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0. Introduction

0.1. Notation

This paper is a continuation of the series of works [FraMaTschi], [BaMa], [MaTschi] devoted to counting points of bounded height on algebraic varieties. We start with recalling some basic notions and notation.

Let V be an projective variety defined over a number field k, L an ample sheaf on V, h_L a height function on $V(k)$ w.r.t. L. We normalize it in such a way that it coincides up to $\exp(0(1)))$ with $H_L(x)^{[k:Q]}$, where $H_L(x)$ is an extension invariant height, defined e.g. in [Se], p. 11. (In the Introduction to [FraMaTschi] we erroneously chose $H_L(x)$ although in the text actually used $h_L(x)$).

For a subset $U \subset V(k)$, put

$$Z_U(L; s) = \sum_{x \in U} h_L(x)^{-s},$$

$\beta_U(L) = \inf\{s \mid Z_U(L; s)\}$ converges$\}$.
   Put also

$$N_U(L; H) = \mathrm{card}\{x \in U \mid h_L(x) \leqslant H\}.$$

If U is finite, then $\beta_U(L) = -\infty$; otherwise

$$\beta_U(L) = \limsup(\log N_U(L; H)/\log H).$$

0.2. Arithmetical stratification

A Zariski closed subset $Z \subset V(k)$ is called *accumulating* (over k, w.r.t. L), if

$$\beta_V(L) = \beta_Z(L) > \beta_{V \setminus Z}(L).$$

For every V, there exists a sequence of Zariski open subsets

$$V = V_0 \supset V_1 \supset V_2 \supset \cdots$$

such that for every i, $V_i \backslash V_{i+1}$ is the minimal accumulating subset in V_i. We shall call this sequence the *arithmetical stratification*, and the numbers $\beta_i = \beta_{V_i}(L)$ its *growth orders*.

We can say that a variety V *has many points* if $\beta_V(L) > 0$ for some (and therefore all) ample L. The calculation of its arithmetical stratification and its growth orders is then the first natural problem. Of course, afterwards we hope to obtain some analytic continuation properties of $Z_{V_i}(L; s)$ and more precise information about $N_{V_i}(L; H)$, for example, asymptotic expressions of the type $CH^{\beta_i}(\log H)^{t_i}$. But this usually involves very subtle arithmetical problems, whereas the structure of the stratification, especially over sufficiently large ground fields, may be guessed (and sometimes proved) using algebro-geometric techniques.

In particular, we can state the following conjectures.

RATIONAL CURVE CONJECTURE (RCC). *If $\beta_U(L) > 0$ for a Zariski open subset $U \subset V$, then U contains an open subset of $\mathbb{P}^1_k$.*

One can construct Enriques and K3 surfaces containing infinitely many k-rational curves (it suffices to furnish one such curve and an infinite automorphism group). In [BaMa], we conjectured that arithmetical stratifications for such a surface must be infinite: unions of rational curves of maximal L-degree are consecutive accumulation subsets.

Over functional ground fields of characteristic zero, the multidimensional Mordell conjecture is proved for manifolds with ample cotangent sheaf [Mar-Des]. They do not contain rational curves.

LINEAR GROWTH CONJECTURE (LGC). *If V is a Fano variety (i.e. the anticanonical sheaf is ample), then the arithmetical stratification is finite, and for sufficiently large ground fields the last growth order for $-K_V$ equals 1.*

In [FraMaTschi], LGC was proved for homogeneous Fano varieties. It was also remarked that the circle method, when it gives asymptotic formulas, establishes LGC for certain Fano complete intersections (see e.g. [Sh1]).

In [BaMa] and [MaTschi], LGC was proved for certain two-dimensional Fano varieties (del Pezzo surfaces).

We can use LGC in order to pinpoint (parts of) accumulating subvarieties in a Fano variety V. In fact, let $W \subset V$ be a Fano subvariety. We will say that $-K_V|W$ is *too small*, if

$$K_V|W = aK_W + Z$$

in the Néron-Severi space of W, where $a < 1$, and Z is effective. For example, if V is a del Pezzo surface, and W is an exceptional curve on it, then $K_V | W = (1/2)K_W$, so that $-K_V | W$ is too small.

If LGC is valid for W, then

$$\beta_V(-K_V) \geqslant \beta_W(-K_V | W) \geqslant a^{-1}\beta_W(-K_W) > 1.$$

Therefore W must be contained in an accumulating subvariety (of some V_i of the arithmetical stratification w.r.t. $-K_V$), if LGC is valid also for V.

0.3. *Results*

In this paper, we discuss from this viewpoint Fano threefolds. After summing up the necessary preliminaries in Section 1, we prove in Section 2 the following lower linear bound:

0.4. **THEOREM.** *Let V be a Fano threefold over a number field k. Then for every Zariski open dense subset U there exists a finite extension k' of k such that if k'' contains k', then $N_{U \otimes k''}(-K; H) > c \cdot H$ for some $c > 0$ and large H, so that*

$$\beta_{U \otimes k''}(-K_V) \geqslant 1.$$

The proof is heavily based upon the classification theory of Fano threefolds, developed by Fano, Iskovskih, Shokurov, Mori and Mukai, and its main interest lies in exhibiting several different reasons for existence of many points on members of various deformation families. Actually, it establishes a stronger inequality $N_U(-K_V; H) > c \cdot H(\log H)^t$, with some $t \geqslant 0$.

In the course of the proof, we also register Fano subvarieties W with too small $-K_V | W$ for most of the families and make conjectures about the structure of the arithmetical stratification. In many cases, the growth order for these (conjecturally) accumulating subvarieties (with respect to $-K_V$) is known, and usually equals 2. Therefore, in order to prove that these varieties are accumulating, it suffices to establish that $\beta_U(-K_V) < 2$ for the complement U.

Section 3 is devoted to a discussion of known methods of estimating $\beta_U(-K)$ from above.

1. Preliminaries on Fano varieties

1.1. *Definition and examples*

In this paper, *a Fano variety V* is a smooth proper variety over a field k whose anticanonical class $-K_V$ is ample. Here are some examples.

(a) Let $V(n; d_1, \ldots, d_s)$ be a smooth complete intersection of hypersurfaces of degrees $d_1, \ldots, d_s$ in $\mathbb{P}^n$. Then the anticanonical sheaf on V is isomorphic to $\mathcal{O}(n + 1 - \Sigma d_i)$. Therefore, V is Fano iff $n + 1 - \Sigma d_i > 0$.

(b) Let G be a semisimple linear algebraic group, P a parabolic subgroup. Then the generalized flag space $P \backslash G$ is Fano. There exist also flag spaces without k-points (e.g. quadrics); they are defined by k-rational conjugacy classes of parabolic subgroups. *Every homogeneous Fano variety is a flag space*: see [Dem2].

(c) Two-dimensional Fano varieties are called *del Pezzo surfaces*. Over an algebraically closed ground field, they form 10 deformation families: (a) $\mathbb{P}^2$; (b) $\mathbb{P}^1 \times \mathbb{P}^1$; (c) S_a = the result of blowing up $1 \leqslant a \leqslant 8$ points of $\mathbb{P}^2$ in sufficiently general position: cf. [Ma].

In the next section, we shall summarily describe all 104 deformation types of Fano threefolds, discovered by prolonged efforts of Fano, Iskovskih, Shokurov, Mori, and Mukai.

1.2. *Basic invariants*

(i) *Index* $r = r(V)$ of a Fano variety is the maximal integer such that K_V is divisible by r in $\text{Pic}(V)$. Generally, *the index is not stable with respect to ground field extensions.* For example, $r(\mathbb{P}^n) = n + 1$, whereas non-trivial (having no k-points) forms of this V have index 1.

Unless stated otherwise, we shall usually give values of index over a closure of the ground field.

We have $r(V(n; d_1, \ldots, d_s)) = n + 1 - d_1 - \cdots - d_s$, while the dimension is $n - s$. The circle method works well if "the number of variables is large in comparison with degrees and number of equations", that is, if index is close to the dimension. Algebro-geometric structure of Fano varieties also tends to simplify for larger values of index.

In particular, for del Pezzo surfaces we have $r(\mathbb{P}^2) = 3$; $r(\mathbb{P}^1 \times \mathbb{P}^1) = 2$; $r(S_a) = 1$.

(ii) *Rank* $\rho = \rho(V) = \text{rk Pic}(V)$. It can also jump after a field extension, so that we shall usually give its values over a closed ground field. Over $\mathbb{C}$ it coincides with B_2. If $\dim V(n; d_1, \ldots, d_2) \geqslant 3$, $\rho(V) = 1$ according to Lefschetz. Furthermore, $\rho(S_a) = a + 1$.

(iii) *Degree* of V is $d = d(V) = (-K_V)^{\dim V}$ (self-intersection index). Usually it is more convenient to use the *reduced degree* $\delta(V) = d/r^{\dim(V)}$. For example, the projective degree $d_1 \cdots d_s$ of $V = V(n; d_1, \ldots, d_s)$ is just $\delta(V)$.

(iv) *Mori invariant* $b(V)$. Here we shall from the start extend the ground field to its closure. Put

$$b(V) = \min\{b \mid \text{through every geometric point of } V \text{ passes a rational curve } C \\ \text{with } (-K_V \cdot C) \leqslant b\}.$$

According to a deep theorem of Mori, $2 \leqslant b(V) \leqslant \dim(V) + 1$. Upper bound is achieved for $\mathbb{P}^n$.

Clearly, $b(V)$ is divisible by the index.

We shall now state some properties of growth orders.

1.3. PROPOSITION. *Let V be a homogeneous Fano variety having a k-point. Then V has no accumulating subvarieties w.r.t. any L. $V(k)$ is dense, and $\beta_U(-K_V) = 1$ for every dense Zariski open subset U if k is sufficiently large.*

Proof. Represent V as $P \backslash G$ where G is a semisimple linear algebraic group, P a parabolic subgroup. According to a theorem of Rosenlicht, G is k-unirational so that $G(k)$ is dense (see e.g. [BoSp]). Hence $V(k)$ is dense. Assume that $\beta_U(L) < \beta_V(L)$ for some dense U and ample L.

Cover V by a finite family of translates Ug_i, $g_i \in G(k)$. Since $g_i^*(L) \cong L$ for every i, we have $\beta_{Ug_i}(L) = \beta_U(L)$. Hence $\beta_V(L)$ cannot be larger.

The statement $\beta_U(-K_V) = 1$ is proved in [FraMaTschi] using Langlands' deep theory of Eisenstein series, in the case when G contains a Borel subgroup defined over k. □

REMARKS. (a) Actually, Langlands' theory can be used to show an asymptotic formula of the type

$$N_V(L; H) = \text{const} \cdot H^{\alpha(L)} P_L(\log H)(1 + O(H^{-\eta}))$$

where $\alpha(L)$ is a rational number defined in [BaMa], P_L a polynomial, $\eta > 0$.

(b) Let V be quasihomogeneous, i.e. having a dense orbit U with respect to an action of a linear group G on V. Then the argument above shows that $U(k)$ is dense, and U has no accumulating subvarieties with respect to any L.

1.4. PROPOSITION. *Let $f: W \to V$ be a birational morphism of Fano varieties, $U \subset W$ an open subset disjoint with the exceptional locus of f. Then*

$$\beta_U(-K_W) \geqslant \beta_{f(U)}(-K_V).$$

Proof. We have $-K_W = f^*(-K_V) - E$ where E is an effective divisor disjoint with U. Hence $h_E(x) > \text{const} > 0$ for $x \in U$, and

$$h_{-K_W}(x) < Ah_{f^*(-K_V)}(x) < Bh_{-K_V}(f(x))$$

for some positive constants A, B. □

1.5. PROPOSITION. *Let V be a Fano variety over a number field k, $b(V)$ its Mori invariant. Then for every Zariski dense open subset U there exists a finite extension k' of k such that if k'' contains k', then*

$$\beta_{U \otimes k''}(-K_V) \geqslant 2/b(V).$$

Proof. Choose k' in such a way that U contains a k'-point and a rational curve C with $(C \cdot -K_V) \leqslant b(V)$ passing through this point and splitting (i.e. birational to $\mathbb{P}^1$) over k'. Then

$$\beta_{U \otimes k''}(-K_V) \geqslant \beta_{C \otimes k''}(-K_V \mid C) = 2/(C \cdot -K_V). \qquad \square$$

1.6. PROPOSITION. *Let U, V be two open subsets in some Fano varieties $\bar{U}, \bar{V}$. Then*

$$\beta_{U \times V}(-K_{\bar{U} \times \bar{V}}) = \max(\beta_U(-K_{\bar{U}}), \beta_V(-K_{\bar{V}})).$$

Proof. This follows directly from the definition of β via the height zeta function, because $K_{\bar{U} \times \bar{V}} = p_1^*(K_{\bar{U}}) + p_2^*(K_{\bar{V}})$. $\qquad \square$

1.7. REMARK. Clearly, the Prop. 1.6 can be stated more precisely: we actually prove that $N_{U \otimes k}(-K_V, H) > \text{const} \cdot H^{2/b(V)}$. Similarly, in Prop. 1.5 we prove that

$$N_U(-K_W; H) > \text{const} \cdot N_{f(U)}(-K_V; H).$$

Therefore, when V is homogeneous, we may gain a power of logarithm.

2. Linear lower bound for Fano threefolds

In this section, we review the classification of Fano threefolds. The following is the most concise statement implying the lower bound $\beta(-K) \geqslant 1$ (stated in the Introduction) with the help of Propositions 1.3–1.6. We shall check it case by case making comments about possible accumulating subvarieties on the way.

2.1. THEOREM. *Every Fano threefold V over a closure of the ground field becomes isomorphic to a member of at least one of the following families:*

(i) *A generalized flag space $P \backslash G$.*
(ii) *A Fano variety with $b(V) = 2$.*
(iii) *A blow up of varieties of the previous two groups.*
(iv) *A direct product of $\mathbb{P}^1$ and a Fano surface.*

REMARK. A similar statement is valid in dimension 2.

Proof. Call a Fano variety *minimal*, if it is not a blow up of a Fano variety along a smooth center. Clearly, it suffices to check, that over a closed ground field of characteristic zero minimal Fano threefolds either are homogeneous, or have $b(V) = 2$. $\qquad \square$

All Fano varieties with $\rho = 1$ are minimal; their list is given in [Is1, 2] (cf. also [Mu]). Besides, there are minimal varieties with $\rho = 2, 3$: their deformation families are described in [MoMu1,2,3] together with information about admissible blowings up. Here is a summary of the relevant information, with the notation we shall use in the sequel.

A. *Varieties V with $r \geqslant 2$, $\rho = 1$.* In this group, there are two homogeneous members, $\mathbb{P}^3$ with $r = 4$, and quadric Q^3 with $r = 3$. The remaining five deformation families have $r = 2$; they are classified according to the value of the reduced degree $\delta = (-K/2)^3$, $1 \leqslant \delta \leqslant 5$. Denote by V_δ a member of the corresponding family. Rational curves C with $(C \cdot -K_V) = 2$ are called *lines* in [Is1-4]. We shall call them $(-K)$-*conics*. According to the Proposition 1.4 of Chapter III, [Is2], every V_δ is covered by $(-K)$-conics, so that $b(V) = 2$. Since K_V is divisible by two, there are no rational curves C with too small $-K_V | C$.

CONJECTURE. *If V belongs to the group A, and the ground field is sufficiently large, then $\beta_V(-K) = 1$ and V contains no accumulating subvarieties.*

We shall denote by $\Sigma(V)$ the scheme parametrizing $(-K)$-conics on V.

B. *Varieties with $r = 1$, $\rho = 1$.* The families $\{W_d\}$ are classified by the degree $d = (-K)^3$, taking values 2, 4, 6, 8, 10, 12, 14, 16, 18, 22. Actually, for $d = 4$ general W_4 are smooth quartics in $\mathbb{P}^4$, ((4; 4) in notation of 1.1a), but they admit flat specialization W'_4, which can be realized as a double cover of a quadric $Q^3 \subset \mathbb{P}^4$ ramified along the intersection of Q^3 with a quartic.

In this family, rational curves with $(-K \cdot C) = 2$ are called *conics*, whereas those with $(-K \cdot C) = 1$ are called *lines*.

A basic result due to Shokurov says that lines always exist (and form one-dimensional family, whose base may well be reducible and non-reduced), and conics on a W_d cover W_d, so that $b = 2$ in this group. Not all details of proofs are spelled out in the literature I was able to trace; especially the so called *hyperelliptic families* W'_4 and W_2 need some more attention; cf. below. The first basic reference is [Is2], Chapter III and Chapter II, sec. 2. Shokurov's approach was carefully worked out by Miles Reid (see [Re2], [Mu]); conics on a general W_2 are investigated in [CeVe].

CONJECTURE. *In the group B, over a sufficiently large ground field the accumulating subvariety is a union of lines and has the growth order 2 w.r.t. $-K$; the complement to it has no accumulating subvarieties, and has the growth order 1.*

C. *Varieties with $\rho = 2, 3$.* Here we refer to [MoMu1]. In this paper, all varieties with a given value of ρ are listed in the Table number ρ. We shall denote the i-th variety in this table by $\rho \cdot i$, writing ρ by Roman numerals, so that III.11 means the eleventh family with $\rho = 3$.

There are 12 minimal families in the Tables (they are easily detected by the

word "none" in the last column meaning that no divisor can be blown down to a smooth curve): II: 2, 6, 8, 18, 24, 32, 34, 36; III: 1, 2, 27, 31. We exclude II.35, on which a plane can be blown down to a point.

Among these families, the following consist of homogeneous varieties: II.32 (complete flags in a projective plane), II.34 $= \mathbb{P}^1 \times \mathbb{P}^2$, III.27 $= \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Two more are quasihomogeneous: II.36 $= \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$ and III.31 $= \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1,1))$. Undoubtedly, for them and their forms there should exist a theory of height zeta functions as precise as that of Eisenstein series. For the time being, we see at least that $b = 2$ for them because they are line bundles.

The remaining families of this group consist of conic bundles over $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$: see [MoMu2], Theorem 1.6. Therefore, they have Mori invariant 2.

CONJECTURE. *Among the conic bundles in the group C, over sufficiently large ground fields, the accumulating subvarieties are unions of lines, and have the growth order 2, whereas the complement has the growth order 1.*

This completes the proof of the Theorem 2.1 (and 0.4). In particular, we could have omitted mentioning direct products, but since more precise results can be obtained for them, we notice that all Fano threefolds with $\rho \geqslant 6$ are direct products, and hence have $\rho \leqslant 10$. ☐

In this proof of the lower bound $\beta \geqslant 1$, we tried to minimize the algebro-geometric information quoted from the classification theory. In the rest of this section, for the sake of a future finer theory, we shall give some more details about the geometry of lines, conics, centers of admissible blowings up, and definition fields. I thank Prof. J. P. Murre for extensive list of references.

In addition, we recall the known facts about (uni)rationality which can be used to ascertain that $V(k)$ is dense whenever it is non-empty, and about birational automorphisms.

2.2. DEFINITION. (a) *A variety V over a field k is called split if* $\mathrm{Pic}(V) = \mathrm{Pic}(V \otimes \bar{k})$.

(b) *An extension $k' \supset k$ is a splitting extension, if $V \otimes k'$ is split over k'.*

It seems that most algebro-geometric arguments we use for counting points fully work only for split Fano varieties: cf. e.g. [MaTschi], where we treat split del Pezzo surfaces (for them, all exceptional curves are defined over the ground field).

2.3. *Group A*

Let V be a Fano variety with $r \geqslant 2$, $\rho = 1$. It is split over k, iff $-K_V$ is divisible

by r over k. For $r = 4$, that is, $V = $ a k-form of $\mathbb{P}^3$, we have

$$V \text{ is } k\text{-split} \Leftrightarrow V \cong \mathbb{P}^3 \Leftrightarrow V(k) \neq \varnothing.$$

Forms are classified by elements of $Br(k)$ of order 4, and their theory over number fields is well understood. Certainly, quadrics ($r = 3$) can be treated similarly.

I do not know anything about splitting fields for $r = 2$. Iskovskih describes in [Is1, 2, 4] standard models for V_δ corresponding to either $|-K_V|$, or $|H|$ where $|-K_V| = |2H|$. The latter are defined over k only for split V. Let φ_{-K} (resp. φ_H) be the corresponding map. Then we have:

(i) $\delta = 1$ *case.* $\varphi_{-K}: V_1 \rightarrow W$ is a double covering of the threefold W which is a cone over the Veronese surface F_4 in $\mathbb{P}^5$. It is ramified along a smooth divisor D, which is an intersection of W and a cubic hypersurface not passing through the vertex of W. Every such covering is a V_1.

If V_1 is split, $\varphi_H: V_1 \rightarrow \mathbb{P}^2$ is a rational map with one indeterminacy point and irreducible elliptic fibers.

(ii) $\delta = 2$ *case.* If V_2 is split, $\varphi_H: V_2 \rightarrow \mathbb{P}^3$ is a double covering ramified at a smooth quartic D. Every such covering is a V_2.

In [We2] it is proved that $\Sigma(V_2)$ is a connected surface, which is smooth iff D does not contain lines; $Alb(\Sigma(V_2))$ is isomorphic to the intermediate Jacobian of V_2 of dimension 10.

(iii) $\delta = 3$ *case.* If V_3 is split, φ_H defines an isomorphism of V_3 with a smooth cubic $V(4; 3)$.

$\Sigma(V_3)$ is (generically?) smooth and irreducible.

$Alb(\Sigma(V_3))$ is isomorphic to the intermediate Jacobian of W of dimension 5 ([ClGr], [AltKl]).

(iv) $\delta = 4$ *case.* If V_4 is split, φ_H defines an isomorphism of V_4 with a smooth intersection of two quadrics $V(5; 2, 2)$.

$\Sigma(V_4)$ is isomorphic to the Jacobian of a curve of genus 2, which is simultaneously the intermediate Jacobian of V_4 ([New], [Re1]).

(v) $\delta = 5$ *case.* There is only one isomorphism class of split V_5. Its $|H|$-model in $\mathbb{P}^6$ can be described as a birational image of a smooth quadric Q^3 corresponding to the linear system $|2E-C|$, where E is hyperplane section of Q^3, C a non-plane cubic curve on it.

Another description of V_5 identifies it with a section of the Grassmannian of lines in $\mathbb{P}^4$ by a general codimension 2 hyperplane.

$\Sigma(V_5)$ is (a form of) $\mathbb{P}^2$ ([Is2], Chapter III, Prop. 1.6).

Our lower bounds for point count in this group was based exclusively on the fact that every V_δ over a closed field is covered by $(-K)$-lines, i.e. rational curves of $(-K)$-degree 2.

One can in principle quantitatively control the size of the set of *k-rational* lines

on V_δ, and, by implication, the subset $V_0(k) \subset V(k)$ consisting of points lying on rational lines.

PROBLEM. Count points of bounded height in $V_0(k)$.

PROBLEM. Count $V(k)\backslash V_0(k)$. One can try to obtain a lower bound by using birational automorphisms ([Is3], [Pu1], [Pu2]).

EXERCISE. Look at the proofs of (uni)rationality of these varieties in order to understand the ground fields over which this is true: V_4, V_5 (rationality); V_2, V_3 (unirationality). General V_1, V_2 are not rational: this is proved in [Beau]. No V_3 is rational ([ClGr]). I do not know whether V_1 is unirational.

2.4. Group B

Since $r = 1$, $\rho = 1$ in this group, all W_d are split. Another property of these varieties can be read off Mori-Mukai tables: no blowing up of a member of this group can be Fano. (Members of group A have a lot of Fano blowings up).

For a variety W of this class, we denote by $\Gamma(W)$ the base scheme parametrizing lines in W, and by $S(W)$ the subscheme of lines in W.

(i) $d = 2$ case. Here $\varphi_{-K}: W_2 \to \mathbb{P}^3$ is a double solid ramified along a smooth sextic surface.

Arithmetical genus of $\Gamma(W)$ is 6865; the $(-K)$-degree of $S(W)$ is 1248; every line intersects 625 lines. This is proved in [Mar], where it is also conjectured that for general W_2, $\Gamma(W)$ is smooth and irreducible.

For a generic W_2, $\Sigma(W_2)$ is a smooth irreducible surface, and $\text{Alb}(\Sigma(W_2))$ is isomorphic to the intermediate Jacobian of W_2 of dimension 52. This is shown in [CeVe], where also the "non-generic" cases are described in some detail.

(ii) $d = 4$ case. We denote by W_4 a smooth quartic embedded in $\mathbb{P}^4$ by $|-K|$. This family admits a flat specialization on which $|-K|$ ceases to be very ample and gives instead a double covering W_4' of a quadric Q ramified at a smooth surface of degree 8.

Collino ([Co]) shows that $\Gamma(W)$ is always one-dimensional in Char$(k) \neq 2, 3$. On W_4, $p_a(\Gamma(W)) = 1600$, deg$(S(W)) = 320$: see [Te].

For general W_4, $\Gamma(W)$ is a smooth irreducible curve ([Co], [BarVe], [BlMu]).

Tennison also considers the special case of the Fermat quartic $x_0^4 + \cdots + x_3^4 = 0$. Here $\Gamma(W)$ is reducible and non-reduced: it consists of forty plane Fermat quartics of genus 3, each of multiplicity 2; they intersect only pairwise, and every component intersects 12 of the remaining components. In Char$(k) = 3$, $\Gamma(W)$ becomes two-dimensional ([Co]).

For a general W_4, the base scheme of conics $\Sigma(W)$ is two-dimensional ([Te]) and smooth ([CoMuWe], Prop. 3.6). In Char$(k) \neq 2,3$ this dimension cannot be larger ([CoMuWe]). In [Le] it is shown that $\text{Alb}(\Sigma(W))$ is isomorphic to the intermediate Jacobian of W, which is of dimension 30.

[CoMuWe] is devoted to the numerical study of $\Sigma(W_4)$, in particular, 972 conics pass through a general point of W_4.

On W'_4, $\Gamma(W)$ has arithmetical genus 801; $\Sigma(W)$ has degree 320; every line intersects 81 lines ([Mar]).

There are two more cases with nice anticanonical models:

(iii) $d = 6$ *case*: $W_6 = V(5; 2, 3)$.

For a general W_6, lines are parametrized by an irreducible smooth curve (see [BlMu]). It has genus 271; the surface of lines is of degree 180; every line intersects 31 lines ([Mar]).

(iv) $d = 8$ *case*: $W_8 = V(6; 2, 2, 2)$.

For a general W_8, lines are parametrized by a smooth irreducible curve of genus 129, the surface of lines has degree 128. Every line intersects 17 lines ([BlMu] and [Mar]).

Alb($\Sigma(W_8)$) is isogenous to $J(W_8)$ (G. Welters, thesis).

(v) $W_{10} \subset \mathbb{P}^7$ which is a section of the Grassmannian of lines in $\mathbb{P}^5$, embedded into $\mathbb{P}^9$, by $\mathbb{P}^7$ (general case); $W'_{10} \subset \mathbb{P}^7$, a section of a cone over V_5 in $\mathbb{P}^7$ by a quadric (specialization) (see [Gu1] about this special subfamily).

Here $p_a(\Gamma(W)) = 71$, $\deg(S(W)) = 100$; every line intersects 11 lines ([Mar]).

(vi) $W_{12} \subset \mathbb{P}^8$.

(vii) $W_{14} \subset \mathbb{P}^9$: a section of the Grassmannian of lines in $\mathbb{P}^5$, embedded into $\mathbb{P}^{14}$, by $\mathbb{P}^9$ (see [Gu2]).

For a generic W_{14}, $p_a(\Gamma(W)) = 26$, every line intersects 6 lines ([Mar]). Markushevich also investigates a more special family of W_{14} consisting of generic intersections of five Schubert varieties σ_1 in the relevant Grassmannian. For them, $\Gamma(W)$ consists of fifteen irreducible components each of which is a smooth rational curve. It has 40 double points, where the components are pairwise intersecting.

(viii)–(x) $W_{16} \subset \mathbb{P}^{10}$; $W_{18} \subset \mathbb{P}^{11}$; $W_{22} \subset \mathbb{P}^{13}$.

EXERCISE. Study the ground fields over which the following (uni)rationality statements can be proved: W_{12}, W_{16}, W_{18}, some W_{22} are rational; some W_4, W'_4, W_6, W_8, W_{10}, W_{14} are unirational but irrational (the latter statement is known for general W_{10}). We gave above information about lines on irrational W's taken from [Mar].

2.5. Group C

In this group, families of conics conjecturally correspond to conic bundle structures, whereas families of lines conjecturally correspond to discriminant curves on the conic bundle bases. It is worth checking this, because the geometry then is much simpler than in group B.

One can try to estimate $U(k)$ from above, where U is the complement to lines, by looking arithmetically on the families of non-split conics.

Using sieves, Serre in [Se2] proved that most fibers of such a family remain non-split, i.e. have no k-points.

What we need, is an estimate of the "volume" of split conics, that is, the coefficient $c(s)$ in the asymptotic

$$N_{C(s)}(-K_V \mid C(s), H) \approx c(s)H$$

as a function of a height of the base point s.

2.6. Remarks

(i) In the Group B, the conjecturally accumulating surface of lines $S(W)$ is used to calculate the intermediate Jacobian, i.e. essentially the middle motive of W. For $K3$ surfaces, deleting rational curves leads to a mixed motive, extension of W by Tate motives. Can one establish a direct connection between counting points and motives? (Over finite fields, this is of course achieved via étale cohomology).

(ii) Let k be a field of functions on a curve over an algebraically closed constant field. J. Kollàr asked me whether a Fano variety over k always has a k-point.

A natural framework for this problem is the class of C_1-fields k. For rational surfaces S, in my Moscow ICM talk it was conjectured that $S(k) \neq \varnothing$. Using the classification theory, Colliot-Thélène was able to prove this for the cases I left open. The same approach partially works for Fano threefolds V. One knows that $V(k)$ is non-empty for:

(a) Fano complete intersections of any dimension.

(b) Homogeneous Fano varieties (Springer's theorem: cf. J.-P. Serre, Cohomologie Galoisienne, Ch. III, 2.4).

(c) Fano threefolds admitting a structure of a conic bundle over k.

(d) Fano threefolds containing a rational curve or surface defined over k.

It would be interesting to investigate bad cases systematically. This will probably involve some understanding of splitting fields.

3. Non-minimal threefolds and exceptional heights

3.1. Non-minimal Fano threefolds

In the tables of [MoMu], there are many Fano threefolds that can be obtained by blowing up minimal homogeneous models. Since for the latter the linear growth conjecture is known in a strong form, one can try to deduce it for blowings.

In 3.2, we list some Fano blowings of $\mathbb{P}^3$ and Q^3.

In 3.3, we prove by a direct count the linear growth conjecture for $\mathbb{P}^n$ blown up along a $\mathbb{P}^m$ (over $\mathbb{Q}$). In [MaTschi], we succeeded to treat similarly $\mathbb{P}^2$ blown up along $\leqslant 4$ points. Hopefully, such methods can be applied to some more cases listed in 3.2, which was our motivation for including them.

Finally, in 3.4 we make some comments on heights with respect to the exceptional divisors, motivated by the previous discussion.

3.2. *Fano blowings up of* $\mathbb{P}^3$ *and* Q^3

In the following list, a notation of the type $X(Z) \to Y$ means that Y is obtained from X by blowing it up along Z. The list is far from complete.

$\mathbb{P}^3$ (point) $\to$ II.35 (strict transform of the twisted cubic passing through the blown up point) $\to$ III.16.

$\mathbb{P}^3$ (line) $\to$ II.33 (conic) $\to$ III.18

$\mathbb{P}^3$ (conic) $\to$ II.30.

$\mathbb{P}^3$ (plane cubic) $\to$ II.28.

$\mathbb{P}^3$ (twisted cubic) $\to$ II.27.

$\mathbb{P}^3$ (elliptic curve, intersection of two quadrics) $\to$ II.25.

$\mathbb{P}^3$ (intersection of a quadric and a cubic) $\to$ II.15.

$\mathbb{P}^3$ (intersection of two cubics) $\to$ II.4.

$\mathbb{P}^3$ (a curve of degree 6 and genus 3, intersection of cubics) $\to$ II.12.

$\mathbb{P}^3$ (a curve of degree 7 and genus 5, intersection of cubics) $\to$ II.9.

Q^3 (point) $\to$ II.30 (a point) $\to$ III.19.

Q^3 (line) $\to$ II.31.

Q^3 (conic) $\to$ II.29.

Q^3 (elliptic curve of degree 5) $\to$ II.17.

Q^3 (rational quartic spanning $\mathbb{P}^4$) $\to$ II.21.

Now we will treat the first two items of this list.

Put $k = \mathbb{Q}$. Consider a morphism $\pi: V \to \mathbb{P}^n$, blowing up a projective subspace $\mathbb{P}^m \subset \mathbb{P}^n$, $m \leqslant n - 2$. Let X be the exceptional divisor on V, $U = V \backslash X$. We put $\Lambda = [\pi^*(\mathcal{O}(1))]$, $\ell = [\mathcal{O}(X)]$, the square brackets denoting classes in $\mathrm{Pic}(V)$ which we write additively. Choose an ample sheaf L on V, $[L] = a\Lambda - b\ell$, and a Weil height h_L on V. We shall consider h_L, h_Λ, h_ℓ and make the respective point counts only up to $\exp(O(1))$.

The following theorem is a generalization of Serre's result for $n = 2$, $m = 0$ ([Se1]).

3.3. THEOREM. *Up to* $\exp(O(1))$ *we have*

$$N_U(L; H) \cong \begin{cases} H^{(m+2)/(a-b)} & \text{if } (n-m-1)a < (n+1)b; \\ H^{(n+1)/a} \log(H) & \text{if } (n-m-1)a = (n+1)b; \\ H^{(n+1)/a} & \text{if } (n-m-1)a > (n+1)b. \end{cases}$$

In particular, $-K_V = (n+1)\Lambda - (n-m-1)\ell$ is a linear growth point.

Proof. Let $(x_0, \ldots, x_m; x_{m+1}, \ldots, x_n)$ be a coordinate system in which $\mathbb{P}^m$ is given by $x_{m+1} = \cdots = x_n = 0$. For a point $x \in V$ with $\pi(x) = (x_i) \in \mathbb{Z}^{n+1}_{\mathrm{prim}}$, put $d = \gcd(x_{m+1}, \ldots, x_n)$. Then the height $h_{2\Lambda - \ell}(x)$ can be calculated as Weil's height of a point with projective coordinates $(x_i x_j | j > m)$. But the gcd of these coordinates is exactly d. It follows easily that $h_\ell(x)$ is equivalent to

$$h_\ell(x) \cong d \frac{|x_0| + \cdots + |x_n|}{|x_m| + \cdots + |x_n|}$$

(represent h_ℓ as $h_{2\Lambda}/h_{2\Lambda - \ell}$). It follows that if h_0 (resp. h_1) is the height of $(x_0 : \cdots : x_m)$ (resp. $(x_{m+1} : \cdots : x_n)$), then

$$h_{a\Lambda - b_\ell}(x) \cong (h_0 + dh_1)^{a-b} h_1^b.$$

On the other hand, there are about $h_0^m h_1^{n-m-1}$ points with given values h_0 and h_1. It follows that

$$N_U(a\Lambda - b_\ell; H) \cong \sum_{\substack{(h_0, h_1, d) \\ (h_0 + dh_1)^{a-b} h_1^b \leqslant H}} h_0^m h_1^{m-n-1}, \tag{3.1}$$

We split (3.1) into three parts. Part $\Sigma_1 + \Sigma_2$ is taken over $dh_1 \leqslant h_0$. For a fixed $h_1 \leqslant h_0$, there are about $h_0 h_1^{-1}$ values of d. We incorporate this into the respective summand of (3.1) which then becomes $h_0^{m+1} h_1^{m-n-2}$.

Furthermore, in this domain $h_0 + dh_1$ is equivalent to h_0. Hence it suffices to sum over $h_0^{a-b} h_1^b \leqslant H$, which, together with $h_1 \leqslant h_0$, gives $h_1 \leqslant \min\{h_0, H^{1/b} h_0^{-(a-b)/b}\}$. The bound dividing two different minima is $h_0 = H^{1/a}$. Hence we can put

$$\Sigma_1 = \sum_{h_0 = 1}^{H^{1/a}} h_0^{m+1} \sum_{h_1 = 1}^{h_0} h_1^{n-m-2}, \quad \Sigma_2 = \sum_{h_0 = H^{1/a}}^{H^{1/(a-b)}} h_0^{m+1} \sum_{h_1 = 1}^{H^{1/b} h_0^{(b-a)/b}} h_1^{n-m-2}.$$

The sum Σ_3 is taken over $dh_1 > h_0$, where $h_0 + dh_1$ is equivalent to dh_1, so that summation can be taken over the domain $W = \{h_0 < dh_1, d^{a-b} h_1^a \leqslant H\}$. We present the result in the form

$$\Sigma_3 = \sum_{h_1 = 1}^{H^{1/a}} h_1^{n-m-1} \sum_{d \leqslant H^{1/(a-b)} h_1^{a/(b-a)}} \sum_{h_0 < dh_1} h_0^m.$$

Now all three sums can be calculated directly. We obtain $\Sigma_1 \cong H^{(n+1)/a}$. However, the behaviour of $\Sigma_{2,3}$ varies in the different regions of the Pic-plane (a, b). The reason is that when we sum up one or two inner sums, we get a power

of h_1 which may be < -1, -1 or > -1, depending on whether $(n - m - 1)a$ is $<$, $=$, or $>(n + 1)b$. We leave the rest to the reader. $\qquad\square$

3.4. *Exceptional heights*

We will supplement the direct calculation above by certain qualitative arguments. They are centered around a vague notion that the linear growth conjecture for non-minimal Fano varieties is connected with the fact that *exceptional heights are small, at least in average.*

Concretely, let $f: V' \to V$ be a birational morphism of Fano varieties, isomorphic outside a divisor $E \subset V'$, $C = f(E)$. Let $U \subset V \setminus C$, $U' = f(U)$, $K_V = K$, $K_{V'} = K'$. We have $-K' = -f^*(K) - D$, supp$(D) \subset E$. We put $h_D = h_{\mathcal{O}(D)}$ etc.

3.4.1. PROPOSITION. *Assume that $\beta_U(-K) = 1$. Put for $\varepsilon > 0$, $N \to \infty$,*

$$\sum_{\substack{x \in U' \\ h_{-K}(f(x)) \leqslant N}} h_D^{1+\varepsilon}(x) = N\psi_\varepsilon(N)$$

(notice that the number of summands is $N_U(-K; N)$ which equals approximately N). Then

$$\beta_{U'}(-K') = 1 \Leftrightarrow \sum_{N=1}^{\infty} \frac{\psi_\varepsilon(N)}{N^{1+\varepsilon}} \text{ converges for all } \varepsilon > 0.$$

Proof. This is essentially Abel summation. Clearly, $\beta_{U'}(-K') \geqslant \beta_U(-K)$. Hence equality means that the series $\sum_{x \in U'} h_{-K'}(x)^{-(1+\varepsilon)}$ converges for all $\varepsilon > 0$. This series can be rewritten as

$$\sum_{x \in U'} h_{-f^*(K)-D}(x)^{-(1+\varepsilon)} = \sum_{x \in U'} h_D(x)^{1+\varepsilon} h_{-K}(f(x))^{-1-\varepsilon}. \tag{3.2}$$

We may and will assume that $h_{-K}(f(x))$ takes values $1, 2, 3, \ldots$. Put

$$a(n) = \sum_{\substack{x \in U' \\ h_{-K}(f(x)) = n}} h_D(x)^{1+\varepsilon}$$

Then (3.2) equals

$$\sum_{n=1}^{\infty} a(n)n^{-(1+\varepsilon)} = \sum_{N=1}^{\infty} \left(\sum_{n=1}^{N} a(n) \right) (N^{-1-\varepsilon} - (N+1)^{-1-\varepsilon}),$$

and the statement becomes clear. $\qquad\square$

Notice that exceptional heights can be small only in average.

In fact, the discussion in [MaTschi], 2.3, shows that on split del Pezzo surfaces of degree $\leqslant 6$, exceptional heights infinitely often are comparable with ample heights, because the sum of all exceptional curves is proportional to the anticanonical class.

On the other hand, the argument in [MaTschi], sec. 4 shows, that exceptional heights can be infinitely often smaller than an arbitrarily small power of an ample height.

Here is an intuitive reason, explaining why for most points x, an exceptional height h_E should be small. Let us use an Arakelov definition of $h_E(x)$ representing it as a product of local heights

$$h_E(x) = \prod_{v \in M_k} h_{\mathscr{E},v}(x).$$

Here $\mathscr{E}$ denotes a divisor on an O_k-model $\mathscr{V}'$ of V', inducing E on its generic fiber V', and $h_{\mathscr{E},v}(x)$ is an exponentiated local intersection index of $\mathscr{E}$ with the section $\bar{x} \colon \mathrm{Spec}(O_k) \to \mathscr{V}'$ corresponding to x.

Now, if we were in a geometric situation (imagining $\mathrm{Spec}(O_k)$ as, say, a curve), for most x's the local intersection index would vanish, because on a blowing down $\mathscr{V}$ of $\mathscr{V}'$ the non-vanishing of an intersection index would mean that $\bar{x}$ should pass through a subset of codimension > 1.

Can one make this argument quantitative and applicable in Arakelov geometry?

Note finally that a different type of statement about smallness of exceptional heights plays a crucial role in the latest proof of Mordell's conjecture (Bombieri's version of the Vojta-Faltings argument).

It is based upon a refinement of an earlier Mumford's inequality. Let V be a smooth projective curve.

3.4.2. LEMMA. (a) *The diagonal divisor $\Delta \subset V \times V$ can be blown down iff $g = $ genus of $V \geqslant 2$.*

(b) *There exists a map $j \colon V(k) \to F$ with finite fibers of bounded cardinality into an Euclidean space F such that for some constant c and all $(x, y) \in (V \times V \backslash \Delta)(k)$ we have*

$$\log h_\Delta(x, y) \leqslant g^{-1}(|j(x)|^2 + |j(y)|^2) - 2(j(x) \cdot j(y)) + c,$$

where the norms and the scalar product are taken in F.

Moreover, $|j(x)|$ as a function of x up to $O(1)$ coincides with the logarithm of an ample height function.

It would be quite interesting to try to extend Mumford's argument to multidimensional manifolds with ample tangent bundle.

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