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Lawson homology for quasiprojective varieties

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1. Introduction

For a long time algebraic cycles have been used in the study of algebraic varieties and are themselves a fascinating primary object of study. Recently, new and interesting invariants for algebraic varieties, arising from the structure of the algebraic cycles supported on them, have been studied, generalizing many well-known invariants. Examples are found in the theory of *Higher Chow Groups* by S. Bloch [3] and in the *Lawson Homology* introduced by E. Friedlander in [8].

This paper introduces Lawson homology for quasi-projective varieties, using compactifications together with a relative version of Lawson homology (cf. [24]). Lawson homology, as defined by Friedlander (and subsequently developed in his work [9], and also in [11] and [23]), is a set of invariants attached to a closed projective algebraic variety X derived from the homotopy properties of the Chow monoid $\mathscr{C}_p(X)$ of effective p-cycles supported on X.

The precursor of this theory is B. Lawson's foundational paper [21], where he establishes a homotopy equivalence

$$\Sigma\!\!\!/ : \tilde{\mathscr{C}}_p(X) \to \tilde{\mathscr{C}}_{p+1}(\Sigma\!\!\!/ X)$$

between the space of algebraic p-cycles (after a certain "completion") supported on a complex projective variety X and the space of $(p+1)$-cycles on the projective cone $\Sigma\!\!\!/ X$ over X. The cone $\Sigma\!\!\!/ X$ is called "the complex suspension" of X in Lawson terminology. The techniques used in his work constitute a beautiful combination of complex geometry, geometric measure theory and homotopy theory. Lawson's paper has interesting derivations in various directions such as the work of Lawson–Michelsohn [22] on algebraic cycles and the Chern characteristic map and the work of Boyer–Lawson–Lima(-Filho)–Mann–Michelsohn [5] on algebraic cycles and infinite loop spaces. More recently, Friedlander and Lawson [10] have introduced a cohomology theory (morphic cohomology) which pairs to the present homology theory in the case of closed varieties.

In [9] Friedlander works with varieties over algebraically closed fields of arbitrary characteristic, and in this broader context he first proves the l-adic

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analog of Lawson's complex suspension theorem, using techniques of étale homotopy theory. Then Lawson homology is introduced in this general setting. For complex closed projective varieties it is simply the homotopy groups of the group completion $\tilde{\mathscr{C}}_p(X)$ of the Chow monoid, more precisely

$$L_pH_n(X) \stackrel{\text{def}}{=} \pi_{n-2p}(\tilde{\mathscr{C}}_p(X)),$$

for $n \geqslant 2p$. This bigraded group is then shown to be a highly interesting invariant of the variety X which encompasses many classical invariants. For example, Friedlander shows that the Néron–Severi group $NS(X)$, $H^1(\mathrm{Pic}^0(X))$ and the classical Chow groups of algebraic cycles modulo algebraic equivalence are particular cases of Lawson homology. Furthermore, with the use of an "algebraic version" of the Dold–Thom Theorem [6] he goes further and shows that étale homology is also obtained via Lawson homology. (In the complex case one obtains the singular homology immediately from the classical Dold–Thom theorem, op. cit.)

More recently, Friedlander and Mazur [11], using the complex suspension theorem, provided the bigraded Lawson homology with operations, which make it into a bigraded module over the polynomial ring in two variables $\mathbf{Z}[h, s]$. An iteration of the action of $s \in \mathbf{Z}[h, s]$ defines a "cycle map" into singular homology. Although this module structure depends on the embedding into a projective space, the cycle map is a natural transformation of functors, cf. [23] and [25]. The image of the successive iterations of s provide a filtration in singular homology, called topological filtration which is shown in [11] to be finer than the geometric filtation in case X is smooth. The geometric filtration is the analog in homology of Grothendieck's [15] arithmetic filtration.

We start, in Section 2, by introducing the relative Lawson homology $L_*H_*(X, X')$ of a pair $X' \subset X$ of algebraic sets as a functor of algebraic pairs and present basic properties and background information.

In Section 3 we proceed showing the existence of long exact sequences, in Lawson homology, for triples of closed projective varieties.

In Section 4 we present the main result of this work, which enables one to define Lawson homology for quasiprojective varieties using compactifications, namely:

THEOREM 4.3. *A relative isomorphism* $\Psi: (X, X') \to (Y, Y')$ *induces an isomorphism of topological groups*:

$$\Psi_*: \tilde{\mathscr{C}}_p(X, X') \stackrel{\cong}{\to} \tilde{\mathscr{C}}_p(Y, Y'),$$

for all $p \geqslant 0$, *where* $\tilde{\mathscr{C}}_p(X, X') \stackrel{\text{def}}{=} \tilde{\mathscr{C}}_p(X)/\tilde{\mathscr{C}}_p(X')$ *is the quotient group.*

This result enables us to define the Lawson homology $L_pH_n(U)$ of a quasiprojective set as the homotopy group $\pi_{n-2p}(\tilde{\mathscr{C}}_p(X, X'))$, for $n \geqslant 2p$. This is a covariant functor for proper maps satisfying:

PROPOSITION 4.8. (a) *For any quasiprojective variety U there is a natural isomorphism*

$$L_pH_{2p}(U) \cong A_p(U),$$

where $A_p(U)$ is the group of algebraic cycles in U modulo algebraic equivalence [13].

(b) *Let V be a closed subset of the quasiprojective variety U. Then there is a (localization) long exact sequence in Lawson homology:*

$$\cdots \to L_pH_n(V) \to L_pH_n(U) \to L_pH_n(U - V) \to L_pH_{n-1}(V) \to \cdots$$

ending at

$$\cdots L_pH_{2p+1}(U - V) \to A_p(V) \to A_p(U) \to A_p(U - V) \to 0.$$

The main results and technical lemmas are proven separately in Section 5. In Section 6 we use the excision theorem to compute some examples in which we also make use of the "generalized cycle map", cf. [11] and [25], making the computation more natural and elegant.

2. Cycle groups and Lawson homology for closed pairs

In this section we provide basic definitions and a concise summary of the preliminary results which originated the theory in study.

An (effective) algebraic p-cycle in the complex projective space $\mathbf{P}^N$ is a finite formal sum $\sigma = \Sigma_\lambda n_\lambda V_\lambda$, where the n_λ's are positive integers and the V_λ's are (irreducible) subvarieties of dimension p in $\mathbf{P}^N$.

The degree of $\sigma = \Sigma_\lambda n_\lambda V_\lambda$ is defined as $\deg(\sigma) = \Sigma_\lambda n_\lambda \deg(V_\lambda)$, where $\deg V_\lambda$ is the degree of the irreducible subvariety V_λ in $\mathbf{P}^N$, and the support of σ is the algebraic subset $\bigcup_\lambda V_\lambda$ of $\mathbf{P}^N$.

The set of p-cycles of a fixed degree d in $\mathbf{P}^N$ can be given the structure of an algebraic set, which we denote by $\mathbf{C}_{p,d}(\mathbf{P}^N)$, cf. [28] and [30]. In case $j: X \hookrightarrow \mathbf{P}^N$ is an algebraic subset of $\mathbf{P}^N$, the subset $\mathbf{C}_{p,d}(X) \subset \mathbf{C}_{p,d}(\mathbf{P}^N)$, consisting of those cycles whose support is contained in X, is an algebraic subset of $\mathbf{C}_{p,d}(\mathbf{P}^N)$ whose algebraic structure depends on the embedding j. See [9] and [28].

DEFINITION 2.1. The set

$$\mathscr{C}_p(X) = \coprod_{d \geqslant 0} \mathbf{C}_{p,d}(X) = \{0\} \coprod \left\{ \coprod_{d > 0} \mathbf{C}_{p,d}(X) \right\}$$

of all effective p-cycles in X, together with an isolated point 0 ("zero cycle") is an abelian topological monoid, called the Chow monoid of effective p-cycles in X. Here we are considering the analytic topology on $\mathscr{C}_p(X)$. Its associated Grothendieck group (or naïve group completion) is denoted $\tilde{\mathscr{C}}_p(X)$, and will, henceforth, be called the cycle group or the group of p-cycles in X.

Although the algebraic structure of each individual $\mathbf{C}_{p,d}(X)$ depends on the embedding of X into a projective space, the homeomorphism type of $\mathscr{C}_p(X)$ does not, cf. [9] or [18].

Given X as above, endow $\tilde{\mathscr{C}}_p(X)$ with the topology induced by the quotient map:

$$\begin{aligned} p\colon \mathscr{C}_p(X) \times \mathscr{C}_p(X) &\to \tilde{\mathscr{C}}_p(X) \\ (\sigma,\tau) &\mapsto \sigma - \tau. \end{aligned} \tag{1}$$

This makes $\tilde{\mathscr{C}}_p(X)$ into an abelian topological group having the universal property with respect to topological monoid mappings from $\mathscr{C}_p(X)$ into topological groups, cf. [24].

REMARK 2.2. (a) There is a continuous embedding [21] of $\tilde{\mathscr{C}}_p(X)$ into the group of integral cycles in X (in the sense of geometric measure theory) with the flat-norm topology [7]. In particular $\tilde{\mathscr{C}}_p(X)$ is a Hausdorff topological group.

(b) This topology makes $\tilde{\mathscr{C}}_p(X)$ also into a homology group-completion for $\mathscr{C}_p(X)$, meaning that in the level of homology the natural map

$$\mathscr{C}_p(X) \hookrightarrow \tilde{\mathscr{C}}_p(X)$$

is the localization of the Pontrjagin ring of $\mathscr{C}_p(X)$ with respect to the action of $\pi_0(\mathscr{C}_p(X))$. See [24].

DEFINITION 2.3. For a closed algebraic subset X' of X, the cycle group $\tilde{\mathscr{C}}_p(X')$ is naturally a closed subgroup of the topological group $\tilde{\mathscr{C}}_p(X)$. Define the relative cycle group $\tilde{\mathscr{C}}_p(X, X')$ as the quotient group $\tilde{\mathscr{C}}_p(X)/\tilde{\mathscr{C}}_p(X')$ endowed with the quotient topology.

Our investigation is centered on the invariants for algebraic sets obtained as the homotopy groups of the cycle groups above. More precisely:

DEFINITION 2.4. The (relative) Lawson homology (or L-homology for short) of the pair (X, X') is defined as

$$L_pH_n(X, X') \stackrel{\text{def}}{=} \pi_{n-2p}(\tilde{\mathscr{C}}_p(X, X')),$$

for $n \geqslant 2p$. Whenever X' is empty it is denoted simply $L_pH_n(X)$.

Using Remark 2.2(b) above, we see [24] that our definition for the absolute case coincides with Friedlander's over the complex numbers. Also notice that since the homeomorphism type of $\mathscr{C}_p(X)$ is independent of the embedding $j: X \hookrightarrow \mathbf{P}^N$, Lawson homology becomes an invariant of X as an abstract variety, cf. [9].

In case $p=0$ we have $\mathscr{C}_0(X)$ equal to $\amalg_{d \geqslant 0} SP^d(X)$, where $SP^d(X)$ is the d-fold symmetric product of X. It follows, e.g. from [24] or from an alternative description of the Lawson homology also given in [9], that for X connected there is a natural homotopy equivalence

$$\tilde{\mathscr{C}}_0(X)_\circ \overset{\text{h.eq.}}{\cong} SP^\infty(X),$$

where $\tilde{\mathscr{C}}_0(X)_\circ \subseteq \tilde{\mathscr{C}}_0(X)$ is the connected component of the identity and $SP^\infty(X)$ is the infinite symmetric product of X. Therefore the Dold–Thom [6] theorem gives an isomorphism

$$L_0H_i(X) \cong H_i(X, \mathbf{Z}),$$

between $L_0H_i(X)$ and the ith singular homology group of X with Z coefficients. Actually, the latter isomorphism holds for arbitrary X [6], [24].

Being belian topological groups, the cycle spaces $\tilde{\mathscr{C}}_p(X)$ are products of Eilenberg–MacLane spaces [33], and hence they are determined, up to homotopy, by the Lawson homology. In other words, there is a homotopy equivalence

$$\tilde{\mathscr{C}}_p(X) \cong \prod_i K(L_pH_i(X), i).$$

Functoriality of Lawson homology

DEFINITION 2.5. Let $X = \amalg_\alpha X_\alpha$ and $Y = \amalg_\beta Y_\beta$ be disjoint unions of algebraic sets (not necessarily finite unions) taken with the disjoint union topology (of the Zariski topology of their components). We say that a continuous map $f: X \to Y$ is a morphism of X into Y if the restriction of f to any component X_α is a morphism of algebraic sets. A proper morphism $f: X \to Y$ is a bicontinuous algebraic morphism if it is a set theoretic bijection and for every $y \in Y$ the induced map on residue fields $\mathbf{C}(y) \to \mathbf{C}(f^{-1}(y))$ is an isomorphism. A continuous algebraic map $f: X \to Y$ is a correspondence, i.e., a pair $\{g: Z \to X, h: Z \to Y\}$ in which g is a birational, bicontinuous morphism. In other words, a continuous algebraic map is a rational map which is well defined and continuous at all points. Here we follow Friedlander's [9] terminology closely.

We see that a bicontinuous algebraic morphism $f: X \to Y$, with X and Y as in the definition above, induces birational equivalences (in the sense of [16]) between the irreducible components of X and Y. Furthermore, taking X and Y with the analytic topology, it follows that f induces a homeomorphism between $(X)^{an}$ and $(Y)^{an}$, whose restriction to an irreducible component of X is a homeomorphism onto a corresponding component of Y. Observe that a rational continuous map $f: X \to Y$ is semi-algebraic in the sense of [17]. In particular one sees that the monoid operation in the Chow monoids is algebraic continuous in the above sense.

With the notions just introduced, it now makes sense to talk about continuous algebraic maps between Chow monoids, and we present the first functorial properties of Lawson homology.

PROPOSITION [9]. *Let X, Y and W be closed projective algebraic sets.*

(a) *For any morphism $f: X \to Y$ and integer $0 \leqslant p \leqslant \dim X$, there exists a continuous algebraic map*

$$f_{\#}: \mathscr{C}_p(X) \to \mathscr{C}_p(Y)$$

defined by

$$f_{\#}\left(\sum_i n_i V_i\right) = \sum_i n_i \cdot \deg(V_i/f(V_i)) \cdot f(V_i).$$

The map $f_{\#}$ is a morphism of abelian topological monoids in the analytic topology and induces a morphism f_* on Lawson homology for all $n \geqslant 2p$

$$f_*: L_pH_n(X) \to L_pH_n(Y).$$

(b) *For any flat morphism* [16] *$f: W \to X$ of relative dimension $r \geqslant 0$, and any integer p, $0 \leqslant p \leqslant \dim X$, there exists a continuous algebraic map*

$$f^{\#}: \mathscr{C}_p(X) \to \mathscr{C}_{p+r}(W)$$

which is a morphism of topological monoids in the analytic topology. Therefore, the map $f^{\#}$ induces a morphism on Lawson homology, for all $n \geqslant 2p$:

$$f^*: L_pH_n(X) \to L_{p+r}H_{n+2r}(W).$$

Here, for a morphism $f: X \to Y$ and subvariety $V \subset X$, $\deg(V/f(V))$ is defined as

$$\deg(V/f(V)) = \begin{cases} 0, & \text{if } \dim V > \dim f(V) \\ [\mathbf{C}(V): \mathbf{C}(f(V))], & \text{if } \dim V = \dim f(V), \end{cases}$$

where $\mathbf{C}(V)$, $\mathbf{C}(f(V))$ are the function fields of V and $f(V)$ respectively.

REMARK 2.6. From now on we sometimes use the word algebraic indistinctly meaning either algebraic morphisms in the usual sense or algebraic continuous maps, depending on the context.

From the above one sees that Lawson homology is a covariant functor from the category of algebraic sets and morphisms to the category of bigraded groups; also, it is a contravariant functor from the category of algebraic varieties and flat morphisms (with relative dimension) to bigraded groups.

Joins and suspensions

DEFINITION 2.7. Let $X \hookrightarrow \mathbf{P}^N$ and $Y \hookrightarrow \mathbf{P}^M$ be algebraic sets. Embed $\mathbf{P}^N$ and $\mathbf{P}^M$ linearly in $\mathbf{P}^{N+M+1}$ as two disjoint linear subspaces. Define the complex join (also called the ruled join) $i \# j: X \# Y \hookrightarrow \mathbf{P}^{N+M+1}$ of X and Y as the algebraic subset of $\mathbf{P}^{N+M+1}$ obtained as the union of all projective lines joining points of X to points of Y in $\mathbf{P}^{N+M+1}$. In the particular case where Y is a point $\mathbf{P}^0 \in \mathbf{P}^{N+1}$ not lying in $\mathbf{P}^N$, the complex join $\mathbf{P}^0 \# X$ of $\mathbf{P}^0 \in \mathbf{P}^{N+1}$ with $X \subset \mathbf{P}^N \subset \mathbf{P}^{N+1}$ is simply the projective cone over X. In [21], Lawson calls it the complex suspension of X and denotes it by $\Sigma\!\!\!/ X$. The m-fold complex suspension of X, $\Sigma\!\!\!/^m X$, is

$$\underbrace{\Sigma\!\!\!/(\Sigma\!\!\!/(\cdots(\Sigma\!\!\!/ X)\cdots))}_{m-\text{times}}.$$

Observe that the complex suspension can also be seen as the Thom space of the hyperplane bundle $\mathcal{O}(1)$ over X, and its structure as an algebraic set does not depend on the point $\mathbf{P}^0 \in \mathbf{P}^{N+1} - \mathbf{P}^N$.

It is immediate that whenever V is a subvariety of $X \overset{i}{\hookrightarrow} \mathbf{P}^N$ having dimension p and degree d, and W is a subvariety of $Y \overset{j}{\hookrightarrow} \mathbf{P}^M$ with dimension q and degree e (in other words, $V \in \mathbf{C}_{p,d}(X)$ and $W \in \mathbf{C}_{q,e}(Y)$), then the join $V \# W$ is a subvariety of $X \# Y$ having dimension $p+q+1$ and degree $d \cdot e$.

Notice that the m-fold suspension $\Sigma\!\!\!/^m X$ of X can also be viewed as the join $\mathbf{P}^{m-1} \# X$ of $\mathbf{P}^{m-1}$ with X. From the above we conclude that the m-fold suspension takes irreducible cycles in $\mathbf{C}_{p,d}(X)$ to irreducible cycles in $\mathbf{C}_{p+m,d}(\Sigma\!\!\!/^m X)$. Actually the join operation can be extended linearly to the cycle spaces as follows.

PROPOSITION [9]. *Let $i: X \hookrightarrow \mathbf{P}^N$, $j: Y \hookrightarrow \mathbf{P}^M$ be algebraic sets. The external join induces a continuous algebraic map*

$$\mathbf{C}_{r,d}(X) \times \mathbf{C}_{s,e}(Y) \to \mathbf{C}_{r+s+1,d\cdot e}(X \# Y)$$

for any $r \leqslant \dim X$, $s \leqslant \dim Y$, *d and e. Up to bicontinuous algebraic equivalence, this pairing is independent of the embeddings i and j. These continuous algebraic maps induce a bi-additive continuous rational map*

$$\# : \mathscr{C}_r(X) \times \mathscr{C}_s(Y) \to \mathscr{C}_{r+s+1}(X \# Y)$$

which sends $\mathscr{C}_r(X) \times \{0\}$ *and* $\{0\} \times \mathscr{C}_r(Y)$ *to* $\{0\} \in \mathscr{C}_{r+s+1}(X \# Y)$.

Considering $\mathbf{P}^{m-1}$ as a cycle in $\mathbf{C}_{m-1,1}(\mathbf{P}^{m-1})$ we obtain an algebraic map

$$\Sigma\!\!\!/^m X : \mathscr{C}_p(X) \to \mathscr{C}_{p+m}(\Sigma\!\!\!/^m X)$$

which is defined in such a way that it sends a cycle $\sigma = \Sigma_\lambda n_\lambda V_\lambda$ in $\mathbf{C}_{p,d}(X)$ to $\Sigma\!\!\!/^m \sigma = \Sigma_\lambda n_\lambda(\Sigma\!\!\!/^m V_\lambda) = \Sigma_\lambda n_\lambda(\mathbf{P}^{m-1} \# V_\lambda)$. The map induced on the cycle spaces (by functoriality) $\Sigma\!\!\!/^m : \tilde{\mathscr{C}}_p(X) \to \tilde{\mathscr{C}}_{p+m}(\Sigma\!\!\!/^m X)$ is remarkably well behaved and satisfies the following theorem, which is the foundation stone of the theory, proven by Lawson in [21]:

THEOREM CST (the complex suspension theorem). *The m-fold complex suspension*

$$\Sigma\!\!\!/^m : \tilde{\mathscr{C}}_p(X) \to \tilde{\mathscr{C}}_{p+m}(\Sigma\!\!\!/^m X)$$

is a homotopy equivalence for every integer p, with $0 \leqslant p \leqslant \dim X$ *and every positive integer m.*

Equivalently:

COROLLARY. *The m-fold complex suspension* $\Sigma\!\!\!/^m$ *induces an isomorphism*

$$\Sigma\!\!\!/^m_* : L_pH_n(X) \xrightarrow{\cong} L_{p+m}H_{n+2m}(\Sigma\!\!\!/^m X)$$

for every $n \geqslant 2p$.

Concluding this section we observe that the cycle spaces of the complex projective space $\mathbf{P}^t$ are completely determined by the complex suspension theorem. Namely

$$\tilde{\mathscr{C}}_p(\mathbf{P}^t) \cong \tilde{\mathscr{C}}_0(\mathbf{P}^{t-p}) \cong K(\mathbf{Z}, 2) \times \cdots \times K(\mathbf{Z}, 2(t-p)),$$

for all $0 \leqslant p \leqslant \dim X$, the later equivalence being a consequence of the Dold–Thom theorem, as observed above. Equivalently, one has isomorphisms

$$L_pH_{i+2p}(\mathbf{P}^t) \cong L_0H_i(\mathbf{P}^{t-p}, \mathbf{Z}) \cong H_i(\mathbf{P}^{t-p}, \mathbf{Z}).$$

3. Fibrations and long exact sequences

The exceptional features of the Chow monoids endow the relative cycle groups with interesting and fundamental properties, such as the following result, proven in slightly greater generality in [24] for a certain category of filtered monoids.

THEOREM 3.1. *For a pair of algebraic sets (X, X') the quotient map $\tilde{\mathscr{C}}_p(X) \to \tilde{\mathscr{C}}_p(X)/\tilde{\mathscr{C}}_p(X')$ admits a local cross-section. In particular one has a principal fibration*:

$$\tilde{\mathscr{C}}_p(X') \xrightarrow{i} \tilde{\mathscr{C}}_p(X) \to \tilde{\mathscr{C}}_p(X, X').$$

Observe that the natural morphism

$$f_{\#}: \tilde{\mathscr{C}}_p(X, X') \to \tilde{\mathscr{C}}_p(Y, Y')$$

induced by a given morphism of pairs of algebraic sets $f:(X, X') \to (Y, Y')$ fits into a morphism of principal fibrations.

As an immediate consequence of the above theorem one obtains the existence of long exact sequences for the L-homology of triples, as follows.

PROPOSITION 3.2. *For a triple of closed algebraic sets (X, X', X'') there is a natural exact sequence*

$$0 \to \tilde{\mathscr{C}}_p(X', X'') \xrightarrow{\nu} \tilde{\mathscr{C}}_p(X, X'') \xrightarrow{\pi} \tilde{\mathscr{C}}_p(X, X') \to 0$$

which is a principal fibration. In particular there is a long exact sequence in Lawson homology:

$$\cdots \to L_pH_n(X', X'') \xrightarrow{\nu} L_pH_n(X, X'') \xrightarrow{\pi} L_pH_n(X, X') \xrightarrow{\delta} L_pH_{n-1}(X', X'') \to \cdots$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \to & \tilde{\mathscr{C}}_p(X'') & \xrightarrow{j} & \tilde{\mathscr{C}}_p(X') & \xrightarrow{p_1} & \tilde{\mathscr{C}}_p(X', X'') & \to & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \nu} & & \\
0 & \to & \tilde{\mathscr{C}}_p(X'') & \xrightarrow{i} & \tilde{\mathscr{C}}_p(X) & \xrightarrow{p_2} & \tilde{\mathscr{C}}_p(X, X'') & \to & 0 \\
 & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \pi} & & \\
0 & \to & \tilde{\mathscr{C}}_p(X') & \xrightarrow{i} & \tilde{\mathscr{C}}_p(X) & \xrightarrow{p_3} & \tilde{\mathscr{C}}_p(X, X') & \to & 0. \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

The existence of ν and π follows from the universal properties of the projections p_i, $i = 1, 2, 3$ and exactness for the third column follows from the exactness of the rows and a diagram chasing.

Now, if $\sigma: U \to \tilde{\mathscr{C}}_p(X)$ is a local cross-section defined on an open set $U \subseteq \tilde{\mathscr{C}}_p(X, X')$, then the composition $p_2 \circ \sigma: U \to \tilde{\mathscr{C}}_p(X, X'')$ is also a cross-section for π. The proposition now follows, cf. [31]. □

For a given pair of (closed) algebraic sets (X, X'), the complex suspension $\Sigma\!\!\!/ : \mathscr{C}_p(X) \to \mathscr{C}_{p+1}(\Sigma\!\!\!/ X)$ restricts to $\Sigma\!\!\!/ : \mathscr{C}_p(X') \to \mathscr{C}_{p+1}(\Sigma\!\!\!/ X')$, and hence it naturally induces a morphism of the corresponding principal fibrations associated to the pairs (X, X') and $(\Sigma\!\!\!/ X, \Sigma\!\!\!/ X')$ as in Theorem 3.1. From this one obtains a morphism of long exact sequences for the Lawson homology of the pairs, which, together with the complex suspension theorem (CST) and the five lemma yields the following:

COROLLARY 3.3. *The relative complex suspension*

$$\Sigma\!\!\!/ : \tilde{\mathscr{C}}_p(X, X') \to \tilde{\mathscr{C}}_{p+1}(\Sigma\!\!\!/ X, \Sigma\!\!\!/ X')$$

is a homotopy equivalence for all $p \geqslant 0$.

4. Excision and quasiprojective varieties

The notation of this section is slightly abusive but it will prove to be useful.

DEFINITION 4.1. We say that a pair (X, X') of closed projective sets is a compactification of a quasiprojective set U if $X - X'$ is isomorphic to U as a quasiprojective set. Two pairs (X, X') and (Y, Y') of closed algebraic sets are relatively isomorphic if they are compactifications of the same quasiprojective set.

Note that we are not requiring U to be dense in X for a projectivization (X, X') of U.

REMARK 4.2. Let $\Psi: U \to V$, be a proper map between two quasiprojective varieties, and let (X, X') and (Y, Y') be two compactifications of U, V, respectively, as above. Let

$$\Gamma \overset{\text{def}}{=} \overline{\text{Graph}(\Psi)} \subset X \times Y$$

be the closure of the graph of Ψ and let $f: \Gamma \to X$, $g: \Gamma \to Y$ be the restrictions to Γ of the projections onto the first and second factors of $X \times Y$, respectively.

Define $\Gamma' \subset \Gamma$ to be $\Gamma - \text{Graph}(\Psi)$. Then there is a "correspondence of pairs"

$$\begin{array}{ccc} & (\Gamma, \Gamma') & \\ {}^{f}\swarrow & & \searrow^{g} \\ (X, X') & & (Y, Y') \end{array}$$

where $f: (\Gamma, \Gamma') \to (X, X')$ is a relative isomorphism and $g: (\Gamma, \Gamma') \to (Y, Y')$ is an actual morphism of pairs, by the properness of Ψ.

In particular, whenever (X, X') and (Y, Y') are relatively isomorphis pairs, the maps f and g are relative isomorphisms.

Our main goal is to prove the following "localization theorem":

THEOREM 4.3. *A relative isomorphism* $\Psi: (X, X') \to (Y, Y')$ *induces an isomorphism of topological groups*:

$$\Psi_*: \tilde{\mathscr{C}}_p(X, X') \xrightarrow{\cong} \tilde{\mathscr{C}}_p(Y, Y'),$$

for all $p \geqslant 0$.

As a consequence we obtain the equivalent of an excision theorem for the Lawson homology of pairs of closed projective algebraic sets:

COROLLARY 4.4. *A relative isomorphism* $\Psi: (X, X') \to (Y, Y')$ *gives an isomorphism in Lawson homology*:

$$\Psi_*: L_pH_n(X, X') \xrightarrow{\cong} L_pH_n(Y, Y'),$$

for all p *and* $n \geqslant 2p$.

The relevance of the theorem above lies in the fact that it allows one to define unambiguously a topology on the group of all p-cycles supported on any quasiprojective set U as well as to define its Lawson homology. More precisely:

DEFINITION 4.5. Given a quasiprojective set U define the group of p-cycles on U as

$$\tilde{\mathscr{C}}_p(U) \stackrel{\text{def}}{=} \tilde{\mathscr{C}}_p(X, X'),$$

where (X, X') is any compactification of U. This is a topological group whose homotopy groups define the Lawson homology of U, namely

$$L_pH_n(U) \stackrel{\text{def}}{=} \pi_{n-2p}(\tilde{\mathscr{C}}_p(U)) = L_pH_n(X, X').$$

From Remark 4.2 we obtain, as expected, the following:

COROLLARY 4.6. *Lawson homology is covariant for proper morphisms of quasiprojective varieties.*

REMARK 4.7. (a) When $p=0$, we still obtain $L_0H_n(U) \cong H_n(X, X')$, where (X, X') is any compactification of U, cf. [24]. Since $X' \hookrightarrow X$ is a cofibration, one sees that $H_n(X, X')$ is isomorphic to the Borel–Moore homology of U, cf. [4], [12].

(b) For arbitrary characteristic one has to consider homotopy group completions instead of the naïve ones as above. In this case one has to prove an excision theorem similar to Corollary 4, where the relative homology of a pair is defined as the homotopy groups of the homotopy quotient of the respective group completions. This homotopy theoretic formulation over the complex numbers is equivalent to the present one, and the excision theorem can be proven directly in this context without the use of naïve completions. See [24] for a proof of the equivalence of both approaches as well as a proof of Corollary 4 in the homotopy theoretic context. We believe that the techniques of [24] can be carried out to the context of varieties over arbitrary algebraically closed fields.

PROPOSITION 4.8. (a) *For any quasiprojective variety U there is a natural isomorphism*

$$L_pH_{2p}(U) \cong A_p(U),$$

where $A_p(U)$ is the group of algebraic cycles in U modulo algebraic equivalence [13].

(b) *Let V be a closed subset of the quasiprojective variety U. Then there is a (localisation) long exact sequence in Lawson homology:*

$$\cdots \to L_pH_n(V) \to L_pH_n(U) \to L_pH_n(U-V) \to L_pH_{n-1}(V) \to \cdots$$

ending at

$$\cdots \to L_pH_{2p+1}(U-V) \to A_p(V) \to A_p(U) \to A_p(U-V) \to 0.$$

Proof. (a) It follows from the fact that for a closed projective variety X, there is a natural isomorphism $L_pH_{2p}(X) \cong A_p(X)$ sending the connected component of a cycle to its class modulo algebraic equivalence. Combining this with the exact sequence in [13], Section 1.8, concludes the argument.

(b) Given a compactification (X, X'') of U, let $\bar{V}$ be the closure of V in X and define $X' = \bar{V} \cup X''$. Since V is closed in U we have $X' - X'' = V$ and also $X - X' = U - V$. Applying item (a) and Proposition 3.2 to the triple (X, X', X'') we conclude the proof. □

REMARK 4.9. A last remark should be the following interpretation of Lawson's complex suspension theorem in terms of quasiprojective varieties. Let

X be a closed projective variety and let $\tau: H \to X$ be the hyperplane bundle over X. Composing the complex suspension $\Sigma\!\!\!\!/\,: \mathscr{C}_p(X) \to \tilde{\mathscr{C}}_{p+1}(\Sigma\!\!\!\!/\, X)$ with the isomorphism $\pi: \tilde{\mathscr{C}}_{p+1}(\Sigma\!\!\!\!/\, X) \to \tilde{\mathscr{C}}_{p+1}(H) = \tilde{\mathscr{C}}_{p+1}(\Sigma\!\!\!\!/\, X, pt)$ one obtains the pull-back map $\tau_{\#}: \tilde{\mathscr{C}}_p(X) \to \tilde{\mathscr{C}}_{p+1}(H)$. In this context, the complex suspension theorem stands as a restricted form (only for very ample line bundles) of the homotopy axiom (see [13], Theorem 3.3) for Lawson homology.

This axiom actually holds in a more general form, namely, the pull-back of cycles in arbitrary vector bundles induces an isomorphism in Lawson homology. A proof of this fact as well as some of its deep consequences will appear in a forthcoming paper*.

5. Proof of Theorem 4.3

We start introducing a bit of notation in order to interpret the topology of $\tilde{\mathscr{C}}_p(X, X')$, for a given pair (X, X') of closed projective algebraic sets.

DEFINITION 5.1. Given (X, X') as above define:

$$\mathbf{C}_{p,\leqslant D}(X) \overset{\text{def}}{=} \bigcup_{d \leqslant D} \mathbf{C}_{p,d}(X);$$

$$\Upsilon_{p,d}(X, X') \overset{\text{def}}{=} \{\sigma \in \mathbf{C}_{p,d}(X) \colon \sigma \text{ has no component in } X'\};$$

$$\Upsilon_{p,\leqslant D}(X, X') \overset{\text{def}}{=} \bigcup_{d \leqslant D} \Upsilon_{p,d}(X, X');$$

$$\Upsilon_p(X, X') \overset{\text{def}}{=} \bigcup_{D} \Upsilon_{p,\leqslant D}(X, X');$$

$$\mathscr{K}_D(X) \overset{\text{def}}{=} \mathbf{C}_{p,\leqslant D}(X) \times \mathbf{C}_{p,\leqslant D}(X) \subset \mathscr{C}_p(X) \times \mathscr{C}_p(X).$$

Notice that $\Upsilon_p(X, X')$ is a submonoid of $\mathscr{C}_p(X)$ and that $C_{p,\leqslant D}(X, j)$ is an algebraic set. Using the canonical projections $p: \mathscr{C}_p(X) \times \mathscr{C}_p(X) \to \tilde{\mathscr{C}}_p(X)$ and $\pi: \tilde{\mathscr{C}}_p(X) \to \tilde{\mathscr{C}}_p(X)/\tilde{\mathscr{C}}_p(X') \equiv \tilde{\mathscr{C}}_p(X, X')$ we also define the sets:

$$\tilde{\mathscr{K}}_D(X) \overset{\text{def}}{=} p(\mathscr{K}_D(X)) \subset \tilde{\mathscr{C}}_p(X);$$

$$\tilde{\Upsilon}_{p,\leqslant D}(X, X') \overset{\text{def}}{=} p(\Upsilon_{p,\leqslant D}(X, X') \times \Upsilon_{p,\leqslant D}(X, X')) \subset \tilde{\mathscr{C}}_p(X);$$

$$\tilde{\Upsilon}_p(X, X') \overset{\text{def}}{=} \bigcup_{D} \tilde{\Upsilon}_{p,\leqslant D}(X, X');$$

$$Q_D(X, X') \overset{\text{def}}{=} \pi(\tilde{\mathscr{K}}_D(X)).$$

Here, the sets $\{\tilde{\mathscr{K}}_D(X)\}_{D=1}^{\infty}$, $\{Q_D(X, X')\}_{D=1}^{\infty}$ form a filtering family of compact subsets of $\tilde{\mathscr{C}}_p(X)$ and $\tilde{\mathscr{C}}_p(X, X')$, respectively. Also notice that $\tilde{\Upsilon}_p(X, X')$ is the

*We have learned, after this paper was submitted for publication, that E. Friedlander and O. Gaber have obtained this result prior to us.

subgroup of $\tilde{\mathscr{C}}_p(X)$ obtained as the Grothendieck group associated to the monoid $\Upsilon_p(X, X')$. Summarizing we have the following commutative diagram:

$$\begin{array}{ccccc} \Upsilon_{p,\leqslant D}(X, X') \times \Upsilon_{p,\leqslant D}(X, X') & \to & \mathscr{K}_D(X) & \to & \mathscr{C}_p(X) \times \mathscr{C}_p(X) \\ \downarrow p & & \downarrow p & & \downarrow p \\ \tilde{\Upsilon}_{p,\leqslant D}(X, X') & \to & \tilde{\mathscr{K}}_D(X) & \to & \tilde{\mathscr{C}}_p(X) \\ & & \downarrow \pi & & \downarrow \pi \\ & & Q_D(X, X') & \to & \tilde{\mathscr{C}}_p(X, X'), \end{array}$$

where the horizontal arrows are inclusions and the vertical ones are proclusions (i.e. they are quotient maps, in Steenrod's terminology [32].)

From the above picture we draw the following

LEMMA 5.2. *Let (X, X') be a pair of closed algebraic sets. Then:*

(a) *The topology of $\mathscr{C}_p(X) \times \mathscr{C}_p(X)$ is the weak topology induced by the filtering family of compact sets $\mathscr{K}_1(X) \subset \mathscr{K}_2(X) \subset \cdots \mathscr{K}_D(X) \subset \cdots$. The same holds for $\tilde{\mathscr{C}}_p(X)$ and $\tilde{\mathscr{C}}_p(X, X')$ with respect to the families $\tilde{\mathscr{K}}_1(X) \subset \tilde{\mathscr{K}}_2(X) \subset \cdots$ and $Q_1(X, X') \subset Q_2(X, X') \subset \cdots$, respectively.*

(b) *The composition $\tilde{\Upsilon}_p(X, X') \hookrightarrow \tilde{\mathscr{C}}_p(X) \to \tilde{\mathscr{C}}_p(X, X')$ is an abstract group isomorphism which takes $\tilde{\Upsilon}_{p\leqslant D}(X, X')$ onto $Q_D(X, X')$.*

The proof of this lemma is elementary and follows from general facts about graded monoids, cf. [24].

The fundamental key for the theorem is the following result:

PROPOSITION 5.3. *Given a relative isomorphism of algebraic pairs $f: (X, X') \to (Y, Y')$, the induced morphism $f_{\#}: \mathscr{C}_p(X) \to \mathscr{C}_p(Y)$ of Chow monoids restricts to an isomorphism of submonoids*

$$f_{\#}: \Upsilon_p(X, X') \to \Upsilon_p(Y, Y'),$$

with the subspace topology. Equivalently, $f_{\#}$ restricts to a bijection and for every $d > 0$, there exists $D > 0$ such that

$$f_{\#}^{-1}(\Upsilon_{p,\leqslant d}(Y, Y')) \subseteq \Upsilon_{p,\leqslant D}(X, X'),$$

for all $p \geqslant 0$.

Proof. Take a p-dimensional subvariety $V \subset X$ not contained in X'. Since $f_{|X-X'}: X - X' \xrightarrow{\cong} Y - Y'$ is aniomorphism, we see that $f(V)$ is a p-dimensional subvariety of Y not contained in Y'. Also $\deg(V/f(V)) = 1$, since $f_{|X-X'}$ restricts to an isomorphism between $V - (V \cap X')$ and $f(V) - (f(V) \cap Y')$. Therefore

$$f_{\#}(V) \overset{\text{def}}{=} \deg(V/f(V)) \cdot f(V) = f(V) \in \Upsilon_p(Y, Y'),$$

and hence $f_\#$ sends $\Upsilon_p(X, X')$ to $\Upsilon_p(Y, Y')$. This immediately implies that the restriction of $f_\#$ to $\Upsilon_p(X, X')$ is an isomorphism of discrete monoids, since the monoids are freely generated by the irreducible varieties.

In order to prove the remaining part of the proposition we need the following technical result.

LEMMA 5.4. *Given $f: (X, X') \to (Y, Y')$ as in Proposition 5.3, there can not exist a sequence $\{V_n\}_{n=1}^{\infty}$ of p-dimensional subvarieties of X satisfying:*

(a) $\limsup \deg(V_n) = \infty$;
(b) $\deg f_\#(V_n) \leqslant M$, *for some constant M, for all n*;
(c) $V_n \not\subset X'$, *for all n.*

Assume the lemma for a while and suppose the proposition does not hold. In other words, there exists $d > 0$ such that

$$f_\#^{-1}(\Upsilon_{p,\leqslant d}(Y, Y')) \nsubseteq \Upsilon_{p,\leqslant n}(X, X')$$

for all $n > 0$. This allows one to select a sequence of p-cycles $\{\sigma_n\}_{n=1}^{\infty}$ contained in $\Upsilon_p(X, X')$ satisfying:

$$\deg \sigma_n \geqslant n \quad \text{and} \quad \deg f_\#\sigma_n \leqslant d,$$

for all n. Write $\sigma_n = \Sigma_{i=1}^{l_n} k_n^i V_i^n$, where the V_i^n's are irreducible subvarieties of X not contained in X'. As we pointed out at the beginning of the proof, we have $\deg(V_i^n/f(V_i^n)) = 1$, and hence

$$f_\#(\sigma_n) \overset{\text{def}}{=} \sum_{i=1}^{l_n} k_n^i \cdot \deg(V_i^n/f(V_i^n)) \cdot f(V_i^n) = \sum_{i=1}^{l_n} k_n^i \cdot f(V_i^n).$$

Suppose that $\deg(V_i^n) \leqslant B$ for all n and $1 \leqslant i \leqslant l_n$, where B is some constant. In this case one has

$$n \leqslant \deg \sigma_n = \sum_{i=1}^{l_n} k_n^i \cdot \deg(V_i^n) \leqslant \sum_{i=1}^{l_n} k_n^i \cdot B$$

and hence

$$\sum_{i=1}^{l_n} k_n^i \geqslant \frac{1}{B} \cdot n$$

for all n. On the other hand, by hypothesis

$$M \geqslant \deg f_\#(\sigma_n) = \sum_{i=1}^{l_n} k_n^i \cdot \deg(f(V_i^n)) \geqslant \sum_{i=1}^{l_n} k_n^i \geqslant \frac{1}{B} \cdot n,$$

which is a contradiction. Therefore no such constant B can exist. In other words

$$\limsup_i \deg(V_i^n) = \infty$$

and $\deg f_{\#}(V_i^n) \leqslant \deg f_{\#}(\sigma_n) \leqslant M$, which contradicts the lemma, and the proposition is proven. □

Let us prove the lemma now:

Proof of Lemma 5.4. It suffices to assume that X and Y are irreducible, for if $\{V_n\}$ is an infinite sequence of irreducible subvarieties of X (not contained in X') we can extract a subsequence $\{V_{n_i}\}$ so that all V_{n_i}'s are contained in a unique irreducible component X_1 of X. Define $X_1' = X \cap X'$ and observe that $f(X_1)$ must be contained in an irreducible component Y_1 of Y. Define $Y_1' = Y_1 \cap Y'$, in doing so we obtain a morphism of pairs $f:(X_1, X_1') \to (Y_1, Y_1')$. Since $f_{|X-X'}$ is an isomorphism of quasiprojective sets, its restriction to $X_1 - X_1' \subset X - X'$ establishes an isomorphism between $X_1 - X_1'$ and $Y_1 - Y_1'$. Therefore $f:(X_1, X_1') \to (Y_1, Y_1')$ is a relative isomorphism and the sequence $\{V_{n_i}\}$ satisfies the hypothesis of the lemma with X_1 and Y_1 irreducible.

We now use induction to prove the lemma with X and Y irreducible.

For $p=0$ it is immediate, since a 0-dimensional variety has always degree 1.

Consider the case $p=1$. Take a sequence of irreducible curves $V_n \subset X$ satisfying the hypothesis of the lemma. We may assume $\deg V_n \geqslant n$. Observe that the set $V_n' \stackrel{\text{def}}{=} V_n \cap X'$ is finite (or empty) since V_n is irreducible. Now use the following facts

(i) The generic hyperplane intersection of an irreducible subvariety of $\mathbf{P}^N$ of dimension $\geqslant 2$ is irreducible.
(ii) The generic hyperplane intersects the curve V_n transversely and misses the finite subset $V_n' \subset V_n$,

to obtain (by Baire category arguments) a hyperplane $H \subset \mathbf{P}^N$ satisfying:

(a) H is transversal to V_n, $\forall\, n$;
(b) $H \cap V_n' \equiv H \cap V_n \cap X' = \varnothing$, $\forall\, n$;
(c) $H \cap X$ is irreducible and is not contained in X'.

By definition, the cardinality of the intersection of V_n with H is its degree, and hence,

$$\#(H \cap V_n) \stackrel{\text{def}}{=} \deg(V_n) \geqslant n,$$

by hypothesis. On the other hand, since $H \cap X$ is irreducible and not contained in X', we have

$$f_{\#}(H \cap X) = f(H \cap X),$$

as observed at the beginning of the proof of Proposition 5.3. Call $D = f(H \cap X) \subset Y \subset \mathbf{P}^M$. As a subvariety of $\mathbf{P}^M$, D is a set-theoretic intersection of a finite number of (irreducible) hypersurfaces $H_1, \dots, H_k \subset \mathbf{P}^M$. Consequently,

$$f(H \cap V_n) = f(H \cap X \cap V_n) \subseteq f(H \cap X) \cap f(V_n) = D \cap f(V_n).$$

Since $f(V_n) \not\subset D$, there must be one hypersurface H_{j_0} (among $H_1, \dots, H_k$) not containing $f(V_n)$. From this we get:

$$\#(D \cap f(V_n)) \leqslant \#(H_{j_0} \cap f(V_n)) \leqslant \sum_{x \in H_{j_0} \cap f(V_n)} i(H_{j_0}, f(V_n); x)$$
$$= \deg(H_{j_0}) \cdot \deg(f(V_n)) \leqslant \sup_j \{\deg(H_j)\} \cdot \deg(f(V_n)),$$

where $i(H_{j_0}, f(V_n); x)$ denotes the multiplicity of the intersection of H_{j_0} and $f(V_n)$ along x. Again by hypothesis $\deg f_{\#}(V_n) = \deg f(V_n) \leqslant M$, and $\# f(V_n \cap H) = \#(V_n \cap H)$. Hence:

$$n \leqslant \deg(V_n \cap H) = \# f(V_n \cap H) \leqslant \#(D \cap f(V_n)) \leqslant S \cdot M,$$

where $S = \sup\{\deg(H_j)\}$. This is a contradiction.

Suppose that the lemma is true for subvarieties of dimension $\leqslant p-1$, $p \geqslant 2$. Let $\{V_n\}$ be a sequence of p-dimensional subvarieties satisfying the hypothesis of the lemma, and suppose that $\deg V_n \geqslant n$. Using the same general position arguments as before we can choose a generic hyperplane $H \subset \mathbf{P}^N$ so that:

(a′) $H \cap V_n$ is an irreducible (p-1)-dimensional subvariety of $\mathbf{P}^N$;
(b′) $H \cap V_n \not\subset X'$;
(c′) $H \cap X$ is also irreducible and $H \cap X \not\subset X'$.

Define $D = f_{\#}(H \cap X) = f(H \cap X)$, and let $H_1, H_2, \dots, H_k$ be irreducible hypersurfaces in $\mathbf{P}^M$ whose set theoretic intersection is D. Since

$$f(V_n \cap H) \subset f(H \cap X) \cap f(V_n) = f(V_n) \cap D = f(V_n) \cap H_1 \cap \cdots \cap H_k$$

and $f(V_n \cap H)$ is irreducible, we know that $f(V_n \cap H)$ is an irreducible component of the intersection $f(V_n) \cap H_{j_0}$, for some j_0 such that $f(V_n) \not\subset H_{j_0}$. Write $f(V_n) \cap H_{j_0} = \bigcup_r Z_r$, Z_r irreducible. Therefore

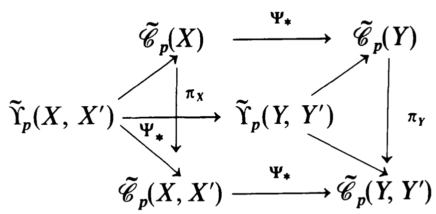
$$\deg f(V_n \cap H) \leqslant \sum_r i(H_{j_0}, f(V_n); Z_r) \cdot \deg Z_r$$
$$= \deg H_{j_0} \cdot \deg f(V_n) \leqslant \sup\{\deg H_j\} \cdot M.$$

However $\deg(V_n \cap H) = \deg V_n \geqslant n$, which contradicts the induction hypothesis, and proves the lemma.

Proof of Theorem 4.3. In Proposition 5.3 we saw that $\Psi_{\#}$ takes $\Upsilon_p(X, X')$ into $\Upsilon_p(Y, Y')$, and hence $\Psi_* : \tilde{\mathscr{C}}_p(X) \to \tilde{\mathscr{C}}_p(Y)$ restricts to a group homomorphism $\Psi_* : \tilde{\Upsilon}_p(X, X') \to \tilde{\Upsilon}_p(Y, Y')$, since $\tilde{\Upsilon}_p(X, X')$, (respectively $\tilde{\Upsilon}_p(Y, Y')$), is the group completion of $\Upsilon_p(X, X')$, (respectively $\Upsilon_p(Y, Y')$).

In a similar fashion to the proof of Proposition 5.3 one sees that $\Psi_* : \tilde{\Upsilon}_p(X, X') \to \tilde{\Upsilon}_p(Y, Y')$ is an isomorphism of discrete groups.

The whole situation is summarized in the following diagram:



We have observed that $\Psi_{*|\tilde{\Upsilon}_p(X,X')}$ is an isomorphism and Lemma 5.2(b) shows that both $\pi_{X|\tilde{\Upsilon}_p(X,X')}$ and $\pi_{Y|\tilde{\Upsilon}_p(Y,Y')}$ are isomorphisms. By the commutativity of the diagram we conclude that Ψ_* is an (continuous) isomorphism.

In Lemma 5.2 we have shown further that $\pi_X(\tilde{\Upsilon}_{p,\leqslant d}(X, X')) = Q_d(X, X')$ and $\pi_Y(\tilde{\Upsilon}_{p,\leqslant d}(Y, Y')) = Q_d(Y, Y')$. By Proposition 5.3 we know that for every $d > 0$ there exists $D > 0$ such that

$$\Psi_{\#}^{-1}(\Upsilon_{p,\leqslant d}(Y, Y')) \subseteq \Upsilon_{p,\leqslant D}(X, X'),$$

and hence

$$\Psi_*^{-1}(\tilde{\Upsilon}_{p,\leqslant d}(Y, Y')) \subseteq \tilde{\Upsilon}_{p,\leqslant D}(X, X').$$

Consequently

$$\begin{aligned}\Psi_*^{-1}(Q_d(Y, Y')) &= (\pi_Y \circ \Psi_* \circ \pi^{-1}_{X|\tilde{\Upsilon}_p(X,X')})^{-1}(Q_d(Y, Y')) \\ &= \pi_X(\Psi_*^{-1}(\pi_Y^{-1}(Q_d(Y, Y')))) = \pi_X(\Psi_*^{-1}(\tilde{\Upsilon}_{p,\leqslant d}(Y, Y'))) \\ &\subseteq \pi_X(\tilde{\Upsilon}_{p,\leqslant D}(X, X')) = Q_D(X, X').\end{aligned}$$

Now let F be a closed subset of $\tilde{\mathscr{C}}_p(X, X')$, and, given $d > 0$ choose $D > 0$ as above so that

$$\begin{aligned}\Psi_*(F) \cap Q_d(Y, Y') &= \Psi_*(F \cap \Psi_*^{-1}(Q_d(Y, Y'))) \cap Q_d(Y, Y') \\ &= \Psi_*(F \cap Q_D(X, X')) \cap Q_d(Y, Y').\end{aligned}$$

Since $Q_D(X, X')$ is compact and F is closed in $\tilde{\mathscr{C}}_p(X, X')$, then $F \cap Q_D(X, X')$ is compact and hence $\Psi_*(F \cap Q_D(X, X'))$ is compact. From this we conclude that $\Psi_*(F) \cap Q_d(Y, Y')$ is closed, for all d. Therefore $\Psi_*(F)$ is closed, by Lemma 5.2(a). This last conclusion shows that Ψ_* is a closed map, and hence an isomorphism of topological groups. □

6. Examples

The results we have proven, together with the functorial properties of L-homology, allow the immediate computation of several examples.

In [23] and [25] examples are computed using the "generalized cycle map" defined by Friedlander and Mazur in [11]. This map is a homomorphism $s_X^p: L_pH_n(X) \to H_n(X)$ from L-homology to singular homology, which coincides with the classical cycle map [13] from the group of cycles modulo algebraic equivalence to singular homology in the case $n = 2p$, and realizes the Dold–Thom isomorphism when $p = 0$ and n is arbitrary.

The two simple cases presented here can be computed without the use of cycle maps, however this presentation is more natural and elegant.

We need the following facts which were proven in [25]:

(1) There is a relative cycle map

$$s_{X,X'}^p: L_pH_n(X, X') \to H_n(X, X')$$

for pairs of closed projective algebraic varieties. If (X, X') and (Y, Y') are relatively isomorphic and $s_{X,X'}^p$ is an isomorphism, then so is $s_{Y,Y'}^p$.

(2) The cycle maps are natural transformation of (covariant) functors and are compatible with long exact sequences.

(3) Given a projective variety $X \hookrightarrow \mathbf{P}^n$ the following diagram commutes:

$$\begin{array}{ccc} L_pH_n(X) & \xrightarrow{s_X^p} & H_n(X) \\ \Sigma\!\!\!/ \downarrow & & \downarrow \tau \\ L_{p+1}H_{n+2}(\Sigma\!\!\!/ X) & \xrightarrow[s_{\Sigma\!\!\!/ X}^{p+1}]{} & H_{n+2}(\Sigma\!\!\!/ X), \end{array}$$

where τ is the Thom isomorphism for the hyperplane bundle over X.

EXAMPLE 1. (Affine spaces). Let $\mathbf{A}^n$ denote the n-dimensional affine space and let $(\mathbf{P}^n, \mathbf{P}^{n-1})$ be its usual compactification.

In [22] it is shown that the inclusion $\mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^n$ induces an injection in the level of homotopy of cycle groups taking generator to generator. Since

$$\tilde{\mathscr{C}}_p(\mathbf{P}^t) \cong \tilde{\mathscr{C}}_0(\mathbf{P}^{t-p}) \cong K(\mathbf{Z}, 2) \times \cdots \times K(\mathbf{Z}, 2(t-p)),$$

as seen in Section 1, the long exact sequence for the Lawson homology of the pair $(\mathbf{P}^n, \mathbf{P}^{n-1})$ shows that

$$\tilde{\mathscr{C}}_p(\mathbf{A}^n) \cong K(\mathbf{Z}, 2(n-p)),$$

in other words

$$L_pH_k(\mathbf{A}^n) = \begin{cases} 0, & \text{if } k \neq 2n \\ \mathbf{Z}, & \text{if } k = 2n \end{cases}$$

Note that the same result can be obtained using the properties of the cycle map described above, long exact sequences and induction on dimension.

EXAMPLE 2: (Products of projective spaces). We show, by induction, that the cycle map $s^p_{\mathbf{P}^n\times\mathbf{P}^m}$: $L_pH_n(\mathbf{P}^n\times\mathbf{P}^m)\to H_n(\mathbf{P}^n\times\mathbf{P}^m)$ is an isomorphism for all n, m.

The induction is on the sum $N=n+m$. For $N=1$ we have that $\mathbf{P}^n\times\mathbf{P}^m=\mathbf{P}^1$. In this case the result follows from the Dold–Thom theorem, since $\tilde{\mathscr{C}}_0(\mathbf{P}^1)\cong SP^\infty(\mathbf{P}^1)$ and $\tilde{\mathscr{C}}_1(\mathbf{P}^1)\cong \mathbf{Z}$. See [21] for details.

Assume the result true for any product $\mathbf{P}^r\times\mathbf{P}^s$ with $r+s\leqslant N-1$, and consider $\mathbf{P}^n\times\mathbf{P}^m$ with $n+m=N$, $N\geqslant 2$.

Embed $\mathbf{P}^n\times\mathbf{P}^m$ into $\mathbf{P}^{nm+m+n}$ via the Segre embedding j: $\mathbf{P}^n\times\mathbf{P}^m\hookrightarrow\mathbf{P}^{nm+m+n}$. This is the embedding provided by the complete linear system associated to the divisor $D=H_1+H_2$, where $H_1=\mathbf{P}^{n-1}\times\mathbf{P}^m$ and $H_2=\mathbf{P}^n\times\mathbf{P}^{m-1}$ are two effective generators of $\mathrm{Div}(\mathbf{P}^n\times\mathbf{P}^m)$, and $\mathbf{P}^{n-1}=\{pt\}$ when $n\leqslant 1$. Observe that $H_1\cap H_2=\mathbf{P}^{n-1}\times\mathbf{P}^{m-1}$ and that D is the divisor obtained by a hyperplane section of $\mathbf{P}^n\times\mathbf{P}^m$ in $\mathbf{P}^{nm+m+n}$.

REMARK 6.1. Since $D=H_1+H_2$ has no weights on its irreducible components, we use D to denote both the divisor and the algebraic set $D=H_1\cup H_2$, indistinctly. Also we assume $n\leqslant m$, $p\geqslant 1$.

Let us compute the cycle spaces $\tilde{\mathscr{C}}_p(D)$ associated to the algebraic set D, noticing that both $(H_1, H_1\cap H_2)$ and (D, H_2) are compactifications of the affine space $\mathbf{A}^{n+m-1}$ and that i: $(H_1, H_1\cap H_2)\hookrightarrow(D, H_2)$ is a relative isomorphism.

Since

$$H_1=\mathbf{P}^{n-1}\times\mathbf{P}^m, \qquad H_2=\mathbf{P}^n\times\mathbf{P}^{m-1} \quad\text{and}\quad H_1\cap H_2=\mathbf{P}^{n-1}\times\mathbf{P}^{m-1}$$

satisfy the induction hypothesis, fact (2) above, the five lemma and the diagram below

$$\begin{array}{ccccccc}
\cdots\to L_pH_n(H_1) & \to & L_pH_n(H_1, H_1\cap H_2) & \to & L_pH_{n-1}(H_1\cap H_2) & \to\cdots \\
\cong\downarrow s^p_{H_1} & & \downarrow s^p_{H_1,H_1\cap H_2} & & \cong\downarrow s^p_{H_1\cap H_2} & \\
\cdots\to H_n(H_1) & \to & H_n(H_1, H_1\cap H_2) & \to & H_{n-1}(H_1\cap H_2) & \to\cdots
\end{array}$$

show that $s^p_{H_1,H_1\cap H_2}$ is an isomorphism. Since $i:(H_1, H_1 \cap H_2) \hookrightarrow (D, H_2)$ is a relative isomorphism, we conclude that

$$s^p_{D,H_2}: L_pH_n(D, H_2) \to H_n(D, H_2) \tag{2}$$

is also an isomorphism, according to fact (1).

Applying (2), the induction hypothesis and the five lemma again, one obtains that

$$s_D: L_pH_n(D) \to H_n(D) \tag{3}$$

is an isomorphism, for all i and p.

Since

$$\mathbf{P}^n \times \mathbf{P}^m - D \cong \mathbf{A}^{n+m} \cong \mathbf{P}^{n+m} - \mathbf{P}^{n+m-1}, \tag{4}$$

fact (1) together with (3) and (4), the five lemma and the well-known isomorphisms for the pair $(\mathbf{P}^{n+m}, \mathbf{P}^{n+m-1})$ complete the proof.

EXAMPLE 3: Hyperquadrics $\mathcal{Q}_n \subset \mathbf{P}^{n+1}$. Let $\mathcal{Q}_n \subset \mathbf{P}^{n+1}$ be a smooth quadric, that is, a quadric of rank $n+1$ (and dimension n), and let H be a hyperplane in $\mathbf{P}^{n+1}$ which is tangent to $\mathcal{Q}_n$ at some point $p_0 \in \mathcal{Q}_n$. Recall the following facts:

(1) The intersection $H \cap \mathcal{Q}_n$ is a singular quadric of rank $n-1$ in H, and hence it is isomorphic to the complex suspension $\Sigma\mathcal{Q}_{n-2} = p_0 \# \mathcal{Q}_{n-2}$, where $\mathcal{Q}_{n-2}$ is the intersection $H \cap \mathcal{Q}_n \cap H'$, with H' being any hyperplane in $\mathbf{P}^{n+1}$ not containing p_0.

(2) With p_0, H and H' chosen as before, consider the projection $\pi: \mathbf{P}^{n+1} - p_0 \to H'$, away from p_0. Let $p: \mathcal{Q}_n - (\mathcal{Q}_n \cap H) \to H'$ be the restriction of π to $\mathcal{Q}_n - (\mathcal{Q}_n \cap H)$. A standard argument shows that p is, actually, an isomorphism onto $H' - (H \cap H') \equiv \mathbf{A}^n$. From now on, denote $\mathcal{Q}_n \cap H$ by $\Sigma\mathcal{Q}_{n-2}$.

Using excision it follows that

$$\tilde{\mathscr{C}}_p(\mathcal{Q}_n, \Sigma\mathcal{Q}_{n-2}) = \tilde{\mathscr{C}}_p(\mathbf{A}^n) \cong K(\mathbf{Z}, 2(n-p)), \tag{5}$$

and hence from fact (1) one has an isomorphism

$$s^p_{\mathcal{Q}_n, \Sigma\mathcal{Q}_{n-2}}: L_pH_n(\mathcal{Q}_n, \Sigma\mathcal{Q}_{n-2}) \to H_n(\mathcal{Q}_n, \Sigma\mathcal{Q}_{n-2})$$

for all $p \geqslant 1$, $i \geqslant 0$. The commutative diagram of fact (3) now reads:

$$\begin{array}{ccc} L_pH_n(\mathcal{Q}_{n-2}) & \xrightarrow{s^p_{\mathcal{Q}_{n-2}}} & H_n(\mathcal{Q}_{n-2}) \\ \cong\downarrow\Sigma & & \cong\downarrow\tau \\ L_{p+1}H_{n+2}(\Sigma\mathcal{Q}_{n-2}) & \xrightarrow{s^p_{\Sigma\mathcal{Q}_{n-2}}} & H_{n+2}(\Sigma\mathcal{Q}_{n-2}), \end{array} \tag{6}$$

where τ is the Thom isomorphism of the hyperplane bundle over $\mathcal{Q}_{n-2}$.

Using (5) and (6), the exact sequences for the pair $(\mathcal{Q}_n, \Sigma\mathcal{Q}_{n-2})$, the previous example and the fact that $\mathcal{Q}_1 \cong \mathbf{P}^1$ and $\mathcal{Q}_2 \cong \mathbf{P}^1 \times \mathbf{P}^1$, it follows by induction that the cycle map $s^p_{\mathcal{Q}_n}$ is an isomorphism for all p and n.

REMARK 6.2. A singular quadric $\mathcal{Q}^k_n \subset \mathbf{P}^{n+1}$, of rank k, is isomorphic to the iterated complex suspension $\Sigma^{n-k}\mathcal{Q}_k$, where $\mathcal{Q}_k$ is a smooth hyperquadric contained in a linear subspace $\mathbf{P}^{k+1} \subset \mathbf{P}^{n+1}$. Therefore, the complex suspension theorem asserts that

$$\tilde{\mathscr{C}}_p(\mathcal{Q}^k_n) \cong \tilde{\mathscr{C}}_{p+k-n}(\mathcal{Q}_k),$$

which, combined with the above results yields

$$\tilde{\mathscr{C}}_p(\mathcal{Q}^k_n) \overset{\text{h.eq.}}{\cong} \tilde{\mathscr{C}}_{p+k-n}(\mathcal{Q}_{k-2}) \times K(\mathbf{Z}, 2(n-p)).$$

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