TAMOTSU IKEDA

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On the location of poles of the triple L-functions

TAMOTSU IKEDA*
Kyoto University, Kitashirakawa, Kyoto, 606 Japan

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Introduction

Let $\mathbf{K}$ be a semi-simple abelian algebra of degree 3 over a global field k. In [22], I. I. Piatetski-Shapiro and S. Rallis constructed the triple L-functions for irreducible cuspidal automorphic representations of $GL_2(\mathbf{K} \otimes \mathbf{A}_k)$ by means of Rankin-type integrals following P. B. Garrett [3]. The purpose of this paper is to determine the location of the poles of these L-functions. To describe our main result, assume, for simplicity, $\mathbf{K} = k \oplus k \oplus k$. Let α be the standard idele norm: $\mathbf{A}_k^\times \to \mathbf{R}_+^\times$. Given three irreducible cuspidal automorphic representations π_1, π_2, and π_3 of $GL_2(\mathbf{A}_k)$, let ω be the product of the central quasi-characters of these representations. Let σ be the 8-dimensional representation of the L-group $GL_2(\mathbf{C})^3$ obtained by the tensor product of the standard representations of $GL_2(\mathbf{C})$. The triple L-function $L(s, \Pi, \sigma)$ is the L-function associated to $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ and σ. This is defined by the Euler product:

$$L(s, \Pi, \sigma) = \prod_v L(s, \Pi_v, \sigma).$$

If k_v is non-archimedean and Π_v is of class 1, then

$$L(s, \Pi_v, \sigma) = \det(\mathbf{1}_8 - A_1 \otimes A_2 \otimes A_3 \cdot q_v^{-s})^{-1},$$

where q_v is the order of the residue field of k_v, and A_i is the Langlands class of $\pi_{i,v}$ $(i = 1, 2, 3)$. Then our main theorem in the case $\mathbf{K} = k \oplus k \oplus k$ can be stated as follows.

THEOREM 2.7. *Suppose that $\mathbf{K} = k \oplus k \oplus k$, and $L(s, \Pi, \pi)$ has a pole somewhere. Then the following two assertions hold:*

(a) *Let Π', ω' be the objects obtained by twisting π_1 by α^{s_0}, $s_0 \in \mathbf{C}$. Then $\omega'^2 = 1$, $\omega' \neq 1$, and $L(s, \Pi', \sigma)$ has a simple pole at $s = 1$, for some $s_0 \in \mathbf{C}$.*

(b) *Assume that $\omega^2 = 1$, $\omega \neq 1$, and $L(s, \Pi, \sigma)$ has a pole at $s = 1$. Let K be the*

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quadratic extension of k corresponding to ω by class field theory. Let θ be the generator of Gal (K/k). Then there exist quasi-characters χ_1, χ_2, and χ_3 of $\mathbf{A}_K^\times/K^\times$ such that $\pi_1 = \pi(\chi_1)$, $\pi_2 = \pi(\chi_2)$, $\pi_3 = \pi(\chi_3)$, and $\chi_1\chi_2\chi_3 = 1$. Moreover, the triple L-function is equal to

$$\zeta_K(s)L_K(s, \chi_1^{-1}\chi_1^\theta)L_K(s, \chi_2^{-1}\chi_2^\theta)L_K(s, \chi_3^{-1}\chi_3^\theta).$$

Note that our results are consistent with "the Langlands philosophy". Assume that for each π_i, there is a 2-dimensional complex representation ρ_i of *Gal* $(\bar{k}/k)$ such that $L(s, \pi_i) = L(s, \rho_i)$. Then our main theorem implies that, up to twist by α^{s_0} for some $s_0 \in \mathbf{C}$, $L(s, \Pi, \sigma)$ has a pole if and only if $\rho_1 \otimes \rho_2 \otimes \rho_3$ has a trivial constituent.

A significant point of this result is its possible application to the construction of the lift $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_4$ of automorphic representations by means of "the converse theorem". The author hopes to treat this problem in the future.

Let us now describe the contents of this paper. Section 1 is devoted to the theory of Eisenstein series on symplectic group Sp_n. Assume, for simplicity, k is a number field. Consider the representation space $I(\omega, s)$ of the representation $\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}\omega\alpha^s$ induced from a quasi-character ω of the parabolic subgroup

$$P_n = \left\{ \begin{pmatrix} A & * \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix} \in \mathrm{Sp}_n \right\}$$

of Sp_n. Let $f^{(s)}$ be a meromorphic section of $I(\omega, s)$, which roughly means that $f^{(s)}$ belongs to $I(\omega, s)$ for each $s \in \mathbf{C}$ and is meromorphic in s. In order to make use of the Rankin-Selberg convolution, we require that the family $\{f^{(s)}\}$ has the following properties:

(i) $E(h; f^{(s)})$ has finite number of poles.
(ii) The family $\{f^{(s)}\}$ is stable under the intertwining operator M_{w_0} with respect to the longest Weyl group element w_0.
(iii) The family $\{f^{(s)}\}$ is the restricted tensor product of families of meromorphic sections $\{f_v^{(s)}\}$ of induced representations $I(\omega_v, s)$ on $\mathrm{Sp}_n(k_v)$.
(iv) The family $\{f_v^{(s)}\}$ contains all holomorphic sections.

Moreover, to get a good local functional equation, we need a normalization $M_{w_0}^*$ of the local intertwining operator such that

(v) $M_{w_0}^* \circ M_{w_0}^* = \text{const.}$
(vi) The family $\{f_v^{(s)}\}$ is stable under the normalized intertwining operator $M_{w_0}^*$.

We shall construct this normalized intertwining operator, and the family $\{f_v^{(s)}\}$ in Section 1.2. A function $f^{(s)}$ in this family is called a good section. Our normalized intertwining operator is different from Langlands's normalization [16, Appendix 2]. In Section 1.3 we shall determine the location of the poles of the Eisenstein series $E(h; f^{(s)})$ associated to a good section $f^{(s)}$. In Section 1.4 we calculate the residue of the Eisenstein series $E(h; f^{(s)})$ at $s = \frac{n-1}{2}$.

Section 2 is devoted to the theory of the triple L-functions. We shall define the local L-factor and ε-factor, and give the functional equation for the triple L-functions. The location of the poles is then determined. The key lemma is that if $\omega = 1$, then $L(s, \Pi, \sigma)$ does not have a pole at $s = 1$ (Proposition 2.5). The main theorem will be proved by showing that the base change of Π to $GL_2(\mathbf{A}_K)^3$ is not cuspidal.

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Notation

The $n \times n$ zero and identity matrices are denoted by $\mathbf{0}_n$ and $\mathbf{1}_n$, respectively. If X is a matrix, det X stands for its determinant. For a function f on a group G and $x \in G$, we denote by $\rho(x)f$ the right translation of f by x, i.e., $\rho(x)f(g) = f(gx)$. When G is locally compact, the Schwartz-Bruhat space of G is denoted by $\mathcal{S}(G)$. If G is an algebraic group defined over a field k, the group of k-valued points of G is denoted by $G(k)$ or G. If π is a representation of G, its contragredient is denoted by $\tilde{\pi}$. When k is a global field, the adele ring (resp. the idele group) of k is denoted by $\mathbf{A}_k$ or $\mathbf{A}$ (resp. $\mathbf{A}_k^\times$ or $\mathbf{A}^\times$). We fix a non-trivial additive character ψ of $\mathbf{A}/k$ (resp. k), if k is a global field (resp. local field). The standard idele norm: $\mathbf{A}^\times \to \mathbf{R}_+^\times$ is denoted by $| \, |$ or α. When k is a local field, the normalized absolute value: $k^\times \to \mathbf{R}_+^\times$ is denoted by $| \, |$ or α. When k is a global (resp. local) field, a quasi-character χ of $\mathbf{A}^\times$ (resp. $k^\times$) is called principal if $\chi = \alpha^{s_0}$ for some $s_0 \in \mathbf{C}$. When k is a global function field, the order of the coefficient field of k is denoted by q. When k is a non-archimedean local field, $\mathcal{O}$, ϖ, and q are the maximal order of k, a prime element of $\mathcal{O}$, and the order of the residue field of k, respectively. The multiplicative Haar measure $d^\times x$ of $k^\times$ is normalized so that $\mathrm{Vol}(\mathcal{O}^\times) = 1$.

1. Analytic theory of Eisenstein series

1.1. Definitions

Let H_n be the symplectic group Sp_n:

$$H_n = \mathrm{Sp}_n = \left\{ h \in \mathrm{GL}_{2n} \,\middle|\, h \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix} {}^t h = \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \right\}.$$

We define parabolic subgroups P_n and B_n of H_n by

$$P_n = \left\{ \begin{pmatrix} A & * \\ \mathbf{0}_n & {}^t A^{-1} \end{pmatrix} \in H_n \right\},$$

$$B_n = \left\{ \begin{pmatrix} A & * \\ \mathbf{0}_n & {}^t A^{-1} \end{pmatrix} \in P_n \,\middle|\, A \text{ is upper triangular} \right\}.$$

Let M_m (resp. T_n) be a Levi factor of P_n (resp. B_n) given by

$$M_n = \left\{ \begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^t A^{-1} \end{pmatrix} \,\middle|\, A \in \mathrm{GL}_n \right\},$$

$$T_n = \left\{ \begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^t A^{-1} \end{pmatrix} \,\middle|\, A \text{ is diagonal} \right\}.$$

We denote by U_n (resp. N_n) the unipotent radical of P_n (resp. B_n):

$$U_n = \left\{ \begin{pmatrix} \mathbf{1}_n & B \\ \mathbf{0}_n & \mathbf{1}_n \end{pmatrix} \,\middle|\, B = {}^t B \right\},$$

$$N_n = \left\{ \begin{pmatrix} A & * \\ \mathbf{0}_n & {}^t A^{-1} \end{pmatrix} \in H_n \,\middle|\, A \text{ is unipotent upper triangular} \right\}.$$

Let P_n^- and B_n^- be the opposite parabolic subgroups of P_n and B_n, respectively. We denote by U_n^- (resp. N_n^-) the unipotent radical of P_n^- (resp. B_n^-).

Let x_i $(1 \leqslant i \leqslant n)$ be the character of T_n given by

$$\begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_1^{-1} & & \\ & & & & \ddots & \\ & & & & & t_n^{-1} \end{pmatrix} \mapsto t_i.$$

Let $\mathrm{Norm}(T_n)$ be the normalizer of T_n in H_n. We denote the Weyl group $\mathrm{Norm}(T_n)/T_n$ by W_{H_n}. We shall often use the same symbol for an element of $\mathrm{Norm}(T_n)$ and its image in W_{H_n}. Let Φ_{H_n} (resp. Φ_{M_n}) be the set of roots of H_n (resp. M_n) with respect to T_n. We denote by N_α the unipotent group associated to a root $\alpha \in \Phi_{H_n}$. Each N_α is isomorphic to k in the natural way (by the coordinate). We denote by w_α the reflection determined by α. Let α_i be the simple root:

$$\alpha_i = x_i - x_{i+1}, \quad (1 \leqslant i \leqslant n-1)$$

$$\alpha_n = 2x_n.$$

Let Ω_n be the complete set of representatives for W_{H_n}/W_{M_n} obtained by choosing the unique element of minimal length in each coset. For each subset $I = \{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$, we define an element w_I of W_{H_n} by

$$x_1 \to x_{j_1}, \ldots, x_{n-k} \to x_{j_{n-k}},$$

$$x_{n-k+1} \to -x_{i_k}, \ldots, x_n \to -x_{i_1},$$

where $J = \{j_1, j_2, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\} - I$, $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_{n-k}$. The element w_I belongs to Ω_n and each element of Ω_n is obtained in this way (cf. [20]). We also denote by Ω_n a set of representatives of Ω_n in $\mathrm{Norm}(T_n)$. The length $l(w_I)$ of w_I is given by

$$\begin{aligned} l(w_I) &= \#\{\alpha \in \Phi_{H_n} \mid \alpha > 0,\ w_I\alpha < 0\} \\ &= \sum_{r=1}^{k} (n+1-i_r). \end{aligned}$$

Put

$$w_0 = w_{\{1,2,\dots,n\}} = \left(\begin{array}{ccc|ccc} & & & & & -1 \\ & \mathbf{0}_n & & & \cdot^{\cdot^{\cdot}} & \\ & & & -1 & & \\ \hline & & 1 & & & \\ & \cdot^{\cdot^{\cdot}} & & & \mathbf{0}_n & \\ 1 & & & & & \end{array}\right).$$

This is the longest element in Ω_n. For $w \in \mathrm{Norm}(T_n)$ and a character χ of T_n, we put

$$\chi^w(t) = \chi(w^{-1}tw).$$

Obviously χ^w depends only upon the class of w in W_{H_n}, so we shall use the same notation χ^w for $w \in W_{H_n}$. We often regard a character of T_n as a character of B_n by the isomorphism $B_n/N_n \simeq T_n$.

1.2. *Local theory*

In this subsection, k is a local field. We define the standard maximal compact subgroup K_n of H_n as follows.

When k is non-archimedean, we put $K_n = H_n(\mathcal{O})$. When $k = \mathbf{R}$, we put

$$K_n = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in H_n \,\middle|\, A^tB = B^tA,\ A^tA + B^tB = \mathbf{1}_n \right\}.$$

When $k = \mathbf{C}$, we put

$$K_n = \left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in H_n \,\middle|\, A^tB = B^tA,\ A^t\bar{A} + B^t\bar{B} = \mathbf{1}_n \right\}.$$

When k is non-archimedean, we put $R = \mathbf{C}[q^s, q^{-s}]$. When k is archimedean, we let R be the ring of entire functions on $\mathbf{C}$. Let ω be a quasi-character of $k^\times$ and let s denote a complex number. Let $I(\omega, s) = \mathrm{Ind}_{P_n}^{H_n}(\omega\alpha^s)$ be the space of functions f on H_n which satisfy the following two conditions:

(i) f is right K_n-finite.

(ii) For any $p = \begin{pmatrix} A & * \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix} \in P_n$,

$$f(ph) = \omega(\det A)|\det A|^{s+(n+1)/2} f(h).$$

We say that a function $f^{(s)}(h)$ on $H_n \times \mathbf{C}$ is a holomorphic section of $I(\omega, s)$ if the following three conditions are satisfied:

(1) For each $s \in \mathbf{C}$, $f^{(s)}(h)$ belongs to $I(\omega, s)$ as a function of $h \in H_n$.
(2) For each $h \in H_n$, $f^{(s)}(h)$ belongs to R as a function of $s \in \mathbf{C}$.
(3) $f^{(s)}(h)$ is right K_n-finite.

We say that a meromorphic function $f^{(s)}(h)$ on $H_n \times \mathbf{C}$ is a meromorphic section of $I(\omega, s)$, if there is $\alpha(s) \in R$ such that $\alpha(s) \not\equiv 0$, and $\alpha(s) f^{(s)}(h)$ is a holomorphic section of $I(\omega, s)$. Note that a holomorphic section of $I(\omega, s)$ is determined by its restriction to $K_n \times \mathbf{C}$. We say that a holomorphic section $f^{(s)}(h)$ is a standard section if its restriction to $K_n \times \mathbf{C}$ does not depend on $s \in \mathbf{C}$. Obviously the space of holomorphic sections is generated by standard sections over R.

For a quasi-character χ of T_n, we define $\mathrm{Ind}_{B_n}^{H_n}(\chi)$ to be the space of right K_n-finite functions $f(h)$ on H_n such that

$$f(bh) = \chi(b)\delta_{B_n}^{1/2}(b) f(h),$$

where δ_{B_n} is the modulus quasi-character of B_n. Put

$$\chi_s(t) = \prod_{i=1}^{n} \omega(t_i)|t_i|^{s-(n+1)/2+i},$$

Then $I(\omega, s) \subset \mathrm{Ind}_{B_n}^{H_n}(\chi_s)$. We define holomorphic sections, meromorphic sections, and standard sections of $\mathrm{Ind}_{B_n}^{H_n}(\chi_s)$ similarly.

For $w \in \mathrm{Norm}(T_n)$ and a quasi-character χ of T_n, we define the intertwining operator

$$M_w = M(w, \chi) \colon \mathrm{Ind}_{B_n}^{H_n}(\chi) \to \mathrm{Ind}_{B_n}^{H_n}(\chi^w)$$

by

$$M_w f(h) = \int_{N_n \cap w N_n^- w^{-1}} f(w^{-1} n h) dn.$$

Here the Haar measure dn is determined as follows. For each $\alpha \in \Phi_{H_n}$, the Haar measure dn_α on N_α is given by the self dual measure on k with respect to ψ by the natural isomorphism $N_\alpha \simeq k$. Then the measure dn is the product measure: $dn = \Pi\, dn_\alpha$. The integral is absolutely convergent if χ belongs to some open set and can be meromorphically continued to all χ (cf. [8], [25]).

If $l(w_1) + l(w_2) = l(w_1 w_2)$, then $M_{w_1} \circ M_{w_2} = M_{w_1 w_2}$. When $w = w_\alpha$ is a reflection with respect to a simple root α, then $M(w, \chi)$ can be regarded as an intertwining operator on SL_2 as follows: let $\iota_\alpha: \mathrm{SL}_2 \to H_n$ be a homomorphism corresponding to α. We may assume $w = \iota_\alpha\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$. Then for any $f \in \mathrm{Ind}_{B_n}^{H_n}(\chi)$,

$$\iota_\alpha^*(M(w, \chi)f) = M\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \iota_\alpha^*\chi\right)(\iota_\alpha^* f), \tag{1.2.1}$$

as a function on SL_2. Since $M(w, \chi)$ commutes with right translations (or actions of Hecke operators), it follows from (1.2.1) that the whole property of $M(w, \chi)$ is reduced to that of $M\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \iota_\alpha^*\chi\right)$. When ω is unramified, there exists a unique standard section $\phi_{\omega,s}$ of $I(\omega, s)$ such that $\phi_{\omega,s}|_{K_n} \equiv 1$. Similarly, there exists a unique standard section $\phi_{\omega,s}^w$ of $\mathrm{Ind}_{B_n}^{H_n}(\chi_s^w)$ such that $\phi_{\omega,s}^w|_{K_n} \equiv 1$, for any $w \in \Omega_n$. Note that $\phi_{\omega,s}^{w_0} = \phi_{\omega^{-1},-s}$.

Let us recall some known results concerning $\mathrm{SL}_2 \simeq H_1$. Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $M_w = M(w, \omega) = M(w, \omega, s): I(\omega, s) \to I(\omega^{-1}, -s)$. Then:

(1.2.2) $L(s, \omega)^{-1} M_w$ is holomorphic.
(1.2.3) $M(w^{-1}, \omega^{-1}) \circ M(w, \omega) = \varepsilon'(s, \omega, \psi)^{-1} \varepsilon'(-s, \omega^{-1}, \psi)^{-1} \cdot \mathrm{id}$.
(1.2.4) If ω is unramified, and ψ is of order 0,

$$M_w \phi_{\omega,s} = \frac{L(s, \omega)}{L(s+1, \omega)} \phi_{\omega^{-1},-s}.$$

(1.2.5) If k is non-archimedean and $\omega = 1$, the kernel and the image of $M(w, 1, 1)$: $I(1, 1) \to I(1, -1)$ are the Steinberg representation and the trivial representation, respectively.
(1.2.6) If k is non-archimedean and $\omega = 1$, the kernel and the image of $M(w, 1, -1)$: $I(1, -1) \to I(1, 1)$ are the trivial representation and the Steinberg representation, respectively.
(1.2.7) If $\omega = 1$, then $\mathrm{Res}_{s=0} M(w, 1, s)$ is a non-zero scalar multiplication.

If $w \in \Omega_n$, then the restriction of M_w to $I(\omega, s) \subset \mathrm{Ind}_{B_n}^{H_n}(\chi_s)$ is well defined (except for countably many values of s). If $f^{(s)}$ is a holomorphic section of $I(\omega, s)$, then $M_w f^{(s)}$ is a meromorphic section of $\mathrm{Ind}_{B_n}^{H_n}(\chi_s^w)$. We denote this restriction by $M_w = M(w, \omega) = M(w, \omega, s)$, too. If ω is unramified, $w \in \mathrm{Norm}(T_n) \cap K_n$, and ψ is of order 0, then there exists a meromorphic function $c_w(s) = c_w(\omega, s)$ such that

$$M_w(\phi_{\omega,s}) = c_w(s) \phi_{\omega,s}^w.$$

$$c_w(s) = \prod_{\substack{\alpha \in \Phi_{H_n} \\ w\alpha < 0 \\ \alpha > 0}} \frac{L(\langle \check{\alpha}, \chi_s \rangle)}{L(\langle \check{\alpha}, \chi_s \rangle + 1)},$$

where $\langle\,,\rangle$ is a W_{H_n}-invariant inner product on $X^*(T_n)\otimes_{\mathbf{Z}}\mathbf{C}$, and $\check{\alpha}=2\alpha/\langle\alpha,\alpha\rangle$ is the coroot of α.

In [20], the common denominator of $c_w(s)$ is calculated. Here we proceed in a slightly different way. Let $w=w_I$, $I=\{i_1,i_2,\ldots,i_k\}$. Put

$$\begin{aligned} N(w_I) &= \{\alpha\in\Phi_{H_n}\mid \alpha>0,\ w_I\alpha<0\} \\ &= \{2x_{n-m+1}\mid 1\leqslant m\leqslant k\} \\ &\quad \bigcup\{x_m+x_{n-r+1}\mid 1\leqslant r\leqslant k,\ i_r-r+1\leqslant m\leqslant n-r\} \end{aligned}$$

We divide $N(w_I)$ into a disjoint union $\coprod_{r=0}^{[n/2]} N_r(w_I)$:

$$N_r(w_I)=\begin{cases}\{2x_{n-m+1}\mid 1\leqslant m\leqslant k\}, & \text{if } r=0\\ \varnothing, & \text{if } r>k\\ \{x_m+x_{n-r+1}\mid i_r-r+1\leqslant m\leqslant n-r\}, & \text{if } 1\leqslant r\leqslant k,\ i_r\geqslant 2r\\ \{x_m+x_{n-r+1}\mid r\leqslant m\leqslant n-r\} & \\ \quad\bigcup\{x_m+x_r\mid \mu_w(r)\leqslant m\leqslant n-r\}, & \text{if } 1\leqslant r\leqslant k,\ i_r\leqslant 2r-1.\end{cases}$$

Here

$$\mu_w(r)=\begin{cases}\min\{m\mid n-k+1\leqslant m\leqslant n,\ j_r<i_{n-m+1}\}, & \text{if } 1\leqslant r\leqslant n-k\\ r+1, & \text{if } n-k+1\leqslant r\leqslant [\frac{n}{2}].\end{cases}$$

Put

$$d^r(s)=\begin{cases} L\left(s+\dfrac{n+1}{2},\,\omega\right), & \text{if } r=0\\[2ex] L(2s+n+1-2r,\ \omega^2), & \text{if } 1\leqslant r\leqslant\left[\dfrac{n}{2}\right],\end{cases}$$

$$a_w^r(s)=\begin{cases} L\left(s+\dfrac{n+1}{2}-k,\,\omega\right), & \text{if } r=0\\[2ex] L(2s+n+1-2r,\ \omega^2), & \text{if } k<r\leqslant\left[\dfrac{n}{2}\right]\\[2ex] L(2s+i_r-2r+1,\ \omega^2), & \text{if } 1\leqslant r\leqslant k,\ i_r\geqslant 2r\\ L(2s-n+r+\mu_w(r)-1,\ \omega^2), & \text{if } 1\leqslant r\leqslant k,\ i_r\leqslant 2r-1,\end{cases}$$

$$d(s)=\prod_{r=0}^{[n/2]} d^r(s),\qquad a_w(s)=\prod_{r=0}^{[n/2]} a_w^r(s).$$

Then we have

$$c_w(s) = \prod_{r=0}^{[n/2]} \prod_{\alpha\in N_r(w_I)} \frac{L(\langle\check{\alpha}, \chi_s\rangle)}{L(\langle\check{\alpha}, \chi_s\rangle+1)}$$
$$= \prod_{r=0}^{[n/2]} \frac{a_w^r(s)}{d^r(s)}$$
$$= \frac{a_w(s)}{d(s)}.$$

Thus $d(s)$ is the smallest common denominator of $c_w(s)$, $w\in\Omega_n$. Note that

$$c_w(s) = \prod_{r=0}^{\min(k,[n/2])} \frac{a_w^r(s)}{d^r(s)}.$$

Now, even when ω is not unramified, we define $c_w(s)$, $d(s)$ etc. by formally substituting ω.

DEFINITION. The normalized intertwining operator

$$M_{w_0}^* = M^*(w_0, \omega) = M^*(w_0, \omega; \psi)\colon I(\omega, s)\to I(\omega^{-1}, -s)$$

is given by

$$M_{w_0}^* = \varepsilon'\left(s-\frac{n-1}{2}, \omega, \psi\right)\cdot\prod_{r=1}^{[n/2]} \varepsilon'(2s-n+2r, \omega^2, \psi)\cdot M_{w_0}.$$

LEMMA 1.1.

$$M^*(w_0^{-1}, \omega^{-1}; \psi)\circ M^*(w_0, \omega; \psi) = \omega(-1)^{n+1}\cdot\mathrm{id},$$
$$M^*(w_0, \omega^{-1}; \bar{\psi})\circ M^*(w_0, \omega; \psi) = \mathrm{id}.$$

Proof. The second formula is just a reformulation of the first formula. We will prove the first formula. When $n=1$, this is (1.2.3). Since

$$\varepsilon'(-s, \omega^{-1}, \psi)\varepsilon'(s+1, \omega, \psi) = \omega(-1),$$

the right-hand side of (1.2.3) is equal to

$$\omega(-1)\frac{\varepsilon'(s+1, \omega, \psi)}{\varepsilon'(s, \omega, \psi)}\cdot\mathrm{id}.$$

For general n, take a minimal expression of w_0 in W_{H_n} by simple reflections

$$w_0 = w_1 w_2 \cdots w_k.$$

By using (1.2.1) and (1.2.3) successively,

$$\begin{aligned}
M_{w_0^{-1}} \circ M_{w_0} &= M_{w_k^{-1}} \circ \cdots \circ M_{w_2^{-1}} \circ M_{w_1^{-1}} \circ M_{w_1} \circ M_{w_2} \circ \cdots \circ M_{w_k} \\
&= \omega(-1)^n \prod_{\substack{\alpha \in \Phi^+_{H_n} \\ \alpha \notin \Phi_{M_n}}} \frac{\varepsilon'(\langle \check{\alpha}, \chi_s \rangle + 1, \psi)}{\varepsilon'(\langle \check{\alpha}, \chi_s \rangle, \psi)} \cdot \mathrm{id} \\
&= \omega(-1)^n \frac{\varepsilon'(s+(n+1)/2, \omega, \psi)}{\varepsilon'(s-(n-1)/2, \omega, \psi)} \\
&\quad \times \prod_{r=1}^{[n/2]} \frac{\varepsilon'(2s+n+1-2r, \omega^2, \psi)}{\varepsilon'(2s-n+2r, \omega^2, \psi)} \cdot \mathrm{id} \\
&= \omega(-1)^{n+1} \varepsilon'\left(s - \frac{n-1}{2}, \omega, \psi\right)^{-1} \varepsilon'\left(-s - \frac{n-1}{2}, \omega^{-1}, \psi\right)^{-1} \\
&\quad \times \prod_{r=1}^{[n/2]} \varepsilon'(2s-n+2r, \omega^2, \psi)^{-1} \varepsilon'(-2s-n+2r, \omega^{-2}, \psi)^{-1} \cdot \mathrm{id}.
\end{aligned}$$

Hence the lemma.

DEFINITION. A meromorphic section $f^{(s)}(h)$ of $I(\omega, s)$ is a good section of $I(\omega, s)$ if for any $w \in \Omega_n$,

$$[d(s)c_w(s)]^{-1} M_w f^{(s)}$$

is holomorphic.

In particular, if ω is unramified, $d(s)\phi_{\omega,s}$ is a good section of $I(\omega, s)$.

LEMMA 1.2. *$f^{(s)}$ is a good section of $I(\omega, s)$ if and only if $M^*_{w_0} f^{(s)}$ is a good section of $I(\omega^{-1}, -s)$.*

Proof. It will suffice to prove that for each $w_I \in \Omega_n$, there exists an entire function $\varepsilon(s)$ with no zeros such that

$$\begin{aligned}
&[d(\omega, s)c_{w_I}(\omega, s)]^{-1} M_{w_I} f^{(s)}(h) \\
&\quad = \varepsilon(s)[d(\omega^{-1}, -s)c_{w_J}(\omega^{-1}, -s)]^{-1} M_{w_J} \circ M^*_{w_0} f^{(s)}(h). \qquad (1.2.8)
\end{aligned}$$

We shall proceed by induction on $l(w_J)$. Obviously, (1.2.8) holds when $l(w_J) = 0$.

Suppose $l(w_J) > 0$. There are two cases:

(1) $j_{n-k} = n$.
(2) $j_{n-k} = m < n$.

In case (1), put $I' = I \cup \{n\}$, $J' = J - \{n\}$. Then

$$l(w_{I'}) = l(w_I) + 1, \qquad l(w_{J'}) = l(w_J) - 1,$$

$$w_J = w_{\alpha_n} \cdot w_{J'}, \qquad M_{w_J} = M_{w_{\alpha_n}} \circ M_{w_{J'}},$$

$$w_{I'} = w_{\alpha_n} \cdot w_I, \qquad M_{w_{I'}} = M_{w_{\alpha_n}} \circ M_{w_I},$$

$$c_{w_J}(\omega^{-1}, -s) = c_{w_{J'}}(\omega^{-1}, -s) \frac{L\left(-s + \frac{-n+1}{2} + k, \omega^{-1}\right)}{L\left(-s + \frac{-n+1}{2} + k + 1, \omega^{-1}\right)},$$

$$c_{w_I}(\omega, s) = c_{w_{I'}}(\omega, s) \frac{L\left(s + \frac{n+1}{2} - k, \omega\right)}{L\left(s + \frac{n+1}{2} - k - 1, \omega\right)}.$$

On the other hand, by (1.2.1) and (1.2.3),

$$\begin{aligned} M_{w_{\alpha_n}} \circ M_{w_{I'}} &= M_{w_{\alpha_n}} \circ M_{w_{\alpha_n}} \circ M_{w_I} \\ &= C \cdot \varepsilon'\left(s + \frac{n-1}{2} - k, \omega, \psi\right)^{-1} \varepsilon'\left(-s - \frac{n-1}{2} + k, \omega^{-1}, \psi\right)^{-1} \cdot M_{w_I}, \end{aligned}$$

where C is some non-zero constant. We have

$$\begin{aligned} &[d(\omega, s) c_{w_I}(\omega, s)]^{-1} M_{w_I} f^{(s)} \\ &= [d(\omega, s) c_{w_{I'}}(\omega, s)]^{-1} \frac{L\left(s + \frac{n+1}{2} - k - 1, \omega\right)}{L\left(s + \frac{n+1}{2} - k, \omega\right)} \\ &\quad \times C^{-1} \cdot \varepsilon'\left(s + \frac{n-1}{2} - k, \omega, \psi\right) \varepsilon'\left(-s - \frac{n-1}{2} + k, \omega^{-1}, \psi\right) \cdot M_{w_{\alpha_n}} \circ M_{w_{I'}} f^{(s)}. \end{aligned}$$

By the induction assumption, this is equal to

$$\varepsilon_1(s)\frac{L\left(s+\frac{n+1}{2}-k-1,\omega\right)}{L\left(s+\frac{n+1}{2}-k,\omega\right)}\frac{L\left(1-s-\frac{n-1}{2}+k,\omega^{-1}\right)}{L\left(s+\frac{n-1}{2}-k,\omega\right)}$$

$$\times\frac{L\left(s+\frac{n+1}{2}-k,\omega\right)}{L\left(-s-\frac{n-1}{2}+k,\omega^{-1}\right)}$$

$$\times[d(\omega^{-1},-s)c_{w_{J'}}(\omega^{-1},-s)]^{-1}M_{w_{\alpha_n}}\circ M_{w_{J'}}\circ M^*_{w_0}f^{(s)}$$

$$=\varepsilon_1(s)[d(\omega^{-1},-s)c_{w_J}(\omega^{-1},-s)]^{-1}M_{w_J}\circ M^*_{w_0}f^{(s)}.$$

Here $\varepsilon_1(s)$ is some entire function with no zeros.

In case (2), put $I'=I-\{m\}\cup\{m+1\}$, $J'=J-\{m+1\}\cup\{m\}$. Then

$$l(w_{I'})=l(w_I)+1,\qquad l(w_{J'})=l(w_J)-1,$$

$$w_J=w_{\alpha_m}\cdot w_{J'},\qquad M_{w_J}=M_{w_{\alpha_m}}\circ M_{w_{J'}},$$

$$w_{I'}=w_{\alpha_m}\cdot w_I,\qquad M_{w_{I'}}=M_{w_{\alpha_m}}\circ M_{w_I}.$$

By a calculation similar to case (1), (1.2.8) for I is reduced to (1.2.8) for I'. Thus the lemma follows.

The following lemma is crucial for our theory.

LEMMA 1.3. *Every holomorphic section of $I(\omega,s)$ is a good section.*

REMARK. If $k\neq\mathbf{C}$, and ω is unramified, this lemma is nothing but [22, Theorem 4.2].

Proof of Lemma 1.3. Here we assume k is non-archimedean. We may assume ω is ramified. If ω^2 is ramified, then $d(s)=c_w(s)=1$, for any $w\in\Omega_n$. Take a minimal expression of w by simple reflections:

$$w=w_1w_2\cdots w_r,\qquad M_w=M_{w_1}\circ M_{w_2}\circ\cdots\circ M_{w_r}.$$

Each M_{w_i} $(1\leqslant i\leqslant r)$ is holomorphic by (1.2.1) and (1.2.2). So the lemma is obvious in this case.

Now we assume ω is ramified and $\omega^2=1$. Let $w=w_I$, $I=\{i_1,i_2,\dots,i_k\}$. Recall

$$a_w(s)=d(s)c_w(s)=\prod_{r=0}^{[n/2]} a_w^r(s).$$

It suffices to prove

$$\left[\prod_{r=0}^{\min(k,[n/2])} a_w^r(s)\right]^{-1} M_w f^{(s)} \tag{1.2.9}$$

is holomorphic. Put

$$A_w(s)=\prod_{r=0}^{\min(k,[n/2])} a_w^r(s).$$

We proceed by induction on $l(w)$. If $l(w)=0$, (1.2.9) is obviously holomorphic.

(I) When $i_k=n$: put $I'=I-\{n\}$, $w'=w_{I'}$. Then

$$M_w=M_{w_{\alpha_n}}\circ M_{w'},\qquad A_w(s)=A_{w'}(s).$$

Since $M_{w_{\alpha_n}}$ is entire, the holomorphy of (1.2.9) for w is reduced to that for w'.

(II) When $i_r+2=i_{r+1}+1<i_{r+2}$, for some $1\leqslant r\leqslant k-2$: put $i_r=m$, $I'=I-\{m+1\}\cup\{m+2\}$, $I''=I-\{m\}\cup\{m+2\}$, $w'=w_{I'}$, $w''=w_{I''}$. We reduce the holomorphy of (1.2.9) for w to that for w'. By definition, we have

$$A_{w'}(s)A_w(s)^{-1}=\zeta(2s+m-2r+2)\zeta(2s+m-2r+1)^{-1},$$

$$M(w,\chi_s)=M(w_{\alpha_m},\chi_s^{w'})\circ M(w',\chi_s).$$

Since $\zeta(2s+m-2r+1)^{-1}M(w_{\alpha_m},\chi_s^{w'})$ is entire, it will suffice to prove that $2s\equiv -m+2r-2\left(\bmod \dfrac{2\pi\sqrt{-1}}{\log q}\mathbf{Z}\right)$ are not poles of (1.2.9). We now prove that the residue vanishes. By (1.2.7),

$$\zeta(2s+m-2r+1)^{-1}M(w_{\alpha_m},\chi_s^{w'})$$

is holomorphic at these points. The residue is

$$\begin{aligned}&\mathrm{Res}_{2s\equiv -m+2r-2}(A_w(s)^{-1}M_wf^{(s)})\\ &=c\cdot M(w_{\alpha_m},\chi_s^{w'})\circ \mathrm{Res}_{2s\equiv -m+2r-2}[\zeta(2s+m-2r+2)A_{w'}(s)^{-1}M_{w'}f^{(s)}]\\ &=c'\cdot M(w_{\alpha_m},\chi_s^{w'})\circ[A_{w'}(s)^{-1}M_{w'}f^{(s)}]_{2s\equiv -m+2r-2},\end{aligned}$$

for some non-zero constants c, c'. By (1.2.6), it is sufficient to prove that

$$[A_{w'}(s)^{-1}M_{w'}f^{(s)}]_{2s\equiv -m+2r-2} \tag{1.2.10}$$

is left $\iota_{\alpha_m}(\mathrm{SL}_2)$-invariant. We first observe

$$A_{w'}(s)^{-1}M_{w'}f^{(s)}$$
$$=\zeta(2s+m-2r+3)\zeta(2s+m-2r+2)^{-1}A_{w''}(s)^{-1}M(w_{\alpha_{m+1}}, \chi_s^{w''})M(w'', \chi_s)f^{(s)}.$$

Since $\zeta(2s+m-2r+3)$ and $\zeta(2s+m-2r+2)^{-1}M(w_{\alpha_{m+1}}, \chi_s^{w''})$ is holomorphic at $2s \equiv -m+2r-2 \left(\mathrm{mod}\, \frac{2\pi\sqrt{-1}}{\log q}\mathbf{Z}\right)$, this is equal to

$$c''\cdot[\zeta(2s+m-2r+2)^{-1}M(w_{\alpha_{m+1}}, \chi_s^{w''})]_{2s\equiv -m+2r-2}\circ A_{w''}(s)^{-1}M(w'', \chi_s)f^{(s)},$$

for some non-zero constant c''. By the induction assumption,

$$A_{w''}(s)^{-1}M(w'', \chi_s)f^{(s)}$$

is holomorphic. Moreover this is left $\iota_{\alpha_m}(\mathrm{SL}_2)$-invariant since

$$w''^{-1}\iota_{\alpha_m}(\mathrm{SL}_2)w'' \subset M_n.$$

By (1.2.7),

$$[\zeta(2s+m-2r+2)^{-1}M(w_{\alpha_{m+1}}, \chi_s^{w''})]_{2s\equiv -m+2r-2}$$

is a scalar multiplication. Thus (1.2.10) is left $\iota_{\alpha_m}(\mathrm{SL}_2)$-invariant.

(III) When $i_k=n-1$, $i_{k-1}=n-2$: this case can be treated by the same technique as in the case (II) by putting

$$I'=I-\{n-1\}\cup\{n\}, \qquad I''=I-\{n-2\}\cup\{n\}.$$

(IV) When $i_k < n-1$. This case can be treated by a similar technique as in the case (II) by putting

$$I'=I-\{i_k\}\cup\{i_k+1\}, \qquad I''=I-\{i_k\}\cup\{i_k+2\}.$$

Now we may assume $i_k=n-1$, by (I) and (IV). Moreover, we may assume $k\leqslant[\frac{n}{2}]$, since otherwise the assumption of (II) or (III) holds. To see this, assume

$k > [\frac{n}{2}]$ and neither the assumption of (II) nor that of (III) holds. Then

$$i_k = n-1,\ i_{k-1} \leqslant n-3, \ldots, i_k \leqslant n-2k+2m-1, \ldots, i_1 \leqslant n-2k+1 \leqslant 0.$$

This is a contradiction.

(V) When $k \leqslant [\frac{n}{2}]$: put $I' = I - \{n-1\}$, $w' = w_{I'}$.. Then

$$M_w = M(w_{\alpha_{n-1}}, \chi_s^{w_{\alpha_n} w'}) \circ M(w_{\alpha_n}, \chi_s^{w'}) \circ M(w', \chi_s),$$
$$A_w(s) = A_{w'}(s) \cdot \zeta(2s+n-2k).$$

By the induction assumption, $A_{w'}(s)^{-1} M_{w'} f^{(s)}$ is entire. Since both $M(w_{\alpha_n}, \chi_s^{w'})$ and $\zeta(2s+n-2k)^{-1} \cdot M(w_{\alpha_{n-1}}, \chi_s^{w_{\alpha_n} w'})$ are entire, $A_w(s)^{-1} M_w f^{(s)}$ is entire. Thus the proof for non-archimedean local field is complete.

Appendix 1. Proof for Lemma 1.3 for archimedean case

In this appendix, we give a proof for Lemma 1.3 for an archimedean local field k. We may assume that ω is unitary.

SUBLEMMA 1. *If* $w = w_0$, *then* (1.2.9) *is holomorphic.*

Proof. If $k = \mathbf{R}$, and $\omega = 1$, this is proved in [22 §4 Appendix 1]. Their proof is valid for $k = \mathbf{R}$, $\omega = \mathrm{sgn}$. If $k = \mathbf{C}$, we have to show that the first part of [22 §4 Appendix 1, Theorem (p. 106)] holds for our situation, i.e., we have to show that

$$a_{w_0}(\omega, s)^{-1} \int_{\mathrm{Sym}^n(\mathbf{C})} \varphi(z) |\det z\bar{z}|^{s-(n+1)/2} \omega(\det z)\, dz \tag{1.2.11}$$

is entire for any $\varphi \in \mathcal{S}(\mathrm{Sym}^n(\mathbf{C}))$. We may assume that $\omega(z) = z^k$ or $(\bar{z})^k$, $k \geqslant 0$. But the case $\omega(z) = (\bar{z})^k$ is reduced to the case $\omega(z) = z^k$ by taking complex conjugate. Put

$$\partial = \det \begin{vmatrix} \dfrac{\partial}{\partial z_{11}} & \dfrac{1}{2}\dfrac{\partial}{\partial z_{12}} & \cdots & \dfrac{1}{2}\dfrac{\partial}{\partial z_{1n}} \\ \dfrac{1}{2}\dfrac{\partial}{\partial z_{12}} & \dfrac{\partial}{\partial z_{22}} & & \vdots \\ \vdots & & \ddots & \\ \dfrac{1}{2}\dfrac{\partial}{\partial z_{1n}} & \cdots & & \dfrac{\partial}{\partial z_{nn}} \end{vmatrix}$$

Then it is known that

$$\partial(|\det z\bar{z}|^s(\det z)^k) = \prod_{i=0}^{n-1}\left(s+k+\frac{i}{2}\right)\cdot(|\det z\bar{z}|^s(\det z)^{k-1}).$$

Repeating partial integration, we have

$$\prod_{j=1}^{m}\prod_{i=0}^{n-1}\left(s+k+j+\frac{i-n-1}{2}\right)\int_{\mathrm{Sym}^n(\mathbf{C})}\varphi(z)|\det z\bar{z}|^{s-(n+1)/2}(\det z)^k\,dz$$
$$=(-1)^{mn}\int_{\mathrm{Sym}^n(\mathbf{C})}\partial^m\varphi(z)|\det z\bar{z}|^{s-(n+1)/2}(\det z)^{k+m}\,dz$$

for $\mathrm{Re}(s)\gg 0$. Since the right-hand side is absolutely convergent for $\mathrm{Re}(s)>\dfrac{n-k-m-1}{2}$, we have

$$\prod_{i=0}^{n-1}\Gamma\left(s+k-\frac{i}{2}\right)^{-1}\int_{\mathrm{Sym}^n(\mathbf{C})}\varphi(z)|\det z\bar{z}|^{s-(n+1)/2}(\det z)^k\,dz$$

is entire. So (1.2.11) is entire.

Let Q (resp. Q') be the maximal parabolic subgroup of GL_n given by

$$Q=\left\{\begin{pmatrix}a_1 & *\\ 0 & a_2\end{pmatrix}\middle|\, a_1\in\mathrm{GL}_{n-1},\ a_2\in k^\times\right\}$$
$$\left(\text{resp. } Q'=\left\{\begin{pmatrix}a_1 & *\\ 0 & a_2\end{pmatrix}\middle|\, a_1\in k^\times,\ a_2\in\mathrm{GL}_{n-1}\right\}\right).$$

Let $I_Q(\omega,s)$ (resp. $I_{Q'}(\omega,s)$) be the representation of GL_n induced from the character of Q (resp. Q') given by

$$\begin{pmatrix}a_1 & *\\ 0 & a_2\end{pmatrix}\mapsto\omega(\det a_1)|\det a_1|^{s/n}|a_2|^{-[(n-1)/n]s}$$
$$\left(\text{resp. }\begin{pmatrix}a_1 & *\\ 0 & a_2\end{pmatrix}\mapsto\omega^{-1}(\det a_2)|a_1|^{[(n-1)s]/n}|\det a_2|^{-s/n}\right).$$

We define standard sections, holomorphic sections, and meromorphic sections as usual. We define the intertwining operator $M_w\colon I_Q(\omega,s)\mapsto I_{Q'}(\omega^{-1},-s)$

(resp. $M_{w'}: I_{Q'}(\omega, s) \mapsto I_Q(\omega^{-1}, -s)$). Here

$$w = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & \end{pmatrix}, \quad w' = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}.$$

SUBLEMMA 2. $L\left(s-\frac{n-2}{2}, \omega\right)^{-1} M(w, s)$ *and* $L\left(s-\frac{n-2}{2}, \omega\right)^{-1} M(\omega', s)$ *are holomorphic.*

Proof. This can be proved in the same way as [22, §4]. (See also [12 §5].)

SUBLEMMA 3.

$$\begin{aligned} &M(w', \omega^{-1}) \circ M(w, \omega) \\ &\quad = \omega(-1)^{n+1} \varepsilon'\left(s-\frac{n-2}{2}, \omega, \psi\right)^{-1} \varepsilon'\left(-s-\frac{n-2}{2}, \omega^{-1}, \psi\right)^{-1} \cdot \mathrm{id}. \end{aligned}$$

Proof. This can be proved in the same way as the proof of Lemma 1.1.

We now return to the proof of Lemma 1.3. Let $w = w_I$ be an element of Ω_n. We prove that

$$[d(\omega, s) c_w(\omega, s)]^{-1} M_w f(s)$$

is holomorphic. M_w can be considered as an intertwining operator of $I\left(\omega, s+\frac{i_1-1}{2}\right)$ on Sp_{n-i_1+1}. We may assume $i_1=1$ by replacing n by $n-i_1+1$ and I by $\{i_r - i_1 + 1 \mid 1 \leqslant r \leqslant k\}$. We proceed by the induction on $\delta(w) = n-k$. When $n=k$, this is Sublemma 1. Assume $n-k \geqslant 1$. Put

$$\begin{aligned} m &= \max\{r \mid i_r < n-k+r\}, \\ I' &= I \cup \{n-k+m\}, \\ w' &= w_{I'}. \end{aligned}$$

Then $\sharp I' = k+1$, $l(w') = l(w) + k - m + 1$ and

$$w' = w_{\alpha_n} w_{\alpha_{n-1}} \cdots w_{\alpha_{n-k+m}} w.$$

Put

$$w_{(0)} = w,$$
$$w_{(r)} = w' = w_{\alpha_{n-k+m+r-1}} \cdots w_{\alpha_{n-k+m+1}} w_{\alpha_{n-k+m}} w, \quad 1 \leqslant r \leqslant k-m+1.$$

Then

$$M_{w_{(r)}} = M(w_{\alpha_{n-k+m+r-1}}, \chi_s^{w_{(r-1)}}) \circ M_{w_{(r)}}, \quad 1 \leqslant r \leqslant k-m+1$$

$$c_{w_{(r)}}(s) = c_{w_{(r-1)}}(s) \times \begin{cases} \dfrac{L(2s+n-k-m-r, \omega^2)}{L(2s+n-k-m-r+1, \omega^2)}, & 1 \leqslant r \leqslant k-m \\[2ex] \dfrac{L\left(s+\dfrac{n-1}{2}-k, \omega\right)}{L\left(s+\dfrac{n+1}{2}-k, \omega\right)}, & r = k-m+1 \end{cases}$$

We have

$$c_{w'}(s) = \frac{L(2s+n-2k, \omega^2)}{L(2s+n-k-m, \omega^2)} \frac{L\left(s+\dfrac{n+1}{2}-k, \omega\right)}{L\left(s+\dfrac{n+1}{2}-k, \omega\right)} c_w(s).$$

It is easy to see that

$$M(w_{\alpha_{n-1}}, \chi_s^{w_{(k-m-1)}}) \circ \cdots \circ M(w_{\alpha_{n-k+m}}, \chi_s^{w})$$

is an intertwining operator on GL_{k-m}. By (1.2.3) and Sublemma 3,

$$M(w_{\alpha_{n-k+m}}, \chi_s^{w_{(1)}}) \circ \cdots \circ M(w_{\alpha_{n-1}}, \chi_s^{w_{(k-m)}}) \circ M(w_{\alpha_n}, \chi_s^{w'}) \circ M_{w'}$$
$$= \omega(-1) \varepsilon'\left(s+\frac{n-1}{2}-k, \omega, \psi\right)^{-1} \varepsilon'\left(-s-\frac{n-1}{2}+k, \omega^{-1}, \psi\right)^{-1}$$
$$\times \varepsilon'(2s+n-2k, \omega^2, \psi)^{-1} \varepsilon'(-2s-n+k+m+1, \omega^{-2}, \psi)^{-1} M_w$$

By (1.2.2), Sublemma 2, and the induction assumption,

$$L\left(-s-\frac{n-1}{2}+k, \omega^{-1}\right)^{-1} M(w_{\alpha_n}, \chi_s^{w'}),$$
$$L(-2s-n+k+m+1, \omega^{-2})^{-1} M(w_{\alpha_{n-k+m}}, \chi_s^{w_{(1)}}) \circ \cdots \circ M(w_{\alpha_{n-1}}, \chi_s^{w_{(k-m)}})$$

and

$$[d(\omega, s)c_{w'}(\omega, s)]^{-1}M_{w'}$$

are holomorphic. Thus we have

$$L\left(-s-\frac{n-3}{2}+k, \omega^{-1}\right)^{-1} L(-2s-n+2k+1, \omega^{-2})^{-1}[d(\omega, s)c_w(\omega, s)]^{-1}M_w$$

is holomorphic.

On the other hand, put

$$w_k = \left(\begin{array}{cc|cc} \mathbf{1}_{n-k} & & & \\ & & & -\mathbf{1}_k \\ \hline & & \mathbf{1}_{n-k} & \\ & \mathbf{1}_k & & \end{array}\right),$$

$$w = w'w_k.$$

Then $M_w = M_{w'} \circ M_{w_k}$. Here, as in [22 §4], $M_{w'}$ is an intertwining operator on certain induced representation of GL_n. As in [22 §4], we can prove

$$\prod_{r=1}^{k} L(2s+i_r-2r+1, \omega^2)^{-1}M_{w'}$$

is holomorphic (cf. [22, Remark 4.1]). As for M_{w_k}, by Sublemma 1,

$$L\left(s+\frac{n+1}{2}-k, \omega\right)^{-1} \prod_{r=1}^{[k/2]} L(2s+n-2k+2r, \omega^2)^{-1}M_{w_k}$$

is holomorphic. Putting together, we can easily deduce

$$\prod_{r=[k+1/2]}^{k} L(2s+n-2r, \omega^2)^{-1}[d(\omega, s)c_w(\omega, s)]^{-1}M_w$$

is holomorphic. Since

$$L\left(-s-\frac{n-3}{2}+k, \omega^{-1}\right)L(-2s-n+2k+1, \omega^{-2})$$

has no poles in $\mathrm{Re}(s) < -\frac{n}{2} + k + \frac{1}{2}$, and

$$\prod_{r=[k+1/2]}^{k} L(2s+n-2r, \omega^2)$$

has no poles in $\mathrm{Re}(s) > -\frac{n}{2} + k$, it follows that

$$[d(\omega, s)c_w(\omega, s)]^{-1} M_w$$

is holomorphic. Thus Lemma 1.3 is proved.

REMARK. Our definition of good section is different from that of [22]. But we can prove that "germs" of good section of $I(\omega, s)$ at $s = s_0$ are generated by the following two families:

(1) germs of holomorphic sections of $I(\omega, s)$ at $s = s_0$,
(2) $\{M^*_{w_0} f^{(s)} \mid f^{(s)}$ is a germ of holomorphic section of $I(\omega^{-1}, -s)$ at $s = s_0\}$.

In fact, we may assume ω is unitary and $\mathrm{Re}(s_0) \geqslant 0$, by Lemma 1.2. Since $d(\omega, s)$ does not have zero at $s = s_0$, any good section of $I(\omega, s)$ is holomorphic at $s = s_0$. It is easy to see that when k is non-archimedean, our definition agrees to that of [22] because there are essentially finite number of singularities.

Appendix 2. An interpretation of the normalizing factor

We give an interpretation of the normalizing factor $d(\omega, s)$ in terms of Arthur's conjecture [1]. Let G be a reductive group, P be a maximal parabolic subgroup of G, M be a Levi factor of P, N be the unipotent radical of P, and A be the maximal split torus of the center of M. Let π be an irreducible discrete automorphic representation of M. Then, according to Arthur's conjecture, π is associated to a homomorphism

$$\varphi_\pi \colon \mathscr{L} \times \mathrm{SL}_2(\mathbf{C}) \to {}^L M.$$

Here $\mathscr{L}$ is the conjectual Langlands group. Let ${}^L\mathscr{N}$ be the Lie algebra of ${}^L N$. Decompose ${}^L\mathscr{N}$ as in Shahidi [24].

$${}^L\mathscr{N} = \prod_{i=1}^{r} {}^L\mathscr{N}_i.$$

Consider the induced representation $\mathrm{Ind}_M^G \pi\tilde{\alpha}^s$. Here $\tilde{\alpha}$ is as in [24]. Let $\mathrm{Ad}_{{}^L\mathscr{N}_i}$ be the adjoint action of ${}^LM$ on ${}^L\mathcal{N}_i$. If π is cuspidal and φ_π is trivial on $SL_2(\mathbf{C})$, then the normalizing factor should be given by

$$\prod_{i=1}^{r} L(1+is,\ \varphi_\pi \circ \mathrm{Ad}_{{}^L\mathcal{N}_i}).$$

(cf. Shahidi [24], Langlands [15].) Consider the general case where $\varphi_\pi \circ \mathrm{Ad}_{{}^L\mathcal{N}_i}$ is not trivial on $SL_2(\mathbf{C})$. In this case, decompose $\varphi_\pi \circ \mathrm{Ad}_{{}^L\mathcal{N}_i}$ into irreducible representation:

$$\varphi_\pi \circ \mathrm{Ad}_{{}^L\mathcal{N}_i} = \bigoplus_{j=1}^{m_i} \varphi_{ij} \otimes \mathrm{sym}^{r_{ij}},$$

where φ_{ij} is an irreducible representation of $\mathcal{L}$, and $\mathrm{sym}^{r_{ij}}$ is the r_{ij}th symmetric power of the standard representation of $SL_2(\mathbf{C})$. Then we claim the normalizing factor should be

$$\prod_{i=1}^{r} \prod_{j=1}^{m_i} L\left(is + \frac{r_{ij}}{2} + 1,\ \varphi_{ij}\right).$$

In fact, the c-function $c_{w_0}(\pi, s)$ for the longest element w_0 of the Weyl group is given by

$$\begin{aligned}
c_{w_0}(\pi, s) &= \prod_{i=1}^{r} \frac{L(is,\ \varphi_\pi \circ \mathrm{Ad}_{{}^L\mathcal{N}_i})}{L(1+is,\ \varphi_\pi \circ \mathrm{Ad}_{{}^L\mathcal{N}_i})} \\
&= \prod_{i=1}^{r} \prod_{j=1}^{m_i} \frac{L(is,\ \varphi_{ij} \otimes \mathrm{sym}^{r_{ij}})}{L(1+is,\ \varphi_{ij} \otimes \mathrm{sym}^{r_{ij}})} \\
&= \prod_{i=1}^{r} \prod_{j=1}^{m_i} \prod_{a=0}^{r_{ij}} \frac{L\left(is - \frac{r_{ij}}{2} + a,\ \varphi_{ij}\right)}{L\left(is - \frac{r_{ij}}{2} + a + 1,\ \varphi_{ij}\right)} \\
&= \prod_{i=1}^{r} \prod_{j=1}^{m_i} \frac{L\left(is - \frac{r_{ij}}{2},\ \varphi_{ij}\right)}{L\left(is + \frac{r_{ij}}{2} + 1,\ \varphi_{ij}\right)},
\end{aligned}$$

at least up to bad primes. If π is cuspidal, this is the only non-trivial c-function. This means at least when π is cuspidal, our claim is justified, since the normalizing factor should be the least common denominator of the c-functions. One can expect that the least common denominator of the c-functions is equal to the denominator of the c-function for the longest Weyl element even when π is not cuspidal.

Observe that in our case, $G=\mathrm{Sp}_n$, $M=\mathrm{GL}_n$, $\pi=\omega$, $\varphi_\pi=\omega\otimes\mathrm{sym}^{n-1}$, $\mathrm{Ad}_{{}^L\mathcal{N}_1}=\rho$, $\mathrm{Ad}_{{}^L\mathcal{N}_2}=\Lambda^2\rho$. Here ρ is the standard representation of GL_n. Therefore,

$$\varphi_\pi\circ\mathrm{Ad}_{{}^L\mathcal{N}_1}=\omega\otimes\mathrm{sym}^{n-1}$$

gives $L\left(s+\dfrac{n+1}{2},\omega\right)$, and

$$\varphi_\pi\circ\mathrm{Ad}_{{}^L\mathcal{N}_2}=\bigotimes_{j=1}^{[n/2]}(\omega^2\otimes\mathrm{sym}^{2n-4j})$$

gives $\Pi_{r=1}^{[n/2]}L(2s+n+1-2r,\,\omega^2)$.

1.3. *Eisenstein series*

In this subsection, we assume k to be a global field. We will investigate the poles of Eisenstein series associated to good sections.

Let ω be a quasi-character of $\mathbf{A}^\times/k^\times$. Put $K_n=\Pi_v K_{n,v}$. Let $I(\omega,s)$ be the space of functions $f(h)$ on $H_n(\mathbf{A})$ which satisfy (1) and (2):

(1) f is right K_n-finite.
(2) For any $p=\begin{pmatrix}A & *\\ \mathbf{0}_n & {}^tA^{-1}\end{pmatrix}\in P_n(\mathbf{A})$,

$$f(ph)=\omega(\det A)|\det A|^{s+(n+1)/2}f(h).$$

Clearly, $I(\omega,s)=\otimes_v I(\omega_v,s)$. We also define holomorphic sections and meromorphic sections similarly. We say that a meromorphic section of $I(\omega,s)$ is a good section if it is a finite sum of decomposable elements $f^{(s)}=\Pi_v f_v^{(s)}$ satisfying following (i) and (ii).

(i) For almost all unramified v, $f_v^{(s)}=d(\omega_v,s)\phi_{\omega_v,s}$.
(ii) $f_v^{(s)}$ is a good section of $I(\omega_v,s)$ for all v.

In other words, the space of global good sections is the restricted tensor product of the local good sections with respect to $d(\omega_v,s)\phi_{\omega_v,s}$. Note that the product $f^{(s)}=\Pi_v f_v^{(s)}$ is absolutely convergent for $\mathrm{Re}(s)>\dfrac{n+1}{2}$, and can be meromorphically continued to $\mathbf{C}$.

We define the Eisenstein series $E(h; f^{(s)})$ associated to $f^{(s)}$ by

$$E(h; f^{(s)}) = \sum_{\gamma \in P_n \backslash H_n} f^{(s)}(\gamma h).$$

This is absolutely convergent for $\mathrm{Re}(s) \gg 0$, and can be meromorphically continued to $\mathbf{C}$. The functional equation of $E(h; f^{(s)})$ is given by

$$E(h; f^{(s)}) = E(h; M_{w_0} f^{(s)}).$$

Here M_{w_0} is the global intertwining operator:

$$M_{w_0} = \bigotimes_v (M_{w_0})_v.$$

The global intertwining operator M_{w_0} does not depend on the choice of representative of $w_0 \in W_{H_n}$ in $\mathrm{Norm}(T_n)$.

LEMMA 1.4. *If $f^{(s)}$ is a good section of $I(\omega, s)$, then $M_{w_0} f^{(s)}$ is a good section of $I(\omega^{-1}, -s)$.*

Proof. Let S be a finite set of places of k such that if $v \notin S$, then ω_v is unramified, ψ_v is of order 0, and $f_v^{(s)} = d(\omega_v, s)\phi_{\omega_v, s}$. Then

$$\begin{aligned} M_{w_0} f^{(s)} &= \prod_{v \notin S} d(\omega_v, s) c_{w_0}(\omega_v, s) \phi_{\omega_v^{-1}, -s} \times \prod_{v \in S} M_{w_0} f_v^{(s)} \\ &= \prod_{v \notin S} a_{w_0}(\omega_v, s) \phi_{\omega_v^{-1}, -s} \times \prod_{v \in S} M_{w_0} f_v^{(s)} \\ &= \prod_{v \notin S} d(\omega_v^{-1}, -s) \phi_{\omega_v^{-1}, -s} \times \prod_{v \in S} M_{w_0}^* f_v^{(s)}. \end{aligned}$$

By Lemma 1.2, the lemma follows.

LEMMA 1.5. *Suppose that $n = 1$, and $\omega = 1$. Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then the global intertwining operator $M_w\colon I(1, s) \to I(1, -s)$ is holomorphic at $s = 0$, and is equal to the scalar multiplication by -1 at $s = 0$.*

Proof. Put $f^{(s)} = \Pi_v \phi_{1,s}$, and $\xi(s) = |D|^{s/2}\zeta(s)$. Here D is the discriminant of k (resp. $D = q^{2g-2}$, g is the genus of k) if k is a number field (resp. if k is a function field). Then

$$M_w f^{(s)} = \frac{\xi(s)}{\xi(s+1)} \prod_v \phi_{1,-s}. \tag{1.3.1}$$

Since $\xi(1-s) = \xi(s)$ and $\xi(s)$ has a simple pole at $s = 0$, 1, the right-hand side of

(1.3.1) is holomorphic at $s=0$, and

$$M_w f^{(0)} = -f^{(0)}.$$

Since $I(1, s)$ is irreducible on some neighbourhood of $s=0$, the lemma follows.

PROPOSITION 1.6. *Suppose that k is a number field. If $f^{(s)}$ is a good section of $I(\omega, s)$, then the pole of $E(h; f^{(s)})$ are at most simple. The set of possible poles is as follows.*

(1) *When ω is principal: we may assume $\omega=1$. Then the set of possible poles is:*

$$\left\{\frac{n+1}{2} - m \mid m \in \mathbf{Z},\ 0 \leqslant m \leqslant n+1,\ m \neq \frac{n+1}{2}\right\}$$

(2) *When ω is not principal, and ω^2 is principal: we may assume $\omega^2=1$. Then the set of possible poles is:*

$$\left\{\frac{n-1}{2} - m \mid m \in \mathbf{Z},\ 0 \leqslant m \leqslant n-1,\ m \neq \frac{n-1}{2}\right\}$$

(3) *If ω^2 is not principal, then $E(h; f^{(s)})$ is entire.*

Proof. As in [22], the constant term $E^0(h; f^{(s)})$ of $E(h; f^{(s)})$ along $U_n(\mathbf{A})$ is given by

$$\begin{aligned} E^0(h; f^{(s)}) &= \int_{U_n(k)\backslash U_n(\mathbf{A})} E(uh; f^{(s)})\,du \\ &= \sum_{w \in \Omega_n} M_w f^{(s)}. \end{aligned}$$

Let S be as in the proof of Lemma 1.4. Then

$$\begin{aligned} M_w f^{(s)} &= \prod_{v \notin S} d(\omega_v, s) c_w(\omega_v, s) \phi^w_{\omega_v, s} \times \prod_{v \in S} M_w f_v^{(s)} \\ &= d(\omega, s) c_w(\omega, s) \prod_{v \notin S} \phi^w_{\omega_v, s} \\ &\quad \times \prod_{v \in S} [d(\omega_v, s) c_w(\omega_v, s)]^{-1} M_w f_v^{(s)}. \end{aligned}$$

Therefore the poles of $E(h; f^{(s)})$ comes from the poles of $d(\omega, s)c_w(\omega, s)$. In particular, if ω^2 is not principal, $E(h; f^{(s)})$ is entire.

We may assume $\omega^2=1$, without loss of generality. When $\omega=1$, (resp. $\omega^2=1$,

$\omega \neq 1$), the possible poles of $d(\omega, s)c_w(\omega, s)$ are integral or half-integral points in

$$\left[-\frac{n+1}{2}, \frac{n+1}{2}\right]\left(\text{resp.}\left[-\frac{n-1}{2}, \frac{n-1}{2}\right]\right).$$

We first prove the proposition for the case $n=1$ or $n=2$. If $n=1$, $\omega \neq 1$, then (2) is obvious since $d(\omega, s)c_w(\omega, s)$ are entire. If $n=1$, $\omega=1$, then we have to show that $s=0$ is not a pole of $E^0(h; f^{(s)})$. Note that $f^{(s)}$ may have a simple pole at $s=0$. Let w be as in Lemma 1.5. Then by Lemma 1.5,

$$\begin{aligned}\lim_{s\to 0} sE^0(h; f^{(s)}) &= (1+M_w)\left[\lim_{s\to 0} sf^{(s)}\right]\\ &= 0.\end{aligned}$$

Thus $E^0(h; f^{(s)})$ is holomorphic at $s=0$.

If $n=2$, the possible poles of $d(\omega, s)c_w(\omega, s)$ are as follows:

	I	$l(w)$	$d(\omega, s)c_w(\omega, s)$	poles ($\omega=1$)	poles ($\omega^2=1, \omega\neq 1$)
w_1	$\varnothing$	0	$L(s+\frac{3}{2})\zeta(2s+1)$	$\{-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, 0\}$	$\{-\frac{1}{2}, 0\}$
w_2	$\{2\}$	1	$L(s+\frac{1}{2})\zeta(2s+1)$	$\{-\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}\}$	$\{-\frac{1}{2}, 0\}$
w_3	$\{1\}$	2	$L(s+\frac{1}{2})\zeta(2s)$	$\{-\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}\}$	$\{0, \frac{1}{2}\}$
w_4	$\{1, 2\}$	3	$L(s-\frac{1}{2})\zeta(2s)$	$\{0, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$	$\{0, \frac{1}{2}\}$

Here, $L(s)=L(s, \omega)$. By functional equation, we may assume $\mathrm{Re}(s)\geqslant 0$, so what we have to prove are reduced to the following two statements.

(1.3.2) If $\omega=1$,

$$\lim_{s\to 1/2} (s-\tfrac{1}{2})^2(M_{w_3}+M_{w_4})f^{(s)}=0.$$

(1.3.3) If $\omega^2=1$,

$$\lim_{s\to 0} s(1+M_{w_2}+M_{w_3}+M_{w_4})f^{(s)}=0.$$

Proof of (1.3.2)

$$\lim_{s\to 1/2} (s-\tfrac{1}{2})^2 M_{w_4} f^{(s)} = \lim_{s\to 1/2} M(w_{\alpha_2}, \chi_s^{w_3})\circ[(s-\tfrac{1}{2})^2 M_{w_3} f^{(s)}].$$

We know that $(s-\frac{1}{2})^2 M_{w_3} f^{(s)}$ is holomorphic at $s=\frac{1}{2}$. Moreover, by (1.2.1) and

Lemma 1.5, $M(w_{\alpha_2}, \chi_s^{w_3})$ is holomorphic and is equal to the scalar multiplication by -1 at $s=\frac{1}{2}$. Hence (1.3.2).

Proof of (1.3.3). By the same way as above, we can prove

$$\lim_{s\to 0} s(M_{w_2}+M_{w_3})f^{(s)}=0.$$

But the proof that

$$\lim_{s\to 0} s(1+M_{w_4})f^{(s)}=0$$

is more delicate. We have

$$M_{w_4}f^{(s)}=M(w_{\alpha_2}, \chi_s^{w_3})\circ M(w_{\alpha_1}, \chi_s^{w_2})\circ M(w_{\alpha_2}, \chi_s)f^{(s)}.$$

By (1.2.1) and Lemma 1.5, $M(w_{\alpha_1}, \chi_s^{w_2})$ is holomorphic and is equal to the scalar multiplication by -1 at $s=0$. Moreover, by (1.2.1), $M(w_{\alpha_2}, \chi_s)$ (resp. $M(w_{\alpha_2}, \chi_s^{w_3})$ is essentially the intertwining operator

$$M\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, s+\frac{1}{2}\right): I\left(\omega, s+\frac{1}{2}\right)\to I\left(\omega, -s-\frac{1}{2}\right)$$
$$\left(\text{resp. } M\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, -s-\frac{1}{2}\right): I\left(\omega, -s-\frac{1}{2}\right)\to I\left(\omega, s+\frac{1}{2}\right)\right)$$

on SL_2. Moreover, these two are mutually the inverse of the other except for their singular points. Since the representations $I(\omega, s+\frac{1}{2})$ and $I(\omega, -s-\frac{1}{2})$ of $SL_2(\mathbf{A})$ are irreducible on some neighbourhood of $s=0$, there is an integer α such that

$$s^{-\alpha}M\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, s+\frac{1}{2}\right) \quad\text{and}\quad s^{\alpha}M\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, -s-\frac{1}{2}\right)$$

are holomorphic, and are mutually the inverse of each other at $s=0$. In fact, it is easy to see that $\alpha=\operatorname{ord}_{s=1/2} L(s,\omega)$. We have

$$\lim_{s\to 0} sM_{w_4}f^{(s)}=\lim_{s\to 0}[s^{\alpha}M(w_{\alpha_2}, \chi_s^{w_3})]\circ[M(w_{\alpha_1}, \chi_s^{w_2})]\circ[s^{-\alpha}M(w_{\alpha_2}, \chi_s)][sf^{(s)}].$$

Each term is holomorphic at $s=0$, so the exchange of limit and the composition is possible. Hence (1.3.3).

Now we assume $n\geqslant 3$. By the functional equation, it is enough to investigate the integral or half-integral points in $\left[0, \frac{n+1}{2}\right]$. Note that $f^{(s)}$ is holomorphic on the right half plane $\mathrm{Re}(s) \geqslant 0$ except for the case n is even and $s=0$. In particular, if n is odd, $s=0$ is not a pole of $E(h; f^{(s)})$, by [16].

We recall the theory of degenerate Eisenstein series on GL_n (see [12, §5]). Let Q be the maximal parabolic subgroup of GL_n given by

$$Q = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \middle| a_1 \in \mathrm{GL}_{n-1}, a_2 \in k^\times \right\}.$$

Let $I_Q(s)$ be the representation of GL_n induced from the character of Q given by

$$\begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \mapsto |\det a_1|^{s/n} |a_2|^{-(n-1)s/n}.$$

We define standard sections, holomorphic sections etc. as usual. For each prime v of k, let $F_{0,v}^{(s)}$ be the meromorphic section of $I_{Q,v}(s)$ which takes value $\zeta_v(s+\frac{n}{2})$ on the standard maximal compact subgroup of $\mathrm{GL}_{n,v}$.

Taking any finite set S of primes of k, put

$$F^{(s)} = \prod_{v \notin S} F_{0,v}^{(s)} \times \prod_{v \in S} F_v^{(s)}$$

where $F_v^{(s)}$, $v \in S$ are arbitrary holomorphic sections of $I_{Q,v}(s)$. Define degenerate Eisenstein series on GL_n by

$$E(g; F^{(s)}) = \sum_{\gamma \in Q \backslash \mathrm{GL}_n} F^{(s)}(\gamma g).$$

Then the possible poles of $E(g; F^{(s)})$ are $s = \pm\frac{n}{2}$. Moreover, each pole is at most simple and the residue is a constant function. The functional equation is given by

$$E(g; F^{(s)}) = E(g; M_w F^{(s)}).$$

Here

$$w = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & \end{pmatrix}.$$

$M_w F^{(s)}$ is a meromorphic section of the representation induced from the character

$$\begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \mapsto |a_1|^{-(n-1)s/n} |\det a_2|^{s/n}$$

of the parabolic subgroup

$$Q' = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \middle| a_1 \in k^\times,\ a_2 \in \mathrm{GL}_{n-1} \right\}.$$

$M_w F^{(s)}$ has at most simple poles at $s = \frac{n}{2}, \frac{n}{2} - 1$.

We return to the proof of Proposition 1.6. Let

$$f^{(s)} = \prod_{v \notin S} d(\omega_v, s) \phi_{\omega_v, s} \times \prod_{v \in S} f_v^{(s)}$$

be a good section. We may assume each $f_v^{(s)}$, $v \in S$ is a standard section, since $d(\omega_v, s)$ has no pole in $\mathrm{Re}(s) \geqslant 0$.

Let P_1^* be the parabolic subgroups of H_n given by

$$P_1^* = \left\{ \left(\begin{array}{cc|cc} a & * & * & * \\ 0 & A & * & * \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}_n$}} & a^{-1} & 0 \\ & & * & {}^tA^{-1} \end{array} \right) \in H_n \middle| a \in k^\times,\ A \in \mathrm{GL}_{n-1} \right\}.$$

Let $t = (t_1, t_2) \in \mathbf{C}^2$. Let $I_{P_1^*}(\omega_v, t)$, be the space of right K_v-finite function $f_{P_1^*}^{(t)}$ on $H_{n,v}$ such that

$$f_{P_1^*}^{(t)}(p_1 h) = \omega(a \det A)|a|^{t_1 + n} |\det A|^{t_2 + n/2} f_{P_1^*}^{(t)}(h),$$

where

$$p_1 = \left(\begin{array}{cc|cc} a & * & * & * \\ 0 & A & * & * \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}_n$}} & a^{-1} & 0 \\ & & * & {}^tA^{-1} \end{array} \right) \in P_1^*.$$

For each $v \in S$, let $\tilde{f}_v^{(t)}$ be a standard section (of two variables) of $I_{P_1^*}(\omega_v, t)$ defined by

$$\tilde{f}_v^{(t)}(p_1 k) = |a^{n-1} \det A^{-1}|^{(t_1 - t_2)/n + 1/2} f_v^{(s)}(k),$$

where p_1 is as above, $k \in K_v$, and

$$s = \frac{t_1 + (n-1)t_2}{n}.$$

When $v \notin S$, let $\phi_{P_1^*, \omega_v, t}$ be the standard section of $I_{P_1^*}(\omega_v, t)$ which is identically 1 on K_v. Put

$$\tilde{f}^{(t)} = \prod_{v \notin S} \left[L_v(t_1+1)\zeta_v\left(t_1 - t_2 + \frac{n}{2}\right)\zeta_v\left(t_1 + t_2 + \frac{n}{2}\right) L_v\left(t_2 + \frac{n}{2}\right) \prod_{r=1}^{[(n-1)/2]} \zeta_v(2t_2 + n - 2r) \right]$$

$$\times \prod_{v \notin S} \phi_{P_1^*, \omega_v, t} \times \prod_{v \in S} \tilde{f}_v^{(t)}.$$

Here $L_v(s)$ stands for $L(\omega_v, s)$. Put

$$E(h; \tilde{f}^{(t)}) = \sum_{\gamma \in P_1^* \backslash H_n} \tilde{f}^{(t)}(\gamma h)$$

$$= \sum_{\gamma \in P_n \backslash H_n} \sum_{\gamma_1 \in P_1^* \backslash P_n} \tilde{f}^{(t)}(\gamma_1 \gamma h). \tag{1.3.4}$$

The inner sum in the last expression is a degenerate Eisenstein series on GL_n. In particular, the residue of this inner Eisenstein series along $t_1 - t_2 = \frac{n}{2}$ is, up to non-zero constant, equal to

$$L_S\left(s + \frac{n+1}{2}\right)\zeta_S(s+n-1)L_S\left(s + \frac{n-1}{2}\right) \prod_{r=1}^{[(n-1)/2]} \zeta_S(2s+n+1-2r)$$

$$\times \prod_{v \notin S} \phi_{\omega_v, s} \times \prod_{v \in S} f_v^{(s)}(\gamma h).$$

Here $s = t_2 + \frac{1}{2}$. So, the residue of $E(h; \tilde{f}^{(t)})$ along $t_1 - t_2 = \frac{n}{2}$ is, up to non-zero constant, equal to

$$\begin{cases} L_S\left(s + \dfrac{n-1}{2}\right)\zeta_S(2s)E(h; f^{(s)}), & \text{if } n \text{ is even} \\ L_S\left(s + \dfrac{n-1}{2}\right)E(h; f^{(s)}), & \text{if } n \text{ is odd.} \end{cases} \tag{1.3.5}$$

Put

$$D_1 = \left\{(t_1, t_2) \mid \mathrm{Re}(t_1) > \mathrm{Re}(t_2) + \frac{n}{2},\ \mathrm{Re}(t_2) > \frac{n}{2}\right\}.$$

Then $\tilde{f}^{(t)}$ is holomorphic on D_1, and the summation (1.3.4) is absolutely convergent on D_1, so $E(h; \tilde{f}^{(t)})$ is holomorphic on D_1. Put

$$P_2^* = \left\{ \left(\begin{array}{cc|cc} a & * & * & * \\ 0 & A & * & B \\ \hline 0 & 0 & a^{-1} & 0 \\ 0 & C & * & D \end{array} \right) \in H_n \,\middle|\, a \in k^\times, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H_{n-1} \right\}.$$

Then

$$E(h; \tilde{f}^{(t)}) = \sum_{\gamma \in P_2^* \backslash H_n} \sum_{\gamma_1 \in P_1^* \backslash P_2^*} \tilde{f}^{(t)}(\gamma_1 \gamma h). \tag{1.3.6}$$

The inner sum of (1.3.6) is

$$L_S(t_1+1)\zeta_S\left(t_1 - t_2 + \frac{n}{2}\right)\zeta_S\left(t_1 + t_2 + \frac{n}{2}\right)$$

times an Eisenstein series on H_{n-1} associated to a good section of $I(\omega, t_2)$. By the induction assumption, the poles of this Eisenstein series is

$$\begin{cases} \left\{ t_2 = \frac{n}{2} - m \,\middle|\, m \in \mathbf{Z},\ 0 \leqslant m \leqslant n,\ n \neq \frac{n}{2} \right\} & \text{if } \omega = 1 \\ \left\{ t_2 = \frac{n-2}{2} - m \,\middle|\, m \in \mathbf{Z},\ 0 \leqslant m \leqslant n-2,\ n \neq \frac{n-2}{2} \right\} & \text{if } \omega \neq 1 \end{cases} \tag{1.3.7}$$

By the functional equation of the inner Eisenstein series, $E(h; \tilde{f}^{(t)})$ is holomorphic on the domain

$$D_2 = \left\{ (t_1, t_2) \mid \mathrm{Re}(t_1) > \mathrm{Re}(t_2) + \frac{n}{2},\ \mathrm{Re}(t_1) > -\mathrm{Re}(t_2) + \frac{n}{2},\ \mathrm{Re}(t_2) > \frac{n}{2} \right\}.$$

Therefore $E(h; \tilde{f}^{(t)})$ can be meromorphically continued to the convex closure of $D_1 \cup D_2$, and the singularities in this domain are given by (1.3.7).

Similarly, by the functional equation of degenerate Eisenstein series on GL_n, $E(h; \tilde{f}^{(t)})$ is holomorphic on the domain

$$D_3 = \left\{ (t_1, t_2) \,\middle|\, \mathrm{Re}(t_1) > 1,\ \mathrm{Re}(t_2) > \mathrm{Re}(t_1) + \frac{n}{2} \right\}$$

and can be meromorphically continued to the convex closure of $D_1 \cup D_3$. The singularities in this domain are given by

$$\left\{t_1 - t_2 = \pm\frac{n}{2}\right\}. \tag{1.3.8}$$

By the same reason, $E(h; \tilde{f}^{(t)})$ is holomorphic on

$$D_4 = \left\{(t_1, t_2) \,|\, \mathrm{Re}(t_1) < -1,\ \mathrm{Re}(t_2) > -\mathrm{Re}(t_1) + \frac{n}{2}\right\}$$

and can be meromorphically continued to the convex closure of $D_2 \cup D_4$. The singularity in this domain is

$$\left\{t_1 + t_2 = \pm\frac{n}{2}\right\}. \tag{1.3.9}$$

Thus $E(h; \tilde{f}^{(t)})$ can be meromorphically continued to the convex closure of $D_1 \cup D_2 \cup D_3 \cup D_4$ and the singularity in this domain is the union of (1.3.7), (1.3.8) and (1.3.9). Therefore (1.3.5) has at most simple poles at

$$\begin{cases} s = \dfrac{1}{2}, \dfrac{3}{2}, \ldots, \dfrac{n+1}{2}, & \text{if } n \text{ is even} \\ s = \dfrac{1}{2}, 1, 2, \ldots, \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

for $\mathrm{Re}(s) \geqslant 0$. Here $\frac{n+1}{2}$ is a pole only if $\omega = 1$. If n is even, $L_S\left(s + \frac{n-1}{2}\right)$ has neither poles nor zeros for $\mathrm{Re}(s) \geqslant 0$. If n is odd, $L_S\left(s + \frac{n-1}{2}\right)\zeta_S(2s)$ has a simple pole at $s = \frac{1}{2}$ and has no zero at positive integral or half-integral points. Note that we already know that $s = 0$ is not a pole if n is odd. Thus we have proved Proposition 1.6.

COROLLARY. *Let $f^{(s)}$ be a global holomorphic section of $I(\omega, s)$. Let S be a finite set of places of k such that $f^{(s)}$ is invariant under K_v, $v \notin S$. Then the set of poles of*

$$d_S(\omega, s) E(h; f^{(s)})$$

is given by Proposition 1.6.

This result is also proved in [14].

If k is a function field, we can prove the following proposition similarly.

PROPOSITION 1.7. *Suppose k is a function field. If $f^{(s)}$ is a good section of $I(\omega, s)$, then the poles of $E(h; f^{(s)})$ are at most simple. The set of possible poles is as follows.*

(1) *When ω is principal: we may assume $\omega = 1$. The set of possible poles is:*

$$\left\{\pm\frac{n+1}{2}+\frac{2\pi\sqrt{-1}}{\log q}\mathbf{Z}\right\}$$
$$\cup\left\{\frac{n-1}{2}-m+\frac{\pi\sqrt{-1}}{\log q}\mathbf{Z}\,|\, m\in\mathbf{Z},\ 0\leqslant m\leqslant n-1,\ m\neq\frac{n-1}{2}\right\}$$

(2) *When ω is not principal, and ω^2 is principal: we may assume $\omega^2 = 1$. Then the set of possible poles is:*

$$\left\{\frac{n-1}{2}-m+\frac{\pi\sqrt{-1}}{\log q}\mathbf{Z}\,\middle|\, m\in\mathbf{Z},\ 0\leqslant m\leqslant n-1,\ m\neq\frac{n-1}{2}\right\}$$

(3) *If ω^2 is not principal, then $E(h; f^{(s)})$ is entire.*

REMARK. Proposition 1.6 or 1.7 implies that the possible poles of Langlands L-function of irreducible cuspidal automorphic representations of Sp_n attached to the standard representation of the L-group ${}^L\mathrm{Sp}_n \simeq \mathrm{SO}(2n+1)$ are

$$\{-n+1,\ -n+2,\ldots,n-1,\ n\}$$

or

$$\left\{-n+1+\frac{\pi\sqrt{-1}}{\log q}\mathbf{Z},\ -n+2+\frac{\pi\sqrt{-1}}{\log q}\mathbf{Z},\ldots,n-1+\frac{\pi\sqrt{-1}}{\log q}\mathbf{Z},\ n+\frac{\pi\sqrt{-1}}{\log q}\mathbf{Z}\right\},$$

and all of them are at most simple (cf. [14], [20], [21]).

1.4. *Calculation of the residue at $s = \dfrac{n-1}{2}$*

In this subsection, we assume $\omega = 1$. Then there exists a class 1 element of $I(\omega, s)$. Take $\phi_s \in I(\omega, s)$ such that $\phi_s|_{K_n} \equiv 1$. Put

$$E(h, s) = E(h; \phi_s),$$
$$\tilde{E}(h, s) = \xi\left(s+\frac{n+1}{2}\right)\prod_{r=1}^{[n/2]}\xi(2s+n+1-2r)E(h, s).$$

$\tilde{E}(h, s)$ satisfies the following functional equation:

$$\tilde{E}(h, s) = \tilde{E}(h, -s).$$

We will determine the residue of $E(h; s)$ at $s = \frac{n-1}{2}$. Let $P_{n,r}$ be a parabolic subgroup of H_n given by

$$P_{n,r} = \left\{ \left(\begin{array}{cc|cc} a & * & * & * \\ 0 & A & * & B \\ \hline 0 & 0 & {}^t a^{-1} & 0 \\ 0 & C & * & D \end{array}\right) \in H_n \,\middle|\, a \in \mathrm{GL}_{n-r}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_r \right\}.$$

Let $s \in \mathbf{C}$ and $t = (t_1, t_2, \ldots, t_n) \in \mathbf{C}^n = X^*(T_n) \otimes_{\mathbf{Z}} \mathbf{C}$. Let $\phi(h; P_{n,r}; s)$, $\phi(h; B_n; t) = \phi(h; B_n; t_1, t_2, \ldots, t_n)$ be the functions on $H_n(\mathbf{A})$ given by

$$\phi(pk; P_{n,r}; s) = |a|^{s+(n+r+1)/2}$$

$$\phi(bk; B_n; s) = \prod_{i=1}^{n} |b_i|^{t_i+n+1-i},$$

where $k \in K_n$,

$$p = \left(\begin{array}{cc|cc} a & * & * & * \\ 0 & A & * & B \\ \hline 0 & 0 & {}^t a^{-1} & 0 \\ 0 & C & * & D \end{array}\right) \in P_{n,r}(\mathbf{A}),$$

$$b = \left(\begin{array}{cccc|cccc} b_1 & & & * & & & & \\ & b_2 & & & & * & & \\ & & \ddots & & & & & \\ 0 & & & b_n & & & & \\ \hline & & & & b_1^{-1} & & & 0 \\ & \mathbf{0}_n & & & & b_2^{-1} & & \\ & & & & & & \ddots & \\ & & & & * & & & b_n^{-1} \end{array}\right) \in B_n(\mathbf{A}).$$

Put

$$E_{P_{n,r}}(h, s) = \sum_{\gamma \in P_{n,r}\backslash H_n} \phi(\gamma h; P_{n,r}; s),$$
$$E_{B_n}(h, t) = \sum_{\gamma \in B_n\backslash H_n} \phi(\gamma h; B_n; t).$$

For any $\alpha \in \Phi^+_{H_n}$, let $l^\pm_\alpha(t)$ and $\mathscr{F}^\pm_\alpha$ be linear forms and hyperplanes of $\mathbf{C}^n$ given by

$$l^+_\alpha(t) = \langle \check{\alpha}, t\rangle - 1, \qquad l^-_\alpha(t) = \langle \check{\alpha}, t\rangle + 1,$$
$$\mathscr{F}^+_\alpha = \{t \in \mathbf{C}^n \mid l^+_\alpha(t) = 0\}, \qquad \mathscr{F}^-_\alpha = \{t \in \mathbf{C}^n \mid l^-_\alpha(t) = 0\}.$$

It is easy to see that the residue along $\mathscr{F}^+_{\alpha_1}, \ldots, \mathscr{F}^+_{\alpha_{n-r-1}}, \mathscr{F}^+_{\alpha_{n-r+1}}, \ldots, \mathscr{F}^+_{\alpha_n}$ in the sense of [9, p. 195] is

$$R^{n-1} \prod_{i=2}^{n-r} \xi(i)^{-1} \prod_{i=1}^{r} \xi(2i)^{-1} E_{P_{n,r}}\left(h, t_{n-r} + \frac{n-r-1}{2}\right),$$

where $R = \mathrm{Res}_{s=1}\, \xi(s)$. Put

$$\begin{aligned}\tilde{E}_{B_n}(h, t) &= \prod_{\alpha \in \Phi^+_{H_n}} \xi(\langle \check{\alpha}, t\rangle + 1) E_{B_n}(h, t)\\ &= \prod_{1 \leqslant i < j \leqslant n} \xi(t_i + t_j + 1)\xi(t_i - t_j + 1) \prod_{i=1}^{n} \xi(t_i + 1) E_{B_n}(h, t).\end{aligned}$$

Then it is known that

$$\prod_{\alpha \in \Phi^+_{H_n}} l^+_\alpha(t) l^-_\alpha(t) E_{B_n}(h, t) \tag{1.4.6}$$

is entire and invariant under $t \to wtw^{-1}$ for any $w \in W_{H_n}$.

The value of (1.4.6) at $t = \left(s + \frac{n-1}{2}, s + \frac{n-3}{2}, \ldots, s - \frac{n-1}{2}\right)$ is

$$\begin{aligned}&(2R)^{n-1} \prod_{i=2}^{n-1} \{(i-1)(i+1)\xi(i)\}^{n-i}\\ &\times \prod_{i=1}^{n} \left(s + \frac{n+3}{2} - i\right)\left(s + \frac{n-1}{2} - i\right) \xi\left(s + \frac{n+3}{2} - i\right)\\ &\times \prod_{1 \leqslant i < j \leqslant n} (2s + n + 2 - i - j)(2s + n - i - j)\xi(2s + n + 2 - i - j)\\ &\times E_{P_{n,0}}(h, s).\end{aligned}$$

So the value of (1.4.6) at $t=(n-1, n-2, \dots, 1, 0)$ is

$$(2R)^{n-1}\prod_{i=2}^{n-1}\{(i-1)(i+1)\xi(i)\}^{n-i}$$

$$\times(-R)n!(n-2)!\prod_{i=2}^{n}\xi(i)$$

$$\times 2\xi(2)\prod_{i=2}^{n-1}\prod_{j=1}^{i}\xi(i+j)$$

$$\times 2\,\mathrm{Res}_{s=(n-1)/2}E_{P_{n,0}}(h, s).$$

On the other hand, the value of (1.4.6) at $t=(s, n-1, n-2, \dots, 1)$ is

$$(2R)^{n-1}\prod_{i=2}^{n-1}\{(i-1)(i+1)\xi(i)\}^{n-i}$$

$$\times\prod_{1\leqslant i<j\leqslant n-1}(i+j+1)(i+j-1)\xi(i+j)$$

$$\times\prod_{i=1}^{2n-1}(s-n+i+1)(s-n+i-1)\xi(s-n+i+1)$$

$$\times E_{P_{n,n-1}}(h, s).$$

It follows that $E_{P_{n,n-1}}(h, s)$ is holomorphic at $s=0$, and the value of (1.4.6) at $t=(0, n-1, n-2, \dots, 1)$ is

$$(2R)^{n-1}\prod_{i=2}^{n-1}\{(i-1)(i+1)\xi(i)\}^{n-i}$$

$$\times\prod_{1\leqslant i<j\leqslant n-1}(i+j+1)(i+j-1)\xi(i+j)$$

$$\times(-R^2)(n!)^2\{(n-2)!\}^2\prod_{i=2}^{n}\xi(i)\prod_{i=2}^{n-1}\xi(i)$$

$$\times E_{P_{n,n-1}}(h, 0).$$

Thus we get the following proposition.

PROPOSITION 1.8.

$$\mathrm{Res}_{s=(n-1)/2} E_{P_{n,0}}(h, s)$$
$$= \frac{1}{2} R \prod_{i=1}^{[n/2]-1} \xi(2i+1) \prod_{i=1}^{[n/2]} \xi(2n-2i)^{-1} E_{P_{n,n-1}}(h, 0),$$

or, equivalently

$$\mathrm{Res}_{s=(n-1)/2} \tilde{E}_{P_{n,0}}(h, s)$$
$$= \frac{1}{2} R\xi(n) \prod_{i=1}^{[n/2]-1} \xi(2i+1) E_{P_{n,n-1}}(h, 0).$$

LEMMA 1.9. $I\left(1, \frac{n-1}{2}\right)$ *is generated by class* 1 *vectors.*

Proof. Let χ be a character of T_n given by

$$\chi(t) = \prod_{i=1}^{n} |t_i|^{n-i}.$$

Then $I\left(1, \frac{n-1}{2}\right)$ is a quotient of $\mathrm{Ind}_{B_n}^{H_n}\chi$. It is sufficient to prove that $\mathrm{Ind}_{B_n}^{H_n}\chi$ is generated by class 1 vectors. Let P be the standard parabolic subgroup of H_n corresponding to α_n. Then

$$\mathrm{Ind}_{B_n}^{H_n}\chi = \mathrm{Ind}_{P}^{H_n}(\mathrm{Ind}_{B_n}^{P}\chi).$$

The restriction of $\mathrm{Ind}_{B_n}^{P}\chi$ to $\iota_{\alpha_n}(\mathrm{SL}_2)$ is an irreducible tempered representation. Let M be the standard Levi factor of P and w be the longest element of $W_M \backslash W_{H_n}$, i.e.,

$$w = \left(\begin{array}{cc|cc} & & -\mathbf{1}_{n-1} & \\ & 1 & & \\ \hline \mathbf{1}_{n-1} & & & \\ & & & 1 \end{array}\right).$$

By the well-known theory of Langlands quotient, $\mathrm{Ind}_{P}^{H_n}(\mathrm{Ind}_{B_n}^{P}\chi)$ is generated by any element f such that $M_w f \neq 0$. It is easy to check that a non-zero class 1 vector satisfies this condition.

Let $f^{(s)}$ be any good section of $I(1,s)$. Put

$$w = w_{\{2,\cdots,n\}} = \left(\begin{array}{cccc|cccc} 1 & & & & 0 & & & \\ & & & & & & & -1 \\ & & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & & -1 & & \\ \hline 0 & & & & 1 & & & \\ & & & 1 & & & & \\ & & \cdot^{\cdot^{\cdot}} & & & & & \\ & 1 & & & & & & \end{array}\right). \tag{1.4.7}$$

It is easy to check that $M_w f^{(s)}$ has at most a simple pole at $s=\dfrac{n-1}{2}$ and

$$\mathrm{Res}_{s=(n-1)/2} M_w f^{(s)}$$

is in $\mathrm{Ind}_{P_{n,n-1}}^{H_n} 1$. An easy calculation shows

$$\begin{aligned} &\mathrm{Res}_{s=(n-1)/2} M_w \phi(h;\ P_{n,0};\ s) \\ &= R \prod_{i=1}^{[n/2]-1} \xi(2i+1) \prod_{i=1}^{[n/2]} \xi(2n-2i)^{-1} \phi(h;\ P_{n,n-1};\ 0). \end{aligned}$$

Thus by Proposition 1.8,

$$\begin{aligned} &\mathrm{Res}_{s=(n-1)/2} E_{P_{n,0}}(h,\ \phi(h;\ P_{n,0};\ s)) \\ &= \tfrac{1}{2} E_{P_{n,n-1}}(h,\ \mathrm{Res}_{s=(n-1)/2} M_w \phi(h;\ P_{n,0};\ s)). \end{aligned}$$

PROPOSITION 1.10.

$$\mathrm{Res}_{s=(n-1)/2} E_{P_{n,0}}(h;\ f^{(s)}) = \tfrac{1}{2} E_{P_{n,n-1}}(h;\ \mathrm{Res}_{s=(n-1)/2} M_w f^{(s)}).$$

Proof. By Proposition 1.8, this equation holds for a non-zero class 1 vector. Since both sides are H_n-equivariant, it holds for any $f^{(s)}$.

2. Triple L-functions

Let k be a global field. Let $\mathbf{K}$ be a semi-simple abelian algebra of degree 3 over k. There are three cases:

Case (1) $\mathbf{K} = k \oplus k \oplus k$.
Case (2) $\mathbf{K} = k \oplus k'$, k' is a quadratic extension of k.
Case (3) $\mathbf{K} = k''$, k'' is a cubic extension of k.

Let G be an algebraic group defined over k given by

$$G = \{g \in \mathrm{GL}_2(\mathbf{K}) \,|\, \det g \in k^\times\}.$$

Thus G is

Case (1) $\{(g^{(1)}, g^{(2)}, g^{(3)}) \in (\mathrm{GL}_2)^3 \,|\, \det g^{(1)} = \det g^{(2)} = \det g^{(3)}\}$,
Case (2) $\{(g^{(1)}, g^{(2)}) \in \mathrm{GL}_2 \times R_{k'/k}\mathrm{GL}_2 \,|\, \det g^{(1)} = \det g^{(2)}\}$,
Case (3) $\{g \in R_{k''/k}\mathrm{GL}_2 \,|\, \det g \in k^\times\}$.

As in [22, §0], we take an 8-dimensional representation σ of the L-group of $\mathrm{GL}_2(\mathbf{K})$. The L-group is the semi-direct product of $\mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C})$ and W_k. W_k acts by permuting the three $\mathrm{GL}_2(\mathbf{C})$ factors. The restriction of σ to $\mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C})$ is $\sigma_2 \otimes \sigma_2 \otimes \sigma_2$, where σ_2 is the standard 2-dimensional representation of $\mathrm{GL}_2(\mathbf{C})$. The restriction of σ to W_k is the permutation of the three factors.

We denote by Z the connected component of the center of G. Z is naturally isomorphic to GL_1. We embed G into

$$\mathrm{GSp}_3 = \left\{ h \in \mathrm{GL}_6 \,\middle|\, h \begin{pmatrix} \mathbf{0}_3 & -\mathbf{1}_3 \\ \mathbf{1}_3 & \mathbf{0}_3 \end{pmatrix} {}^t h = m(h) \begin{pmatrix} \mathbf{0}_3 & -\mathbf{1}_3 \\ \mathbf{1}_3 & \mathbf{0}_3 \end{pmatrix},\ m(h) \in k^\times \right\}$$

as in [22, §1]. We denote this embedding by ι.

Let Π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A} \otimes \mathbf{K})$, i.e.,

Case (1) $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$, where π_1, π_2, and π_3 are irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_k)$,
Case (2) $\Pi = \pi_1 \otimes \pi_2$, where π_1 (resp. π_2) is an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_k)$ (resp. $\mathrm{GL}_2(\mathbf{A}_{k'})$),
Case (3) Π is an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_{k''})$.

Let Ω_Π be the central quasi-character of Π, and ω_Π be the restriction of Ω_Π to

$Z(\mathbf{A})$. Put $\omega = \omega_\Pi$. Let $\mathscr{W}(\Pi, \psi)$ be the Whittaker model of Π, i.e.,

Case (1) $\mathscr{W}(\Pi, \psi) = \mathscr{W}(\pi_1, \psi) \otimes \mathscr{W}(\pi_2, \psi) \otimes \mathscr{W}(\pi_3, \psi)$,
Case (2) $\mathscr{W}(\Pi, \psi) = \mathscr{W}(\pi_1, \psi) \otimes \mathscr{W}(\pi_2, \psi \circ \mathrm{tr}_{k'/k})$,
Case (3) $\mathscr{W}(\Pi, \psi) = \mathscr{W}(\Pi, \psi \circ \mathrm{tr}_{k''/k})$.

If φ is a cusp form belonging to Π, then there exists $W \in \mathscr{W}(\Pi, \psi)$ such that

$$\varphi(g) = \sum_{\alpha \in \mathbf{K}^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

We assume that W is decomposable: $W = \Pi_v W_v$. Here, v runs over all places of k. Put

$$P = \left\{ \begin{pmatrix} mA & * \\ \mathbf{0}_3 & {}^tA^{-1} \end{pmatrix} \in \mathrm{GSp}_3 \right\}.$$

By [22, §1], the double cosets $P \backslash \mathrm{GSp}_3 / \iota(G)$ contains one open coset and the other cosets are all negligible in the terminology of [20]. We choose a representative η_0 of the open double coset and put

$$R_0 = \{ g \in G \mid \eta_0 \iota(g) \eta_0^{-1} \in P \}.$$

We can choose η_0 so that

$$R_0 = \left\{ \begin{pmatrix} a & n \\ 0 & a \end{pmatrix} \in \mathrm{GL}_2(\mathbf{K}) \mid a \in k^\times,\ \mathrm{tr}_{\mathbf{K}/k} n = 0 \right\}.$$

Let v be a place of k. Let $J(\omega_v, s)$ be the space of functions $f_v(h)$ on $\mathrm{GSp}_3(k_v)$ which satisfy the following (i) and (ii):

(i) f_v is right finite by the standard maximal compact subgroup of $\mathrm{GSp}_3(k_v)$.

(ii) For $p = \begin{pmatrix} mA & * \\ \mathbf{0}_3 & {}^tA^{-1} \end{pmatrix} \in P(k_v)$,

$$f_v(ph) = \omega_v(m)|m|^{3s+(3/2)} \omega_v(\det A) |\det A|^{2s+1} f_v(h).$$

Observe that if $f_v \in J(\omega_v, s)$, then $f_v|_{\mathrm{Sp}_3(k_v)} \in I(\omega_v, 2s-1)$. We define holomorphic sections and meromorphic sections of $J(\omega_v, s)$ in the same way as in Section 1. The intertwining operator M_w can be defined similarly. We define a meromorphic section $f_v^{(s)}$ is good if

$$[d(\omega_v, 2s-1) c_w(\omega_v, 2s-1)]^{-1} M_w f_v^{(s)}$$

is holomorphic for all $w \in \Omega_3$. Obviously this condition is equivalent to say that $\rho(\phi) f_v^{(s)}|_{\mathrm{Sp}_3(k_v)}$ is a good section of $I(\omega_v, 2s-1)$ for each Hecke operator ϕ on $\mathrm{GSp}_3(k_v)$. By Lemma 1.2, $f_v^{(s)}(h)$ is a good section of $J(\omega_v, s)$ if and only if $\omega_v(m(h)) M_{w_0}^* f_v^{(s)}(h)$ is a good section of $J(\omega_v^{-1}, 1-s)$, where $m(h)$ is the multiplier of h, and by Lemma 1.3, any holomorphic section of $J(\omega_v, s)$ is a good section.

For each meromorphic section $f_v^{(s)} \in J(\omega_v, s)$, and $W_v \in \mathscr{W}(\Pi_v, \psi_v)$, put

$$\Psi_s(f_v^{(s)};\, W_v) = \int_{R_{0,v}\backslash G_v} f_v^{(s)}(\eta_0 \iota(g)) W_v(g)\, \mathrm{d}g.$$

In [7], [22], it is proved that $\Psi_s(f_v^{(s)}; W_v)$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$, and has meromorphic continuation to $\mathbf{C}$, and if v is non-archimedean, $\Psi_s(f_v^{(s)}; W_v)$ is a rational function of q_v^{-s}. By [22, Proposition 3.3], for each $s_0 \in \mathbf{C}$, there exists a holomorphic section $f_v^{(s)}$ of $J(\omega_v, s)$, and $W_v \in \mathscr{W}(\Pi_v, \psi_v)$ such that

$$\Psi_{s_0}(f_v^{(s_0)};\, W_v) \neq 0.$$

Put $\tilde{W}_v(g) = \Omega_v(\det g)^{-1} W_v(g)$, where Ω_v is the central quasi-character of Π_v. Then $\tilde{W}_v \in \mathscr{W}(\tilde{\Pi}_v, \psi_v)$. It is proved in [7], [22], that there exists a meromorphic function $\varepsilon'(s, \Pi_v, \sigma, \psi_v)$ such that

$$\Psi_{1-s}(\omega_v(m(h)) M_{w_0}^* f_v^{(s)};\, \tilde{W}_v) = \varepsilon'(s, \Pi_v, \sigma, \psi_v) \Psi_s(f_v^{(s)};\, W_v).$$

For a non-archimedean place v, we consider the fractional ideal I_v of $R_v = \mathbf{C}[q_v^{-s}, q_v^s]$, generated by $\Psi_s(f_v^{(s)}; W_v)$ attached to good sections $f_v^{(s)}$ of $J(\omega_v, s)$ and $W_v \in \mathscr{W}(\Pi_v, \psi_v)$. Then by [22, Appendix 3 to §3], I_v admits a common denominator and $1 \in I_v$. Thus I_v has a generator of the form $P(q_v^{-s})^{-1}$, $P(X) \in \mathbf{C}[X]$, $P(0) = 1$. We let

$$L(s, \Pi_v, \sigma) = P(q_v^{-s})^{-1},$$

$$\varepsilon(s, \Pi_v, \sigma, \psi_v) = \varepsilon'(s, \Pi_v, \sigma, \psi_v) L(s, \Pi_v, \sigma) L(1-s, \tilde{\Pi}_v, \sigma)^{-1},$$

then $\varepsilon(s, \Pi_v, \sigma, \psi_v)$ is of the form aq^{bs}, $a \in \mathbf{C}$, $b \in \mathbf{Z}$, and

$$\frac{\Psi_{1-s}(\omega_v(m(h)) M_{w_0}^* f_v^{(s)};\, \tilde{W}_v)}{L(1-s, \tilde{\Pi}_v, \sigma)} = \varepsilon(s, \Pi_v, \sigma, \psi_v) \frac{\Psi_s(f_v^{(s)};\, W_v)}{L(s, \Pi_v, \sigma)}. \tag{2.1}$$

When v is unramified, this definition agrees to usual definition $\det(\mathbf{1}_8 - \sigma(g_v, \mathrm{Fr}) q_v^{-s})^{-1}$, where g_v is the Langlands class of Π_v. For a holomorphic section $f_v^{(s)}$ and $W_v \in \mathscr{W}(\Pi_v, \psi_v)$, a careful calculation of denominator of

$\Psi_s(f_v^{(s)}; W_v)$ shows that the denominator divides $\det(\mathbf{1}_8 - \sigma(g_v, \mathrm{Fr})q_v^{-s})$ (cf. [22, Appendix 3 to §3]). It follows that $L(s, \Pi_v, \sigma)^{-1}$ is a divisor of $d(\omega_v, 2s-1)^{-1}$ $\det(\mathbf{1}_8 - \sigma(g_v, \mathrm{Fr})q_v^{-s})$. On the other hand, there are a good section $f_v^{(s)}$ of $J(\omega_v, s)$ and $W_v \in \mathscr{W}(\Pi_v, \psi_v)$ such that $\Psi_s(f_v^{(s)}; W_v) = \det(\mathbf{1}_8 - \sigma(g_v, \mathrm{Fr})q_v^{-s})^{-1}$. This shows that $L(s, \Pi_v, \sigma)^{-1}$ is a multiple of $\det(\mathbf{1}_8 - \sigma(g_v, \mathrm{Fr})q_v^{-s})$. Moreover we know

$$\varepsilon'(s, \Pi_v, \sigma, \psi_v) = \frac{\det(\mathbf{1}_8 - \sigma(g_v, \mathrm{Fr})q_v^{-s})}{\det(\mathbf{1}_8 - \sigma(g_v, \mathrm{Fr})^{-1} q_v^{s-1})}$$

Since $d(\omega_v, 2s-1)^{-1}$ and $d(\omega_v^{-1}, 1-2s)^{-1}$ have no common divisor, we have $L(s, \Pi_v, \sigma) = \det(\mathbf{1}_8 - \sigma(g_v, \mathrm{Fr})q_v^{-s})^{-1}$, as we expected.

When k_v is archimedean, we define L-factor $L(s, \Pi_v, \sigma)$ as follows. The proof of [7, Proposition 5.1] shows that there is a meromorphic function $\alpha(s) \not\equiv 0$ such that

$$\alpha(s)^{-1} \Psi_s(f_v^{(s)}; W_v)$$

is holomorphic for any holomorphic section $f_v^{(s)}$ and $W_v \in \mathscr{W}(\Pi_v, \psi_v)$. Though [7] has dealt with only case (1), it is not difficult to generalize the result to the case $k_v = \mathbf{R}$, $\mathbf{K}_v = \mathbf{R} \oplus \mathbf{C}$. We have only to use the local functional equation of Asai-type L-functions instead of the results of [8]. By Weierstrass theorem, there is a meromorphic function $\lambda(s)$ such that

$$\lambda(s)^{-1} \Psi_s(f_v^{(s)}; W_v) \tag{2.2}$$

is holomorphic for any good section $f_v^{(s)}$ and $W_v \in \mathscr{W}(\Pi_v, \psi_v)$ and if $\lambda'(s)$ is another function with this property, then $\lambda(s)\lambda'(s)^{-1}$ is holomorphic. Obviously, for each $s_0 \in \mathbf{C}$, there exists a good section $f_v^{(s)}$ and $W_v \in \mathscr{W}(\Pi_v, \psi_v)$ such that (2.2) does not have a zero at $s = s_0$. By Lemma 1.3 and [22, Proposition 3.3], $\lambda(s)$ has no zeros. We define $L(s, \Pi_v, \sigma) = \lambda(s)$. Then (2.1) holds with some entire function $\varepsilon(s, \Pi_v, \sigma, \psi_v)$ which have no zeros. Note that $L(s, \Pi_v, \sigma)$ and $\varepsilon(s, \Pi_v, \sigma, \psi_v)$ is determined only up to entire functions which have no zeros.

Let v be any place of k. Assume Π_v is unitary. We define a non-negative real number $\lambda(\Pi_v)$ as follows.

Case (1) $\Pi_v = \pi_1 \otimes \pi_2 \otimes \pi_3$: When π_i is tempered, put $\lambda(\pi_i) = 0$. When π_i is the complementary series $\pi(\mu\alpha^{\lambda}, \mu\alpha^{-\lambda})$, ($\mu$ is a unitary character of $k_v^{\times}$), put $\lambda(\pi_i) = |\lambda|$. Put $\lambda(\Pi_v) = \lambda(\pi_1) + \lambda(\pi_2) + \lambda(\pi_3)$.

Case (2) $\Pi_v = \pi_1 \otimes \pi_2$: let $\lambda(\pi_i)$ be as above, and put $\lambda(\Pi_v) = \lambda(\pi_1) + 2\lambda(\pi_2)$.
Case (3) $\Pi_v = \pi_1$: let $\lambda(\pi_1)$ be as above, and put $\lambda(\Pi_v) = 3\lambda(\pi_1)$.

LEMMA 2.1. *If Π_v is unitary, then $L(s, \Pi_v, \sigma)$ has no poles on the domain $\mathrm{Re}(s) > \lambda(\Pi_v)$.*

Proof. By an argument similar to [7, Theorem 1], [22, Proposition 3.2], we can show that if $f_v^{(s)}$ is a holomorphic section of $J(\omega_v, s)$ and $W_v \in \mathscr{W}(\Pi_v, \psi_v)$, then $\Psi_s(f_v^{(s)}; W_v)$ is absolutely convergent for $\mathrm{Re}(s) > \lambda(\Pi_v)$. Since $d(\omega_v, s)$ has no poles for $\mathrm{Re}(s) > 0$, a good section $f_v^{(s)}$ is holomorphic for $\mathrm{Re}(s) > 0$. This proves the lemma.

LEMMA 2.2. *Assume* **K** *is not a cubic extension of k. Assume Π_v is unitary. Assume each component is a subquotient of a principal series, and $\lambda(\Pi_v) < 1/2$. Then $L(s, \Pi_v, \sigma)$ (resp. $\varepsilon(s, \Pi_v, \sigma, \psi_v)$) agrees to L-factor (resp. ε-factor) associated to the 8-dimensional representation of the Weil group W_{k_v} determined by Π_v, and σ.*

Proof. By [7, Proposition 5.1], $\varepsilon'(s, \Pi_v, \sigma, \psi_v)$ coincides ε'-factor determined by the Weil group. The proof of [7] Proposition 5.1 works for case (2). By the assumption, $L(s, \Pi_v, \sigma)$ has no poles on the domain $\mathrm{Re}(s) > \lambda(\Pi_v)$ and $L(1-s, \tilde{\Pi}_v, \sigma)$ has no poles on the domain $\mathrm{Re}(s) < 1-\lambda(\Pi_v)$. This proves the lemma.

REMARK. By Lemma 2.2, we can identify the archimedean L-factors and usual Γ-factors if Π is generated by Hilbert modular forms over a totally real field.

COROLLARY. *Assume* **K** *is not a cubic extension of k. Assume Π_v is unitary. Assume no component is extraordinary, and $\lambda(\Pi_v) < 1/2$. Then the conclusion of Lemma 2.2 holds.*

Proof. For simplicity, we assume $\mathbf{K} = k \oplus k \oplus k$, $\Pi_v = \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v}$, and all of $\pi_{1,v}$, $\pi_{2,v}$ and $\pi_{3,v}$ are supercuspidal. $\pi_{i,v} = \pi(\chi_{i,v})$ $(i = 1, 2, 3)$ for some quasi-character $\chi_{i,v}$ of some quadratic extension $K_{i,v}$ of k_v. Choose global quadratic extension K_i of k such that $K_i k_v = K_{i,v}$. It is easy to check that there exists global quasi-character χ_i of $\mathbf{A}_{K_i}^{\times}$ such that v-part of χ_i is $\chi_{i,v}$ and $\pi(\chi_i)$ is principal series outside of v and all archimedean place. Put $\Pi = \pi(\chi_1) \otimes \pi(\chi_2) \otimes \pi(\chi_3)$. Then $L(s, \Pi, \sigma)$ is L-function associated to 8-dimensional representation of global Weil group. The conclusion of Lemma 2.2 holds outside v, so does at v.

We now consider the global theory. We say that a meromorphic section of $J(\omega, s)$ is a good section if it is a finite sum of decomposable elements $f^{(s)} = \Pi_v f_v^{(s)}$, satisfying the following two conditions:

(i) For almost all unramified places v, $f_v^{(s)}|_{K_v} \equiv d(\omega_v, 2s-1)$.
(ii) $f_v^{(s)}$ is a good section of $J(\omega_v, s)$ for all v.

Note that the infinite product $\Pi_v f_v^{(s)}$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$, and can be meromorphically continued to **C**.

For each good section $f^{(s)}$ of $J(\omega, s)$, put

$$E(h; f^{(s)}) = \sum_{\gamma \in P\backslash \mathrm{GSp}_3} f^{(s)}(\gamma h).$$

Then the restriction of $E(h; f^{(s)})$ to $\mathrm{Sp}_3(\mathbf{A})$ is an Eisenstein series on $\mathrm{Sp}_3(\mathbf{A})$ investigated in Section 1.3. In [7], [22], it is proved that if $f^{(s)} = \Pi_v f_v^{(s)}$ is decomposable, then

$$\int_{Z(\mathbf{A})G(k)\backslash G(\mathbf{A})} E(\iota(g); f^{(s)})\varphi(g)\,\mathrm{d}g = \prod_v \Psi_s(f_v^{(s)}; W_v), \tag{2.3}$$

for $\mathrm{Re}(s) \gg 0$. Set

$$L(s, \Pi, \sigma) = \prod_v L(s, \Pi_v, \sigma)$$

and

$$\varepsilon(s, \Pi, \sigma) = \prod_v \varepsilon(s, \Pi_v, \sigma, \psi_v).$$

Then by Proposition 1.6, (2.1), and (2.3), we have the following propositions.

PROPOSITION 2.3. *$L(s, \Pi, \sigma)$ can be meromorphically continued to $\mathbf{C}$. It is entire if ω^2 is not a principal quasi-character. If $\omega^2 = 1$, and k is a number field, then $L(s, \Pi, \sigma)$ has possible poles at $s = 0, 1$. If $\omega^2 = 1$, and k is a function field with constant field $\mathbf{F}_q$, then $L(s, \Pi, \sigma)$ has possible poles at $s \in \frac{\pi\sqrt{-1}}{2\log q}\mathbf{Z}$, $1 + \frac{\pi\sqrt{-1}}{2\log q}\mathbf{Z}$. All the possible poles are at most simple.*

PROPOSITION 2.4. *$L(s, \Pi, \sigma)$ satisfies the following functional equation*:

$$L(s, \Pi, \sigma) = \varepsilon(s, \Pi, \sigma)L(1-s, \tilde{\Pi}, \sigma).$$

Now we investigate the poles of $L(s, \Pi, \sigma)$. By Proposition 2.3, we may assume $\omega^2 = 1$ and $s = 0$ or 1. By the functional equation, $s = 0$ is reduced to $s = 1$. If $L(s, \Pi, \sigma)$ has a pole at $s = 1$, then there exists a good section $f^{(s)}$ of $J(\omega, s)$ and a cusp form φ belonging to Π such that

$$\int_{Z(\mathbf{A})G(k)\backslash G(\mathbf{A})} [\mathrm{Res}_{s=1} E(\iota(g); f^{(s)})]\varphi(g)\,\mathrm{d}g \neq 0. \tag{2.4}$$

PROPOSITION 2.5. *If $\omega = 1$, then $L(s, \Pi, \sigma)$ is holomorphic at $s = 1$. In* *particular, if k is a number field, $L(s, \Pi, \sigma)$ is entire* (cf. [22, Theorem 5.1]).

Proof. By Proposition 1.10, the restriction of $\mathrm{Res}_{s=1} E(h; f^{(s)})$ to Sp_3 is an Eisenstein series associated to a function in the representation induced from the trivial character of the maximal parabolic subgroup $P_{3,2}$. It is easy to see that each coset in $(\iota(G) \cap \mathrm{Sp}_3) \backslash \mathrm{Sp}_3 / P_{3,2}$ is negligible. It follows that (2.4) is identically zero.

We now assume that $\omega^2 = 1$, $\omega \neq 1$ and $L(s, \Pi, \sigma)$ has a pole at $s = 1$. Let K be the quadratic extension of k corresponding to ω by class field theory, and θ be the non-trivial element of $\mathrm{Gal}(K/k)$.

Suppose that $\mathbf{K} = k''$, k'' is a cubic extension of k. Let Π_K be the base change of Π to $\mathrm{GL}_2(\mathbf{A}_{k''K})$ (cf. [18]). Consider the triple L-function $L(s, \Pi_K, \sigma_K)$ of Π_K over K. Here, σ_K is the restriction of σ to the semi-direct product of $\mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C})$ and W_K. Then an easy calculation shows

$$L(s, \Pi_K, \sigma_K) = L(s, \Pi \otimes \tilde{\omega}, \sigma) L(s, \Pi, \sigma).$$

Here, $\tilde{\omega}$ is any extension of ω to $\mathbf{A}_{k''}^{\times}$. Note that G is a Levi subgroup of the quasi-split simply connected group Spin(8) of either type ${}^3D_4$ or ${}^6D_4$ according as k''/k is cyclic or not (see Shahidi [23]). Then [23, Theorem 5.1] implies

$$L(1 + 2s, \omega) L(1 + s, \Pi \otimes \tilde{\omega}, \sigma) \neq 0$$

for $\mathrm{Re}(s) = 0$. Since ω is a non-trivial unitary character of $A_k^{\times}$, this implies the non-vanishing of $L(s, \Pi, \sigma)$ at $s = 1$. So, $L(s, \Pi_K, \sigma_K)$ has a pole at $s = 1$. But since $\omega_{\Pi_K} = 1$, Π_K cannot be cuspidal by Proposition 2.5. It follows that there is a quasi-character χ of $\mathbf{A}_{k''K}^{\times}$ such that $\Pi = \pi(\chi)$. By a simple calculation, the triple L-function $L(s, \pi(\chi), \sigma)$ is given by

$$L(s, \pi(\chi), \sigma) = L_K(s, \chi|_{\mathbf{A}_K^{\times}}) L_{k''K}(s, (\chi \circ N_{k''K/K}) \chi^{-1} \chi^{\theta}). \tag{2.5}$$

Here, θ is regarded as an element of $\mathrm{Gal}(k''K/k'')$, by the natural isomorphism $\mathrm{Gal}(k''K/k'') \simeq \mathrm{Gal}(K/k)$. This equality holds up to bad prime factors. But in fact, (2.5) is an equality of global L-functions. To see this, observe that

$$\prod_{v \in S} \varepsilon'(s, \Pi_v, \sigma, \psi_v)$$

has no zero on $\mathrm{Re}(s) > 0$, and has no poles on $\mathrm{Re}(s) < 1$, by comparing the functional equation as a triple L-function and that as a L-function associated to 8-dimensional representation of the Weil group. By Lemma 2.1,

$$\prod_{v \in S} L(s, \Pi_v, \sigma)$$

coincides with the product of L-factors of the right-hand side, since $\lambda(\Pi_v)=0$ for $\Pi=\pi(\chi)$. It follows that (2.5) is an equality of global L-functions.

Let us prove $\chi|_{\mathbf{A}_K^\times}=1$. First observe that $\chi|_{\mathbf{A}_k^\times}=1$, since $\omega_{\pi(\chi)}=\omega\cdot\chi|_{\mathbf{A}_k^\times}$. Suppose $\chi|_{\mathbf{A}_K^\times}\neq 1$. Then $L_{k''K}(s,(\chi\circ N_{k''K/K})\chi^{-1}\chi^\theta)$ has a pole at $s=1$, therefore we have

$$\chi\circ N_{k''K/K}=\chi(\chi^\theta)^{-1}.$$

Put $I=\mathrm{Im}(N_{k''K/K}:\mathbf{A}_{k''K}^\times\to\mathbf{A}_K^\times)$. Then the index $[\mathbf{A}_K^\times:I\cdot K^\times]$ is 1 or 3, by the class fields theory. Let $y\in\mathbf{A}_{k''K}^\times$, $x=N_{k''K/K}(y)$. Then

$$\begin{aligned}\chi^\theta(x)&=\chi(y^\theta)\chi(y^{-1})\\&=\chi(x)^{-1}.\end{aligned}$$

It follows that

$$\begin{aligned}\chi(x^3)&=\chi(N_{k''K/K}(x))\\&=\chi(x)\chi^\theta(x)^{-1}\\&=\chi(x^2).\end{aligned}$$

So χ is trivial on $I\cdot K^\times$. It follows that $\chi|_{\mathbf{A}_K^\times}=1$, since $I\cdot K^\times\cdot\mathbf{A}_k^\times=\mathbf{A}_K^\times$. Thus we have proved the following theorem.

THEOREM 2.6. *Suppose that $\mathbf{K}=k''$, k'' is a cubic extension of k, and $L(s,\Pi,\sigma)$ has a pole somewhere. Then*

(a) *Let Π', ω' be the objects obtained by twisting π_1 by α^{s_0}, $s_0\in\mathbf{C}$. Then $\omega'^2=1$, $\omega'\neq 1$, and $L(s,\Pi',\sigma)$ has a simple pole at $s=1$, for some $s_0\in\mathbf{C}$.*

(b) *Assume that $\omega^2=1$, $\omega\neq 1$, and $L(s,\Pi,\sigma)$ has a pole at $s=1$. Let K be the quadratic extension of k corresponding to ω by class field theory. Let θ be the non-trivial element of $\mathrm{Gal}(k''K/k'')$. Then there exists a quasi-character χ of $\mathbf{A}_{k''K}^\times/k''K^\times$ such that $\Pi=\pi(\chi)$ and $\chi|_{\mathbf{A}_K^\times}=1$. Moreover the triple L-function is given by*

$$L(s,\pi(\chi),\sigma)=\zeta_K(s)L_{k''K}(s,\chi^{-1}\chi^\theta).$$

Next, suppose that $\mathbf{K}=k\oplus k\oplus k$, $\Pi=\pi_1\otimes\pi_2\otimes\pi_3$. By the assumption, $\omega_1\omega_2\omega_3=\omega$. Let $\pi_{i,K}$ $(i=1,2,3)$ be the base change of π_i to $\mathrm{GL}_2(\mathbf{A}_K)$. Put $\Pi_K=\pi_{1,K}\otimes\pi_{2,K}\otimes\pi_{3,K}$. Then,

$$L(s,\Pi_K,\sigma_K)=L(s,\Pi\otimes\omega,\sigma)L(s,\Pi,\sigma).$$

Here, $\Pi \otimes \omega$ means $(\pi_1 \otimes \omega) \otimes \pi_2 \otimes \pi_3$. As is case (3), the left-hand side has a pole at $s=1$, and $\omega_{\Pi_K} = 1$. This time, we can deduce that one of $\pi_{i,K}$ $(i=1,2,3)$, say $\pi_{1,K}$, is not cuspidal. So there is a quasi-character χ of $\mathbf{A}_K^\times/K^\times$ such that $\pi_1 = \pi(\chi)$. Observe that $\chi|_{\mathbf{A}_k^\times} = \omega_2^{-1}\omega_3^{-1}$, since the central quasi-character of $\pi(\chi)$ is $\omega \cdot \chi|_{\mathbf{A}_k^\times}$. The triple L-function $L(s, \Pi, \sigma)$ is given by

$$L(s, \Pi, \sigma) = L_K(s, (\pi_{2,K} \otimes \chi) \times \pi_{3,K}).$$

Let us now prove that neither $\pi_{2,K}$ nor $\pi_{3,K}$ are cuspidal. Suppose that $\pi_{2,K}$ or $\pi_{3,K}$, say $\pi_{2,K}$, is cuspidal. Then

$$\pi_{2,K} \otimes \chi \simeq \tilde{\pi}_{3,K}. \tag{2.6}$$

In particular, $\pi_{3,K}$ is cuspidal, too. Since $\pi_{2,K}$ and $\pi_{3,K}$ are θ-invariant,

$$\pi_{2,K} \otimes \chi^\theta \simeq \tilde{\pi}_{3,K}. \tag{2.7}$$

Put $\varepsilon = \chi(\chi^\theta)^{-1}$. Since $\pi(\chi)$ is cuspidal, $\varepsilon \neq 1$. By (2.6) and (2.7), we have $\pi_{2,K} \otimes \varepsilon \simeq \pi_{2,K}$. It follows that $\varepsilon^2 = 1$. Since $\varepsilon^\theta = \varepsilon^{-1} = \varepsilon$, there is a character ε' of $\mathbf{A}_k^\times/k^\times$ such that $\varepsilon = \varepsilon' \circ N_{K/k}$. Taking the central quasi-character of (2.6), we have

$$(\omega_2 \circ N_{K/k})\chi^2 = (\omega_3 \circ N_{K/k})^{-1}.$$

Put $I = \mathrm{Im}(N_{K/k}\colon \mathbf{A}_K^\times \to \mathbf{A}_k^\times)$. Let $y \in \mathbf{A}_K^\times$, $x = N_{K/k}(y)$. Then

$$\begin{aligned} \omega_2(x) &= \omega_3(x)^{-1}\chi(y)^{-2} \\ &= \omega_3(x)^{-1}\chi(y)^{-1}\chi(y^\theta)^{-1}\varepsilon(y) \\ &= \omega_3(x)^{-1}\chi(x)^{-1}\varepsilon'(x). \end{aligned}$$

It follows that

$$\begin{aligned} \omega_1(x)\omega_2(x)\omega_3(x) &= \chi(x)\omega(x)\omega_3(x)^{-1}\chi(x)^{-1}\varepsilon'(x)\omega_3(x) \\ &= \omega(x)\varepsilon'(x). \end{aligned}$$

This contradicts to the assumption $\omega_1\omega_2\omega_3 = \omega$, since ε' is not trivial on I.

We have proved that there are quasi-characters χ_i $(i=1,2,3)$ of $\mathbf{A}_K^\times$ such that $\pi_i = \pi(\chi_i)$. The triple L-function is given by

$$L(s, \Pi, \sigma) = L_K(s, \chi_1\chi_2\chi_3)L_K(s, \chi_1^\theta\chi_2\chi_3)L_K(s, \chi_1\chi_2^\theta\chi_3)L_K(s, \chi_1\chi_2\chi_3^\theta).$$

In this case, this equality holds for every local L-factor, by Lemma 2.2. Replacing χ_i by χ_i^θ if necessary, we have $\chi_1\chi_2\chi_3=1$. We have proved the following theorem.

THEOREM 2.7. *Suppose that* $\mathbf{K}=k\oplus k\oplus k$, *and* $L(s,\Pi,\sigma)$ *has a pole somewhere. Then the following two assertions hold*:

(a) *Let* Π', ω' *be the objects obtained by twisting* π_1 *by* α^{s_0}, $s_0\in\mathbf{C}$. *Then* $\omega'^2=1$, $\omega'\neq 1$, *and* $L(s,\Pi',\sigma)$ *has a simple pole at* $s=1$, *for some* $s_0\in\mathbf{C}$.

(b) *Assume that* $\omega^2=1$, $\omega\neq 1$, *and* $L(s,\Pi,\sigma)$ *has a pole at* $s=1$. *Let* K *be the quadratic extension of* k *corresponding to* ω *by class field theory. Let* θ *be the generator of* Gal(K/k). *Then there exist quasi-characters* χ_1, χ_2, *and* χ_3 *of* $\mathbf{A}_K^\times/K^\times$ *such that* $\pi_1=\pi(\chi_1)$, $\pi_2=\pi(\chi_2)$, $\pi_3=\pi(\chi_3)$, *and* $\chi_1\chi_2\chi_3=1$. *Moreover, the triple L-function is equal to*

$$\zeta_K(s)L_K(s,\,\chi_1^{-1}\chi_1^\theta)L_K(s,\,\chi_2^{-1}\chi_2^\theta)L_K(s,\,\chi_3^{-1}\chi_3^\theta).$$

Now, suppose that $\mathbf{K}=k\oplus k'$, k' is a quadratic extension of k, $\Pi=\pi_1\otimes\pi_2$. Let ω_1 and ω_2 be the central quasi-characters of π_1 and π_2, respectively. By the assumption, $\omega_1\cdot(\omega_2|_{\mathbf{A}_k^\times})=\omega$.

We first prove $K\neq k'$. Assume that $K=k'$. In this case we have, as in case (3),

$$L(s,\,\Pi\otimes\omega,\,\sigma)L(s,\,\Pi,\,\sigma)=L_K(s,\,\pi_{1,K}\times\pi_2\times\pi_2^\theta),$$

and this has a pole at $s=1$. Here, $\Pi\otimes\omega$ means $(\pi_1\otimes\omega)\otimes\pi_2$. As in case (3), we can prove that $\pi_{1,K}$ is not cuspidal. It follows that there is a quasi-character χ of K such that $\pi_1=\pi(\chi)$. Then

$$L(s,\,\Pi,\,\sigma)=L_K(s,\,(\pi_2\otimes\chi)\times\pi_2^\theta).$$

Therefore we have $\pi_2\otimes\chi\simeq\tilde{\pi}_2^\theta$. Then $\pi_2\otimes\varepsilon\simeq\pi_2$, where $\varepsilon=\chi(\chi^\theta)^{-1}$. As in case (1), we can prove that $\varepsilon^2=1$, $\varepsilon\neq 1$, $\varepsilon^\theta=\varepsilon$ and that there is a character ε' of $\mathbf{A}_k^\times/k^\times$ such that $\varepsilon=\varepsilon'\circ N_{K/k}$. Taking the central character of $\pi_2\otimes\chi\simeq\tilde{\pi}_2^\theta$, we have

$$\omega_2\chi^2=(\omega_2^\theta)^{-1}.$$

Let I, x and y be as in the case (1). Then

$$\begin{aligned}\omega_2(y)&=\omega_2(y^\theta)^{-1}\chi(y)^{-2}\\&=\omega_2(y^\theta)^{-1}\chi(y)^{-1}\chi(y^\theta)^{-1}\varepsilon(y)\\&=\omega_2(y^\theta)^{-1}\chi(x)^{-1}\varepsilon'(x).\end{aligned}$$

It follows that

$$\begin{aligned}\omega_1(x)\omega_2(x) &= \chi(x)\omega(x)\omega_2(yy^{\theta})\\ &= \chi(x)\omega(x)\chi(x)^{-1}\varepsilon'(x)\\ &= \omega(x)\varepsilon'(x).\end{aligned}$$

This contradicts to the assumption $\omega_1 \cdot \omega_2|_{\mathbf{A}_k^\times} = \omega$, since ε' is non-trivial on I. Thus we have proved $K \neq k'$.

Suppose $K \neq k'$. Let $\pi_{1,K}$ (resp. $\pi_{2,K}$) be the base change of π_1 (resp. π_2) to $\mathrm{GL}_2(\mathbf{A}_k)$ (resp. $\mathrm{GL}_2(\mathbf{A}_{k'K})$). In this case we can prove that at least one of $\pi_{1,K}$ and $\pi_{2,K}$ is not cuspidal as in case (1). We first prove that actually $\pi_{2,K}$ is not cuspidal. Suppose that $\pi_{2,K}$ is cuspidal. Then $\pi_{1,K}$ is not cuspidal, so there is a quasi-character χ of $\mathbf{A}_K^\times$ such that $\pi_1 = \pi(\chi)$. Then the triple L-function is given by the Asai-L-function of $\pi_{2,K}$ twisted by χ:

$$L(s, \Pi, \sigma) = L_K(s, \pi_{2,K}, \chi)_{\mathrm{Asai}}.$$

Let η be the character of $\mathbf{A}_K^\times/K^\times$ corresponding to $k'K/K$ by class field theory. Then

$$L_K(s, (\pi_{2,K} \otimes \chi) \times \pi_{2,K}^{\theta}) = L_K(s, \pi_{2,K}, \chi)_{\mathrm{Asai}} L_K(s, \pi_{2,K}, \chi\eta)_{\mathrm{Asai}}.$$

Since $L_K(s, \pi_{2,K}, \chi\eta)_{\mathrm{Asai}}$ is the triple L-function for $\pi(\chi\eta) \times \pi_2$, it does not have a zero at $s = 1$, so $L_K(s, (\pi_{2,K} \otimes \chi) \times \pi_{2,K}^{\theta})$ has a pole at $s = 1$. As in the case $K = k'$, this is impossible.

Thus $\pi_{2,K}$ is not cuspidal, so $\pi_2 = \pi(\chi)$ for some quasi-character χ of $\mathbf{A}_{k'K}^\times$. The triple L-function is given by

$$L(s, \Pi, \sigma) = L(s, \pi_1 \times \pi(\chi|_{\mathbf{A}_K^\times}))L(s, \pi_1 \times \pi(\chi|_{\mathbf{A}_{K'}^\times})),$$

up to finite number of Euler factors. Here, K' is the quadratic extension of k, contained in $k'K$ different from K and k'.

It follows that $\pi_1 \simeq \pi(\chi^{-1}|_{\mathbf{A}_K^\times})$ or $\pi_1 \simeq \pi(\chi^{-1}|_{\mathbf{A}_{K'}^\times})$, but the latter is impossible for the following reason. First we observe the central quasi-character of $\pi(\chi)$, $\pi(\chi^{-1}|_{\mathbf{A}_K^\times})$, and $\pi(\chi^{-1}|_{\mathbf{A}_{K'}^\times})$ are $\chi|_{\mathbf{A}_{k'}^\times} \cdot \omega_{k'K/k'}$, $\chi^{-1}|_{\mathbf{A}_k^\times} \cdot \omega$, and $\chi^{-1}|_{\mathbf{A}_k^\times} \cdot \omega_{K'/k}$, respectively. Here, $\omega_{k'K/k'}$ (resp. $\omega_{K'/k}$) is the character of $\mathbf{A}_{k'}^\times/k'^\times$ (resp. $\mathbf{A}^\times/k^\times$) of order 2 corresponding to $k'K/k'$ (resp. K'/k) by class field theory. If $\pi_1 \simeq \pi(\chi^{-1}|_{\mathbf{A}_{K'}^\times})$, we have

$$\begin{aligned}\omega_1(x)\omega_2(x) &= \chi^{-1}(x)\omega_{K'/k}(x)\chi(x)\omega_{k'K/k'}(x)\\ &= \omega_{K'/k}(x).\end{aligned}$$

This contradicts to the assumption $\omega_1 \cdot (\omega_2|_{\mathbf{A}_K^\times}) = \omega$, so one cannot have $\pi_1 \simeq \pi(\chi^{-1}|_{\mathbf{A}_K^\times})$.

Suppose $\pi_1 \simeq \pi(\chi^{-1}|_{\mathbf{A}_K^\times})$, and $\pi_2 \simeq \pi(\chi)$. Then an easy calculation shows that the triple L-function is equal to

$$\zeta_K(s) L_K(s, (\chi^{-1}\chi^\theta)|_{\mathbf{A}_K^\times}) L_{k'K}(s, \chi^{-1}\chi^\theta).$$

Here, θ is regarded as an element of $\mathrm{Gal}(k'K/k')$, by the natural isomorphism $\mathrm{Gal}(k'K/k') \simeq \mathrm{Gal}(K/k)$. As in case (1), this equation holds for all place v.

Thus we have proved the following theorem.

THEOREM 2.8. *Suppose that* $\mathbf{K} = k \oplus k'$, k' *is a quadratic extension of* k, *and* $L(s, \Pi, \sigma)$ *has a pole somewhere. Then the following two assertions hold*:

(a) *Let* Π', ω' *be the objects obtained by twisting* Π *by* α^{s_0}, $s_0 \in \mathbf{C}$. *Then* $\omega'^2 = 1$, $\omega' \neq 1$, ω' *does not correspond to* k'/k *by class field theory, and* $L(s, \Pi', \sigma)$ *has a simple pole at* $s = 1$, *for some* $s_0 \in \mathbf{C}$.

(b) *Assume that* $\omega^2 = 1$, $\omega \neq 1$, ω *does not correspond to* k'/k *by class field theory, and* $L(s, \Pi, \sigma)$ *has a simple pole at* $s = 1$. *Let* K *be the quadratic extension of* k *corresponding to* ω *by class field theory. Let* θ *be the generator of* $\mathrm{Gal}(k'K/k')$. *Then there exists a quasi-character* χ *of* $\mathbf{A}_{k'K}^\times / k'K^\times$ *such that* $\pi_1 \simeq \pi(\chi^{-1}|_{\mathbf{A}_K^\times})$, *and* $\pi_2 = \pi(\chi)$. *Moreover, the triple L-function is equal to*

$$\zeta_K(s) L_K(s, (\chi^{-1}\chi^\theta)|_{\mathbf{A}_K^\times}) L_{k'K}(s, \chi^{-1}\chi^\theta).$$

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