COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 84, nº 1 (1992), p. 85-89
<http://www.numdam.org/item?id=CM_1992__84_1_85_0>

Compositio Mathematica **84**: 85–89, 1992.

Inducing 'Supercuspidal' Representations of Unipotent p-adic groups from compact-mod-Center Subgroups

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Received 10 January 1991; accepted 20 August 1991

Keywords: nilpotent p-adic Lie group, square integrable irreducible representation, supercuspidal representation, induced representation, compact open subgroup.

Abstract. Let G be a p-adic nilpotent Lie group, π an irreducible unitary representation of G with matrix coefficients that are L^2 functions modulo the center Z of G. It is proved that π is induced from a character on a subgroup that is compact modulo Z.

Let G be a totally disconnected locally compact group, with center Z. We shall say that an irreducible unitary representation π of G is *supercuspidal* if π has matrix coefficients with compact support modulo Z. In the case of reductive p-adic groups, this agrees with the standard definition (see [5]). For these groups, it is a "classical" conjecture that any supercuspidal representation is induced from a finite-dimensional representation on a subgroup that is compact mod Z; while some recent progress has been made, the conjecture is still open.

In this note, we show that the corresponding result for p-adic nilpotent groups is true. More precisely, we prove:

THEOREM. *Let G be the group of $\mathbb{Q}_p$-rational points of a unipotent algebraic group over $\mathbb{Q}_p$, and let Z be the center of G. If π is an irreducible unitary representation of G with square integrable matrix coefficients* mod Z, *then there is an open subgroup $K \supseteq Z$ and a character χ on K such that K/Z is compact and* $\pi = \mathrm{Ind}_K^G \chi$.

REMARK. It is clear that supercuspidal representations have matrix coefficients that are square integrable mod Z. For nilpotent p-adic groups, the converse is true (see below); for reductive p-adic groups, the converse is known to be false.

Before embarking on the proof, we recall some facts about representation theory for G. The basic principle (see [3]) is that the theory is the same as for real nilpotent Lie groups. We write P^n for the compact open subgroup $p^n\mathbb{Z}_p \subseteq \mathbb{Q}_p$.

*Supported by NSF Grant DMS 89-02993.

Let ψ be the standard character on $\mathbb{Q}_p$ (ψ is trivial on $\mathbb{Z}_p$ but not on P^{-1}), let $\mathfrak{g}$ be the Lie algebra of G (over $\mathbb{Q}_p$), and let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$; $\mathfrak{g}^* \simeq \mathfrak{g}^\wedge$ under the correspondence of $l \in \mathfrak{g}^*$ with the homomorphism $X \mapsto \psi(l(X))$ of $\mathfrak{g}^* \to S^1$. G acts on $\mathfrak{g}$ by Ad and on $\mathfrak{g}^*$ by the contragredient action, Ad*, and $G^\wedge \simeq \mathfrak{g}^*/\mathrm{Ad}^*(G)$, just as described by Kirillov theory for real groups. For $l \in \mathfrak{g}^*$, let $\mathfrak{r}_l = \{Y \in \mathfrak{g}: l([X, Y]) = 0 \text{ for all } X \in \mathfrak{g}\}$. There exists a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $l|_{\mathfrak{h}}$ is a Lie homomorphism and $2 \dim \mathfrak{h} = \dim \mathfrak{g} + \dim \mathfrak{r}_l$ (as $\mathbb{Q}_p$-vector spaces). Define χ_l on $M = \exp \mathfrak{h}$ by $\chi_l(\exp Y) = \psi(l(Y))$. (If one realizes $\mathfrak{h}$ as a Lie subalgebra of the strictly upper triangular matrices, then exp is the usual exponential,

$$\exp Y = \sum_{j=0}^{\infty} \frac{Y^j}{j!};$$

the sum is really finite.) Then $\mathrm{Ind}_H^G \chi_l = \pi_l$ is the element of $G^\wedge$ corresponding to l. For $\pi \in G^\wedge$, we let $\mathcal{O}_\pi$ be the corresponding Ad*(G)-orbit; thus $l \in \mathcal{O}_{\pi_l}$.

For $\pi \in G^\wedge$, the following conditions are equivalent:

(a) π has square integrable matrix coefficients mod Z.
(b) For all $l \in \mathcal{O}_\pi$, $\mathfrak{r}_l = \mathfrak{z}$ (the center of $\mathfrak{g}$, and the Lie algebra of Z).
(c) For any $l \in \mathcal{O}_\pi$, $\mathcal{O}_\pi = l + \mathfrak{z}^\perp$.
(d) Let $X_1, \ldots, X_r$ span a subspace of $\mathfrak{g}$ complementary to $\mathfrak{z}$. Then the $r \times r$ matrix $A = (a_{ij}) = (l([X_i, X_j]))$ is invertible.

In fact, if one defines Haar measure on G/Z by exponentiating the Haar measure on $\mathfrak{g}/\mathfrak{z} \simeq \mathbb{Q}_p X_1 + \cdots + \mathbb{Q}_p X_r$ that gives $\mathbb{Z}_p X_1 + \cdots + \mathbb{Z}_p X_r$ mass 1, then the formal degree d_π of π with respect to this measure is $|\mathrm{Det}\, A|^{1/2}$, where $|\cdot|$ denotes the usual p-adic absolute value.

The equivalence of (a)–(d) and the other results on square integrable representations are proved in [4] and [1] for real groups; both proofs can be adapted to the p-adic situation. Further remarks on this matter can be found in [2], and further details will appear in a forthcoming book by F. P. Greenleaf and the author.

Van Dijk proved in [2] that any π satisfying the equivalent conditions (a)–(d) is in fact supercuspidal. (See our earlier remark.) We do not need van Dijk's result to prove the Theorem; thus we get a new proof that if π has square integrable coefficients, then π is supercuspidal. We say more about the results in [2] below.

Proof of the Theorem. We use induction on $\dim G$; when $\dim G = 1$, G is Abelian and the theorem is trivial. Let $Z = \exp \mathfrak{z}$ be the center of G, let π have square-integrable matrix coefficients mod Z, and let $l \in \mathcal{O}_\pi$. If $\dim Z > 1$, then there is a 1-dimensional subgroup $Z_0 \subseteq Z \cap \mathrm{Ker}\, \pi$; this reduces the problem for G to one for G/Z_0, where the inductive hypothesis applies.

We may therefore assume that $\dim Z = 1$ and that l is nontrivial on Z. Let $X_1 \in \mathfrak{z}$ satisfy $l(X_1) = 1$, let $\bar{Y}$ be central in $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{z}$, and let Y be a pre-image of Y in $\mathfrak{g}$ with $l(Y) = 0$. (Since Y and $Y + \alpha X_1$ map to $\bar{Y}$, this is possible.) Just as in the real case (see, e.g., Lemma 1.1.12 of [1]), the centralizer $\mathfrak{g}_0$ of Y in $\mathfrak{g}$ is an ideal of codimension 1. We can pick a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ such that:

(i) for all j, $\mathfrak{g}_j = \operatorname{span}\{X_1, \ldots, X_j\}$ is an ideal of $\mathfrak{g}$;
(ii) $\mathfrak{g}_0 = \mathfrak{g}_{n-1}$ and $Y = X_2$.

We may also assume (possibly rescaling elements X_j with $j > 1$) that all structure constants for this basis are in $\mathbb{Z}_p$; i.e., $[X_i, X_j] = \sum_{k=1}^n c_{ijk} X_k$, with all $c_{ijk} \in \mathbb{Z}_p$. (In addition, $c_{ijk} = 0$ if $k \geqslant i$ or $k \geqslant j$, by (i) above.) We use this basis to define Haar measure on G and on G/Z so that $\exp(\sum_{j=1}^n \mathbb{Z}_p X_j)$ has mass 1 in G and its image in G/Z has mass 1 there.

By construction, $[X_2, X_j] = 0$ if $j < n$. Therefore the first row and first column of the matrix $A = (l([X_i, X_j]) \colon 2 \leqslant i,\ j \leqslant n)$ have only one nonzero element each, namely $l([X_2, X_n])$ and $l([X_n, X_2])$ respectively. In $\mathbb{G}_0$, the radical $\mathfrak{r}_{l_0}$ of $l_0 = l|_{\mathfrak{g}_0}$ is $\mathfrak{z}_0 = \mathbb{Q}_p X_1 + \mathbb{Q}_p X_2$, the center of $\mathbb{G}_0$. (Obviously $\mathfrak{z}_0 \subseteq \mathfrak{r}_{l_0}$, but $\dim \mathfrak{r}_{l_0} \leqslant \dim \mathfrak{r}_l + 1 = 2$.) The corresponding matrix $A_0 = (l_0([X_i, X_j]) \colon 3 \leqslant i,\ j \leqslant n-1)$ is nonsingular because $\operatorname{Det} A = l([X_2, X_n])^2 \operatorname{Det} A_0$. Let σ be the representation corresponding to l_0. The above computation of $\operatorname{Det} A_0$ shows that σ_0 is square-integrable; Kirillov theory says that σ induces to π. In fact, the computation also shows that if we normalize Haar measure on G_0/Z_0 (where $Z_0 = \exp \mathfrak{z}_0$) by giving $\exp(\Sigma_{j=3}^{n-1} \mathbb{Z}_p X_j)$ mass 1, then the formal degrees of π, σ are related by $d_\pi = |l([X_2, X_n])| d_\sigma$.

Let K_0 be a compact-mod-center subgroup of G_0 such that σ is induced from χ_0 on K_0. Then $K_0 = H_0 \exp(\mathbb{Q}_p X_1 + \mathbb{Q}_p X_2)$, where $\log H_0 \subseteq \operatorname{span}\{X_3, \ldots, X_{n-1}\}$. Let $K_1 \subseteq G_0$ be the group generated by $H_0 \exp \mathbb{Z}_p X_2 \exp \mathbb{Q}_p X_1$. K_1 is compact mod Z; the reason is that if we use exponential coordinates on G, we need only to worry that the X_2 coordinates of elements stay bounded, and the compactness of H_0 insures this. We may assume (by perhaps increasing K_1) that $K_1 = H_0 \exp(\mathbb{Q}_p X_1 + P^{g_0} X_2)$. Choose an open subgroup $P^h \subseteq \mathbb{Q}_p$, $h \geqslant 0$, such that $\exp(P^h X_n)$ normalizes K_1 and fixes χ_0 there. (Clearly $\exp tX^n$ normalizes K_1 if it normalizes $K_1 \bmod Z_1$ and it fixes χ_0 if it fixes χ on the compact open group generated by $H_0 \mathbb{Z}_p X_2 \exp \mathbb{Z}_p X_1$. Because K_1/Z_1 is open and χ is locally constant, for every $\bar{x} \in K_1/Z_1$ there is an integer $n(\bar{x})$ and a neighborhood $U_{\bar{x}}$ of $\bar{x}$ such that conjugation by any element of $\exp(P^{n(\bar{x})} X_n)$ maps $U(\bar{x})$ into K_1/Z_1. Let $U(\bar{x}_1), \ldots, U(\bar{x}_m)$ cover K_1/Z_1, and let $h_1 = \max(n(x_1), \ldots, n(x_m))$. Then $\exp(P^{h_1} X_n)$ normalizes K_1/Z_1 and hence K_1. The proof that we can choose $h \geqslant h_1$ so that $\exp(P^h X_n)$ also fixes χ_0 is similar.) Suppose that $[X_n, X_2] = aX_1$, so that $l([X_n, X_2]) = a$; choose an integer g so that $aP^{g+h} = \mathbb{Z}_p$. Since $h \geqslant 0$ and $a \in \mathbb{Z}_p$, $g \leqslant 0$; also, $g \leqslant g_0$. Then $|l(a)| = p^{-(g+h)}$. We know that χ_0 is trivial on $\exp \mathbb{Q}_p X_2$, since $\exp \mathbb{Q}_p X_2 \subseteq \operatorname{Ker} \sigma_0$. Furthermore, $\exp(tX_n) \exp(uX_2) \exp(-tX_n) = \exp$

$(uX_2 + atuX_1)$; this shows that $\exp tX_n$ fixes χ_0 on $K_1 \exp P^g X_2 = K_0 \exp P^g X_2$. Define χ on $K = K_0 \exp P^g X_2 \exp P^h X_n$ by letting $\chi(k_0 \exp tX_2 \exp uX_n) = \chi_0(k_0)$; the definition of h and the above remarks show that χ is a character.

Proving directly that $\mathrm{Ind}_K^G \chi \simeq \pi$ presents some problems (though a direct proof does exist); the argument that follows has its own interest. Let $\rho = \mathrm{Ind}_K^G \chi$. Then $\rho|_Z$ is a multiple of $\chi|_Z$, and the same is true for $\pi|_Z$. Since the Kirillov orbit of π is $l + \mathfrak{z}^\perp$, any irreducible agreeing with π on Z must be π. Therefore ρ is a multiple of π.

Realize σ as $\mathrm{Ind}_{K_0}^{G_0} \chi_0$; since $K_0 \backslash G_0$ is discrete, counting measure on cosets is an invariant measure. The function $\varphi: G_0 \to \mathbb{C}$ defined by

$$\varphi_0(x_0) = \begin{cases} \chi_0(x_0) & \text{if } x_0 \in K_0, \\ 0 & \text{if } x_0 \notin K_0 \end{cases}$$

is clearly in $\mathscr{H}_{\pi_0}$, and $\|\varphi_0\|_2^2 = 1$. It is easy to see that $\langle \sigma(x)\varphi_0, \varphi_0 \rangle = \varphi_0(x)$. Therefore the matrix coefficient $f_0 = f_{\varphi_0, \varphi_0}$ is equal to φ_0, and

$$\|f_0\|^2 = \int_{G_0} |\varphi_0(x)|^2 \, dx = \bar{m}_0(K_0),$$

where $\bar{m}_0$ is Haar measure on G_0/Z_0. Since $\|f_0\|^2 = d_{\pi_0}^{-1} \|\varphi_0\|^4 = d_{\pi_0}^{-1}$, this means that $d_{\pi_0}^{-1} = \bar{m}_0(K_0)$. Hence $d_\pi^{-1} = |a|^{-1} \bar{m}_0(K_0)$, where a was defined above. We have $K_0 = H_0 Z_0$ and $K = (H_0 \exp(P^g X_2) \exp(P^h X_n))Z$; a Fubini-type argument (using the fact that the map of $\mathfrak{g}_0 \times \mathbb{Q}_p$ to G taking (X_0, t) to $\exp X_0 \exp tX_n$ preserves Haar measure; see §1.2 of [1] for the corresponding result in the real case) shows that

$$\bar{m}(K/Z) = \bar{m}_0(K_0/Z_0)|p^{g+h}|,$$

where $\bar{m}$ is Haar measure on G/Z. Since $|p^{g+h}| = |a|^{-1}$, we have

$$d_\pi^{-1} = \bar{m}(K/Z).$$

Because K/G is discrete, counting measure for cosets gives an invariant measure that we use to define $\mathscr{H}_\rho$. Define $\varphi: G \to \mathbb{C}$ by

$$\varphi(x) = \begin{cases} \chi(x) & \text{if } x \in K, \\ 0 & \text{if } x \notin K. \end{cases}$$

Then $\varphi \in \mathscr{H}_\rho$ and $\|\varphi\|_2^2 = 1$. The same calculation as with φ_0 shows that the matrix coefficient $f = f_{\varphi,\varphi}$ satisfies $\|f\|^2 = \bar{m}(K/Z) = d_\pi^{-1}$.

The vector φ is clearly cyclic for ρ. Suppose that ρ is not irreducible and that $\mathscr{H}_\rho = \bigoplus_{j=1}^r \mathscr{H}_j$ is a decomposition of ρ into irreducibles (all equivalent to π; r may be ∞). Write $\varphi = \sum_{j1}^r \varphi_j$ correspondingly, so that $\sum_{j=1}^\infty \|\varphi_j\|^2 = 1$, and let $f_j = f_{\varphi_j,\varphi_j}$ be the matrix coefficient corresponding to φ_j. Clearly $\langle \rho(x)\varphi_i, \varphi_j\rangle = 0$ if $i \neq j$; therefore

$$f = \sum_{j=1}^r f_j.$$

Now $\|f\| = d_\pi^{-1/2}$ and $\|f_j\| = d_\pi^{-1/2}\|\varphi_j\|^2$. Hence

$$\|f\| = \sum_{j=1}^r \|f_j\|.$$

This is possible only if the f_j are all nonnegative multiples of f. Hence if U_j: $\mathscr{H}_1 \to \mathscr{H}_j$ gives the unitary equivalence of ρ_1 with ρ_j, then $U_j(\varphi_1) = c_j\varphi_j$ for some constants c_j. But it is then obvious that $\varphi = (\varphi_1, \varphi_2, \dots)$ is not cyclic in $\mathscr{H}_\rho$. This contradiction completes the proof.

REMARK. Van Dijk proved in [2] not only that square-integrable π have compact-mod-center matrix coefficients, but, that if $v, w \in \mathscr{H}_\pi$ and $\varphi, \psi \in C_0^\infty(G)$ (the space of locally constant functions with compact support), then $f_{\pi(\varphi)v,\pi(\psi)w}$ has compact support mod Z. One can also prove this by mimicking the proof of the corresponding result for real groups (see Theorem 4.5 of [1]); we omit the details. There does not seem to be an easy way to modify the above proof to yield Van Dijk's result as well.

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