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p-adic Whittaker functions and vector bundles on flag manifolds

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This paper is about Whittaker models of umramified principal series represen-
tations of *p*-adic groups, and their connection with the geometry of complex flag
manifolds.

Let G be a split reductive group over a local field F, with Iwasawa
decomposition $G = NAK$. Here N is a maximal unipotent subgroup, A is a
maximal F-split torus, and K is a certain maximal compact subgroup. Let $\mathcal{T}$ be
the complex torus of unramified (quasi-)characters of A. Then the free abelian
group $A/A \cap K$ is canonically isomorphic to the group $X(\mathcal{T})$ of rational
characters of $\mathcal{T}$. If $a \in A$, we let λ_a be the corresponding rational character. For
$\tau \in \mathcal{T}$, we can form the (unitarily) induced principal series representation

$$I(\tau) = \text{Ind}_{NA}^G \tau.$$

A Whittaker model of $I(\tau)$ is an intertwining map $\mathscr{W}_\tau: I(\tau) \to \text{Ind}_N^G \sigma$, where σ lies
in an open A-orbit in the space of characters of N. In [Ro], Rodier proved that,
for each σ, the space of such intertwining maps is one dimensional.

Let Φ_+^τ be the K-spherical function in $I(\tau)$, normalized by the condition
$\Phi_+^\tau(e) = 1$. There is a well-known formula, due (in increasing levels of generality)
to Shintani, Kato and Casselman-Shalika for the function $\mathscr{W}_\tau(\Phi_+^\tau)$ on G. Clearly
this function is completely determined by its restriction to A, and there it is
really a function on $A/A \cap K \simeq X(\mathcal{T})$. The formula says, roughly speaking (we
are ignoring a normalizing factor), that as a function of τ, $\mathscr{W}_\tau(\Phi_+^\tau)(a)$ is given by
the character of the irreducible representation $V(\lambda_a)$ of the Langlands dual $\mathscr{G}$ of
G with highest weight λ_a, provided this is a dominant weight. Here "dominant"
is referring to a Weyl chamber in $X(\mathcal{T})$ determined by the choice of N. If λ_a is not
a dominant weight, then $\mathscr{W}_\tau(\Phi_+^\tau)(a) = 0$ for all $\tau \in \mathcal{T}$.

The Borel-Weil theorem says that $V(\lambda_a)$ may be realized on the space of global
sections of a certain line bundle determined by λ_a on the flag manifold X of $\mathscr{G}$. In
this paper we show, among other things, that the values of various Iwahori-

spherical functions in the image of $\mathscr{W}_\tau$ are given by Lefschetz traces of τ on various cohomology groups of sheaves on X.

To be more precise, let B be the Iwahori subgroup of G determined by the choice of N. We set $M(\tau) = I(\tau)^B$. This is a representation of the affine Hecke algebra $\mathscr{H}$ associated to G. Recall (c.f. [L]) that as a vector space, $\mathscr{H}$ is the tensor product of two subalgebras

$$\mathscr{H}_0 \otimes \Theta,$$

where $\mathscr{H}_0$ is the Hecke algebra of the finite Weyl group W of G and Θ is isomorphic to the coordinate ring of $\mathscr{T}$.

We consider certain functions Φ^τ_{J+}, Φ^τ_{J-} and $\tilde{f}^\tau_w$ belonging to $M(\tau)$. Here J is a subset of the simple roots Σ, and $w \in W$. The $\Phi^\tau_{J\pm}$'s are eigenfunctions of parabolic subalgebras of $\mathscr{H}_0$, and the $\tilde{f}^\tau_w$'s are eigenfunctions of Θ. The spherical vector Φ^τ_+ equals $\Phi^\tau_{\Sigma,+}$, and we abbreviate $\Phi^\tau_- = \Phi^\tau_{\Sigma,-}$.

To each subset J there corresponds a flag sub-manifold X_J in X. In (5.1) and (5.4) below, the values of $\mathscr{W}_\tau(\Phi^\tau_{J-})$ are expressed in terms of global sections of line bundles on X_J, and the values of $\mathscr{W}_\tau(\Phi^\tau_{J-})$ are given by Lefschetz numbers on the $\bar{\partial}$-cohomology of line bundles on X_J. Finally, the values of $\mathscr{W}_\tau(\tilde{f}^\tau_w)$ are given by the formal characters of local cohomology groups supported on the Schubert cell corresponding to w. These cohomology groups are the duals of Verma modules, and they are the individual terms in the "Cousin resolution" of $V(\lambda_a)$, found by Kempf, which is dual to the BGG resolution (see [K]). The spherical vector can be written (see [C])

$$\Phi^\tau_+ = \sum_{w \in W} \tilde{f}^\tau_w,$$

and the Cousin resolution is a geometric interpretation of this formula.

Our approach is to first explicitly compute $\mathscr{W}_\tau(\tilde{f}^\tau_w)(a)$, using the results of Casselman-Shalika ([C-S]). This leads to a formula (3.2) for $\mathscr{W}_\tau(\Phi)(a)$ for an arbitrary $\Phi \in M(\tau)$. This general formula is a bit complicated, but becomes more explicit if Φ is some $\Phi^\tau_{J\pm}$. Here there is some overlap with independent work of Jian-Shu Li [Li], who also computed $\mathscr{W}_\tau(\Phi^\tau_-)$, as well as some other interesting Whittaker functions occuring when there are different root lengths. Curiously, the formula for $\mathscr{W}_\tau(\Phi^\tau_-)$ is very similar to Macdonald's formula for the zonal spherical function.

Next we compute, in section 5, the characters of the various cohomology groups mentioned above and compare them with our explicit formulas. The most involved one is for Φ^τ_-. It is easy to make a sign error, so we do that computation in two ways, first using the Borel-Weil-Bott theorem, and then using the Atiyah-Bott fixed point theorem for holomorphic vector bundles. The

former method decomposes the alternating sum of the $\bar{\partial}$-cohomology into irreducible representations. This means that, up to a normalizing factor, $\mathscr{W}_\tau(\Phi^\tau_-)(a)$ has been written as a linear combination of irreducible characters of $\mathscr{G}$. The second proof is similar to that used by Macdonald ([M]) to derive a well-known formula for the Poincare polynomial of a Coxeter group. Our two formulas for $\mathscr{W}_\tau(\Phi^\tau_-)(a)$ (the explicit and the cohomological) may be viewed as a "twisted" version of Macdonald's, in which the cohomology groups do not disappear into an Euler characteristic. To handle $\Phi^\tau_{J_\pm}$ for a proper subset $J \subset \Sigma$, we use a factorization of the Whittaker map $\mathscr{W}_\tau$ (see §4) corresponding to induction in stages.

Next, we apply our explicit formulas to prove two injectivity theorems for $\mathscr{W}_\tau$. The first, (8.1), gives necessary and sufficient conditions for

$$\mathscr{W}_\tau : I(\tau) \to \operatorname{Ind}_N^G \sigma$$

to be injective. Earlier, these were found for GL_n by Bernstein-Zelevinsky, ([B-Z]) and for the case of regular τ by the author ([R1]).

The second theorem (8.2) says, at least if the R-group is trivial, that a nonzero function in $\mathscr{W}_\tau(M(\tau))$ cannot vanish on A. This is perhaps surprising because an Iwahori invariant Whittaker function is seemingly determined by its values on aw, $a \in A$, $w \in W$. However, in the theorem, we have the additional condition that the function belongs to an irreducible representation. This is some justification for only computing the values of our Whittaker functions on A. In fact, finding their values on some aw, $w \neq 1$, can be difficult.

To prove these two theorems, we need more information about the unramified principal series than seems to exist in the literature, e.g., criteria for the generalized eigenspaces in the Jacquet module of $I(\tau)$ to be indecomposable. In our proof of this, it is clear that the spherical and generic constituents of $I(\tau)$ coincide if and only if $I(\tau)$ is irreducible. This last fact has been proven earlier by Barbasch-Moy and Li. It is not always true if the R-group if nontrivial. Here, we have worked out the connection between the R-group and different Whittaker models. This issue arises if G is not of adjoint type. The necessary modifications (of [C-S] and our earlier work), not serious, are found in section 7.

1. Notation and background

We begin with a list of most of the notation to be used throughout this paper, some of which was mentioned in the introduction.

First of all, F is a nonarchimedean local field and G is the group of F-rational points of a reductive algebraic group **G** defined and split over the ring of integers

$\mathcal{O}$ of F. We use similar notational distinctions between other groups over $\mathcal{O}$ and their F-rational points. Next, we have

$\mathbf{P}$ = a Borel subgroup of $\mathbf{G}$ defined over $\mathcal{O}$

$\mathbf{N}$ = the unipotent radical of $\mathbf{P}$

$\mathbf{A}$ = a maximal split torus of $\mathbf{P}$

δ = the modulus of P

$K = \mathbf{G}(\mathcal{O})$

B = the inverse image in K of $\mathbf{P}(k)$, where

k = the residue field of F

q = the number of elements in k

W = the normalizer of A in K

l = the length function on W

ε = the sign character of W

w_0 = a fixed representative of the longest element in W

$N_0 = N \cap K$

$A_0 = A \cap K$

Δ_F^+ = the roots of A in N

$A^- = \{a \in A : |\alpha(a)|_F \leqslant 1 \; \forall \alpha \in \Delta_F^+\}$

$\mathcal{T}$ = the group of unramified quasicharacters of A.

The group $\mathcal{T}$ is a complex torus, isomorphic to $\mathbb{C}^\times \otimes Y$, where Y is the rational character group of $\mathbf{A}$. Using this identification, we have an isomorphism between A/A_0 and the group $X(\mathcal{T})$ of rational characters of $\mathcal{T}$ under which $a \in A$ corresponds to the weight λ_a defined by

$$\lambda_a(z \otimes \mu) = z^{\text{val}(\mu(a))}, \quad \text{for } z \in \mathbb{C}^\times, \mu \in Y.$$

The action of W on A induces actions on $\mathcal{T}$ and $X(\mathcal{T})$ such that for $a \in A$, $\tau \in \mathcal{T}$ and $w \in W$, we have

$$\lambda_{waw^{-1}}(\tau) = w\lambda_a(\tau) = \lambda_a(\tau w).$$

The torus $\mathcal{T}$ is also a maximal torus in the complex Lie group $\mathcal{G}$ whose Weyl group is isomorphic to W and whose root system Δ is the co-root system of G. In this viewpoint, the Weyl chamber $X_+(\mathcal{T}) := \{\lambda_a : a \in A^-\}$ determines a Borel subgroup $\mathcal{B}$ of $\mathcal{G}$ containing $\mathcal{T}$, along with positive roots and simple roots Δ^+

and Σ, respectively, of $\mathcal{T}$ in $\mathcal{B}$. We also set $\Delta^- = \Delta - \Delta^+$. Usually we will think of $X(\mathcal{T})$ multiplicatively.

For an element $w \in W$, we set

$$R(w) = \{\alpha \in \Delta^+ : w\alpha \in \Delta^-\} \quad R'(w) = \{\alpha \in \Delta^+ : w\alpha \in \Delta^+\} = \Delta - R(w).$$

We also have the ubiquitous rational functions

$$C_w = \prod_{\alpha \in R(w^{-1})} \frac{1 - q^{-1}\alpha}{1 - \alpha} \in \mathbb{C}(\mathcal{T}).$$

Finally, for an element $\tau \in \mathcal{T}$, we set

$$W_\tau = \{w \in W : \tau w = \tau\}$$
$$\Delta_\tau^+ = \{\alpha \in \Delta^+ : \alpha(\tau) = 1\}$$
$$R_\tau = \{w \in W_\tau : w\Delta_\tau^+ = \Delta_\tau^+\}.$$

We next discuss the unramified principal series representations of G, for which the best published introduction is still [Car]. For $\tau \in \mathcal{T}$, we set

$$I(\tau) = \mathrm{Ind}_P^G \tau.$$

This is the space of functions on G which are right invariant under some compact open subgroup of G and transform under left multiplication by P according to the character $\delta^{1/2}\tau$. The group G acts by $\pi(g)f(x) = f(xg)$, for $f \in I(\tau)$ and $g, x \in G$. This "extends" to an action of the algebra of locally constant compactly supported functions ψ on G by

$$\pi(\psi)f(x) = \int_G f(xg)\psi(g)dg,$$

where the Haar measure dg gives B volume one.

We are only interested in the subalgebra $\mathcal{H}$ of compactly supported functions on G which are bi-invariant under the Iwahori subgroup B. It is known that taking B-invariants gives a bijection between the G-subquotients of $I(\tau)$ and the $\mathcal{H}$-subquotients of

$$M(\tau) := I(\tau)^B.$$

In particular, every subquotient of $I(\tau)$ is generated by its Iwahori-fixed vectors.

We also know that if $I(\tau)$ and $I(\tau_1)$ have a common subquotient, then τ and τ_1 belong to the same W-orbit in $\mathcal{T}$.

We have the following "standard" basis $\{\Phi_w^\tau : w \in W\}$ of $M(\tau)$, where Φ_w^τ is defined by

$$\text{support of } \Phi_w^\tau = PwB$$

$$\Phi_w^\tau(anwb) = \delta^{1/2}\tau(a), \ a \in A, \ n \in N, \ b \in B.$$

This makes sense because

$$G = \coprod_{w \in W} PwB.$$

For more than typographical reasons, we will often suppress the τ on Φ_w.

Any intertwining map $I(\tau) \to I(\tau w)$ is determined by its effect on $M(\tau)$. If τ is regular, there is a unique

$$\mathcal{A}_w^\tau \in \operatorname{Hom}_{\mathcal{H}}(M(\tau), M(\tau w))$$

which can be defined as follows. First, $\mathcal{A}_{xy}^\tau = \mathcal{A}_y^{\tau x} \circ \mathcal{A}_x^\tau$ if $\ell(xy) = \ell(x) + \ell(y)$. Second, for a simple reflection $s = s_\alpha$ we have ([C, (3.4)]),

$$\mathcal{A}_s \Phi_w^\tau = (C_\alpha(\tau) - 1)\Phi_w^{\tau s} + q^{-1}\Phi_{sw}^{\tau s} \quad \text{if } sw > w,$$ (1.1)

$$\mathcal{A}_s \Phi_w^\tau = (C_\alpha(\tau) - q^{-1})\Phi_w^{\tau s} + \Phi_{sw}^{\tau s} \quad \text{if } sw < w.$$

Each operator $\mathcal{A}_s^\tau$ extends holomorphically to the complement of $\ker \alpha$ in $\mathcal{T}$. More generally, if $w \in W$ has a minimal expression $w = s_{\alpha_n} \cdots s_{\alpha_1}, \alpha_i \in \Sigma$, with the property that $s_{\alpha_1} \cdots s_{\alpha_i}\alpha_{i+1} \notin \Delta_{\tau'} \forall i$, then the operator

$$\mathcal{A}_w^\tau = \mathcal{A}_{s_{\alpha_1}}^{\tau s_{\alpha_n} \cdots s_{\alpha_2}} \circ \cdots \circ \mathcal{A}_{s_{\alpha_n}}^\tau$$

is holomorphic at τ'. It is easy to check that within each coset $W_\tau y$, there is a unique R_τ-orbit (under left multiplication) of elements w with the above property. They are also characterized by the condition $w^{-1}\Delta_\tau^+ = \Delta_{\tau w}^+$. We say that such w's are "minimal in their coset".

We turn now to a more detailed description of the affine Hecke algebra $\mathcal{H}$, taken from [L]. By [I-M], $\mathcal{H}$ has a linear basis $\{T_{aw} : a \in A/A_0, w \in W\}$, where T_{aw} is the characteristic function of $Baw B$. We can also write

$$\mathcal{H} = \mathcal{H}_0 \otimes \Theta,$$

where $\mathscr{H}_0$ consists of the functions in $\mathscr{H}$ supported on K, and Θ is the subalgebra linearly spanned by the elements

$$\theta_a = \delta^{1/2}(a) T_{a_1} T_{a_2}^{-1}, \quad a \in A/A_0.$$

where a_1 and a_2 are any two elements of A^- such that $a = a_1 a_2^{-1}$. It turns out that θ_a is well-defined and moreover the correspondence $\theta_a \to \lambda_a$ induces a ring isomorphism $\Theta \simeq \mathbb{C}[\mathscr{T}]$. In particular, elements of $\mathscr{T}$ are algebra homomorphisms $\Theta \to \mathbb{C}$. The two subalgebras $\mathscr{H}_0$ and Θ do not commute.

The action of $\mathscr{H}_0$ on $M(\tau)$ is independent of τ, and is just the regular representation of $\mathscr{H}_0$. More precisely, if s is a simple reflection in W, we have

$$\pi(T_s)\Phi_w = \begin{cases} \Phi_{ws} & \text{if } ws > w \\ q\Phi_{ws} + (q-1)\Phi_w & \text{if } ws < w \end{cases}$$

In particular, Φ_{w_0} always generates $M(\tau)$ over $\mathscr{H}_0$.

The action of Θ is more subtle, and will be discussed later. For now, we mention only that the projection of $M(\tau)$ into the Jacquet module $I(\tau)_N$ of $I(\tau)$ induces an isomorphism of A/A_0 representations

$$M(\tau) \xrightarrow{\cong} I(\tau)_N \otimes \delta^{-1/2}, \tag{1.2}$$

where A acts on $M(\tau)$ via $a \mapsto \pi(\theta_a)$ (c.f. [R]).

We come now to Whittaker models. Let σ be a quasi-character of N which is centralized by no element of A outside the center of G. This amounts to saying that σ belongs to an open A-orbit in the space of such characters. If G is of adjoint type, there is only one such orbit. We take the Haar measure on N such that the volume of N_0 is one. For every $\tau \in \mathscr{T}$, there is a unique-up-to-scalar intertwining map

$$\mathscr{W}_\tau : I(\tau) \to \operatorname{Ind}_N^G \sigma.$$

This can be defined (see [C-S (2.3)]) by

$$\mathscr{W}_\tau(f)(g) = \int_{N_*} \sigma^{-1}(n) f(w_0 ng) dn,$$

where the integral is taken over a sufficiently large compact open subgroup N_* of N, which depends on $f \in I(\tau)$ and $g \in G$. In other words, if we take an exhaustive sequence of compact opens $N(0) \subseteq N(1) \subseteq \cdots$ in N, then, given f and g, there is an integer m_0 such that the integral over $N(m_0)$ equals the integral over any $N(m)$, $m \geq m_0$.

It follows from [C-S, (2.1)]that for fixed $g \in G$, the functions $\tau \mapsto \mathscr{W}_\tau(\Phi_w^\tau)(g)$ are polynomial functions on $\mathscr{T}$. We denote these $\mathbb{C}[\mathscr{T}]$-valued functions on G by $\mathscr{W}(\Phi_w)$. They will be computed in section 3.

For each $\alpha \in \Delta^+$, let $x_\alpha: F^+ \to N$ be the root group which is dilated by A according to the character α. We say σ is "unramified" if $\sigma \circ x_\alpha$ has conductor $\mathcal{O}$, for every simple root α. From now until section 7, we will assume σ is unramified. In this case, we have the following formula ([C-S, (4.3)], see also (7.1) below) for the twisting of the Whittaker map by an intertwining map.

$$\mathscr{W}_{\tau w} \mathscr{A}_w^\tau = \prod_{\alpha \in R(w-1)} \frac{1 - q^{-1}\alpha^{-1}(\tau)}{1 - \alpha(\tau)} \mathscr{W}_\tau \qquad (1.3)$$

Also, in the unramified case, it is easy to see that a B-invariant function $F \in \mathrm{Ind}_N^G \sigma$, when restricted to A, has support in A^-. We don't need it, but a little more work shows, for $a \in A$ and $w \in W$ that

$$F(aw) \neq 0 \Leftrightarrow \begin{cases} |\alpha(a)| \leqslant 1 & \text{if } \check{\alpha} \in R'(w^{-1}) \\ |\alpha(a)| \leqslant q & \text{if } \check{\alpha} \in R(w^{-1}). \end{cases}$$

2. Special Iwahori invariants

We recall some facts about the action of Θ on $M(\tau)$, which can be found in [R]. For any τ, the semisimplification of $M(\tau)$ is $\oplus_{w \in W} \tau w$. If τ is regular, Θ acts diagonalizably, and there is the following formula for the Θ-eigenvectors. Let

$$f_w^\tau = \sum_{y \in W} a_{y,w}(\tau)\Phi_y^\tau$$

where the $a_{y,w} \in \mathbb{C}(\mathscr{T})$ are defined recursively by $a_{x,z} = 0$ if $x \not\leqslant z$, $a_{x,x} = 1$, and $sz > z$ $(s = s_\alpha)$ implies

$$a_{x,sz} + (C_\alpha - 1)a_{x,z} = \begin{cases} C_\alpha C_{-\alpha} s \cdot a_{sx,z} & \text{if } sx > x \\ s \cdot a_{sx,z} & \text{if } sx < x. \end{cases}$$

It turns out ([R, (6.3)]) that for any $y \in W$, $a_{y,w}$ is holomorphic on $\mathscr{T}_w := \{\tau \in \mathscr{T}: w^{-1}\Delta_\tau^+ \subseteq \Delta^-\}$, and for each $\tau \in \mathscr{T}_w$, $f_w^\tau \in M(\tau)$ affords the Θ-character $\tau\omega$. Finally, if τ is regular, we have the relation ([R, (4.5)])

$$\mathscr{A}_y^\tau f_w^\tau = \left[\prod_{\alpha \in R(y^{-1}) \cap w\Delta^+} C_\alpha(\tau) C_{-\alpha}(\tau) \right] f_{y^{-1}w}^\tau \qquad (2.1)$$

(The product equals 1 if $R(y^{-1}) \cap w\Delta^+$ is empty). This implies that

$$\mathscr{A}_y^\tau f_w^\tau(1) = \begin{cases} 1 & \text{if } y = w \\ 0 & \text{if } y \neq w. \end{cases}$$

By continuity, (2.1) remains true if $\tau \in \mathscr{T}_w$ and y is minimal in its coset $W_\tau y$. In certain circumstances (see (6.3) below), all Θ-eigenvectors are linear combinations of the f_w^τ's. Note that $\Phi_{w_0}^\tau = f_{w_0}^\tau$.

By [C-S, (2.1)], for any $g \in G$, the function

$$\tau \longmapsto \mathscr{W}_\tau(f_w^\tau)(g)$$

is holomorphic (in fact polynomial) on $\mathscr{T}_w$. We denote this element of $\mathbb{C}[\mathscr{T}_w]$ by $\mathscr{W}(f_w)(g)$.

We will need the inverse transpose of the matrix $[a_{y,w}]$. This is the matrix $[b_{y,w}]$ of rational functions defined for regular $\tau \in \mathscr{T}$ by

$$b_{y,w}(\tau) = \mathscr{A}_y^\tau(\Phi_w^\tau)(1),$$

or equivalently by the equation

$$\Phi_w^\tau = \sum_{y \in W} b_{y,w}(\tau) f_y^\tau.$$

(2.2) PROPOSITION. *The functions $b_{y,w}$ belong to $\mathbb{C}[\mathscr{T}_w]$ and are given recursively by $b_{y,w} = 0$ if $y \not\geqslant w$, $b_{w,w} = 1$, and $sy > y$ ($s = s_\alpha$, $\alpha \in \Sigma$) implies*

$$s \cdot b_{sy,w} = \begin{cases} (q^{-1} - C_\alpha)b_{y,w} + q^{-1}b_{y,sw} & \text{if } sw > w \\ (1 - C_\alpha)b_{y,w} + b_{y,sw} & \text{if } sw < w. \end{cases}$$

Proof. The homomorphicity follows from that of the $a_{y,w}$'s, since the eigenvalues of the matrix $[a_{y,w}]$ are all one. The two simple equations for $b_{y,w}$ are proved in [R, (4.4)]. For the complicated part, we take τ regular in $\mathscr{T}$ and compute $\mathscr{A}_s^\tau \Phi_w^\tau$ in two ways.

First, by (2.1), we have

$$\mathscr{A}_s^\tau \Phi_w^\tau = \sum_z b_{z,w} \mathscr{A}_s^\tau f_z^\tau = \sum_{sz > z} b_{z,w}(\tau)C_\alpha(\tau)C_\alpha(\tau)C_{-\alpha}(\tau)f_{sz}^{\tau s} + \sum_{sz < z} b_{z,w}(\tau)f_{sz}^{\tau s}.$$

On the other hand, we can use the formula (1.1), and write the result in terms of $f_z^{\tau s}$'s. It remains to compare the coefficients of $f_y^{\tau s}$ when $sy > y$. $\qquad\square$

3. Explicit formulas

In this section we compute $\mathscr{W}$ on the Θ-eigenvectors f_w, use this to give a somewhat explicit formula for $\mathscr{W}$ on the standard basis elements Φ_w, and finally derive a better formula for $\mathscr{W}(\Phi_-)$.

(3.1) PROPOSITION. *For all $w \in W$ and $a \in A^-$, we have*

$$\mathscr{W}(f_w)(a) = \delta^{1/2}(a) \prod_{\alpha \in R'(w-1)} \frac{1 - q^{-1}\alpha}{1 - \alpha^{-1}} \, w\lambda_a \in \mathbb{C}[\mathscr{T}_w].$$

Proof. We will show that both sides agree for $\tau \in \mathscr{T}_w$ on which no root takes the value 1 or $q^{\pm 1}$. First we use (1.3), which says

$$\mathscr{W}_{\tau w w_0} \mathscr{A}^\tau_{w w_0}(f^\tau_w)(a) = \prod_{\alpha \in R'(w-1)} \frac{1 - q^{-1}\alpha^{-1}(\tau)}{1 - \alpha(\tau)} \, \mathscr{W}_\tau(f^\tau_w)(a).$$

but we also know from (2.1) that

$$\mathscr{A}^\tau_{w w_0}(f^\tau_w) = \prod_{\alpha \in R(w_0 w - 1)} C_\alpha(\tau) C_{-\alpha}(\tau) f^{\tau w w_0}_{w_0},$$

and moreover, the Whittaker map is easily computed on $f^\tau_{w_0}$ (see [C-S, (4.1)] or (4.1) below). We have

$$\mathscr{W}_{\tau w w_0}(f^{\tau w w_0}_{w_0})(a) = \mathscr{W}_{\tau w w_0}(\Phi^{\tau w w_0}_{w_0})(a) = \delta^{1/2}(a) w\lambda_a(\tau).$$

Putting it together, we get

$$\delta^{1/2}(a) w\lambda_a(\tau) \prod_{\alpha \in R'(w-1)} C_\alpha(\tau) C_{-\alpha}(\tau) = \prod_{\alpha \in R'(w-1)} \frac{1 - q^{-1}\alpha^{-1}(\tau)}{1 - \alpha(\tau)} \, \mathscr{W}_\tau(f^\tau_w)(a),$$

and some easy manipulations finish the proof.   □

(3.2) COROLLARY. *For $w \in W$ and $a \in A^-$, we have*

$$\mathscr{W}(\Phi_w)(a) = \delta^{1/2}(a) \sum_{y \geq w} b_{y,w} y\left[\lambda_a \sum_{\alpha \in R'(y-1)} \frac{1 - q^{-1}\alpha}{1 - \alpha^{-1}} \right] \in \mathbb{C}[\mathscr{T}],$$

where the $b_{y,w} \in \mathbb{C}(\mathscr{T})$ are given in (2.2).

The $b_{y,w}$'s are complicated, but I believe they have deep meaning for the unramified principal series. In [R], they are shown to be related to Kazhdan-

Lusztig polynomials. It seems hopeless to compute the $b_{y,w}$'s in closed form, but we can give some relations between them in (4.4).

Let

$$\Phi_- = \sum_{w \in W} (-q)^{-l(w)} \Phi_w.$$

It is easy to check that Φ_- is an eigenfunction for $\mathcal{H}_0$, and affords the sign character $T_w \mapsto \varepsilon(w)$. This means Φ_- is also an eigenfunction for the intertwining operators $\mathcal{A}_w$, in the following sense.

(3.3) LEMMA. *If w is minimal in its coset $W_\tau w$, then*

$$\mathcal{A}_w^\tau(\Phi_-) = D_w(\tau)\Phi_-^{\tau w},$$

where

$$D_w = \prod_{\alpha \in R(w^{-1})} \frac{\alpha - q^{-1}}{1 - \alpha}.$$

   Proof. Let $s = s_\alpha$ be a simple reflection. Using the formulas for $\mathcal{A}_s \Phi_w$, it is straightforward to compute that

$$\mathcal{A}_s^\tau(\Phi_-^\tau) = D_s.$$

The result follows by induction on the length of w.   ☐

(3.4) LEMMA.

$$\Phi_-^\tau = \sum_{w \in W} D_w(\tau) f_w^\tau$$

for all regular $\tau \in \mathcal{T}$.

   Proof. This is because the f_w^τ's are dual to the functionals $\Phi \mapsto \mathcal{A}_w^\tau(\Phi)(1)$ on $M(\tau)$, and the coefficient of Φ_1 in Φ_- is one.   ☐

   Analogously, the spherical function

$$\Phi_+ = \sum_{w \in W} \Phi_w$$

can be written (see [C])

$$\Phi_+^\tau = \sum_{w \in W} C_w(\tau) f_w^\tau$$

and satisfies $\mathcal{A}_w^\tau \Phi_w^\tau = C_w(\tau) \Phi_+^{\tau w}$.

Let ρ be the square root of the product of the roots in Δ^+.

(3.5) PROPOSITION. *For $a \in A^-$, we have*

$$\mathscr{W}(\Phi_-)(a) = \delta^{1/2}(a)(-1)^{|\Delta^+|}\rho \sum_{w \in W} w(\lambda_a \rho C_{w_0}) \in \mathbb{C}[\mathscr{T}].$$

Proof. Let $\tau \in \mathscr{T}$ be regular. By (3.3) and (3.1), we have

$$\mathscr{W}_\tau(\Phi_-^\tau)(a) = \sum_{w \in W} D_w(\tau) \prod_{\alpha \in R'(w-1)} \frac{1 - q^{-1}\alpha(\tau)}{1 - \alpha^{-1}(\tau)} \delta^{1/2}(a)w\lambda_a(\tau),$$

so

$$\mathscr{W}(\Phi_-)(a) = \frac{\delta^{1/2}(a)}{d} \sum_{w \in W} \varepsilon(w) \prod_{\alpha \in R(w-1)} (1 - q^{-1}\alpha^{-1}) \prod_{\alpha \in R'(w-1)} (1 - q^{-1}\alpha)w\lambda_a$$

where we have put $d = \Pi_{\alpha > 0}(1 - \alpha^{-1})$. We set

$$Q_w = \prod_{\alpha \in R(w-1)} (1 - q^{-1}\alpha^{-1}) \cdot \prod_{\alpha \in R'(w-1)} (1 - q^{-1}\alpha).$$

If $s = s_\alpha$ is a simple reflection such that $sw > w$, then

$$R(w^{-1}s) = sR(w^{-1}) \cup \{\alpha\} \quad \text{and} \quad R'(w^{-1}s) = s(R'(w^{-1}) - \{\alpha\}),$$

from which it follows that $Q_{sw} = s \cdot Q_w$. Thus

$$Q_w = w \cdot Q_1 = w \cdot \prod_{\alpha > 0} (1 - q^{-1}\alpha)$$

for all $w \in W$. We arrive at our final formula by moving d inside the sum, using the relation $d = \varepsilon(w)\rho^{-1}w(\rho d)$. $\qquad\square$

REMARKS. 1. Although the right hand side belongs to $\mathbb{C}[\mathscr{T}]$, the factorization is occuring in $\mathbb{C}[\tilde{\mathscr{T}}] \supseteq \mathbb{C}[\mathscr{T}]$, where $\tilde{\mathscr{T}}$ is a maximal torus in the simply-connected cover of $\mathscr{G}$.

2. We have the following functional equation:

$$\rho^{-1}(\tau)\mathscr{W}_\tau(\Phi_-^\tau)(a) = \rho^{-1}(\tau\omega)\mathscr{W}_{\tau w}(\Phi_-^{\tau w})(a) \quad \text{for all } w \in W.$$

It follows from (8.2) below that this holds with a replaced by an arbitrary $g \in G$.

4. Parabolic subgroups

In this section, we use induction in stages to consider a different model of the unramified principal series. In the new model, the Whittaker map is easily computed on functions with certain support. This leads to a reduction formula for $\mathcal{W}(\Phi_w)$ for w that differ from the long word w_0 by something in a parabolic subgroup of W. Using the Casselman-Shalika formula and (3.5), we can then give formulas for $\mathcal{W}$ on eigenfunctions of parabolic subalgebras of $\mathcal{H}_0$, without $b_{y,w}$'s.

For $J \subseteq \Sigma$, we let $P_J = L_J N_J$ (Levi decomposition) be the parabolic subgroup of G containing P whose Levi factor is generated by coroots from J. Put $N_{J,0} = N_J \cap K$. Let δ_J be the modular function of P_J. The subgroup of W generated by J is denoted W_J, and W^J is the set of w in W such that $w^{-1}J \subseteq \Delta^+$. The longest element of W_J is called y_0 and the longest element of W^J is called w_1. Also, let $J' = -w_0 J$ (additive notation) be the dual subset of J.

Let (ρ, V) be an admissible representation of L_J, and consider the induced representation π on $\mathrm{Ind}_{P_J}^G V$. Assume V has a Whittaker map

$$\mathcal{W}_J : V \to \mathrm{Ind}_{N \cap L_J}^{L_J}(\sigma|_{N \cap L_J}).$$

Set

$$I_J = \{\varphi \in \mathrm{Ind}_{P_J}^G V : \mathrm{supp}(\varphi) \subseteq P_J w_1 P\}.$$

We have a map $\varphi \mapsto \varphi^J : I_J \to V$ given by

$$\varphi^J = \int_{N_{J'}} \sigma^{-1}(n)\varphi(w_1 n)dn.$$

In [C-S] it is shown that there is a unique intertwining map $\tilde{\mathcal{W}} : \mathrm{Ind}_{P_J}^G V \to \mathrm{Ind}^{GN} \sigma$ such that

$$\tilde{\mathcal{W}}(\varphi)(e) = \mathcal{W}_J(\varphi^J)(e)$$

for all $\varphi \in I_J$.

(4.1) LEMMA. *Suppose* $\varphi \in (\mathrm{Ind}_{P_J}^G V)^B$ *has support in* $P_J w_1 B = P_J w_1 N_{J',0}$. *Then*

$$(\pi(a)\varphi)^J = \delta_{J'}(a)^{1/2}\rho(w_1 a w_1^{-1})\varphi(w_1)$$

if $a \in A^-$, *and* $(\pi(a)\varphi)^J = 0$ *if* $a \notin A^-$.

Proof. The left hand side is

$$\int_{N_{J'}} \sigma^{-1}(n)\varphi(w_1 na)dn = \delta_{J'}(a) \int_{N_{J'}} \sigma^{-1}(ana^{-1})\varphi(w_1 an)dn$$

$$= \delta_{J'}(a)\delta_J(w_1 aw_1^{-1})^{1/2}\rho(w_1 aw_1^{-1}) \int_{N_{J'}} \sigma^{-1}(ana^{-1})\varphi(w_1 n)dn.$$

The decomposition $P_J w_1 N_J$ has uniqueness of expression. This implies that the function $n \mapsto \varphi(w_1 n)$ is supported on $N_{J',0}$, where it is constant. Hence

$$\int_{N_{J'}} \sigma^{-1}(n)\varphi(w_1 na)dn = \delta_{J'}(a)^{1/2} \left(\int_{N_{J',0}} \sigma^{-1}(ana^{-1})dn \right) \rho(w_1 aw_1^{-1})\varphi(w_1).$$

Finally, the last integral is one if $a \in A^-$, zero otherwise.   □

From now on, we take

$$V = \operatorname{Ind}_{P \cap L_J}^{L_J} \tau \underset{L_J}{\simeq} \operatorname{Ind}_P^{P_J} \tau.$$

We have a G-isomorphism

$$\varphi \mapsto \tilde{\varphi} : I(\tau) = \operatorname{Ind}_P^G \tau \to \operatorname{Ind}_{P_J}^G V$$

where $\tilde{\varphi}(g)(p) = \varphi(pg)$ for $g \in G$, $p \in P_J$.

Let $\{\Phi_{J,y} : y \in W_J\}$ be the standard basis of V^B, defined as in section 3. If $y \in W_J$, $w \in W^J$, then $\tilde{\Phi}_{yw} \in (\operatorname{Ind}_{P_J}^G V)^B$ is characterized by (i) $\operatorname{supp}(\tilde{\Phi}_{yw}) \subseteq P_J wB$ and (ii) $\tilde{\Phi}_{yw}(w) = \Phi_{J,y}$. By (4.1), we have $(\tilde{\Phi}_{yw_1})^J = \tilde{\Phi}_{yw_1}(w_1) = \Phi_{J,y}$ for $y \in W_J$.

(4.2) LEMMA. *For all $\varphi \in I(\tau)$, we have $\widetilde{\mathscr{W}}(\tilde{\varphi}) = \mathscr{W}(\varphi)$.*

Proof. We know the two sides agree up to a constant independent of φ. Recall that y_0 and w_0 are the longest elements of W_J and W, respectively. Then $w_0 = y_0 w_1$ so

$$\widetilde{\mathscr{W}}(\tilde{\Phi}_{w_0})(e) = \mathscr{W}_J(\Phi_{J,y_0})(e) = 1 = \mathscr{W}(\Phi_{w_0})(e).$$

Hence the constant is one.   □

Putting everything together, we get the following reduction formula

(4.3) PROPOSITION. *For all $a \in A^-$ and $y \in W_J$, we have*

$$\mathscr{W}(\Phi_{yw_1})(a) = \delta_{J'}(a)^{1/2}\mathscr{W}_J(\Phi_{J,y})(w_1 aw_1^{-1}).$$

REMARK. For later use, we observe that if σ is taken to be ramified, the calculation of (4.1) will still lead to the equation

$$\mathscr{W}(\Phi_{yw_1})(e) = \mathscr{W}_J(\Phi_{J,y})(e).$$

For each $J \subseteq \Sigma$, we define two elements of $M(\tau)$

$$\Phi_{J,+} = \sum_{y \in W_J} \Phi_{yw_1} = \sum_{z \geqslant w_1} \Phi_z$$

$$\Phi_{J,-} = \sum_{y \in W_J} (-q)^{-l(y)} \Phi_{yw_1} = (-q)^{-l(w_1)} \sum_{z \geqslant w_1} (-q)^{-l(z)} \Phi_z.$$

These are eigenfunctions for the subalgebra of $\mathscr{H}_0$ generated by $\{T_{s_\alpha} : \alpha \in J'\}$, affording the (q-analogues of the) trivial and sign characters, respectively.

(4.4) PROPOSITION. *For regular τ, we have*

$$\Phi_{J,+} = \sum_{y \in W_J} C_y(\tau) f^\tau_{yw_1}, \quad \Phi_{J,-} = \sum_{y \in W_J} D_y(\tau) f^\tau_{yw_1}.$$

Equivalently, we have the relations

$$\sum_{z \leqslant y} b_{yw_1, zw_1} = C_y, \quad \sum_{z \leqslant y} (-q)^{-l(z)} b_{yw_1, zw_1} = D_y.$$

Proof. For a simple reflection s in J, we have $syw_1 < yw_1 \Leftrightarrow sy < y$. Hence (1.1) and induction on the length of $y \in W_J$ imply

$$\mathscr{A}^\tau_y \Phi^\tau_{J,+} = C_y(\tau) \Phi^{\tau y}_{J,+}, \quad \text{and} \quad \mathscr{A}^\tau_y \Phi^\tau_{J,-} = D_y(\tau) \Phi^{\tau y}_{J,-}.$$

Also, $b_{w_1, yw_1}(\tau y)$ equals one if $y = 1$, zero otherwise, and this implies the result. $\square$

Using either of these propositions and the formulas in section 2, we now have

(4.5) PROPOSITION. *For $a \in A^-$ and $J \subseteq \Sigma$, we have*

$$\mathscr{W}(\Phi_{J,+})(a) = \delta^{1/2}(a)\rho_1(-1)^{|\Delta_J^+|} C_{y_0} \sum_{y \in W_J} \varepsilon(y)y(\rho_1 w_1 \lambda_a),$$

$$\mathscr{W}(\Phi_{J,-})(a) = \delta^{1/2}(a)\rho_1(-1)^{|\Delta_J^+|} \sum_{y \in W_J} y(\rho_1 C_{y_0} w_1 \lambda_a),$$

where ρ_1 is the square root of the product of the roots in Δ_J^+.

5. Flag manifolds

Let $\mathscr{G}$ be the complex reductive Lie group whose root datum is dual to that of the p-adic group G. We assume here that $\mathscr{G}$ is simply connected. The torus $\mathscr{T}$ from before is a maximal torus in $\mathscr{G}$. Let $\mathscr{B}$ be the Borel subgroup containing $\mathscr{T}$ corresponding to the dominant chamber in $X(\mathscr{T})$ determined by A^-. Then Δ^+ is the set of roots of $\mathscr{T}$ in $Lie(\mathscr{B})$, and $\Sigma \subset \Delta^+$ is set of the corresponding simple roots.

Let $X = \mathscr{G}/\mathscr{B}$ be the flag manifold associated to $\mathscr{G}$. If ψ is a finite dimensional holomorphic representation of $\mathscr{B}$, we set

$$\mathbf{E}(\psi) = \mathscr{G} \times_\mathscr{B} \psi.$$

This is the unique $\mathscr{G}$-equivariant homomorphic vector bundle on X whose fiber over $g\mathscr{B}$ is the conjugate representation ${}^g\psi$. Conversely, any $\mathscr{G}$-equivariant holomorphic vector bundle on X is of the form $\mathbf{E}(\psi)$ where ψ is the fiber over $\mathscr{B}$.

More generally, for every subset J of Σ, we let X_J be the $\mathscr{L}_J$ orbit in X through $\mathscr{B}$, where $\mathscr{L}_J$ is the Levi subgroup generated by the roots in J. Then $X_J \simeq \mathscr{L}_J/\mathscr{B} \cap \mathscr{L}_J$ is the flag manifold of $\mathscr{L}_J$. If ψ is a holomorphic representation of $\mathscr{B} \cap \mathscr{L}_J$, we let

$$\mathbf{E}_J(\psi) = \mathscr{L}_J \underset{\mathscr{B}\cap\mathscr{L}_J}{\times} \psi.$$

We identify this with the unique homogeneous holomorphic vector bundle over X_J whose fiber over $\mathscr{B}$ affords ψ.

The $\bar{\partial}$-cohomology groups

$$H^{r,s}(X_J, \mathbf{E}(\psi))$$

are finite dimensional holomorphic $\mathscr{L}_J$ representations, and in particular, may be viewed as elements of $\mathbb{C}[\mathscr{T}] = R[\mathscr{T}]$, the representation ring of $\mathscr{T}$.

Finally, let ρ_1 be the square root of the product of the roots Δ_J^+ of $\mathscr{T}$ in $\mathscr{B} \cap \mathscr{L}_J$. As before, w_1 is the longest element of the Weyl group W having the property that $w_1^{-1}\Delta_J^+ \subseteq \Delta^+$. We can now state

(5.1) THEOREM. *As elements of $R[\mathscr{T}]$, we have, for every $a \in A^-$,*

$$\mathscr{W}(\Phi_{J,-})(a) = \delta^{1/2}(a)\rho_1(-1)^{|\Delta_J^+|}\sum_{r,s} (-1)^{r+s}q^{-r}H^{r,s}(X_J, \mathbf{E}(\rho_1 w_1 \lambda_a)).$$

Proof. The reduction formula (4.3) makes it sufficient to prove this for $J = \Sigma$. We use the Dolbeault isomorphism

$$H^{r,s}(X, \mathbf{E}(\psi)) \simeq H^s(X, \mathbf{E}(\psi) \otimes \Omega^r),$$

where Ω^r is the bundle of holomorphic r-forms on X and the right side is the cohomology of the sheaf of germs of holomorphic sections of the bundle $E(\psi) \otimes \Omega^r$. Let $\mathfrak{u}$ be the nilradical of $\mathrm{Lie}(\mathcal{B})$. As a $\mathcal{B}$ module (under the adjoint action), $\mathfrak{u}$ is isomorphic to the cotangent space $T_{\hat{e}}^* X$ of X at $\mathcal{B}$. Hence, $\Omega^r \simeq E(\Lambda^r \mathfrak{u})$, and

$$E(\rho\lambda) \otimes \Omega^r \simeq E(\rho\lambda \otimes \Lambda^r\mathfrak{u}).$$

For the first proof, we need a

(5.2) LEMMA. *As a virtual representation of* $\mathcal{G}$,

$$\sum_s (-1)^s H^s(X, E(\psi))$$

depends only on the $\mathcal{T}$ *action on* ψ, *not on the full* $\mathcal{B}$-*action.*

Proof. [BT, (1.10)].

Thus we can replace ψ by its semisimplification. We denote the virtual representation in (5.2) by $F(\psi)$. It follows from the Borel-Weil-Bott theorem (c.f. [DM]) that the restriction to $\mathcal{T}$ of $F(\psi)$ is given by

$$\sum_{w \in W} w \left(\frac{\psi}{D'} \right),$$

where $D' = \Pi_{\alpha \in \Delta^+}(1 - \alpha)$. Also, as a $\mathcal{T}$ representation,

$$\Lambda^r\mathfrak{u} \simeq \bigoplus_{\substack{S \subseteq \Delta^+ \\ |S|=r}} \delta_S,$$

where δ_S is the product of the roots in S.

Thus,

$$\sum_s (-1)^s H^s(X, E(\rho\lambda) \otimes \Omega^r) = \sum_s (-1)^s \sum_{\substack{S \subseteq \Delta^+ \\ |S|=r}} H^s(X, E(\rho\lambda\delta_S)) = \sum_{\substack{S \subseteq \Delta^+ \\ |S|=r}} F(\rho\lambda\delta_S),$$

so

$$\sum_{r,s} (-1)^{r+s} q^{-r} H^s(X, E(\rho\lambda) \otimes \Omega^r) = \sum_{S \subseteq \Delta^+} (-q)^{-|S|} F(\lambda\rho\delta_S)$$

$$= \sum_W w \left[\frac{\lambda\rho}{D'} \sum_{S \subseteq \Delta^+} (-q)^{-|S|}\delta_S \right].$$

The inner sum is $\Pi_{\alpha \in \Delta^+}(1 - q^{-1}\alpha)$, so we have

$$\sum_{r,s} (-1)^{r+s} q^{-r} H^{r,s}(X, L(\lambda_a \rho)) = \sum_W w(\lambda_a \rho C_{w_0})$$

$$= (-1)^{|\Delta^+|} \delta^{-1/2}(a)\rho^{-1} \mathcal{W}(\Phi_-)(a)$$

by (3.5).

Now for the second proof. By continuity, we may assume τ is regular. Let $h: X \to X$ be left multiplication by τ^{-1}, and let $\tilde{h}$ be the bundle morphism

$$\tilde{h}: h^*(\mathbf{E}(\rho\lambda) \otimes \Omega^r) \to (\mathbf{E}(\rho\lambda) \otimes \Omega^r)$$

given by left multiplication by τ. The fixed point set of h is $\{w\mathcal{B}: w \in W\}$, so the Atiyah-Bott fixed point theorem ([A-B, (4.12)]) says the Lefschetz number of τ on $H^{\cdot}(X, \mathbf{E}(\rho\lambda) \otimes \Omega^r)$ is given by

$$\sum_{w \in W} \frac{\text{trace } \tilde{h}_w}{\det_{\mathbb{C}}(1 - dh_w)}$$

where $\tilde{h}_w$ is the endomorphism of the fiber of $\mathbf{E}(\rho\lambda) \otimes \Omega^r$ over $w\mathcal{B}$ and $dh_w \in \text{End}(T_w X)$ is the derivative of h at $w\mathcal{B}$. We have

$$\text{trace } \tilde{h}_w = \sum_{\substack{A \subseteq \Delta^+ \\ |A|=r}} w(\lambda\rho\delta_A)(\tau) \quad \text{and} \quad \det_{\mathbb{C}}(1 - dh_w) = wD'(\tau).$$

Hence

$$\sum_r (-q)^{-r} \sum_s (-1)^s H^s(X, \mathbf{E}(\rho\lambda) \otimes \Omega^r)$$

$$= \sum_r (-q)^{-r} \sum_W w\left(\sum_{A \subseteq \Delta^+} \frac{\lambda\rho\delta_A}{D'}\right).$$

Now finish as in the first proof.   □

(5.3) COROLLARY. *There exists $\varepsilon > 0$ such that for all $a \in A(\varepsilon) :=$ $\{x \in A : |\alpha(x)|_F \leq \varepsilon \ \forall \alpha \in \Delta_F^+\}$, we have*

$$\mathcal{W}(\Phi_-)(a) = \delta^{1/2}(a)\rho \sum_{S \subseteq \Delta^+} (-q)^{-|S|} chV(\lambda_a \delta_S \rho^{-1}),$$

where $chV(\psi)$ is the character of the representation of $\mathcal{G}$ with highest $\mathcal{B}$-dominant weight ψ.

Proof. We can find ε such that $a \in A(\varepsilon)$ implies $\lambda_a \delta_S \rho^{-1}$ is dominant for all S. Then Bott's extension of the Borel-Weil theorem says $F(\lambda_a \delta_S \rho) = (-1)^{|\Delta^+|} V(\lambda_a \delta_S \rho^{-1})$. The claim now follows from the first proof of (5.1).   □

(5.4) THEOREM. *As elements of $R[\mathcal{T}]$, we have, for every $a \in A^{-}$,*

$$\mathcal{W}(\Phi_{J,+})(a) = \delta^{1/2}(a)\Gamma(X_J, \mathbf{E}(w_0\lambda_a)|_{X_J}) \otimes \sum_r (-q)^{-r}\Lambda^r T_e^* X_J$$

$$= \delta^{1/2}(a) \sum_{r,s} (-1)^{r+s} q^{-r} H^s(X_J, \mathbf{E}(w_0\lambda_a)|_{X_J}) \otimes \Lambda^r T_e^* X.$$

Proof. First, note that as $\mathcal{T}$-representations we have

$$\sum_r (-1)^r (-q^{-r}) \Lambda^r T_e^* X_J \simeq \prod_{\alpha \in \Delta_J^+} (1 - q^{-1}\alpha),$$

where Δ_J^+ denotes the positive roots in the span of J.

Let, as before, y_0 be the longest element of W_J. The space of sections $\Gamma(X_J, \mathbf{E}(y_0w_1\lambda)|_{X_J})$ affords the irreducible $\mathcal{L}_J$ representation with $\mathcal{B} \cap \mathcal{L}_J$-highest weight $w_1\lambda$. Since $w_0 = y_0w_1$, the reduction formula (4.3) reduces us to the case $J = \Sigma$. But here $w_1 = e$, and the result is just [C-S] combined with the Weyl character formula. $\qquad\square$

We now review the Cousin resolution of $\Gamma(X, \mathbf{E}(w_0\lambda))$, following [K]. First, if A is a topological space, $B \subseteq A$ is a closed subset, and S is a sheaf on A, then $\Gamma_B(A, S)$ denotes the sections of S with support in B. The higher derived functors of the functor $S \mapsto \Gamma_B(A, S)$ are the "local cohomology groups" $H_B^i(A, S)$. They are called 'local" because if $B \subseteq C \subseteq A$ where C is open in A, then restriction induces an isomorphism

$$H_B^i(A, S) \to H_B^i(C, S|_C).$$

We will consider these groups when B is a Bruhat cell in the flag manifold X, A is an appropriate open set in X, and S is the sheaf of germs of holomorphic sections of the line bundle $\mathbf{E}(w_o\lambda)$ (we give the sheaf and the bundle the same notation).

For $w \in W$, let X_w be the $\mathcal{B}$-orbit in X through $w\mathcal{B}$. We let U_w denote any $\mathcal{T}$-stable Zariski open set in X such that $X_w \subseteq U_w \subseteq X - \partial\bar{X}_w$. In other words, X_w is supposed to be closed in U_w. For example, we could take

$$U_w = \bigcup_{l(y) \geqslant l(w)} X_y.$$

If $w = w_0$, we have no choice but $X_{x_0} = U_{w_0}$. Let λ be a $\mathcal{B}$-dominant weight and set $\mu = w_0\lambda$ (so $\mathbf{E}(\mu)$ has global sections). Evidently the restriction gives an injection

$$\Gamma(X, \mathbf{E}(\mu)) \hookrightarrow \Gamma_{X_{w_0}}(X_{w_0}, \mathbf{E}(\mu)) = H_{X_{w_0}}^0(X_{w_0}, \mathbf{E}(\mu)).$$

The main theorem in [K] says this can be continued to a resolution of $\Gamma(X, \mathbf{E}(\mu))$ by $(\mathscr{B}, \mathfrak{g})$-modules whose ith term is

$$\bigoplus_{\mathrm{codim}(X_w)=i} H^i_{X_w}(U_w, \mathbf{E}(\mu)).$$

Mainly we are interested in the $\mathscr{T}$ action on these spaces, which is induced by the action of $\mathscr{T}$ on X_w, U_w, and $\mathbf{E}(\mu)$. It is further shown in [K] that $H^i_{X_w}(U_w, \mathbf{E}(\mu))$ is nonzero only when $i = \mathrm{codim}(X_w) = l(ww_0)$, in which case its formal character is

$$\mathrm{ch}\, H^i_{X_w}(U_w, \mathbf{E}(\mu)) = \frac{\rho \cdot w(\mu\rho)}{\prod_{\alpha > 0} 1 - \alpha} = ww_0\lambda \frac{\prod_{\alpha \in R(w^{-1})}\alpha}{\prod_{\alpha > 0} 1 - \alpha}.$$

We can now give our final cohomological interpretation of our earlier formulas. Set $\tilde{f}_w = C_{ww_0}f_{ww_0}$. By (3.1), we have

$$\mathscr{W}(\tilde{f}_w)(a) = \delta^{1/2}(a)\varepsilon(w)(ww_0\lambda_a) \prod_{\alpha \in R(w^{-1})} \alpha \prod_{\alpha > 0} \frac{1 - q^{-1}\alpha}{1 - \alpha},$$

so we get

(5.5) THEOREM. *As elements of the quotient field of $R[\mathscr{T}]$, we have, for every $a \in A^-$,*

$$\mathscr{W}(C_w f_w)(a) = \delta^{1/2}(a)(-1)^{|\Delta^+|} \sum_{r,s} (-1)^{r+s} q^{-r} \mathrm{ch} H^s_{w_0}(U_w, \mathbf{E}(w_0\lambda_a)) \otimes \Lambda^r T_e^* X$$

$$= \delta^{1/2}(a)\varepsilon(w)\mathrm{ch}\, H^{l(ww_0)}_{X_w}(U_w, \mathbf{E}(w_0\lambda_a)) \otimes \sum_r (-q)^{-r}\Lambda^r T_e^* X.$$

6. More on the unramified principal series

Recall that for $\tau \in \mathscr{T}$, the R-group is $R_\tau = \{r \in W : \tau r = \tau, r\Delta_\tau^+ = \Delta_\tau^+\}$. It is known that R_τ is isomorphic to the component group of the centralizer of τ in $\mathscr{G}$. By a theorem of Steinberg, $R_\tau = 1$ if $\mathscr{G}$ is simply connected. In the general case, we let $\tilde{\mathscr{G}}$ be the simply connected covering group of $\mathscr{G}$, $\tilde{\mathscr{T}}$ be the maximal torus in $\tilde{\mathscr{G}}$ which projects onto $\mathscr{T}$, and Z be the kernel of $\tilde{\mathscr{T}} \to \mathscr{T}$. Since Z is contained in the center of $\mathscr{G}$, the map $\chi_\tau : R_\tau \to Z$ given by $\chi_\tau(r) = (\tilde{\tau}r)(\tilde{\tau}^{-1})$, where $\tilde{\tau}$ is a fixed lift of τ in $\tilde{\mathscr{T}}$, is a group homomorphism.

(6.1) LEMMA. *χ_τ is injective. In particular, R_τ is abelian for unramified characters of A.*

Proof. Choose a representative $r' \in \tilde{\mathscr{G}}$ of r. If $\tilde{\tau}r = \tilde{\tau}$, then r' belongs to the centralizer of $\tilde{\tau}$ in $\tilde{\mathscr{G}}$ which is connected, so r' projects into the identity component of the centralizer of τ in $\mathscr{G}$. This implies $r = 1$. $\qquad\square$

Set

$$S_\tau = \{\alpha \in \Delta : \alpha(\tau) = q^{-1}\}.$$

These roots, along with R_τ, control the submodule structure of $M(\tau)$, in ways that are not completely understood. The influence of R_τ is shown in the following.

For $r \in R_\tau$, we set $\tilde{\mathscr{A}}_r = [1/C_r(\tau)]\mathscr{A}_r^\tau$. (It is easy to see that $C_r(\tau) \neq 0$.) We have a representation $R_\tau \to \mathrm{End}_{\mathscr{H}}(M(\tau)$ given by $r \mapsto \tilde{\mathscr{A}}_r$ (see [Ke])), hence for each η in the group of characters $\hat{R}_\tau$ of R_τ, the image $M_\eta(\tau)$ of

$$p_\eta := \sum_{r \in R_\tau} \eta(r)\tilde{\mathscr{A}}_r \in \mathrm{End}_{\mathscr{H}}(M(\tau))$$

is an $\mathscr{H}$- direct summand of $M(\tau)$ and there is a decomposition

$$M(\tau) = \bigoplus_{\eta \in \hat{R}_\tau} M_\eta(\tau).$$

It follows from the next result that each $M_\eta(\tau)$ is nonzero.

(6.2) LEMMA. *Assume all roots in S_τ have the same sign. If y is any element of W such that f_y^τ is defined, then $p_\eta f_y^\tau$ is nonzero.*

Proof. We have

$$p_\eta f_y^\tau = \sum_{r \in R_\tau} \frac{\bar{\eta}(r)}{C_r(\tau)} \prod_{\alpha \in R(r^{-1})} C_\alpha(\tau)C_{-\alpha}(\tau)f_{r^{-1}y}^\tau.$$

The coefficient of f_y^τ in $p_\eta f_y^\tau$ is nonzero, and the $f_{r^{-1}y}^\tau$'s are linearly independent, hence the assertion. $\qquad\square$

(6.3) PROPOSITION. *Assume all roots in S_τ have the same sign. Then each $M_\eta(\tau)$ is an $\mathscr{H}$ invariant subspace having unique irreducible submodule $U_\eta(\tau)$ and unique maximal proper submodule $N_\eta(\tau)$.*

Proof. We first show that we can assume the stronger condition that $|\alpha(\tau)| - 1$ has the same sign for all $\alpha \in \Delta^+$ for which this is not zero. Suppose S_τ consists of positive roots. We can find (c.f. [R1 (8.2)]) $w \in W$ such that $|\alpha(\tau w)| \leqslant 1$ for all positive roots α. We may and do take w to be minimal in its coset $W_\tau w$.

(6.4) LEMMA. *For w as above, the intertwining map*

$$\mathcal{A}_w^\tau : M(\tau) \to M(\tau w)$$

is an isomorphism.

Proof. First note that the minimality of w in $W_\tau w$ implies $w_0 w^{-1} \Delta_\tau^+ \subseteq w_0 \Delta^+ \subseteq \Delta^-$, so that the Θ-eigenvector $f_{ww_0}^\tau$ is defined. Moreover, (2.1) says that

$$\mathcal{A}_w^\tau f_{ww_0}^\tau = \prod_{R(w^{-1})} C_\alpha(\tau) C_{-\alpha}(\tau) f_{w_0}^{\tau w}.$$

The conditions $S_\tau \subseteq \Delta^+$, $|\alpha(\tau w)| \leqslant 1$ for all $\alpha \in \Delta^+$ imply that $R(w^{-1}) \cap \pm S_\tau = \varnothing$, so $f_{w_0}^{\tau w} \in \operatorname{im} \mathcal{A}_w^\tau$. Thus $\mathcal{A}_w^\tau$ is surjective, hence an isomorphism since $M(\tau)$ and $M(\tau w)$ have the same dimension.

If $S_\tau \subseteq \Delta^-$, we instead choose w so that $|\alpha(\tau w)| \geqslant 1$ for all positive roots, and the analogous lemma completes the proof in this case.

We now turn to the proof of (6.3). If V is an irreducible submodule of $I(\tau)$ then Frobenius reciprocity says that $\tau \delta^{1/2}$ appears in the Jacquet module V_N. By (1.2), τ appears as a Θ-eigencharacter in V^B. If all $|\alpha(\tau)| - 1$ have the same sign or are zero, Theorem (8.3) of [R1] says the multiplicity of τ as an eigencharacter in $M(\tau)$ is exactly $|R_\tau|$. Since R_τ is abelian, $|\hat{R}_\tau| = |R_\tau|$. It follows that the socle $U_\eta(\tau)$ of $M_\eta(\tau)$ is irreducible for all η. The same is true for the socle of each $M_\eta(\tau^{-1})$, so $M_\eta(\tau)$ also has a unique irreducible quotient. $\square$

We have shown that each $M_\eta(\tau)$ contains a unique-up-to-scalar Θ-eigenvector of weight τ, and this vector in fact lies in the irreducible submodule $U_\eta(\tau)$. By (6.2), this vector is $p_\eta f_{z_0}^\tau$, where $z_0 \in W_\tau$ is chosen so that $z_0^{-1} \Delta_\tau^+ \subseteq \Delta^-$.

Another consequence of (6.2) is

(6.5) PROPOSITION. *Let $x_0 = z_0^{-1} w_0$. Then*

$$\mathcal{A}_{x_0}^{\tau w_0} : M(\tau w_0) \to M(\tau)$$

is defined. Moreover,

$$\operatorname{im} \mathcal{A}_{x_0}^{\tau w_0} = \bigoplus_{\eta \in \hat{R}_\tau} U_\eta(\tau)$$

and

$$\ker \mathcal{A}_{x_0}^{\tau w_0} = \bigoplus_{\eta \in \hat{R}_\tau} N_\eta(\tau w_0).$$

Proof. It is straightforward to check that x_0 is minimal in its coset $W_{\tau w_0} x_0$.

Since $f_{w_0}^{\tau w_0}$ generates $M(\tau w_0)$, the image of $\mathscr{A}_{x_0}^{\tau w_0}$ is generated by $\mathscr{A}_{x_0}^{\tau w_0} f_{w_0}^{\tau w_0} = f_{z_0}^{\tau}$, so $p_\eta(\mathrm{im}\,\mathscr{A}_{x_0}^{\tau w_0})$ is generated by $p_\eta f_{z_0}^{\tau} \in U_\eta(\tau)$. ☐

Write $U_+(\tau)$ (respectively $U_-(\tau)$) for $U_1(\tau)$ (resp. $U_\varepsilon(\tau)$). These two spaces coincide if ε is trivial on R_τ. The notation $M_\pm(\tau)$ and $p_\pm$ have analogous meaning. By [C], we have $\widetilde{\mathscr{A}}_r \Phi_+ = \Phi_+$, which means $\Phi_+ \in M_+(\tau)$.

(6.6) PROPOSITION. *The following are equivalent:*

$$\Phi_\pm \in U_\pm(\tau) \Leftrightarrow \Phi_\mp \text{ generates } M_\mp(\tau) \Leftrightarrow S_\tau \subseteq \Delta^\mp.$$

Proof. We first show that in either case, we have $\widetilde{\mathscr{A}}\Phi_- = \varepsilon(r)\Phi_-$. This means $\Phi_- \in M_-(\tau)$.

(6.7) LEMMA. *If all roots in S_τ have the same sign, then*

$$\prod_{\alpha \in R(r-1)} 1 - q^{-1}\alpha(\tau) = \prod_{\alpha \in R(r-1)} 1 - q^{-1}\alpha^{-1}(\tau),$$

for all $r \in R_\tau$. Neither side is zero. Finally, we have

$$\prod_{\alpha \in R(r-1)} \alpha(\tau) = \varepsilon(r).$$

Proof. It seems crazy to use Whittaker models for this, but I see no other way. We may assume $S_\tau \subseteq \Delta^+$. By [C-S, (4.3), (5.4)], we have

$$0 \neq \mathscr{W}_\tau(\Phi_+) = \mathscr{W}_\tau \widetilde{\mathscr{A}}_r(\Phi_+) = \prod_{\alpha \in R(r-1)} \frac{1 - q^{-1}\alpha(\tau)}{1 - q^{-1}\alpha^{-1}(\tau)} \mathscr{W}_\tau(\Phi_+),$$

hence the first equation. For the nonvanishing, observe that $|\alpha(\tau)| = 1$ for every $\alpha \in R(r^{-1})$. To finish the lemma, use the fact that a set $\{a_1, \ldots, a_l\}$ of complex numbers which does not contain zero or one and is closed under taking inverses has the property that $\prod a_i = (-1)^l$.

Using (3.3) and (6.7), we compute

$$\widetilde{\mathscr{A}}_r \Phi_- = \frac{D_r(\tau)}{C_r(\tau)} \Phi_- = \prod_{\alpha \in R(r-1)} \frac{\alpha(\tau) - q^{-1}}{1 - q^{-1}\alpha^{-1}(\tau)} \Phi_-$$

$$= \prod_{\alpha \in R(r-1)} \alpha(\tau)\Phi_- = \varepsilon(r)\Phi_-.$$

To finish the proof of (6.6), use (3.3) and [C] to check that

$$\mathscr{A}_{x_0}^{\tau w_0} \Phi_\pm^{\tau w_0} \neq 0 \Leftrightarrow R'(z_0) \cap \pm S_\tau = .$$

The assertion now follows easily from (6.5).   □

As we have seen, it is useful to know the number of linearly independent Θ-eigenvectors in a subquotient of $M(\tau)$. We can give the answer for $M_{\pm}(\tau)$, with τ appropriately chosen in its orbit.

(6.8) PROPOSITION. *Assume* $S_\tau \subseteq \Delta^{\pm}$. *Then every generalized* Θ-*eigenspace in* $M_{\pm}(\tau)$ *is indecomposable. In other words, every* Θ-*eigenvector in* $M_{\pm}(\tau)$ *is a scalar multiple of some* $p_{\pm}f_{\bar{\tau}}$ *where* $z^{-1}\Delta_\tau^+ \subseteq \Delta^-$.

Proof. This follows from (6.7) because $\operatorname{Hom}_{\mathscr{H}}(M(\tau), M(\tau w))$ is one dimensional since an intertwining map is determined by its values on $\Phi_{\pm}^\tau$ (whichever generates $M_{\pm}(\tau)$), and the respective K-types have multiplicity one in $M(\tau w)$. (This argument is taken from [Ro2].)   □

7. Whittaker models and the R-group

Let $\hat{N}^0$ be the set of principal characters of N. We want to relate the A-orbits on $\hat{N}^0$ to characters of R_τ. Let Λ and $\tilde{\Lambda}$ be the weight lattices of $\mathscr{T}$ and $\tilde{\mathscr{T}}$ respectively. We have a surjective map $\hat{N}^0/A \to \tilde{\Lambda}/\Lambda$ defined as follows. For $\kappa \in \hat{N}^0$ and $\alpha \in \Sigma$, let $\kappa(\alpha)$ be the conductor of $\kappa \circ x_a : F^+ \to N$, where x_α is the root group for α. Also let λ_α be the fundamental dominant weight in $\tilde{\Lambda}$ which is dual to α. The map is then

$$\kappa \longmapsto \sum_{\alpha \in \Sigma} \kappa(\alpha)\lambda_\alpha.$$

Next, $\tilde{\Lambda}/\Lambda \overset{\approx}{\to} \hat{Z}$ via the map associating a weight to its character of $\tilde{\mathscr{T}}$. In turn the injection $\chi_\tau : R_\tau \to Z$ from (6.1) induces a surjection $\chi_\tau^* : \hat{Z} \to \hat{R}_\tau$. The result is a surjective map

$$\kappa \longmapsto \eta_\kappa : \hat{N}^0/A \to \hat{R}_\tau$$

given by

$$\eta_\kappa(r) = \prod_{\alpha \in \Sigma} (\lambda_\alpha(\chi_\tau(r)))^{\kappa(\alpha)}.$$

This map can also appear in the following disguise.

LEMMA (7.1). *Let* $r \in R_\tau$ *and choose a reduced expression* $r = s_1 \cdots s_n$, *where each* s_i *is the reflection for the simple root* α_i. *Then*

$$\bar{\eta}_\kappa(r) = \prod_{i=1}^{n} (s_1 \cdots s_{i-1}\alpha_i(\tau))^{\kappa(\alpha)}.$$

Proof. This is straightforward from the definitions.   □

Note that

$$r \mapsto \prod_{i=1}^{n} (s_1 \cdots s_{i-1}\alpha_i)^{-\kappa(\alpha)} := \zeta_r$$

is the restriction to R_τ of an $X(\mathscr{T})$-valued cocycle ζ on W, given by the same formula. In more detail, $\zeta_{s_\alpha} = \alpha^{\kappa(\alpha)}$, $\zeta_{xy} = (x\zeta_y)\zeta_x$, and $\zeta_r(\tau) = \bar{\eta}_\kappa(r)$ for $\alpha \in \Sigma$, x, $y \in W$ and $r \in R_\tau$.

We return to Whittaker models. Let $\mathscr{W}^\kappa_\tau : I(\tau) \to \mathrm{Ind}^G_N(\kappa)$ be the Whittaker map corresponding to κ as in section 1. Every A-orbit on the set of principal characters has a representative which is trivial on N_0 (but maybe trivial on an even larger group), and we assume κ has this property. If κ happens to be unramified, we omit the superscript.

(7.2) PROPOSITION. *Let $\tau \in \mathscr{T}$ and $w \in W$. Then*

$$\mathscr{W}^\kappa_{\tau w} \circ \mathscr{A}^\tau_w = \zeta_w(\tau) \prod_{\alpha \in R(w^{-1})} \frac{1 - q^{-1}\alpha(\tau)^{-1}}{1 - \alpha(\tau)} \, \mathscr{W}^\kappa_\tau \tag{1}$$

if w is minimal in $W_\tau w$.

$$\mathscr{W}^\kappa_\tau(f^\tau_w) = \zeta^{-1}_{ww_0}(\tau)\mathscr{W}_\tau(f^\tau_w) \tag{2}$$

if $\tau \in \mathscr{T}_w$.

(3) *As elements of $\mathbb{C}[\mathscr{T}]$,*

$$\mathscr{W}^\kappa(\Phi_+)(a) = \delta^{1/2}(a)(-1)^{|\Delta^+|}C_{w_0}\rho \sum_{w \in W} \varepsilon(w)\zeta^{-1}_{ww_0}w(\lambda_a\rho),$$

if $|\alpha(a)| \leqslant q^{-\kappa(\alpha)}$ for all $\alpha \in \Sigma$ and is zero otherwise.

Proof. Exactly as in (3.1) and (3.5), we see that $(1) \Rightarrow (2) \Rightarrow (3)$. Let $s = s_\alpha$ for $\alpha \in \Sigma$. The proof of (1) reduces, by induction on the length of w, to showing

$$\mathscr{W}^\kappa_{\tau s} \circ \mathscr{A}^\tau_s = \alpha^{\kappa(\alpha)} \frac{1 - q^{-1}\alpha^{-1}}{1 - \alpha}(\tau)\mathscr{W}^\kappa_\tau.$$

In turn, this is proved by evaluating both sides on $\Phi_{sw_0} + \Phi_{w_0}$, as in [C-S, (4.3)]. Thus, everything boils down to the following lemma, which is the analogue of [C-S, (4.2)].

(7.3) LEMMA.

$$\mathscr{W}^\kappa(\Phi_{sw_0} + \Phi_{w_0})(e) = C_\alpha(1 - \alpha^{1-\kappa(\alpha)}) \in \mathbb{C}[\mathscr{T}].$$

Proof. By the remark following reduction formula (4.3), we may assume $G = \mathrm{SL}_2(F)$, with Weyl group $\{e, s\}$. The proof of (4.1) shows that $\mathscr{W}^\kappa(\Phi_s)(e) = 1$, so we are left with $\mathscr{W}^\kappa(\Phi_e)$. By matrix multiplication,

$$s \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in aNB \Leftrightarrow \mathrm{val}(x) \leqslant -1,$$

in which case

$$a = \begin{pmatrix} -t^{-1} & 0 \\ 0 & -t \end{pmatrix} \quad \text{and} \quad \Phi_e\left(s \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = (q^{-1}\alpha)^{-\mathrm{val}(x)}(\tau).$$

Thus,

$$\int_N \kappa^{-1}(n)\Phi_e(sn)dn = \sum_{i=1}^{\infty} (q^{-1}\alpha)^i \int_{\mathrm{val}(x) = -i} \kappa^{-1}(x)dx.$$

In fact the integral is zero for $i > 1 - \kappa(\alpha)$, so the sum is actually finite. (Compare with the definition of $\mathscr{W}_\tau$.) Moreover,

$$\int_{\mathrm{val}(x) = -i} \kappa^{-1}(x)dx = q^i - q^{i-1}$$

if $1 \leqslant i \leqslant -\kappa(\alpha)$ and equals $-q^{-\kappa(\alpha)}$ for $i = 1 - \kappa(\alpha)$. Thus

$$\mathscr{W}^\kappa(\Phi_e)(e) = \sum_{i=1}^{-\kappa(\alpha)} (q^{-1}\alpha)^i q^i (1 - q^{-1}) - (q^{-1}\alpha)^{1-\kappa(\alpha)} q^{-\kappa(\alpha)}$$

$$= C_\alpha(1 - \alpha^{1-\kappa(\alpha)}) - 1.$$

This completes the proof of the lemma and the proposition.   □

By uniqueness of Whittaker models, there is only one summand $I_\eta(\tau)$ on which $\mathscr{W}_\tau^\kappa$ is nonzero. This is determined by κ as follows.

(7.4) PROPOSITION. *$\mathscr{W}_\tau^\kappa$ is nonzero on $I_{\eta_\kappa}(\tau)$.*

Proof. Pick $\eta \in \hat{R}_\tau$ and $f \in I_\eta(\tau)$ such that $\mathscr{W}_\tau^\kappa(f) \neq 0$. Then for all $r \in R_\tau$, we have

$$\bar{\eta}(r)\mathscr{W}_\tau^\kappa(f) = \mathscr{W}_\tau^\kappa \tilde{\mathscr{A}}_r(f) = \frac{\zeta_r(\tau)}{C_r(\tau)} \prod_{\alpha \in R(r-1)} \frac{1 - q^{-1}\alpha^{-1}(\tau)}{1 - q^{-1}\alpha(\tau)} \mathscr{W}_\tau^\kappa(f)$$

$$= \eta_\kappa(r)\mathscr{W}_\tau^\kappa(f),$$

by (7.2) and (6.7).   □

8. Injectivity results

(8.1) THEOREM. *Let κ be an arbitrary principal character of N. Then the Whittaker map*

$$\mathscr{W}^\kappa_\tau \colon I_{\eta_\kappa}(\tau) \to \mathrm{Ind}^G_N \kappa$$

is injective if and only if and $S_\tau \subseteq \Delta^+$.

Proof. Choose, as before, $z_0 \in W_\tau$ maximal in the Bruhat order. If $S_\tau \subseteq \Delta^+$, then $\mathscr{W}^\kappa_\tau(p_{\eta_\kappa} f_{z_0}) = \mathscr{W}^\kappa_\tau(f_{z_0}) \neq 0$ by (7.2). It follows from (6.3) that $\mathscr{W}^\kappa_\tau$ does not vanish on the socle of $I_{\bar\eta_\kappa}$, hence is injective on that space.

Conversely, suppose $\mathscr{W}^\kappa_\tau$ is injective on $I_{\eta_\kappa}(\tau)$. By (6.2), $\mathscr{W}^\kappa_\tau(f_w) = \mathscr{W}^\kappa_\tau(p_{\eta_\kappa} f_w) \neq 0$ for all $w \in W$ with the property that $w^{-1}\Delta^+_\tau \subseteq \Delta^-$. This means, by (7.2), that $R'(w^{-1}) \cap -S_\tau = \varnothing$ for all such w. Choose w so that $|\alpha(\tau w)| \geqslant 1$ for all positive roots α. Now suppose $\beta \in -S_\tau \cap \Delta^+$. Then $w^{-1}\beta < 0$ so $1 \geqslant |w^{-1}\beta(\tau w)| = |\beta(\tau)| = q$, a contradiction. $\qquad\square$

(8.2) THEOREM. *Let $\sigma \in \hat N^0$ be unramified. Suppose F belongs to $\mathscr{W}_\tau(M_+(\tau))$ and F vanishes on A. Then $F = 0$.*

Proof. Choose $w \in W$ such that $S_{\tau w} \subseteq \Delta^+$, and take w to be minimal in its coset $W_\tau w$. By (8.1), we know that $\mathscr{W}_{\tau w}$ is injective on $M_+(\tau)$. By (1.3), $\mathscr{W}_{\tau w} \circ \mathscr{A}^\tau_w$ is a nonzero multiple of $\mathscr{W}_\tau$, so $\mathrm{im}\,\mathscr{W}_\tau \subseteq \mathrm{im}\,\mathscr{W}_{\tau w}$. We may therefore assume $S_\tau \subseteq \Delta^+$.

For $a \in A^-$, we have $aBa^{-1} \cap B = (N^{w_0} \cap B)A_0 aN_0 a^{-1}$, which means the inclusion $N_0 \hookrightarrow B$ induces a bijection

$$N_0/aN_0 a^{-1} \xrightarrow{\sim} B/aBa^{-1} \cap B.$$

Thus, for $F \in (\mathrm{Ind}^G_N \sigma)^B$, and $a_1 \in A^-$, we have

$$\pi(T_a)F(a_1) = \sum_{B/aBa^{-1} \cap B} F(a_1 ba) = \sum_{N_0/aN_0 a^{-1}} F(a_1 na)$$

$$= F(a_1 a) \sum_{N_0/aN_0 a^{-1}} \sigma(a_1 na_1^{-1}).$$

Since Θ is generated by $\{T_a : a \in A^-\}$, it follows that the space

$$J_A(\tau) := \{F \in \mathscr{W}_\tau(M_+(\tau)) : F|_A = 0\}$$

is stable under Θ. Since $J_A(\tau)$ is finite dimensional, it contains a Θ-eigenvector. Now (6.8)) says, if $S_\tau \subseteq \Delta^+$, that all Θ-eigenvectors in $M(\tau)$ are multiples of f^τ_z, for $z^{-1}\Delta^+_\tau \subseteq \Delta^-$, and moreover, each eigenspace has multiplicity one. However, no $\mathscr{W}_\tau(f^\tau_z)$ vanishes on A, by (7.3)(2) and (3.1), so $J_A(\tau)$ must be zero. This completes the proof. $\qquad\square$

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