2} \otimes \mathscr{L}^{2i-N-j}) \xrightarrow{\delta_2} H^0(Y, \omega_Y^{-1} \otimes \mathscr{L}^{i-j})$$

Both maps δ_1 and δ_2 are given by multiplication of global sections, where δ_1 is additionally composed with the map $H^0(Y, \mathscr{L}^{N-j}) \to H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{L}^{-j})$ induced by the prolongation bundle sequence (2.1.4).

REMARK 4.3. The map δ in Proposition 4.2 corresponds to the map $\tilde{\delta}_{n,j,i}$ in Lemma 3.5. The map $\tilde{\delta}_{n,j,i}$ is the starting point for our proof of Theorem 6.1.

Proof. For the first assertion we have to show the exactness of the natural sequence

$$0 \to H^0(Y, \omega_Y^{-1} \otimes \mathscr{L}^{i-j}) \to H^0(Y, \omega_Y \otimes \mathscr{L}^{N-i})^\vee \otimes_{\mathbb{C}} H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{L}^{-j})$$
$$\xrightarrow{\psi} \wedge^2 H^0(Y, \omega_Y \otimes \mathscr{L}^{N-i})^\vee \otimes_{\mathbb{C}} H^1(Y, \Omega_Y^{n-1}\langle X\rangle \otimes \mathscr{L}^{-i-j}) \qquad (4.3.1)$$

at the left and middle place. Since we have by the third assumption

$$H^0(Y, \Sigma_Y \otimes \omega_Y^{-1} \otimes \mathscr{L}^{-k-j}) = 0,$$

the Jacobisystem $J_{\omega^{-1} \otimes \mathscr{L}^{i-j}}$ vanishes. Since N is sufficiently large, by application of Corollary 3.10 we have to show the exactness of the sequence

$$\begin{aligned} 0 &\to H^0(Y, \mathscr{E} \otimes \mathscr{L}^N) \to H^0(Y, \mathscr{M})^\vee \otimes_{\mathbb{C}} H^1(Y, T_Y\langle -X\rangle \otimes \mathscr{L}^{-j}) \\ &\to \wedge^2 H^0(Y, \mathscr{M})^\vee \otimes_{\mathbb{C}} \frac{H^0(Y, \mathscr{M}^2 \otimes \mathscr{E} \otimes \mathscr{L}^N)}{J_{\mathscr{M}^2 \otimes \mathscr{E} \otimes \mathscr{L}^N}} \end{aligned} \tag{4.3.2}$$

at the left and middle place. Here we have put

$$\mathscr{M} = \omega_Y \otimes \mathscr{L}^{N-i},\ \mathscr{E} = \omega_Y^{-1} \otimes \mathscr{L}^{-N+i-j} \quad \text{and} \quad J_{\mathscr{E} \otimes \mathscr{L}^N} = 0.$$

We also use $J_{\mathscr{E} \otimes \mathscr{L}^N} = 0$. By $N \gg 0$ and the second assumption the Macaulay-Duality (Proposition 3.12) holds for $H^0(Y, \mathscr{E} \otimes \mathscr{L}^N)$. The application of Proposition 3.12 and 3.15 gives that the sequence (4.3.2) is dual to the following sequence

$$\begin{aligned} &\wedge^2 H^0(Y, \mathscr{M}) \otimes_{\mathbb{C}} \frac{H^0(Y, \mathscr{M}^{-2} \otimes \mathscr{E}^\vee \otimes \mathscr{L}^{-N} \otimes \omega)}{J_{\mathscr{M}^{-2} \otimes \mathscr{E}^\vee \otimes \mathscr{L}^{-N} \otimes \omega}} \\ &\to H^0(Y, \mathscr{M}) \otimes_{\mathbb{C}} \frac{H^0(Y, \mathscr{M}^{-1} \otimes \mathscr{E}^\vee \otimes \mathscr{L}^{-N} \otimes \omega)}{J_{\mathscr{M}^{-1} \otimes \mathscr{E}^\vee \otimes \mathscr{L}^{-N} \otimes \omega}} \\ &\to \frac{H^0(Y, \mathscr{E}^\vee \otimes \mathscr{L}^{-N} \otimes \omega)}{J_{\mathscr{E}^\vee \otimes \mathscr{L}^{-N} \otimes \omega}} \to 0. \end{aligned} \tag{4.3.3}$$

The exactness of sequence (4.3.3) at the middle and right place follows from Proposition 4.1, because of $N \gg 0$ and the first assumption. Hence the first assertion is true. Since in the previous proof we firstly choose k and then N, we see that the sheaves $\mathscr{M} = \omega_Y \otimes \mathscr{L}^k$ and $\mathscr{E} = \omega_Y^{-1} \otimes \mathscr{L}^{-k-j}$ are not depending on N and therefore we can achieve the necessary vanishings in Proposition 4.1. The proof of the second assertion is similar. We have to show for the sequence

$$\begin{aligned} 0 &\to H^0(Y, \omega_Y^{-2} \otimes \mathscr{L}^{2i-N-j}) \\ &\to H^0(Y, \omega_Y \otimes \mathscr{L}^{N-i})^\vee \otimes_{\mathbb{C}} H^0(Y, \omega_Y^{-1} \otimes \mathscr{L}^{i-j}) \\ &\to \wedge^2 H^0(Y, \omega_Y \otimes \mathscr{L}^{N-i})^\vee \otimes_{\mathbb{C}} H^1(Y, T_Y\langle X\rangle \otimes \mathscr{L}^{-j}) \end{aligned} \tag{4.3.4}$$

the exactness of the dual sequence. The first and third assumption gives

$$H^0(Y, \Sigma_Y \otimes \omega_Y^{-2} \otimes \mathscr{L}^{-2k-j}) = 0.$$

Therefore we get $J_{\omega_Y^{-2} \otimes \mathscr{L}^{2i-N-j}} = 0$. Since N is sufficiently large, it is enough to show for the sequence

$$\begin{aligned}&0 \to H^0(Y, \mathscr{E} \otimes \mathscr{L}^N) \to H^0(Y, \mathscr{M})^\vee \otimes_{\mathbb{C}} H^0(Y, \mathscr{M} \otimes \mathscr{E} \otimes \mathscr{L}^N)\\ &\quad \to \wedge^2 H^0(Y, \mathscr{M})^\vee \otimes_{\mathbb{C}} H^1(Y, T_Y\langle -X \rangle \otimes \mathscr{L}^{-j})\end{aligned} \tag{4.3.5}$$

the exactness at the left and middle place. Here we have put $\mathscr{M} = \omega_Y \otimes \mathscr{L}^{N-i}$ and $\mathscr{E} = \omega_Y^{-2} \otimes \mathscr{L}^{2i-2N-j}$. The application of the Macaulay-Duality in Proposition 3.12 to the sequence (4.3.5) gives by the Propositions 3.12 and 3.15 that the sequence (4.3.5) is dual to (4.3.3). The first assumption and Proposition 4.1 implies the exactness of the sequence (4.3.3) at the middle and right place. Hence the second assertion follows. □

REMARK 4.4. The somewhat technical assumptions in Proposition 4.2 are caused by the fact that we have to choose both k and N sufficiently large. On the other hand k has to be small enough in comparison to N. The interpretation of the assumptions in Proposition 4.2 is that we have firstly to choose k sufficiently large and then N.

5. The $I_{(1,1)}$-criterion

In the theory of IVHS an important step in proving variational Torelli theorems consists in recovering the algebraic variety in question from certain bilinear maps. Here we give a criterion applicable in such a situation. It might also be of general interest.

Let Y be a smooth projective variety of dimension n and $\mathscr{F}$ and $\mathscr{G}$ two very ample invertible sheaves on Y. We use the following conventions:

$$V = H^0(Y, \mathscr{F}) \text{ with } r_1 = \dim V,$$

$$W = H^0(Y, \mathscr{G}) \text{ with } r_2 = \dim W,$$

$$\mathbb{P}_1 = \mathbb{P}(V),\ \mathbb{P}_2 = \mathbb{P}(W) \text{ and } \mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2.$$

Let p_1 and p_2 denote the projections $\mathbb{P} \to \mathbb{P}_1$ and $\mathbb{P} \to \mathbb{P}_2$. The linear systems $|\mathscr{F}|$ and $|\mathscr{G}|$ induce morphisms $\varphi_{|\mathscr{F}|}: Y \to \mathbb{P}_1$ and $\varphi_{|\mathscr{G}|}: Y \to \mathbb{P}_2$. We get the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\Delta"] \arrow[dr, "i"'] & Y \times Y \arrow[d, "\varphi_{|\mathscr{F}|} \times \varphi_{|\mathscr{G}|}"] \\ & \mathbb{P}. \end{tikzcd}$$

Here Δ denotes the diagonal embedding and i the composition of $\varphi_{|\mathscr{F}|} \times \varphi_{|\mathscr{G}|}$ and Δ. Let $I_{(1,1)}$ be the kernel of the natural map $V \otimes_{\mathbb{C}} W \to H^0(Y, \mathscr{F} \otimes \mathscr{G})$ and let $\mathscr{O}_{\mathbb{P}}(a, b)$ be the invertible sheaf $p_1^* \mathscr{O}_{\mathbb{P}^1}(a) \otimes p_2^* \mathscr{O}_{\mathbb{P}_2}(b)$. There exist natural isomorphisms $S^{k_1} V \simeq H^0(\mathbb{P}_1, \mathscr{O}_{\mathbb{P}_1}(k_1))$ and $S^{k_2} W \simeq H^0(\mathbb{P}_2, \mathscr{O}_{\mathbb{P}_2}(k_2))$. For this reason we identify $S^{k_1} V$ with $H^0(\mathbb{P}_1, \mathscr{O}_{\mathbb{P}_1}(k_1))$ and $S^{k_2} W$ with $H^0(\mathbb{P}_2, \mathscr{O}_{\mathbb{P}_2}(k_2))$. Now we state a sufficient condition for the fact that the variety Y as a subvariety of $\mathbb{P}$ is a scheme-theoretic intersection of the divisors $D \in I_{(1,1)}$. The ideal sheaf of Y in $\mathbb{P}$ is denoted by $\mathscr{T}_Y$.

PROPOSITION 5.1. ($I_{(1,1)}$-Criterion). *If $\mathscr{F}^{-n} \otimes \mathscr{G} \otimes \omega_Y^{-1}$ is nef and big, then the natural map*

$$H^0(\mathbb{P}, \mathscr{T}_Y(1, 1)) \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{P}} \to \mathscr{T}_Y(1, 1)$$

is surjective.

REMARK 5.2. The application of Proposition 5.1, we have in mind, is the following. Suppose $\mathscr{G}$ is *sufficiently ample* in comparison to $\mathscr{F}$. Then the embedding $i: Y \to \mathbb{P}$ can be reconstructed from V and W and the map $V \otimes_{\mathbb{C}} W \to H^0(Y, \mathscr{F} \otimes G)$, since the kernel of the multiplication map $V \otimes_{\mathbb{C}} W \to H^0(Y, \mathscr{F} \otimes \mathscr{G})$ is $I_{(1,1)} = H^0(\mathbb{P}, \mathscr{T}_Y(1, 1))$. Hence the surjective evaluation map $I_{(1,1)} \otimes \mathscr{O}_{\mathbb{P}}(-1, -1) \twoheadrightarrow \mathscr{T}_Y$ determines the ideal sheaf $\mathscr{T}_Y$ of Y.

We recall for later use that by Mumford's definition a coherent sheaf $\mathscr{E}$ on $\mathbb{P}^r$ is called m-regular, if $H^i(\mathbb{P}^r, \mathscr{E}(m - i)) = 0$ for all $i > 0$. By [Mu1] an m-regular sheaf $\mathscr{E}$ is also $(m + 1)$-regular and one has a surjective multiplication map

$$H^0(\mathbb{P}^r, \mathscr{E}(m)) \otimes_{\mathbb{C}} H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(1)) \to H^0(\mathbb{P}^r, \mathscr{E}(m + 1)).$$

Now we state some lemmas, we are going to use in the proof of Proposition 5.1.

LEMMA 5.3. $R^i p_{j_*} \mathscr{T}_Y(a, b) = 0$ *for* $a, b > 0, i > 0$ *and* $j = 1, 2$.

Proof. The natural surjection $\mathscr{O}_{\mathbb{P}}(a, b) \to \mathscr{F}^a \otimes \mathscr{G}^b$ induces the map

$$p_{1*} \mathscr{O}_{\mathbb{P}}(a, b) = \mathscr{O}_{\mathbb{P}_1}(a) \otimes_{\mathbb{C}} S^b W \to \mathscr{F}^a \otimes \mathscr{G}^b, \tag{5.3.1}$$

which factorizes over the surjections

$$\mathscr{O}_{\mathbb{P}_1}(a) \otimes_{\mathbb{C}} S^b W \to \mathscr{F}^a \otimes_{\mathbb{C}} S^b W$$

and

$$\mathscr{F}^a \otimes_{\mathbb{C}} S^b W \to \mathscr{F}^a \otimes \mathscr{G}^b.$$

Hence (5.3.1) is surjective. The assertion for $i = 1$ and $j = 1$ follows from

$$R^1 p_{1*}\mathcal{O}_{\mathbb{P}}(a, b) = \mathcal{O}_{\mathbb{P}_1}(a) \otimes_{\mathbb{C}} H^1(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(b)) = 0.$$

Since the restriction of p_1 to the diagonal is an isomorphism, we get $R^i p_{1*}(\mathscr{F}^a \otimes \mathscr{G}^b) = 0$ for $i > 1$. For $i > 1$ and $j = 1$ the assertion follows from

$$R^i p_{1*}(\mathscr{F}^a \otimes \mathscr{G}^b) = 0 \quad \text{and} \quad R^i p_{1*}(p_1^*\mathcal{O}_{\mathbb{P}_1}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}_2}(b)) = 0.$$

Finally for symmetry reasons the assertion is true for $j = 2$. □

LEMMA 5.4. *If $\mathscr{F}^{-n} \otimes \mathscr{G} \otimes \omega_Y^{-1}$ is nef and big, the natural map*

$$H^0(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(1, 1)) \otimes_{\mathbb{C}} S^{k_1} V \to H^0(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(k_1 + 1, 1))$$

is surjective for all $k_1 \geqslant 0$.

Proof. The first thing we show is the 0-regularity of the sheaf $p_{1*}\mathscr{T}_Y(1, 1)$. By Lemma 5.3 the following sequence is exact

$$0 \to p_{1*}\mathscr{T}_Y(1, 1) \to \mathcal{O}_{\mathbb{P}_1}(1) \otimes_{\mathbb{C}} W \to \mathscr{F} \otimes \mathscr{G} \to 0.$$

This implies for $i > 0$ an exact sequence

$$\begin{aligned} \cdots &\to H^{i-1}(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(1 - i)) \otimes_{\mathbb{C}} W \to H^{i-1}(Y, \mathscr{F}^{1-i} \otimes \mathscr{G}) \\ &\to H^i(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(1, 1)(-i)) \to H^i(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(1 - i)) \otimes_{\mathbb{C}} W \to \cdots. \end{aligned}$$

Since the map $H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}) \otimes_{\mathbb{C}} W \to H^0(Y, \mathscr{G})$ is an isomorphism and the cohomology group $H^1(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1})$ vanishes, we get $H^1(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(1, 1)(-1)) = 0$. Now we check the case $i > 1$. Since $\mathscr{F}^{-n} \otimes \mathscr{G} \otimes \omega_Y^{-1}$ is nef and big, the sheaf $\mathscr{F}^{1-i} \otimes \mathscr{G} \otimes \omega^{-1}$ is also nef and big for $i \leqslant n + 1$. By the Kawamata-Viehweg vanishing theorem [Vie1] and [Ka] we get $H^{i-1}(Y, \mathscr{F}^{1-i} \otimes \mathscr{G}) = 0$ for $i > 1$. As the group $H^i(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(1 - i))$ vanishes for $i > 1$, the 0-regularity of the sheaf $p_{1*}\mathscr{F}_Y(1, 1)$ follows. Hence the natural map

$$H^0(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(l, 1)) \otimes_{\mathbb{C}} V \to H^0(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(l + 1, 1)) \tag{5.4.1}$$

is surjective for $l > 0$. This implies by induction on k_1 the surjectivity of the map

$$H^0(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(1, 1)) \otimes_{\mathbb{C}} V^{\otimes k_1} \xrightarrow{(*)} H^0(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(k_1 + 1, 1))$$

for $k_1 > 0$. In the commutative diagram

$$\begin{tikzcd}
H^0(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(1,1)) \otimes_{\mathbb{C}} V^{\otimes k_1} \arrow[r, "(*)"] \arrow[d] & H^0(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(k_1+1,1)) \\
H^0(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(1,1)) \otimes_{\mathbb{C}} S^{k_1}V \arrow[ur, "(**)"'] &
\end{tikzcd}$$

the surjectivity of the map $(*)$ implies the surjectivity of $(**)$. This is the assertion. $\square$

Choosing k_0. We choose a number $k_0 \in \mathbb{N}$ such that the following three conditions hold:

(1) $k_0 \geqslant \min\{k \mid \mathscr{T}_Y(k', k'')$ is generated by global sections for $k', k'' \geqslant k\}$,
(2) the map $S^k H^0(Y, \mathscr{F}) \to H^0(Y, \mathscr{F}^k)$ is surjective for $k \geqslant k_0$,
(3) $\mathscr{F}^{k_0+1} \otimes \mathscr{G}^{-n} \otimes \omega_Y^{-1}$ is ample.

LEMMA 5.5. *For $k_1 > k_0$ the natural map*

$$H^0(\mathbb{P}_2, p_{2*}\mathscr{T}_Y(k_1+1, 1)) \otimes_{\mathbb{C}} S^{k_2}W \to H^0(\mathbb{P}_2, p_{2*}\mathscr{T}_Y(k_1+1, k_2+1))$$

is surjective for $k_2 \geqslant 0$.

Proof. The proof is analogous to the proof of Lemma 5.4. We show the 0-regularity of the sheaf $p_{2*}\mathscr{T}_Y(k_1+1, 1)$. Since by Lemma 5.3 the sheaf $R^1 p_{2*}\mathscr{T}_Y(k_1+1, 1)$ vanishes, we get a short exact sequence

$$0 \to p_{2*}\mathscr{T}_Y(k_1+1, 1) \to S^{k_1+1}V \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{P}_2}(1) \to \mathscr{F}^{k_1+1} \otimes \mathscr{G} \to 0.$$

This implies for $i > 0$ the exact sequence

$$\cdots \to S^{k_1+1}V \otimes_{\mathbb{C}} H^{i-1}(\mathbb{P}_2, \mathscr{O}_{\mathbb{P}_2}(1-i)) \to H^{i-1}(Y, \mathscr{F}^{k_1+1} \otimes \mathscr{G}^{1-i})$$
$$\to H^i(\mathbb{P}_2, p_{2*}\mathscr{T}_Y(k_1+1, 1)(-i)) \to S^{k_1+1}V \otimes_{\mathbb{C}} H^i(\mathbb{P}_2, \mathscr{O}_{\mathbb{P}_2}(1-i)) \to \cdots.$$

Since by the choice of k_0 the natural map

$$S^{k_1+1}V \otimes_{\mathbb{C}} H^0(\mathbb{P}_2, \mathscr{O}_{\mathbb{P}_2}) \to H^0(Y, \mathscr{F}^{k_1+1})$$

is surjective, this implies in connection with $H^1(\mathbb{P}_1, \mathscr{O}_{\mathbb{P}_1}) = 0$ the vanishing of the group $H^1(\mathbb{P}_2, p_{2*}\mathscr{T}_Y(k_1+1, 1)(-1))$. By Kodaira's vanishing theorem the ampleness of $\mathscr{F}^{k_1+1} \otimes \mathscr{G}^{-n} \otimes \omega_Y^{-1}$ implies $H^{i-1}(Y, \mathscr{F}^{k_1+1} \otimes \mathscr{G}^{1-i}) = 0$ for $1 < i \leqslant n$. Since the group $H^i(\mathbb{P}_2, \mathscr{O}_{\mathbb{P}_2}(1-i))$ is zero for $i > 1$, the 0-regularity of the sheaf

$p_{2*}\mathscr{T}_Y(k_1 + 1, 1)$ follows. This gives for $l > 0$ the surjectivity of the map

$$H^0(\mathbb{P}_2, p_{2*}\mathscr{T}_Y(k_1 + 1, l)) \otimes_{\mathbb{C}} W \to H^0(\mathbb{P}_2, p_{2*}\mathscr{T}_Y(k_1 + 1, l + 1).$$

The assertion follows similar as in Lemma 5.4. □

PROOF OF PROPOSITION 5.1. The following commutative diagram

$$\begin{tikzcd}
H^0(\mathbb{P}, \mathscr{T}_Y(1, 1)) \otimes_{\mathbb{C}} H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(k_1, k_2)) \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{P}} \arrow[r] \arrow[d] & H^0(\mathbb{P}, \mathscr{T}_Y(k_1 + 1, k_2 + 1)) \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{P}} \arrow[d] \\
H^0(\mathbb{P}, \mathscr{T}_Y(1, 1)) \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{P}}(k_1, k_2) \arrow[r] & \mathscr{T}_Y(k_1 + 1, k_2 + 1)
\end{tikzcd}$$

implies by the choice of k_0 that it is sufficient to show for two natural numbers k_1 and k_2 with $k_1, k_2 \geqslant k_0$ the surjectivity of the following map

$$H^0(\mathbb{P}, \mathscr{T}_Y(1, 1)) \otimes_{\mathbb{C}} H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(k_1, k_2)) \to H^0(\mathbb{P}, \mathscr{T}_Y(k_1 + 1, k_2 + 1)).$$

In the natural commutative diagram

$$\begin{tikzcd}
H^0(\mathbb{P}, \mathscr{T}_Y(1, 1)) \otimes H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(k_1, k_2)) \arrow[r, "(***)"] \arrow[d, "\wr\|"'] & H_0(\mathbb{P}, \mathscr{T}_Y(k_1 + 1, k_2 + 1)) \\
H^0(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(1, 1)) \otimes H^0(\mathbb{P}_1, p_{1*}\mathscr{O}_{\mathbb{P}}(k_1, k_2)) \arrow[d, "\wr\|"'] & \\
H^0(\mathbb{P}_1, p_{1*}\mathscr{T}_Y(1, 1)) \otimes_{\mathbb{C}} S^{k_1}V \otimes_{\mathbb{C}} S^{k_2}W \arrow[d, "(*)"'] & \\
H^0(\mathbb{P}_1, p_{1*}\mathscr{F}_Y(k_1 + 1, 1)) \otimes_{\mathbb{C}} S^{k_2}W \arrow[d, "\wr\|"'] & \\
H^0(\mathbb{P}_2, p_{2*}\mathscr{T}_Y(k_1 + 1, 1)) \otimes_{\mathbb{C}} S^{k_2}W \arrow[r, "(**)"] & H^0(\mathbb{P}_2, p_{2*}\mathscr{T}_Y(k_1 + 1, k_2 + 1)) \arrow[uuuu, "\wr\|"']
\end{tikzcd}$$

the maps $(*)$ and $(**)$ are surjective by Lemma 5.4 and 5.5. By application of the Leray spectral sequence the other vertical maps are isomorphisms by Lemma 5.3, respectively by the property $R^i p_{1*}\mathscr{O}_{\mathbb{P}}(k_1, k_2) = 0$ for $i > 0$. This implies the surjectivity of $(***)$ and the assertion follows. □

6. A variational Torelli theorem

In this section we prove our variational Torelli theorem for cyclic coverings of high degree. For the notion of IVHS-Isomorphism of cyclic coverings we refer to Definition 3.4.

THEOREM 6.1. *Let $(Y_1, \mathscr{L}_1)$ and $(Y_2, \mathscr{L}_2)$ be two polarized smooth projective varieties of dimension n with polarization given by the ample invertible sheaves $\mathscr{L}_1$ and $\mathscr{L}_2$. If N is a sufficiently large natural number, then for cyclic coverings*

$$f_i: Z_i \to Y_i, \; i = 1, 2$$

with respect to $\mathscr{L}_i^N = \mathscr{O}_{Y_i}(X_i)$, where X_i is a smooth, reduced divisor on Y_i, the following variational Torelli property holds:

If there is an IVHS-isomorphism of cyclic coverings

$$\mathrm{IVHS}(Z_1, f_1^*\mathscr{L}_1) \xrightarrow{\simeq} \mathrm{IVHS}(Z_2, f_2^*\mathscr{L}_2), \tag{6.1.1}$$

then there exists an isomorphism

$$\sigma: Y_1 \xrightarrow{\simeq} Y_2$$

with $\sigma^*\mathscr{L}_2 = \mathscr{L}_1$ and $\sigma^*X_2 = X_1$. As a conclusion this gives an isomorphism of polarized varieties between $(Z_1, f_1^*\mathscr{L}_1)$ and $(Z_2, f_2^*\mathscr{L}_2)$.

Proof of theorem 6.1. Choose a $k > 0$, such that the sheaf $\omega_{Y_i} \otimes \mathscr{L}_i^k$ is very ample for $i = 1, 2$ and the assumptions of Proposition 4.2 are fulfilled. We use the following notation for $i = 1, 2$:

$$U_i = H^0(Y_i, \omega_{Y_i} \otimes \mathscr{L}_i^k),$$
$$T_i = H^1(Y_i, T_{Y_i}\langle -X_i\rangle), \qquad T_i' = H^1(Y_i, T_{Y_i}\langle -X_i\rangle \otimes \mathscr{L}_i^{-1}),$$
$$W_i = H^1(Y_i, \Omega_{Y_i}^{n-1}\langle X_i\rangle \otimes \mathscr{L}_i^{-N+k}), \quad W_i' = H^1(Y_i, \Omega_{Y_i}^{n-1}\langle X_i\rangle \otimes \mathscr{L}_i^{-N+k-1}),$$

The assumption (6.1.1) gives isomorphisms of vectorspaces

$$U_1 \xrightarrow{\simeq} U_2,$$
$$T_1 \xrightarrow{\simeq} T_2, \qquad T_1' \xrightarrow{\simeq} T_2', \tag{6.1.2}$$
$$W_1 \xrightarrow{\simeq} W_2 \quad \text{and} \quad W_1' \xrightarrow{\simeq} W_2',$$

and hence the following commutative diagrams, where the vertical isomorphisms are induced by these isomorphisms (see Definition 3.4):

$$\begin{tikzcd} U_1 \otimes_{\mathbb{C}} T_1 \arrow[r] \arrow[d] & W_1 \arrow[d] \\ U_2 \otimes_{\mathbb{C}} T_2 \arrow[r] & W_2 \end{tikzcd} \tag{6.1.3}$$

and

$$\begin{tikzcd} U_1 \otimes_{\mathbb{C}} T'_1 \arrow[r] \arrow[d] & W'_1 \arrow[d] \\ U_2 \otimes_{\mathbb{C}} T'_2 \arrow[r] & W'_2. \end{tikzcd} \tag{6.1.4}$$

Step 1: *Construction of the isomorphism* $\sigma\colon Y_1 \to Y_2$

We apply Proposition 4.2 (Case $j = 0$) to the diagram (6.1.3). This gives the following commutative diagrams, where the vertical maps are isomorphisms of vectorspaces:

$$\begin{tikzcd} U_1 \otimes_{\mathbb{C}} B_1 \arrow[r] \arrow[d] & T_1 \arrow[d] \\ U_2 \otimes_{\mathbb{C}} B_2 \arrow[r] & T_2 \end{tikzcd} \tag{6.1.5}$$

and

$$\begin{tikzcd} U_1 \otimes_{\mathbb{C}} C_1 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ U_2 \otimes_{\mathbb{C}} C_2 \arrow[r] & B_2 \end{tikzcd} \tag{6.1.6}$$

By Proposition 4.2 we get for $i = 1, 2$ canonical isomorphisms

$$B_i \simeq H^0(Y_i, \omega_{Y_i}^{-1} \otimes \mathscr{L}_i^{N-k}) \quad \text{and} \quad C_i \simeq H^0(Y_i, \omega_{Y_i}^{-2} \otimes \mathscr{L}_i^{N-2k}). \tag{6.1.7}$$

We have chosen N sufficiently large. By Proposition 5.1 we can hence reconstruct for $i = 1, 2$ the embeddings $Y_i \hookrightarrow \mathbb{P}(U_i) \times \mathbb{P}(C_i)$ from the kernel of the linear maps $U_i \otimes_{\mathbb{C}} C_i \to B_i$ in diagram (6.1.6). By (6.1.2) we obtin isomorphisms

$$a\colon \mathbb{P}(U_1) \times \mathbb{P}(C_1) \xrightarrow{\simeq} \mathbb{P}(U_2) \times \mathbb{P}(C_2). \tag{6.1.8}$$

Hence the diagram (6.1.6) induces the following commutative diagram:

$$\begin{tikzcd}
\mathbb{P}(U_1)\times\mathbb{P}(C_1) \arrow[rrr, "a"] \arrow[dd, "p_1"'] & & & \mathbb{P}(U_2)\times\mathbb{P}(C_2) \arrow[dd, "p_2"] \\
& Y_1 \arrow[ul] \arrow[dl] \arrow[r, "\sigma"] & Y_2 \arrow[ur] \arrow[dr] & \\
\mathbb{P}(U_1) \arrow[rrr, "a"] & & & \mathbb{P}(U_2)
\end{tikzcd} \tag{6.1.9}$$

The maps p_1 and p_2 are the natural projections. The isomorphism α is induced by the corresponding isomorphism in (6.1.2). Furthermore for $i = 1, 2$ the morphisms $Y_i \to \mathbb{P}(U_i)$ are embeddings, because we have chosen $\omega_{Y_i} \otimes \mathscr{L}_i^k$ very ample. Since the ideal sheaves of Y_1 and Y_2 are carried over by a into one another, the morphism σ, which we get by restricting a in (6.1.8) to Y_1, is an isomorphism. Since $\omega_{Y_i} \otimes \mathscr{L}_i^k$ is very ample for $i = 1, 2$ and we have for N sufficiently large

$$H^j(Y_i, \omega_{Y_i}^{-j-1} \otimes \mathscr{L}_i^{N-k(j+1)}) = 0$$

for $1 \leqslant j \leqslant n$, by [Mu2, Thm. 2] the map $U_i \otimes_{\mathbb{C}} C_i \to B_i$ in (6.1.6) is surjective. For sufficiently large N the sheaf $\omega_{Y_i}^{-1} \otimes \mathscr{L}_i^{N-k}$ is very ample for $i = 1, 2$. Hence the natural morphisms $\mathbb{P}(B_i) \to \mathbb{P}(U_i \otimes_{\mathbb{C}} C_i)$ and $Y_i \to \mathbb{P}(B_i)$ are embeddings. The diagram (6.1.6) induces the following commutative diagram:

$$\begin{tikzcd}
\mathbb{P}(U_1)\times\mathbb{P}(C_1) \arrow[rrrr, "a"] \arrow[d] & & & & \mathbb{P}(U_2)\times\mathbb{P}(C_2) \arrow[d] \\
\mathbb{P}(U_1 \otimes_{\mathbb{C}} C_1) & Y_1 \arrow[l] \arrow[ul] \arrow[dl] \arrow[rr, "\sigma"] & & Y_2 \arrow[r] \arrow[ur] \arrow[dr] & \mathbb{P}(U_2 \otimes_{\mathbb{C}} C_2) \\
\mathbb{P}(B_1) \arrow[u] \arrow[rrrr, "\beta"] & & & & \mathbb{P}(B_2) \arrow[u]
\end{tikzcd} \tag{6.1.10}$$

For $i = 1, 2$ the morphism $\mathbb{P}(U_i) \times \mathbb{P}(C_i) \to \mathbb{P}(U_i \otimes_{\mathbb{C}} C_i)$ in diagram (6.1.10) is the Serge embedding. Therefore the isomorphism σ is the restriction of α in (6.1.9) as well as the restriction of β in (6.1.10) to Y_1.

By restricting the equation $\beta^*\mathscr{O}_{\mathbb{P}(B_2)}(1) = \mathscr{O}_{\mathbb{P}(B_1)}(1)$ to Y_1 we get

$$\sigma^*(\omega_{Y_2}^{-1} \otimes \mathscr{L}_2^{N-k}) = \omega_{Y_1}^{-1} \otimes \mathscr{L}_1^{N-k}. \tag{6.1.11}$$

Step 2: $\sigma^* \mathscr{L}_2 = \mathscr{L}_1$

We repeat the whole construction in *Step* 1 by using Proposition 4.2 (Case $j = 1$) and diagram (6.1.4). Since the same vectorspaces U_1 and U_2 appear in diagram (6.1.4), we get in particular

$$\sigma^*(\omega_{Y_2}^{-1} \otimes \mathscr{L}_2^{N-k-1}) = \omega_{Y_1}^{-1} \otimes \mathscr{L}_1^{N-k-1}. \tag{6.1.12}$$

By (6.1.11) and (6.1.12) the assertion $\sigma^* \mathscr{L}_2 = \mathscr{L}_1$ follows.

Step 3: $\sigma^* X_2 = X_1$

In *Step* 1 we have seen that the isomorphisms $U_1 \to U_2$ and $B_1 \to B_2$ in (6.1.2) are induced by σ. By (6.1.5) we obtain the following commutative diagram:

$$\begin{tikzcd} & H^0(Y_1, \mathscr{L}_1^N) \arrow[d] \\ U_1 \otimes_{\mathbb{C}} B_1 \arrow[ur] \arrow[r] \arrow[d] & T_1 \arrow[d] \\ U_2 \otimes_{\mathbb{C}} B_2 \arrow[r] \arrow[dr] & T_2 \\ & H^0(Y_2, \mathscr{L}_2^N). \arrow[u] \end{tikzcd} \tag{6.1.13}$$

The map $H^0(Y_i, \mathscr{L}_i^N) \to T_i$ with $i = 1, 2$ in (6.1.13) is the natural map induced by the prolongation bundle sequence (2.1.4) for the sheaf $\mathscr{L}^N$. Since we have chosen N sufficiently large, the maps $U_i \otimes_{\mathbb{C}} B_i \to H^0(Y_i, \mathscr{L}_i^N)$ in (6.1.13) are surjective. For the Jacobi systems $J_{Y_i, \mathscr{L}_i^N} := \ker\{H^0(Y_i, \mathscr{L}_i^N) \to T_i\}$ this implies an isomorphism $J_{Y_1, \mathscr{L}_1^N} \simeq J_{Y_2, \mathscr{L}_2^N}$ induced by σ. By [Gre, S. 153–154] the assertion $\sigma^* X_2 = X_1$ follows. □

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