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Local factorization of determinants of twisted DR cohomology groups

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1. Introduction

We shall obtain some results concerning DR cohomology with coefficients in a vector bundle equipped with an integrable connection regular singular at infinity. Our main result is Thm. 4.3.1 below.

An easily stated consequence of the main result is as follows. Let $U/\mathbf{C}$ be a smooth affine variety embedded as the complement in a smooth projective variety $X/\mathbf{C}$ of an effective divisor Z whose irreducible components Z_i are smooth and cross normally. Let ω be a global section of $\Omega^1_{X/\mathbf{C}}(\log Z)$ with nonvanishing residues $\mathrm{Res}_i\,\omega$ along each component Z_i. Note that under these hypotheses, the zero locus of ω on U is 0-dimensional. Let f be a meromorphic function on X regular and nowhere vanishing on U. Let $\mathbf{C}(s)$ denote the field of rational functions over $\mathbf{C}$ in a variable s and set

$$\Omega^{\cdot} =: \Gamma(U, \Omega^{\cdot}_{U/\mathbf{C}}), \qquad \Omega^{\cdot}(s) =: \Omega^{\cdot} \otimes_{\mathbf{C}} \mathbf{C}(s).$$

On the graded group $\Omega^{\cdot}(s)$ define operators

$$D(\eta \otimes h(s)) =: d\eta \otimes h(s) + \left(\frac{df}{f} \wedge \eta\right) \otimes sh(s),$$

$$B(\eta \otimes h(s)) =: f\eta \otimes h(s+1).$$

Then D is a $\mathbf{C}(s)$-linear map of degree 1 and square 0, B a semi-linear automorphism ($Bh(s)\eta = h(s+1)B\eta$) of degree 0 and

$$BD = DB.$$

It follows that B induces a semi-linear automorphism of $H^*(\Omega^{\cdot}(s), D)$ and, in any case, one knows that $H^*(\Omega^{\cdot}(s), D)$ is finite-dimensional over $\mathbf{C}(s)$. Therefore one can form a determinant

$$\varepsilon(U, f) =: \prod_i \det(B \mid H^i(\Omega^\cdot(s), D))^{(-1)^i}$$

well defined up to a factor of the form $h(s+1)/h(s)$ ($h(s) \in \mathbf{C}(s)^\times$). Let us call elements of $\mathbf{C}(s)^\times$ of the latter form *coboundaries*. One has the following explicit formula.

THEOREM 1.1

$$\varepsilon(U, f) \equiv \left(\prod_u f(u)^{\mathrm{ord}_u \omega}\right)^{(-1)^{\dim(U)}} \times \prod_i \left((\mathrm{Res}_i\, \omega)^{m_i} \frac{\Gamma(m_i s)}{\Gamma(m_i s + m_i)}\right)^{\chi_i}$$

modulo coboundaries.

Here u runs through the closed points of U, $\mathrm{ord}_u\, \omega$ denotes the order of vanishing of ω at u, and for each irreducible components Z_i of Z, m_i denotes the order to which f vanishes along Z_i and χ_i denotes the Euler characteristic of the complement in Z_i of the union of the irreducible components of Z distinct from Z_i. We note that Thm. 1.1 was obtained independently and by a different method by F. Loeser and C. Sabbah [4]. We note that Dwork's theory [2] of generalized hypergeometric functions provides determinant formulas similar to Thm. 1.1 and, as well, p-adic analogues of number-theoretic interest.

The structure of $\mathbf{C}(s)^\times$ modulo coboundaries is easily determined. Each element of $\mathbf{C}(s)^\times$ has a unique expression of the form

$$\theta \prod_{a \in \mathbf{C}} (s-a)^{n_a} (\theta \in \mathbf{C}^\times;\ n_a \in \mathbf{Z})$$

with all but finitely many of the exponents n_a vanishing. It follows that $\mathbf{C}(s)^\times$ modulo coboundaries is the direct sum of a copy of $\mathbf{C}^\times$ and the free abelian group generated by the set underlying $\mathbf{C}/\mathbf{Z}$. The invariant $\varepsilon(U, f)$ is not in general a coboundary.

Here is the plan of the paper. In section 2 the purpose is to review, with certain modifications and simplifications, a small part of the theory of determinants of complexes (cf. [5]). In section 3 we review the theory of coherent sheaves equipped with a regular singular integrable connection and establish the existence of *virtuous filtrations.* Sections 2 and 3 are independent of each other. In section 4 the main result (Thm. 4.3.1) is given. The proof is based on the consideration of certain finite-dimensional graded vectorspaces, arising naturally from coherent sheaves equipped with regular singular integrable connections, that carry *two* distinct structures of acyclic complex. The *ad hoc* formalism set up in section 2 is designed to handle objects of the latter sort efficiently. The paper concludes in section 5 with the formulation and proof of a "semilinear variant" of the main result which is easier to state and to apply, from which finally Thm. 1.1 is deduced as a special case.

2. Sign rules

The purpose of this section is to define a canonical trivialization $\mathbf{t}$ of the determinant of a finite-dimensional acyclic complex of vectorspaces and a canonical isomorphism h from the determinant of a finite-dimensional complex of vectorspaces to the determinant of the cohomology of the complex. It is obvious how to define such things except for a "nasty sign problem" solved by Knudsen and Mumford [5] according to Grothendieck's specifications. This theory *almost* provides us with the machinery we need, the only problem being that *it is difficult to see how the canonical trivialization of an acyclic complex varies with the choice of differential.* In order to remedy this defect we shall build up a formalism from scratch, by and large following Knudsen-Mumford's lead, except that the canonical trivialization $\mathbf{t}$ is given by a simple explicit rule (§2.3.1).

2.1. *Notation*

Throughout this section a field k is fixed and a vectorspace is understood to be a finite-dimensional vectorspace over k. Further, $\otimes$ and Hom are understood to be over k. Given a vectorspace V, let $r(V)$ denote the dimension of V over k and let $\det(V)$ denote the maximal exterior power of V over k. More generally, if V is graded, set

$$r(V) =: \sum_n (-1)^n r(V^n),$$

$$r'(V) =: \sum_n n(-1)^n r(V^n),$$

Let V^+ and V^- denote, respectively, the direct sum of even and odd degree summands of V and set

$$\det(V) =: \mathrm{Hom}(\det(V^-), \det(V^+)).$$

2.2. *The exact sequence constraint*

2.2.1 *Definition*
Given an exact sequence

$$\Sigma\colon 0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0$$

of vectorspaces, let

$$i_\Sigma : \det(A) \otimes \det(C) \to \det(B)$$

be the isomorphism defined by the rule

$$\begin{aligned} &i_\Sigma((a_1 \wedge \cdots \wedge a_{r(A)}) \otimes (pb_1 \wedge \cdots \wedge pb_{r(C)})) \\ &\quad = ja_1 \wedge \cdots \wedge ja_{r(A)} \wedge b_1 \wedge \cdots \wedge b_{r(C)}. \end{aligned}$$

More generally, if Σ is an exact sequence of graded vectorspaces, let

$$\Sigma^\pm : 0 \to A^\pm \to B^\pm \to C^\pm \to 0$$

be the even and odd exact sequences deduced from Σ and set

$$i_\Sigma =: f \otimes g \mapsto (-1)^{r(A^+)r(C^-)} i_{\Sigma^+} \circ (f \otimes g) \circ (i_{\Sigma^-})^{-1} : \det(A) \otimes \det(C) \to \det(B).$$

2.2.2. *The three-by-three rule*

We claim that given a commutative diagram

$$\begin{array}{ccccccccc} & & & 4: & & 5: & & 6: & \\ & & & 0 & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & & \downarrow & \\ 1: & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 \\ & & & \downarrow & & \downarrow & & \downarrow & \\ 2: & 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow 0 \\ & & & \downarrow & & \downarrow & & \downarrow & \\ 3: & 0 & \longrightarrow & G & \longrightarrow & H & \longrightarrow & K & \longrightarrow 0 \\ & & & \downarrow & & \downarrow & & \downarrow & \\ & & & 0 & & 0 & & 0 & \end{array}$$

of graded vectorspaces with exact rows and columns, the induced diagram

$$\begin{array}{ccc}
\det(A)\otimes\det(C)\otimes\det(G)\otimes\det(K) & \xrightarrow{i_1\otimes i_3} & \det(B)\otimes\det(H) \\
\downarrow{\scriptstyle 1\otimes\psi\otimes 1} & & \\
\det(A)\otimes\det(G)\otimes\det(C)\otimes\det(K) & & \downarrow{\scriptstyle i_5} \\
\downarrow{\scriptstyle i_4\otimes i_6} & & \\
\det(D)\otimes\det(F) & \xrightarrow[i_2]{} & \det(E)
\end{array}$$

commutes, where

$$\psi =: c\otimes g \mapsto (-1)^{r(C)r(G)} g\otimes c.$$

Now if all the graded vectorspaces $A,\dots,K$ are concentrated in degree 0, the claim is easily checked. In the general case, the assertion is certainly true up to a sign, i.e. true if ψ is replaced by $\psi' =: c\otimes g\mapsto(-1)^s g\otimes c$ for a suitable integer s. Set

$$a^+ =: r(A^+),\dots,k^- =: r(K^-).$$

Then modulo 2,

$$s\equiv c^+g^+ + c^-g^- + a^+g^- + c^+k^- + d^+f^- + a^+c^- + g^+k^- + b^+h^-.$$

Taking into account that

$$d^+ = g^+ + a^+,\ f^- = c^- + k^-,\quad b^+ = a^+ + g^+,\quad h^- = g^- + k^-,$$

one finds after a brief calculation that

$$s\equiv (c^+ - c^-)(g^+ - g^-)$$

as desired. The claim is proved.

2.3. *The canonical trivialization of an acyclic complex*

2.3.1. *Definition*

Let V be a graded vectorspace equipped with a *differential* ∂, i.e. an endomorphism of degree 1 and square 0, such that the complex (V,∂) is acyclic. We define in this case a basis $\mathbf{t}(V, \partial)$ of the one-dimensional vectorspace $\det(V)$ by

$$\mathbf{t}(V, \partial) =: (-1)^{w(V)} \det(T + \partial) : \det(V^-) \tilde{\to} \det(V^+),$$

where T is any *contracting homotopy*, i.e. an endomorphism of V of degree -1 such that

$$T\partial + \partial T = 1,$$

and

$$w(V) =: r(V^+)(r(V^+) - 1)/2.$$

Since $(T + \partial)^2 = (1 + T^2): V^\pm \to V^\pm$ is unipotent, $(T + \partial): V^- \to V^+$ is indeed invertible. Further, $\mathbf{t}(V, \partial)$ is independent of the choice of T, as can be verified by induction on the *length* of V, i.e. the smallest nonnegative integer n such that V is concentrated in an interval of length n. If the length of V is 0, V vanishes identically, hence $\mathbf{t}(V, \partial)$ well defined. If V is of length 1, then T is unique, hence $\mathbf{t}(V, \partial)$ well defined. If V is of length $n > 1$ concentrated, say, in the interval $[a + 1 - n, a + 1]$, consider the graded subspace

$$W^i =: V^i (i < a)$$
$$W^a =: \ker(\partial: V^a \to V^{a+1})$$
$$W^i =: 0 (i > a).$$

Then W is both ∂- and T-stable, and both W and V/W are acyclic of length strictly less than n. By induction $\det(T + \partial): \det(V^-) \tilde{\to} \det(V^+)$ is independent of T, hence $\mathbf{t}(V, \partial)$ well defined.

2.3.2. *Homogeneity and multiplicativity*

We note that $\mathbf{t}(V, \partial)$ is *homogeneous* of degree $r'(V)$ as a function of ∂, i.e.

$$\mathbf{t}(V, \lambda\partial) = \lambda^{r'(V)} \mathbf{t}(V, \partial) \; (\lambda \in k^\times),$$

because the graded map $V \to V$ given by multiplication by λ^n in degree n induces an isomorphism $(V, \partial) \tilde{\to} (V, \lambda\partial)$ of complexes. Given an exact sequence

$$\Sigma: 0 \to A_1 \to A_2 \to A_3 \to 0$$

of graded vectorspaces equipped with differentials ∂_i rendering A_i acyclic ($i = 1, 2, 3$) in a Σ-compatible way, we claim that $\mathbf{t}$ is *multiplicative*, i.e.

$$i_\Sigma(\mathbf{t}(A_1, \partial_1) \otimes \mathbf{t}(A_3, \partial_3)) = \mathbf{t}(A_2, \partial_2).$$

In order to prove the claim, select contracting homotopies T_i in a Σ-compatible way (possible because Σ is necessarily split exact as a sequence of complexes), let $f_i : A_i^- \to A_i^+$ be the map induced by $\partial_i + T_i$ and set

$$r_i =: r(A_i^+) = r(A_i^-), \qquad w_i =: w(A_i).$$

Then by definition

$$\mathbf{t}(A_i, \partial i) = (-1)^{w_i} \det(f_i)$$
$$i_\Sigma(f_1 \otimes f_3) = (-1)^{r_1 r_3} f_2.$$

The claim is settled by the equation

$$w_1 + w_2 - w_3 = r_1 r_3.$$

2.3.3. *The* $\mathbf{t}$*-ratio formula*

Let V be a graded vectorspace equipped with differentials ∂_1 and ∂_2. Suppose further that there exists a *codifferential* T, i.e. an endomorphism of V of degree -1 and square 0, and $u_1, u_2 \in k^\times$ such that

$$T\partial_i + \partial_i T = u_i \quad (i = 1, 2).$$

Then T/u_i is a contracting homotopy for ∂_i and, in particular, (V, ∂_i) is acyclic. Let $\tilde{V}$ be the graded vectorspace obtained by turning V upsidedown, i.e. $\tilde{V}^n =: V^{-n}$; note that $\det(V) = \det(\tilde{V})$. Then T may be construed as a differential of $\tilde{V}$, and ∂_i/u_i as a contracting homotopy. It follows that

$$\mathbf{t}(V, \partial_1/u_1) = \mathbf{t}(\tilde{V}, T) = \mathbf{t}(V, \partial_2/u_2).$$

Taking into account the homogeneity of $\mathbf{t}$ we get the relation

$$\frac{\mathbf{t}(V, \partial_1)}{\mathbf{t}(V, \partial_2)} = \left(\frac{u_1}{u_2}\right)^{r'(V)}.$$

2.4. *Determinants of special quasi-isomorphisms*

2.4.1. *Definition*

Let $f:(V,\partial) \to (W,\partial)$ be an injective map of graded vectorspaces equipped with differentials ∂ such that the induced map $H^*(f,\partial): H^*(V,\partial) \xrightarrow{\sim} H^*(W,\partial)$ is an isomorphism; we call such a map a *special quasi-isomorphism.* Such a map is automatically a chain homotopy equivalence. We define

$$\det(f,\partial): \det(V) \to \det(W)$$

by the rule

$$\det(f,\partial)(v) = i_\Sigma(v \otimes \mathbf{t}(\operatorname{coker}(f),\partial)),$$

where Σ is the exact sequence

$$\Sigma: 0 \to V \xrightarrow{f} W \to \operatorname{coker}(f) \to 0.$$

Note that when $f: V \to W$ is already an isomorphism of graded vectorspaces,

$$\det(f,\partial) = \det(f).$$

2.4.2. *The determinant-ratio formula*

Let B be a graded vectorspace equipped with differentials ∂_1 and ∂_2 and let A be a graded subspace stable under both differentials. Let $f: A \to B$ be the inclusion and suppose that $f:(A,\partial_i) \to (B,\partial_i)$ is a quasi-isomorphism for $i = 1, 2$. Then it follows directly from the definitions that

$$\frac{\det(f,\partial_1)}{\det(f,\partial_2)} = \frac{\mathbf{t}(\operatorname{coker}(f),\partial_1)}{\mathbf{t}(\operatorname{coker}(f),\partial_2)}.$$

2.4.3. *Compatibility with composition*

Let $f:(A,\partial) \to (B,\partial)$ and $g:(B,\partial) \to (C,\partial)$ be special quasi-isomorphisms. We claim that

$$\det(gf,\partial) = \det(g,\partial)\det(f,\partial): \det(A) \to \det(C).$$

In order to prove the claim, consider the three-by-three exact diagram below.

$$
\begin{array}{ccccccccc}
 & & 4: & & 5: & & 6: & & \\
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1: & 0 \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & \mathrm{coker}(f) & \longrightarrow 0 & \\
 & & \downarrow 1 & & \downarrow g & & \downarrow & & \\
2: & 0 \longrightarrow & A & \xrightarrow{gf} & C & \longrightarrow & \mathrm{coker}(gf) & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
3: & 0 \longrightarrow & 0 & \longrightarrow & \mathrm{coker}(g) & \xrightarrow{1} & \mathrm{coker}(g) & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Noting that

$$i_4(a \otimes \mathbf{t}(0)) = a,$$

$$i_6(\mathbf{t}(\mathrm{coker}(f), \partial) \otimes \mathbf{t}(\mathrm{coker}(g), \partial)) = \mathbf{t}(\mathrm{coker}(gf), \partial),$$

$$i_1(a \otimes \mathbf{t}(\mathrm{coker}(f), \partial) = \det(f, \partial)(a),$$

$$i_3(\mathbf{t}(0) \otimes \mathbf{t}(\mathrm{coker}(g), \partial)) = \mathbf{t}(\mathrm{coker}(g), \partial),$$

one has

$$
\begin{aligned}
\det(gf, \partial)(a) &= i_2(a \otimes \mathbf{t}(\mathrm{coker}(gf), \partial)) \\
&= i_5(\det(f, \partial)(a) \otimes \mathbf{t}(\mathrm{coker}(g), \partial)) \\
&= \det(g, \partial)(\det(f, \partial)(a)),
\end{aligned}
$$

and the claim is proved.

2.4.4. *Homotopy invariance*

Let $f, g: (A, \partial) \to (B, \partial)$ be special quasi-isomorphisms such that $H^*(f, \partial) = H^*(g, \partial)$. We claim that $\det(f, \partial) = \det(g, \partial)$. Let $\pi: (B, \partial) \to (A, \partial)$ be a homotopy inverse to f and let $T: A \to B$ be a homotopy from f to g, i.e. a degree -1 map such that $f - f = \partial T + T\partial$. We may assume without loss of generality that A is a graded subspace of B annihilated by ∂ and that f is the inclusion. In this case

$\pi f = 1$ and $g = f + \partial T$, and consequently

$$g = (1 + \partial T\pi)f.$$

Further, $1 + \partial T\pi$ is a unipotent automorphism of (B, ∂). Therefore

$$\det(g, \partial) = \det(1 + \partial T\pi)\det(f, \partial) = \det(f, \partial),$$

and the claim is proved.

2.4.5. *The natural isomorphism h*

Let V be a graded vectorspace and ∂ a differential of V. We define a canonical isomorphism

$$h(V, \partial): \det(V) \xrightarrow{\sim} \det(H^*(V))$$

by the rule

$$h(V, \partial) =: \det(\phi, \partial)^{-1}$$

where $\phi: (H^*(V, \partial), 0) \to (V, \partial)$ is any map of complexes inducing the identity on $H^*(V, \partial)$; since ϕ is well-defined up to homotopy, $h(V, \partial)$ by virtue of the composition compatibility and homotopy invariance of the determinant of a special quasi-isomorphism.

3. Coherent sheaves with logarithmically singular connections

The basic reference throughout this section is Deligne's book [1]. The basic conventions in force throughout the rest of the paper are these: Given quasicoherent sheaves $\mathscr{E}$ and $\mathscr{F}$ on a scheme S, $\mathscr{E} \otimes \mathscr{F}$ denotes their tensor product over $\mathcal{O}_S$. Given also a point s of S, the stalk of $\mathscr{E}$ at s is denoted by $\mathscr{E}_s$ and the fiber by $\mathscr{E}(s)$. Each locally closed subset of the topological space underlying S is assumed to be equipped with reduced induced subscheme structure.

3.1. *Notation and setting*

3.1.1. We fix an algebraically closed field k of characteristic 0, a smooth quasiprojective k-scheme X and a locally principal reduced closed subscheme Z of X. We assume that the irreducible components of Z are smooth and cross normally. Given an irreducible component Z_i of Z, we denote by Z_i' the union of the irreducible components of Z distinct from Z_i. Let U denote the complement of Z in X. For each nonnegative integer r, let $Z^{(r)}$ denote the union of the r-fold intersections of irreducible components of Z. Note that $Z^{(r)}$ is empty for $r \gg 0$. Let $U^{(r)}$ denote the complement of $Z^{(r+1)}$ in $Z^{(r)}$. Note that $U^{(r)}$ is a smooth locally closed subscheme of X of codimension r.

3.1.2. Let $\Omega^1_{X/k}$ denote the cotangent sheaf of X/k, i.e. the sheaf whose group of sections over an affine open subscheme Spec(A) is the module of Kähler differentials of A/k. A *local coordinate system adapted to* Z is an affine open subscheme V of X together with functions $x_1, \ldots, x_n$ on V such that $dx_1, \ldots, dx_n$ trivialize $\Omega^1_{X/k}$ over V and $x_1 \cdots x_r$ generates the defining ideal of $Z \cap V$ for some $0 \leqslant r \leqslant n$. By hypothesis there exists near each point of X a local coordinate system adapted to Z.

3.1.3. Let $\mathcal{T}_{X/k}$ denote the tangent sheaf of X/k, i.e. the sheaf whose sections over an affine open Spec(A) are the k-linear derivations $A \to A$. By definition $\mathcal{T}_{X/k}$ is canonically dual to $\Omega^1_{X/k}$. Let $\mathcal{T}_{X/k}(-\log Z)$ denote the subsheaf of $\mathcal{T}_{X/k}$ stabilizing the defining ideal of Z. The sheaf $\mathcal{T}_{X/k}(-\log Z)$ is again locally free and, moreover, is closed under Lie brackets. Indeed, in any local coordinate system $x_1, \ldots, x_n$ adapted to Z, the collection of mutually commuting vector-fields $x_1(\partial/\partial x_1), \ldots, x_r(\partial/\partial x_r)$, $\partial/\partial x_{r+1}, \ldots, \partial/\partial x_n$ constitutes a trivialization of $\mathcal{T}_{X/k}(-\log Z)$.

3.1.4. Let $\Omega^1_{X/k}(\log Z)$ denote the $\mathcal{O}_X$-linear dual of $\mathcal{T}_{X/k}(-\log Z)$. Let $\Omega^\bullet_{X/k}(\log Z)$ denote the exterior algebra generated by $\Omega^1_{X/k}(\log Z)$ over $\mathcal{O}_X$. Then $\Omega^\bullet_{X/k}(\log Z)$ is the sheaf of *differential forms* on X with *logarithmic poles* along Z. Without explicit mention to the contrary, $\Omega^\bullet_{X/k}(\log Z)$ is to be regarded as a graded sheaf of $\mathcal{O}_X$-algebras only and *not* as a complex; we shall have occasion to consider more than one structure of complex on this graded sheaf.

3.2. *Review of differential forms with logarithmic poles*

3.2.1. *Fundamental operations.*
The *interior differential* (or *contraction*) is the unique $\mathcal{O}_X$-bilinear pairing

$$(\mathbf{X}, \xi) \mapsto i_{\mathbf{X}}\xi : \mathcal{T}_{X/k}(-\log Z) \times \Omega^\bullet_{X/k}(\log Z) \to \Omega^\bullet_{X/k}(\log Z)$$

of degree -1 in the second variable extending the dual pairing $\mathcal{T}_{X/k}(-\log Z) \times \Omega_{X/k}(\log Z) \to \mathcal{O}_X$ such that

$$i_{\mathbf{X}}(\xi \wedge \eta) = i_{\mathbf{X}}\xi \wedge \eta + (-1)^{\deg(\xi)}\xi \wedge i_{\mathbf{X}}\eta.$$

It follows that

$$(i_{\mathbf{X}})^2 = 0, \quad i_{\mathbf{X}} i_{\mathbf{Y}} = -i_{\mathbf{Y}} i_{\mathbf{X}}.$$

The *Lie derivative* is the unique k-bilinear pairing

$$(\mathbf{X}, \xi) \mapsto L_{\mathbf{X}}\xi : \mathscr{T}_{X/k}(-\log Z) \times \Omega^{\cdot}_{X/k}(\log Z) \to \Omega^{\cdot}_{X/k}(\log Z)$$

of degree 0 in the second variable such that

$$L_{\mathbf{X}} f = \mathbf{X} f$$

and

$$L_{\mathbf{X}} i_{\mathbf{Y}} \xi - i_{\mathbf{Y}} L_{\mathbf{X}} \xi = i_{[\mathbf{X},\mathbf{Y}]} \xi.$$

It follows that

$$L_{\mathbf{X}}(\xi \wedge \eta) = L_{\mathbf{X}} \xi \wedge \eta + \xi \wedge L_{\mathbf{X}} \eta.$$

The *exterior derivative*

$$d : \Omega^{\cdot}_{X/k}(\log Z) \to \Omega^{\cdot}_{X/k}(\log Z)$$

is the unique k-linear homomorphism of degree 1 verifying *Cartan's homotopy formula*

$$d i_{\mathbf{X}} \xi + i_{\mathbf{X}} d\xi = L_{\mathbf{X}} \xi.$$

It follows that

$$d^2 = 0$$

and that

$$d(\xi \wedge \eta) = d\xi \wedge \eta + (-1)^{\deg(\xi)} \xi \wedge d\eta.$$

The *Koszul differential* ∂_ω associated to a global section ω of $\Omega^1_{X/k}(\log Z)$ is the $\mathcal{O}_X$-linear endomorphism of $\Omega^{\cdot}_{X/k}(\log Z)$ given locally by the rule $\xi \mapsto \omega \wedge \xi$. We shall refer to the relation

$$i_{\mathbf{X}} \partial_\omega + \partial_\omega i_{\mathbf{X}} = i_{\mathbf{X}} \omega$$

as the *Koszul homotopy formula.*

3.2.2. *The order of vanishing of a* 1*-form at a closed point*
Let a closed point u of U and a global section ω of $\Omega^1_{U/k}$ be given. We say that ω

has an *isolated zero* at u if there exists a $\mathcal{O}_{U,u}$-basis $\xi_1, \ldots, \xi_{\dim(U)}$ of $(\Omega^1_{U/k})_u$ and a system of parameters $f_1, \ldots, f_{\dim(U)}$ for the local ring $\mathcal{O}_{U,u}$ such that

$$\omega = \sum_i f_i \xi_i.$$

Suppose now that ω has an isolated zero at u. We define the *order of vanishing* $\mathrm{ord}_u \omega$ to be the dimension over k of $\mathcal{O}_{U,u}/(f_1, \ldots, f_{\dim(U)})$. We note that the complex $((\Omega^1_{U/k})_u, \partial_\omega)$ may be identified with the Koszul complex of the regular sequence $f_1, \ldots, f_{\dim(U)}$, whence it follows that

$$\dim_k \mathscr{H}^n(\Omega^\bullet_{U/k}, \partial_\omega)_u = \begin{cases} \mathrm{ord}_i \omega & (n = \dim(U)) \\ 0 & (n \neq \dim(U)) \end{cases}$$

and, in particular,

$$\mathrm{ord}_u \omega = (-1)^{\dim(U)} \sum_n (-1)^n \dim_k \mathscr{H}^n(\Omega^\bullet_{U/k}, \partial_\omega)_u.$$

3.2.3. *Residues, the residue exact sequence and the residual filtration*
Let Z_i be an irreducible component of Z, let $v_i : Z_i \to X$ denote the inclusion and let Z_i' denote the union of the irreducible components of Z distinct from Z_i. There exists a unique $\mathcal{O}_X$-linear homomorphism of degree -1

$$\mathrm{Res}_i : \Omega^\bullet_{X/k}(\log Z) \to \Omega^\bullet_{Z_i/k}(\log Z_i \cap Z_i')$$

called the *residue along* Z_i, such that in any system $z, x_2, \ldots, x_n$ of local coordinates adapted to Z such that z is a local equation for Z_i,

$$\mathrm{Res}_i \left(\frac{dz}{z} \wedge \eta \right) = v_i^* \eta$$

where η is an arbitrary local section of $\Omega^\bullet_{X/k}(\log Z_i')$. Note that the exterior derivative d satisfies

$$\mathrm{Res}_i \, d = -d \, \mathrm{Res}_i.$$

Given a global section ω of $\Omega^1_{X/k}(\log Z_i')$, note that the Koszul differential ∂_ω satisfies

$$\mathrm{Res}_i \, \partial_\omega = -\partial_{v_i^* \omega} \, \mathrm{Res}_i.$$

The induced sequence

$$0 \to \Omega^{\bullet}_{X/k}(\log Z_i') \xrightarrow{\text{inclusion}} \Omega^{\bullet}_{X/k}(\log Z) \xrightarrow{\text{Res}_i} \Omega^{\bullet}_{Z_i/k}(\log Z_i \cap Z_i')[-1] \to 0$$

of graded sheaves is exact, and we refer to it as the *residue exact sequence* associated to the component Z_i of Z. Let $\mathscr{I}_i$ denote the defining ideal of Z_i and note that

$$\Omega^{\bullet}_{X/k}(\log Z_i') \cap \mathscr{I}_i \Omega^{\bullet}_{X/k}(\log Z) = \ker(\nu_i^*: \Omega^{\bullet}_{X/k}(\log Z_i') \to \Omega^{\bullet}_{Z_i/k}(\log Z_i' \cap Z)).$$

Since ν_i^* is surjective, there exists an exact sequence

$$0 \to \Omega^{\bullet}_{Z_i/k}(\log Z \cap Z_i') \to \mathcal{O}_{Z_i} \otimes \Omega^{\bullet}_{X/k}(\log Z) \to \Omega^{\bullet}_{Z_i/k}(\log Z \cap Z_i')[-1] \to 0$$

of graded $\mathcal{O}_{Z_i}$-modules to which we refer as the *residual filtration.*

3.2.4. *The residual retracting homotopy*

Let Z_i, Z_i' and $\mathscr{I}_i$ be as immediately above. The residue map Res_i induces an $\mathcal{O}_{Z_i}$-linear map $\mathcal{O}_{Z_i} \otimes \Omega^1_{X/k}(\log Z) \to \mathcal{O}_{Z_i}$ which corresponds by duality to a global section of $\mathcal{O}_{Z_i} \otimes \mathscr{T}_{X/k}(-\log Z)$ again denoted Res_i. Note that in any local coordinate system $z, x_2, \ldots, x_n$ adapted to Z such that z is the defining equation of Z_i, Res_i is the reduction $\text{mod}\, \mathscr{I}_i$ of $z(\partial/\partial z)$. In general, we say that a local section $\mathbf{X}$ of $\mathscr{T}_{X/k}(-\log Z)$ *represents* Res_i if the reduction of $\mathbf{X}$ modulo $\mathscr{I}_i$ coincides with Res_i. It follows that there exists a unique $\mathcal{O}_{Z_i}$-linear endomorphism T_i of $\mathcal{O}_{Z_i} \otimes \Omega^{\bullet}_{X/k}(\log Z)$ of degree -1 and square 0 which locally is induced by contraction $i_{\mathbf{X}}$ on any local section $\mathbf{X}$ of $\mathscr{T}_{X/k}(-\log Z)$ representing Res_i. We call T_i the *residual retracting homotopy* along Z_i.

3.3. *Review of connections with logarithmic poles*

3.3.1. *Basic definitions*

Let $\mathscr{E}$ be a quasi-coherent sheaf on X. A (*k-linear*) *connection* ∇ for $\mathscr{E}$ with *logarithmic poles* (or *regular singularities*) along Z is a pairing

$$(\mathbf{X}, e) \mapsto \nabla_{\mathbf{X}} e: \mathscr{T}_{X/k}(-\log Z) \times \mathscr{E} \to \mathscr{E}$$

which is k-linear in e and $\mathcal{O}_X$-linear in $\mathbf{X}$ such that

$$\nabla_{\mathbf{X}} fe = (\mathbf{X}f)e + f\nabla_{\mathbf{X}} e.$$

The connection ∇ is said to be *integrable* if

$$\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}e - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}e = \nabla_{[\mathbf{X},\mathbf{Y}]}e.$$

Given another quasi-coherent sheaf $\mathscr{E}'$ on X equipped with a connection ∇ regular singular along Z, the tensor product $\mathscr{E} \otimes \mathscr{E}'$ is equipped with a connection ∇ regular singular along Z by the *Leibniz rule*

$$\nabla_{\mathbf{X}}(e \otimes e') =: \nabla_{\mathbf{X}}e \otimes e' + e \otimes \nabla_{\mathbf{X}}e'.$$

An $\mathcal{O}_X$-linear homomorphism $\mathscr{E} \to \mathscr{E}'$ commuting with ∇ is said to be *horizontal.*

3.3.2. *The twisted de Rham and Koszul complexes*

Given $\mathscr{E}$ equipped with a connection ∇ as above, we define a k-linear map

$$\partial_\nabla : \mathscr{E} \to \Omega^1_{X/k}(\log Z) \otimes \mathscr{E}$$

by the rule

$$(i_{\mathbf{X}} \otimes 1)\partial_\nabla e = \nabla_{\mathbf{X}} e.$$

One then has

$$\partial_\nabla(fe) = df \otimes e + f\partial_\nabla e.$$

There exists a unique extension of ∂_∇ to a k-linear endomorphism of $\Omega^{\cdot}_{X/k}(\log Z) \otimes \mathscr{E}$ of degree 1 such that

$$\partial_\nabla(\xi \otimes e) = d\xi \otimes e + (-1)^{\deg(\xi)}\xi \wedge \partial_\nabla e.$$

The connection ∇ is integrable if and only if $\partial_\nabla^2 = 0$, and in this case we refer to $(\Omega^{\cdot}_{X/k}(\log Z) \otimes \mathscr{E}, \partial_\nabla)$ as the *twisted de Rham complex* associated to $(\mathscr{E}, \nabla)$. In the integrable case one has also the *twisted Cartan homotopy formula*

$$(\partial_\nabla(i_{\mathbf{X}} \otimes 1) + (i_{\mathbf{X}} \otimes 1)\partial_\nabla)(\xi \otimes e) = L_{\mathbf{X}}\xi \otimes e + \xi \otimes \nabla_{\mathbf{X}} e$$

which has a leading role to play in this paper.

Given a global section ω of $\Omega^1_{X/k}(\log Z)$, we define an $\mathcal{O}_X$-linear endomorphism ∂_ω of $\Omega^{\cdot}_{X/k}(\log Z) \otimes \mathscr{E}$ of degree -1 and square 0 by the rule

$$\partial_\omega(\xi \otimes e) =: (\omega \wedge \xi) \otimes e.$$

Then ∂_ω is simply a 'twisted' variant of the Koszul differential defined above. We refer to $(\Omega^{\bullet}_{X/k}(\log Z) \otimes \mathscr{E}, \partial_\omega)$ as the *twisted Koszul complex* associated to $\mathscr{E}$ and ω. From the Koszul homotopy formula noted above, the *twisted Koszul homotopy formula*

$$(\partial_\omega(i_{\mathbf{X}} \otimes 1) + (i_{\mathbf{X}} \otimes 1)\partial_\omega)(\xi \otimes e) = \xi \otimes (i_{\mathbf{X}}\omega)e$$

follows immediately.

3.3.3. *The twisted residue sequence*

Let $\mathscr{E}$ be a locally free coherent sheaf on X and let Y be a smooth divisor of X such that the irreducible components of the divisor $Z \cup Y$ cross normally. The exact sequence

$$0 \to \Omega^{\bullet}_{X/k}(\log Z) \otimes \mathscr{E} \to \Omega^{\bullet}_{X/k}(\log Z \cup Y) \otimes \mathscr{E}$$
$$\to \Omega^{\bullet}_{Y/k}(\log Z \cup Y)[-1] \otimes \mathscr{E} \to 0$$

of graded sheaves deduced from the residue sequence associated to Y by tensoring with $\mathscr{E}$ will be called the *twisted residue sequence* associated to Y and $\mathscr{E}$.

If $\mathscr{E}$ comes equipped with an integrable connection ∇ regular singular along Z, then the twisted residue sequence is compatible with ∇ as follows: The given connection ∇ induces by restriction a pairing

$$\nabla': \mathscr{T}_{X/k}(-\log Z \cup Y) \times \mathscr{E} \to \mathscr{E}$$

which is an integrable connection regular singular along $Z \cup Y$. The connection ∇' in turn induces a pairing

$$\nabla'': \mathscr{T}_{Y/k}(-\log Z \cap Y) \times \mathscr{O}_Y \otimes \mathscr{E} \to \mathscr{O}_Y \otimes \mathscr{E}$$

which is an integrable connection on Y regular singular along $Z \cap Y$. Then the differentials ∂_∇, $\partial_{\nabla'}$ and $-\partial_{\nabla''}$ are compatible with the twisted residue sequence.

Given a global section ω of $\Omega^1_{X/k}(\log Z)$, the twisted residue sequence is compatible with ω in the following sense: Let ω' denote ω viewed as a global section of $\Omega^1_{X/k}(\log Z \cup Y)$, and let ω'' denote the pull-back of ω to Y. Then the differentials ∂_ω, $\partial_{\omega'}$ and $-\partial_{\omega''}$ are compatible with the twisted residue sequence.

3.4. *$\mathscr{O}_{Z_i}$-modules*

Let Z_i be an irreducible component of Z. Let $\mathscr{E}$ be a coherent sheaf equipped with an integrable connection ∇ regular singular along Z and suppose that $\mathscr{E}$ is annihilated by the defining ideal of Z_i. There exists a unique $\mathcal{O}_{Z_i}$-linear endomorphism $\mathbf{a}_i$ of $\mathscr{E}$ which, in any local system of coordinates $z, x_2, \ldots, x_n$ adapted to Z such that z is a defining equation of Z_i, is represented by $\nabla_{z(\partial/\partial z)}$. We refer to $\mathbf{a}_i$ as the *exponent endomorphism* of $\mathscr{E}$ along Z_i. If for another irreducible component Z_j of Z the defining ideal of Z_j annihilates $\mathscr{E}$, then

$$[\mathbf{a}_i, \mathbf{a}_j] = 0,$$

as follows directly from the hypothesis of integrability of the connection ∇. More generally,

$$[\mathbf{a}_i, \nabla] = 0,$$

i.e. $\mathbf{a}_i$ is a horizontal endomorphism of $\mathscr{E}$. From the twisted Cartan homotopy formula, one deduces

PROPOSITION 3.4.1

$$(T_i \otimes 1)\partial_\nabla + \partial_\nabla(T_i \otimes 1) = 1 \otimes \mathbf{a}_i.$$

In particular, the twisted de Rham complex $(\Omega^\bullet_{X/k}(\log Z) \otimes \mathscr{E}, \partial_\nabla)$ *is acyclic provided that the exponent endomorphism* $\mathbf{a}_i$ *is invertible on* Z_i. □

Suppose now that a global section ω of $\Omega^1_{X/k}(\log Z)$ has been given. From the twisted Koszul homotopy formula, one deduces

PROPOSITION 3.4.2

$$(T_i \otimes 1)\partial_\omega + \partial_\omega(T_i \otimes 1) = 1 \otimes (\mathrm{Res}_i\, \omega).$$

In particular, the twisted Koszul complex $(\Omega^\bullet_{X/k}(\log Z) \otimes \mathscr{E}, \partial_\omega)$ *is acyclic provided that* $\mathrm{Res}_i\, \omega$ *is invertible on* Z_i. □

3.5. *Modules with support in* $Z^{(r)}$

Recall that $Z^{(r)}$ is defined to be the union of the r-fold intersections of irreducible components of Z; now fix a positive integer r such that $Z^{(r)}$ is nonempty. Let $\mathscr{E}$ be a coherent sheaf on X equipped with an integrable connection ∇ regular singular along Z such that the support $\mathrm{supp}(\mathscr{E})$ of $\mathscr{E}$ satisfies

$$\mathrm{supp}(\mathscr{E}) \subseteq Z^{(r)}.$$

Fix an irreducible component Y of $Z^{(r)}$ (which is a reduced closed subscheme of codimension r in X) and let $\mathcal{J}$ be the defining ideal of Y. Note that $\mathcal{J}$ is stable under the action of $\mathcal{T}_{X/k}(-\log Z)$.

PROPOSITION 3.5.1. *If* $\mathcal{J}\mathcal{E} = 0$, *then* $\mathcal{E}$, *viewed as an* $\mathcal{O}_Y$-*module, is locally free on* $Y \cap U^{(r)}$. *Moreover, for each irreducible component* Z_i *of* Z *containing* Y, *the coefficients of the characteristic polynomial of the exponent endomorphism* $\mathbf{a}_i$ *of* $\mathcal{E}$ *are locally constant on* $Y \cap U^{(r)}$, *hence belong to* k.

Proof. Cf. [1, Prop. 3.10, p. 79]. As the assertion to be proved is local on $X \backslash Z^{(r+1)}$, we may assume that $Z^{(r+1)}$ is empty. Let $z, x_2, \ldots, x_n$ be a local coordinate system adapted to Z defined on an affine open subscheme V of X such that (*i*) $zx_2 \cdots x_r$ is the defining equation of $V \cap Z$, (ii) z is the defining equation of $V \cap Z_i$ and (iii) $z, x_2, \ldots, x_r$ generate the defining ideal of $V \cap Y$. Since X is covered by such affines V, we may assume that $X = V$. Let $\mathbf{X} \mapsto \sigma\mathbf{X}: \mathcal{T}_{X/k}(-\log Z) \to \mathcal{T}_{Y/k}$ be the evident restriction map. Note that σ is surjective. Let s be the unique $\mathcal{O}_X$-linear endomorphism of $\mathcal{T}_{X/k}(-\log Z)$ such that

$$s\left(z\frac{\partial}{\partial z}\right) = 0,\ s\left(x_2\frac{\partial}{\partial x_2}\right) = 0, \ldots, s\left(x_r\frac{\partial}{\partial x_r}\right) = 0$$

and

$$s\left(\frac{\partial}{\partial x_{r+1}}\right) = \frac{\partial}{\partial x_{r+1}}, \ldots, s\left(\frac{\partial}{\partial x_n}\right) = \frac{\partial}{\partial x_n}.$$

Then there exists a unique nonsingular connection $\tilde{\nabla}: \mathcal{T}_{Y/k} \times \mathcal{E} \to \mathcal{E}$ such that

$$\tilde{\nabla}_{\sigma\mathbf{X}} e = \nabla_{s\mathbf{X}} e.$$

Since $\sigma s = \sigma$ commutes with Lie brackets, $\tilde{\nabla}$ is integrable. The existence of $\tilde{\nabla}$ implies that $\mathcal{E}$ is locally free as an $\mathcal{O}_Y$-module. As $\mathbf{a}_i$ is induced by $\nabla_{z(\partial/\partial z)}$, the former commutes with $\tilde{\nabla}$ by the integrability of ∇. Thus $\mathbf{a}_i$ is a horizontal endomorphism of $\mathcal{E}$ with respect to the integrable connection $\tilde{\nabla}$. In particular, the coefficients of the characteristic polynomial of $\mathbf{a}_i$ must be locally constant on Y. □

PROPOSITION 3.5.2. *If* $\operatorname{supp}(\mathcal{E}) \cap Y \neq Y$, *then* $\operatorname{supp}(\mathcal{E}) \cap Y \subseteq Z^{(r+1)}$.

Proof. For all $n \gg 0$ one has

$$\operatorname{supp}(\mathcal{J}^n\mathcal{E}) \cap Y \subseteq Z^{(r+1)}.$$

Therefore, replacing $\mathcal{E}$ successively by $\mathcal{E}/\mathcal{J}\mathcal{E}, \mathcal{J}\mathcal{E}/\mathcal{J}^2\mathcal{E}, \ldots$, we may assume that $\mathcal{J}\mathcal{E} = 0$. But then the desired conclusion follows immediately from Prop. 3.5.1. □

PROPOSITION 3.5.3. *If* supp($\mathscr{E}$) = Y, *there exists a* ∇-*stable* $\mathcal{O}_X$-*submodule* $\mathscr{E}$ *of* $\mathscr{E}$ *such that* $\mathscr{J}\mathscr{E} \subseteq \mathscr{E}'$, supp($\mathscr{E}/\mathscr{E}'$) = Y *and, for each irreducible component* $Z_i \supseteq Y$ *of* Z, *the exponent endomorphism* $\mathbf{a}_i$ *of* $\mathscr{E}/\mathscr{E}'$ *reduces to multiplication by a constant* $a_i \in k$.

Proof. Replacing $\mathscr{E}$ by $\mathscr{E}/\mathscr{J}\mathscr{E}$ (the latter is nonzero by Nakayama's lemma), we may assume without loss of generality that $\mathscr{J}\mathscr{E} = 0$. Let V denote the stalk of $\mathscr{E}$ and K the stalk of $\mathcal{O}_Y$ at the generic point of Y. Then K is a field and V is a finite dimension K-vectorspace on which the exponent endomorphisms $\mathbf{a}_i$ associated to the irreducible components $Z_i \supseteq Y$ of Z operate in K-linear and mutually commuting fashion. By Prop. 3.5.1 the eigenvalues of $\mathbf{a}_i$ acting on V belong to k, hence there exist $0 \neq v \in V$ and $a_i \in k$ such that

$$\mathbf{a}_i v = a_i v.$$

The subsheaf

$$\mathscr{E}' = \sum_i (\mathbf{a}_i - a_i)\mathscr{E}$$

has the required properties. □

3.6. *Virtuous filtrations*

Let $\mathscr{E}$ be a coherent sheaf on X equipped with an integrable connection ∇ regular singular along Z such that

$$\text{supp}(\mathscr{E}) \subseteq Z.$$

We say that $\mathscr{E}$ is *pure* if there exists a positive integer r such that $Z^{(r)} \neq \emptyset$ and an irreducible component Y of $Z^{(r)}$ such that $\mathscr{E}$ is killed by the defining ideal of Y, supp($\mathscr{E}$) = Y and for each irreducible component $Z_i \supseteq Y$ of Z, the exponential endomorphism $\mathbf{a}_i$ of $\mathscr{E}$ reduces to a constant. We say that a finite chain of ∇-stable subsheaves

$$\mathscr{E} = \mathscr{E}^0 \supseteq \mathscr{E}^1 \supseteq \cdots \supseteq \mathscr{E}^{m-1} \supseteq \mathscr{E}^m = 0$$

is a *virtuous filtration* of $\mathscr{E}$ if each successive quotient $\mathscr{E}^p/\mathscr{E}^{p+1}$ is pure.

PROPOSITION 3.6.1. $\mathscr{E}$ *admits a virtuous filtration.*

Proof. Inductively we define a descending sequence

$$\mathscr{E} = \mathscr{E}^0 \supseteq \mathscr{E}^1 \supseteq \cdots$$

of ∇-stable subsheaves as follows: If $\mathscr{E}^p = 0$, set $\mathscr{E}^{p+1} =: 0$. Otherwise, let r be the largest positive integer such that $\mathrm{supp}(\mathscr{E}^p) \subseteq Z^{(r)}$. Select an irreducible component Y of $Z^{(r)}$ such that $\mathrm{supp}(\mathscr{E}^p) \cap Y \nsubseteq Z^{(r+1)}$. Then by Prop. 3.5.2, $\mathrm{supp}(\mathscr{E}^p) \cap Y = Y$. By Prop. 3.5.3 we can find a ∇-stable subsheaf $\mathscr{E}^{p+1}$ of $\mathscr{E}^p$ such that $\mathscr{E}^p/\mathscr{E}^{p+1}$ is pure, with $\mathrm{supp}(\mathscr{E}^p/\mathscr{E}^{p+1}) = Y$. Our task is to prove that $\mathscr{E}^p = 0$ for $p \gg 0$. It will suffice to show that $\mathrm{supp}(\mathscr{E}^p/\mathscr{E}^{p+1}) = \emptyset$ for all $p \gg 0$. In turn, it will suffice to show that for each r such that $Z^{(r)}$ is nonempty and each irreducible component Y of $Z^{(r)}$, $\mathrm{supp}(\mathscr{E}^p/\mathscr{E}^{p+1}) = Y$ for only finitely many p. Finally, we may assume without loss of generality that $Y = \mathrm{supp}(\mathscr{E}^0/\mathscr{E}^1)$. Let η be the generic point of Y. Then the length of $\mathscr{E}_\eta$ as an $\mathscr{O}_{X,\eta}$-module is finite and this length bounds the number of indices p for which $Y = \mathrm{supp}(\mathscr{E}^p/\mathscr{E}^{p+1})$. □

3.7. *R*[∇]*-modules*

For the construction of invariants in the next subsection, we consider the structure of modules over a certain noncommutative ring $R[\nabla]$ defined as follows. Let R be a discrete valuation ring of residue characteristic 0, let π be a uniformizer of R and let $r \mapsto r'$ be a derivation of R such that $R' \subseteq \pi R$ and $\pi' = \pi$. Let $R[\nabla]$ be the noncommutative ring generated by a copy of R and a variable ∇ subject to the commutation relations

$$[\nabla, r] = r' \quad (r \in R).$$

Given a (left) $R[\nabla]$-module E annihilated by π, note that ∇ operates R/π-linearly on E and let $P(E; t) \in (R/\pi)[t]$ denote the characteristic polynomial of the (R/π)-linear endomorphism of E induced by ∇. Note that the isomorphism classes of simple left $R[\nabla]$-modules are in bijective correspondence with the irreducible monic polynomials $P(t) \in (R/\pi)[t]$ under the correspondence that sends $P(t) = t^n + \Sigma_{i=0}^{n-1} a_i t^i$ to the isomorphism class of $R[\nabla]/(R[\nabla]\pi + R[\nabla](\nabla^n + \Sigma_{i=0}^{n-1} \tilde{a}_i \nabla^i))$, where $\hat{a} \in R$ denotes a lifting of $a \in R/\pi$. The Jordan-Hölder theorem specializes to yield

PROPOSITION 3.7.1. *Let E be a left* $R[\nabla]$*-module of finite length as an R-module. Let*

$$E = E^0 \supseteq \cdots \supseteq E^p \supseteq E^{p+1} \supseteq \cdots \supseteq E^n = 0$$

be a filtration of E by $R[\nabla]$*-submodules such that* $\pi E^p \subseteq E^{p+1}$*, e.g. a composition series of E. Then* $\Pi_p P(E^p/E^{p+1}; t)$ *depends only on the isomorphism class of the* $R[\nabla]$*-module E.* □

Now let E be a left $R[\nabla]$-module free and finitely generated as an R-module and let E' be an $R[\nabla]$-submodule such that $\pi E \subseteq E'$.

PROPOSITION 3.7.2

$$P(E/\pi E;\, t)P(E/E';\, t+1) = P(E'/\pi E',\, t)P(E/E';\, t).$$

Proof. Since one has

$$P(E/E';\, t+1) = P(\pi E/\pi E';\, t),$$

the proposition follows from the existence of the 4-term exact sequence

$$0 \to \frac{\pi E}{\pi E'} \to \frac{E'}{\pi E'} \to \frac{E}{\pi E} \to \frac{E}{E'} \to 0. \qquad \square$$

3.8. *Characteristic polynomials*

Let Z_i be a irreducible component of Z and let $\mathscr{E}$ be a coherent sheaf on X equipped with an integrable connection ∇ regular singular along Z such that

$$\mathrm{supp}(\mathscr{E}) \subseteq Z.$$

We define the *characteristic polynomial* $P_i(\mathscr{E}; t) \in k[t]$ of $\mathscr{E}$ along Z_i as follows: Let

$$\mathscr{E} = \mathscr{E}^0 \supseteq \cdots \supseteq \mathscr{E}^p \supseteq \cdots \supseteq \mathscr{E}^m = 0$$

be any virtuous filtration of $\mathscr{E}$; such exists by Prop. 3.6.1. Let ζ_i denote the generic point of Z_i. Let the *characteristic polynomial* $P_i(\mathscr{E}; t)$ of $\mathscr{E}$ along Z_i be the product of the factors $(t - a_{ip})^{\lambda_{ip}}$, where p ranges over those indices such that $\mathrm{supp}(\mathscr{E}^p/\mathscr{P}^{p+1}) = Z_i$, $a_{ip} \in k$ is the constant by which the exponent endomorphism $\mathbf{a}_i$ operates on $\mathscr{E}^p/\mathscr{E}^{p+1}$, and λ_{ip} is the length of $(\mathscr{E}^p/\mathscr{E}^{p+1})_{\zeta_i}$ as a module over $\mathcal{O}_{X,\zeta_i}$. Then $P_i(\mathscr{E}; t)$ is, so we claim, well-defined. We use the observations of section 3.7 in order to prove the claim: Take $R =: \mathcal{O}_{X,\zeta_i}$, a discrete valuation ring. Selecting a local coordinate system $z, x_2, \ldots, x_n$ adapted to Z defined in a neighborhood of ζ_i such that z is the local equation of Z_i, take $\pi =: z$ and $r' =: z(\partial/\partial z)r$. Take $E^p =: \mathscr{E}^p_{\zeta_i}$ and equip E^p with $R[\nabla]$-module structure by decreeing that $\nabla e =: \nabla_{z(\partial/\partial z)} e$. Then in the notation of section 3.7

$$P_i(\mathscr{E};\, t) = \prod_p P(E^p/E^{p+1};\, t).$$

As the latter depends only on the isomorphism class of E as an $R[\nabla]$-module by Prop. 3.7.1, $P_i(\mathscr{E}; t)$ is indeed well-defined. It follows immediately from the definition that for any ∇-stable coherent subsheaf $\mathscr{E}'$ of $\mathscr{E}$,

$$P_i(\mathscr{E}; t) = P_i(\mathscr{E}'; t)P_i(\mathscr{E}/\mathscr{E}'; t).$$

3.9. *Exponents*

Let $\mathscr{E}$ now be a locally free coherent sheaf on X equipped with an integrable connection ∇ regular singular along Z. The *exponents* of $\mathscr{E}$ along Z_i are defined to be the roots in k of the characteristic polynomial $P_i(\mathcal{O}_{Z_i} \otimes \mathscr{E}; t)$. Note that $P_i(\mathcal{O}_{Z_i} \otimes \mathscr{E}; t)$ may also be described as the characteristic polynomial of the exponential endomorphism $\mathbf{a}_i$ operating on $\mathcal{O}_{Z_i} \otimes \mathscr{E}$. By Prop. 3.7.2 and an evident induction one deduces

PROPOSITION 3.9.1. *For any locally free ∇-stable coherent subsheaf $\mathscr{E}'$ of $\mathscr{E}$ such that $\mathscr{E}/\mathscr{E}'$ is supported in Z, the exponents $\{a_{ij}\}$ of $\mathscr{E}$ and $\{a'_{ij}\}$ of $\mathscr{E}'$ along Z_i can be simultaneously indexed so that $\ell_{ij} =: a'_{ij} - a_{ij}$ is a nonnegative integer for all indices i and j. Moreover, having so indexed the exponents, one has*

$$P_i(\mathscr{E}/\mathscr{E}', t) = \prod_j \prod_{\lambda=0}^{\ell_{ij}-1} (t - a_{ij} - \lambda). \qquad \square$$

3.10. *Twisting*

Let D be a Weil divisor of X supported in Z, i.e. a formal integral linear combination $D = \Sigma_i m_i Z_i$ of the irreducible components of Z. We say that D is *effective* if $m_i \geqslant 0$ for all indices i and, given another such divisor D', we write $D \geqslant D'$ if $D - D'$ is effective. The invertible sheaf $\mathcal{O}_X(D)$ is equipped with an integrable connection ∇ regular singular along Z uniquely determined by the condition that the restriction of ∇ to $\mathcal{O}_X(D)|_U = \mathcal{O}_U$ coincides with the standard connection. Note that the exponent of $\mathcal{O}_X(D)$ along Z_i is $-m_i$.

Let $\mathscr{E}$ a coherent sheaf on X equipped with an integrable connection ∇ regular singular along Z. We define

$$\mathscr{E}(D) =: \mathscr{E} \otimes \mathcal{O}_X(D),$$

equipping $\mathscr{E}(D)$ with an integrable connection by the Leibniz rule and calling $\mathscr{E}(D)$ the *twist* of $\mathscr{E}$ by D. For convenient reference we record some easily proved facts concerning the twisting operation.

PROPOSITION 3.10.1. *If $\mathscr{E}$ is locally free, and the exponents of $\mathscr{E}$ along Z_i of Z are $\{a_{ij}\}$, then the exponents of $\mathscr{E}(D)$ along Z_i are $\{a_{ij} - m_i\}$.* $\square$

PROPOSITION 3.10.2. *If $\mathscr{E}$ is pure and annihilated by the defining ideal of Z_i so that the exponent endomorphism $\mathbf{a}_i$ is defined and operates by the constant a_i, then $\mathscr{E}(D)$ is again pure and $\mathbf{a}_i$ operates on $\mathscr{E}(D)$ by rthe constant $a_i - m_i$.* $\square$

4. Asymptotics

4.1. *Notation and setting*

4.1.1. Fix k, X, Z and U as in section 3, but now assume that X is projective and irreducible and that U is affine. Fix a nonzero global section ω of $\Omega^1_{X/k}(\log Z)$. Assume that for all irreducible components Z_i of Z, the residue $\mathrm{Res}_i\,\omega$ along Z_i (which is a scalar since Z_i is complete) is nonzero. Note that under these hypotheses the zero locus of ω on U is 0-dimensional. Let Y be a hyperplane section of X, hence an ample divisor, in sufficiently general position to be smooth, to cross Z normally and to avoid the zeroes of ω.

4.1.2. Given a property $\mathscr{P}$ of pairs (D, n) consisting of a Weil divisor D of X supported in Z and an integer n, we say that $\mathscr{P}(D, n)$ holds *asymptotically* if there exists a Weil divisor $D_0 = D_0(\mathscr{P})$ of X supported in Z such that for all $D \geqslant D_0$, there exists an integer $n_0 = n_0(\mathscr{P}, D)$ such that for all $n \geqslant n_0$, $\mathscr{P}(D, n)$ holds.

4.1.3. Given a coherent sheaf $\mathscr{E}$ on X equipped with an integrable connection ∇ regular singular along $Z \cup Y$, a Weil divisor D of X supported in Z and an integer n, set

$$\mathscr{G}_{D,n}(\mathscr{E}) =: \Omega^{\bullet}_{X/k}(\log Z \cup Y) \otimes \mathscr{E}(D + nY).$$

Note that $\mathscr{G}_{D,n}$ is an exact functor. Note that the graded sheaf $\mathscr{G}_{D,n}(\mathscr{E})$ is equipped *two* differentials functorially in $(\mathscr{E}, \nabla)$, namely the Koszul differential ∂_ω and the de Rham differential ∂_∇. Note that the functors $\mathscr{G}_{D,n}$ are *nested* in the sense that given $D' \geqslant D$ and $n' \geqslant n$, there is a natural ∂_∇- and ∂_ω-compatible transformation $\mathscr{G}_{D,n} \to \mathscr{G}_{D',n'}$ induced by the inclusion $\mathscr{O}_X(D + nY) \to \mathscr{O}_X(D' + n'Y)$. Set

$$G_{D,n}(\mathscr{E}) =: \Gamma(X, \mathscr{G}_{D,n}(\mathscr{E})).$$

The finite-dimensional graded k-vectorspaces $G_{D,n}(\mathscr{E})$ are likewise equipped with two differentials ∂_∇ and ∂_ω functorially in $(\mathscr{E}, \nabla)$ and are nested ∂_∇- and ∂_ω-compatibly.

4.1.4. Given a locally free coherent sheaf $\mathscr{E}$ on X and a smooth locally closed subscheme W of U set

$$H^*_\omega(W, \mathscr{E}) =: \mathbf{H}^*(W, (\Omega^{\bullet}_{W/k} \otimes \mu^*\mathscr{E}, \partial_{\mu^*\omega})),$$

where $\mathbf{H}^*$ denotes hypercohomology and μ denotes the inclusion $W \to X$. If, moreover, $\mathscr{E}$ is equipped with an integrable connection ∇ regular singular along Z, set

$$H^*_{DR}(W, \mathscr{E}) =: \mathbf{H}^*(W, (\Omega^{\bullet}_{W/k} \otimes \mu^*\mathscr{E}, \partial_{\mu^*\nabla})).$$

4.2. *The functors P_n and Q*

Let $\mathscr{E}$ be a locally free coherent sheaf on X equipped with an integrable connection ∇ regular singular along Z, let D be a Weil divisor of X supported in Z and let n be an integer. Let D_0 be a Weil divisor of X supported in Z such that all exponents of $\mathscr{E}(D_0)$ along the various irreducible components of Z are not nonnegative integers (the existence of such a divisor D_0 follows from Prop. 3.10.1).

PROPOSITION 4.2.1. *For all irreducible components Z_i of Z, the inclusion*

$$\mathscr{G}_{D-Z_i,n}(\mathscr{E}) \to \mathscr{G}_{D,n}(\mathscr{E})$$

is a ∂_ω-quasi-isomorphism and, for all $D \geqslant D_0$, a ∂_∇-quasi-isomorphism as well. For all $n > 0$ the inclusion

$$\mathscr{G}_{D,n-1}(\mathscr{E}) \to \mathscr{G}_{D,n}(\mathscr{E})$$

is a ∂_∇-quasi-isomorphism.

Proof. The graded sheaf

$$\mathscr{G}_{D,n}(\mathscr{E})/\mathscr{G}_{D-Z_i,n}(\mathscr{E}) = \mathscr{G}_{D,n}(\mathscr{E} \otimes \mathscr{O}_{Z_i})$$

is ∂_ω-acyclic by Prop. 3.4.2 and, for $D \geqslant D_0$, ∂_∇-acyclic by Prop. 3.4.1. The graded sheaf

$$\mathscr{G}_{D,n}(\mathscr{E})/\mathscr{G}_{D,n-1}(\mathscr{E}) = \mathscr{G}_{D,n}(\mathscr{E} \otimes \mathscr{O}_Y)$$

is ∂_∇-acyclic for $n > 0$ by Prop. 3.4.1. □

Set

$$P_n(\mathscr{E}) =: H^*\left(\bigcup_D G_{D,n}(\mathscr{E}), \partial_\omega\right), \; Q(\mathscr{E}) =: H^*\left(\bigcup_D \bigcup_n G_{D,n}(\mathscr{E}), \partial_\nabla\right).$$

Note that $P_n(\mathscr{E})$ and $Q(\mathscr{E})$ depend only on $(\mathscr{E}, \nabla)|_U$. More precisely, any horizontal isomorphism

$$B: (\mathscr{E}', \nabla')|_U \xrightarrow{\sim} (\mathscr{E}, \nabla)|_U$$

induces isomorphisms

$$P_n(B)\colon P_n(\mathscr{E}') \xrightarrow{\sim} P_n(\mathscr{E}), \quad Q(B)\colon Q(\mathscr{E}') \xrightarrow{\sim} Q(\mathscr{E}).$$

By definition we have at our disposal natural maps

$$p_{D,n}(\mathscr{E})\colon H^*(G_{D,n}(\mathscr{E}), \partial_\omega) \to P_n(\mathscr{E}),$$
$$q_{D,n}(\mathscr{E})\colon H^*(G_{D,n}(\mathscr{E}), \partial_\nabla) \to Q(\mathscr{E}).$$

PROPOSITION 4.2.2. *$p_{D,n}(\mathscr{E})$ and $q_{D,n}(\mathscr{E})$ are isomorphisms asymptotically in D and n.*

Proof. We may identify $P_n(\mathscr{E})$ with the direct limit over D of the hypercohomology groups $\mathbf{H}^*(X, (\mathscr{G}_{D,n}(\mathscr{E}), \partial_\omega))$ and we may identify $Q(\mathscr{E})$ with the direct limit over D and n of the hypercohomology groups $\mathbf{H}^*(X, (\mathscr{G}_{D,n}(\mathscr{E}), \partial_\nabla))$. Then, by the preceding proposition,

$$P_n(\mathscr{E}) = \mathbf{H}^*(X, (\mathscr{G}_{D,n}(\mathscr{E}), \partial_\omega))$$

for all D and n, and there exists a Weil divisor D_0 of X supported in Z such that

$$Q(\mathscr{E}) = \mathbf{H}^*(X, (\mathscr{G}_{D,n}(\mathscr{E}), \partial_\nabla))$$

for all $D \geqslant D_0$ and $n > 0$. But for any fixed D and all sufficiently large n, by virtue of the ampleness of the divisor Y, the sheaves $\mathscr{G}_{D,n}(\mathscr{E})$ have vanishing positive-dimensional coherent sheaf cohomology and hence

$$\mathbf{H}^*(X, (\mathscr{G}_{D,n}(\mathscr{E}), \partial)) = H^*(G_{D,n}(\mathscr{E}), \partial)$$

for either $\partial = \partial_\omega$ or $\partial = \partial_\nabla$. □

By definition

$$P_n(\mathscr{E}) = H^*(\Gamma(U, \mathscr{G}_{0,n}(\mathscr{E})), \partial_\omega).$$

By considering the twisted residue exact sequence (§3.3.3)

$$0 \to \Omega^{\cdot}_{X/k}(\log Z) \otimes \mathscr{E}(D + nY) \to \mathscr{G}_{D,n}(\mathscr{E})$$
$$\to \Omega^{\cdot}_{Y/k}(\log Z \cup Y)[-1] \otimes \mathscr{E}(D + nY) \to 0$$

one deduces the existence of a natural long exact sequence

$$\cdots \to H^i_\omega(U, \mathscr{E}(nY)) \to P^i_n(\mathscr{E}) \to H^{i-1}_\omega(U \cap Y, \mathscr{E}(nY)) \to \cdots$$

of cohomology groups. By Prop. 4.2.1 and the definitions one has

$$Q(\mathscr{E}) = H^*(\Gamma(U, \mathscr{G}_{0,0}(\mathscr{E})), \partial_\nabla).$$

Consideration of the twisted residue exact sequence with $n = 0$ yields the existence of a natural long exact sequence

$$\cdots \to H^i_{DR}(U, \mathscr{E}) \to Q^i(\mathscr{E}) \to H^{i-1}_{DR}(U \cap Y, \mathscr{E}) \to \cdots$$

of cohomology groups.

4.3. The asymptotic symbol $\varepsilon_{D,n}$

Let $\mathscr{E}$ be a locally free coherent sheaf on X equipped with an integrable connection ∇ regular singular along Z, n an integer and D a Weil divisor of X supported in Z. For all D and n such that $p_{D,n}(\mathscr{E})$ and $q_{D,n}(\mathscr{E})$ are isomorphisms, we define the *asymptotic symbol*

$$\varepsilon_{D,n}(\mathscr{E}) \colon \det(P_n(\mathscr{E})) \xrightarrow{\sim} \det(Q(\mathscr{E}))$$

to be the unique isomorphism rendering the pentagonal diagram

$$\begin{array}{ccc}
\det(H^*(G_{D,n}(\mathscr{E}), \partial_\omega)) & \xrightarrow{\det(p_{D,n}(\mathscr{E}))} & \det(P_n(\mathscr{E})) \\
{\scriptstyle h(G_{D,n}(\mathscr{E}),\, \partial_\omega)} \uparrow & & \\
\det(G_{D,n}(\mathscr{E})) & & \downarrow {\scriptstyle \varepsilon_{D,n}(\mathscr{E})} \\
{\scriptstyle h(G_{D,n}(\mathscr{E}),\, \partial_\nabla)} \downarrow & & \\
\det(H^*(G_{D,n}(\mathscr{E}), \partial_\nabla) & \xrightarrow[\det(q_{D,n}(\mathscr{E}))]{} & \det(Q(\mathscr{E}))
\end{array}$$

commutative, where h is as defined in section 2.4.5.

Now suppose that we are given another locally free coherent sheaf $\mathscr{E}'$ on X equipped with integrable connection ∇' regular singular along Z toether with a horizontal isomorphism

$$B \colon (\mathscr{E}', \nabla')|_U \xrightarrow{\sim} (\mathscr{E}, \nabla)|_U.$$

Then for each D and n such that $\varepsilon_{D,n}(\mathscr{E})$ and $\varepsilon_{D,n}(\mathscr{E}')$ are defined, the diagram

$$\begin{array}{ccc}
\det(P_n(\mathscr{E}')) & \xrightarrow{\det(P_n(B))} & \det(P_n(\mathscr{E})) \\
{\scriptstyle \varepsilon_{D,n}(\mathscr{E}')} \downarrow & & \downarrow {\scriptstyle \varepsilon_{D,n}(\mathscr{E})} \\
\det(Q(\mathscr{E}')) & \xrightarrow[\det(Q(B))]{} & \det(Q(\mathscr{E}))
\end{array}$$

in general *fails to commute*. The failure of commutativity has the following description. For each irreducible component Z_i of Z set

$m_i =$: the multiplicity with which Z_i occurs in D,

$\chi_i =$: the Euler characteristic of $Z_i \backslash Z_i'$,

$\chi_i(Y) =$: the Euler characteristic of $Y \cap (Z_i \backslash Z_i')$,

where Z_i' denotes the union of the irreducible components of Z distinct from Z_i, and let $\{a_{ij}\}$ (resp. $\{a_{ij}'\}$) be the collection of exponents of $\mathscr{E}$ (resp. $\mathscr{E}'$) along Z_i indexed so that $a_{ij}' - a_{ij} \in \mathbf{Z}$, as is possible by Prop. 3.9.1. The main result of this paper is

THEOREM 4.3.1

$$\frac{\det(Q(B)) \circ \varepsilon_{D,n}(\mathscr{E}')}{\varepsilon_{D,n}(\mathscr{E}) \circ \det(P_n(B))} = \prod_i \left(\prod_j (\mathrm{Res}_i(\omega))^{a_{ij}' - a_{ij}} \frac{\Gamma(a_{ij} - m_i)}{\Gamma(a_{ij}' - m_i)} \right)^{\chi_i - \chi_i(Y)}$$

holds asymptotically in D and n.

In order to make sense of the right-hand side of the asserted formula, the expression

$$\frac{\Gamma(s + \ell)}{\Gamma(s)} \quad (s \in k,\ \ell \in \mathbf{Z})$$

is to be construed as a shorthand for

$$\left.\begin{matrix} s(s+1)\cdots(s+\ell-1) & \text{if } \ell \geqslant 0 \\ (s-1)^{-1}\cdots(s-|\ell|)^{-1} & \text{if } \ell < 0 \end{matrix}\right\}.$$

4.4. *The asymptotic* **t**-*ratio* $\rho_{D,n}$

Let $\mathscr{E}$ be a coherent sheaf on X supported in Z equipped with an integrable connection ∇ regular singular along Z, let D be a Weil divisor of X supported in Z and let n be an integer.

PROPOSITION 4.4.1. *$\mathscr{G}_{D,n}(\mathscr{E})$ is ∂_ω-acyclic. There exists a Weil divisor D_0 such that $\mathscr{G}_{D,n}(\mathscr{E})$ is ∂_∇-acyclic if $D \geqslant D_0$.*

Proof. By the existence of virtuous filtrations (Prop. 3.6.1) there is no loss of generality in assuming that $\mathscr{E}$ is pure and annihilated by the defining ideal of some irreducible component Z_i of Z. The asserted ∂_ω-acyclicity follows from Prop. 3.4.2 together with the hypothesis that $\mathrm{Res}_i\,\omega$ is nonvanishing on Z_i. The assorted ∂_∇-acyclicity follows from Prop. 3.4.1 together with Prop. 3.10.2. □

Since Y is ample, one has

$$H^*(G_{D,n}(\mathscr{E}), \partial) = \mathbf{H}^*(X, (\mathscr{G}_{D,n}(\mathscr{E}), \partial)) = 0$$

asymptotically in D and n for either $\partial = \partial_\omega$ or $\partial = \partial_\nabla$. Asymptotically in D and n we define

$$\rho_{D,n}(\mathscr{E}) =: \frac{\mathbf{t}(G_{D,n}(\mathscr{E}), \partial_\nabla)}{\mathbf{t}(G_{D,n}(\mathscr{E}), \partial_\omega)} \in k^\times,$$

where $\mathbf{t}$ is the canonical trivialization of a acyclic complex of finite-dimensional vectorspaces defined in section 2.3.1.

PROPOSITION 4.4.2. *For each ∇-stable coherent subsheaf $\mathscr{E}'$ of $\mathscr{E}$, the formula*

$$\rho_{D,n}(\mathscr{E}) = \rho_{D,n}(\mathscr{E}')\rho_{D,n}(\mathscr{E}/\mathscr{E}')$$

holds asymptotically in D and n.

Proof. Since Y is ample and the functor $\mathscr{G}_{D,n}$ is exact, the sequence

$$0 \to G_{D,n}(\mathscr{E}') \to G_{D,n}(\mathscr{E}) \to G_{D,n}(\mathscr{E}/\mathscr{E}') \to 0$$

is exact asymptotically in D and n. The asserted formula follows by the multiplicativity (§2.3.2) of $\mathbf{t}$. □

PROPOSITION 4.4.3. *If $\mathscr{E}$ is pure and supported in an irreducible component Z_i of Z, then the formula*

$$\rho_{D,n}(\mathscr{E}) = \left(\frac{a_i - m_i}{\mathrm{Res}_i\,\omega}\right)^{r'(G_{D,n}(\mathscr{E}))}$$

holds asymptotically in D and n, where a_i is the scalar by which the exponential endomorphism $\mathbf{a}_i$ operates on $\mathscr{E}$, m_i is the multiplicity with which Z_i appears in D and r' is as defined in section 2.3.3.

Proof. Let T_i be the residual contracting homotopy (§3.2.4). Then the induced codifferential $T_i \otimes 1$ of $\mathscr{G}_{D,n}(\mathscr{E})$ satisfies

$$(T_i \otimes 1)\partial_\nabla + \partial_\nabla(T_i \otimes 1) = a_i - m_i,$$

$$(T_i \otimes 1)\partial_\omega + \partial_\omega(T_i \otimes 1) = \mathrm{Res}_i\,\omega$$

by Prop. 3.4.1 and Prop. 3.10.2 for the first relation and by Prop. 3.4.2 for the second. The graded k-vectorspace $G_{D,n}(\mathscr{E})$ inherits analogous structure. The asserted formula is then an instance of the $\mathbf{t}$-ratio formula (§2.3.3). □

4.5. *Computation of Euler characteristics*

Let $\mathscr{F}$ be a coherent sheaf on X and let $[\mathscr{F}]$ denote the class of $\mathscr{F}$ in the Grothendieck group of (the category of coherent sheaves on) X. Let $\chi(X, ?)$ denote the unique homomorphism from the Grothendieck group of X to the integers such that for all coherent sheaves $\mathscr{F}$,

$$\chi(X, [\mathscr{F}]) =: \sum_n (-1)^n \dim_k H^n(X, \mathscr{F}).$$

Given a graded coherent sheaf $\mathscr{E}$, set

$$[\mathscr{E}] =: \sum_n (-1)^n [\mathscr{E}^n].$$

Note that the Euler characteristic $\chi(X, [\Omega^{\cdot}_{X/k}(\log Z)])$ coincides with the Euler characteristic of $X\backslash Z$ as defined in any reasonable cohomology theory, e.g. when $k = \mathbf{C}$, $\chi(X, [\Omega^{\cdot}_{X/k}(\log Z)]$ coincides with the Euler characteristic defined in terms of singular cohomology with $\mathbf{Q}$-coefficients. Let ξ denote the generic point of X and $k(\xi)$ the function field of X.

PROPOSITION 4.5.1

$$\chi(X, [\Omega^{\cdot}_{X/k}(\log Z) \otimes \mathscr{F}]) = \chi(X, [\Omega^{\cdot}_{X/k}(\log Z)]) \dim_{k(\xi)} \mathscr{F}_{\xi}.$$

Proof. We begin with some reductions of the proof: (*i*) Since $\mathscr{F}$ admits a finite resolution by locally free coherent sheaves, there is no loss of generality in assuming that $\mathscr{F}$ is locally free. (ii) Let Z_i be an irreducible component of Z and let Z_i' denote the union of the irreducible components of Z distinct from Z. We have at our disposal a relation

$$[\Omega^p_{X/k}(\log Z) \otimes \mathscr{F}] = [\Omega^p_{X/k}(\log Z_i') \otimes \mathscr{F}] + [\Omega^{p-1}_{Z_i/k}(\log Z \cap Z_i') \otimes \mathscr{F}]$$

in the Grothendieck group of X by virtue of the existence of the residue exact sequence (§3.2.3). By induction on the dimension of X and a subsidiary induction on the number of irreducible components of Z, we may assume that $Z = \varnothing$. (iii) We may assume $k = \mathbf{C}$ by the Lefschetz principle.

The reductions having been made, we apply the Hirzebruch-Riemann-Roch theorem [3, Thm. 21.1.1]. Write the total Chern classes of the tangent bundle of X and of $\mathscr{F}$ in formally factored forms $\prod_i(1 + \gamma_i)$ and $\prod_j(1 + \delta_j)$, respectively, where i runs from 1 up to the dimension of X and j runs from 1 up to the rank r of $\mathscr{F}$. Then the Chern character of $[\Omega^{\cdot}_{X/\mathbf{C}} \otimes \mathscr{F}]$ is given in formally factored form by

$$\left(\prod_i (1 - e^{-\gamma_i})\right) \sum_j e^{\delta_j},$$

while the Todd class of X is given by

$$\sum_i \frac{\gamma_i}{1 - e^{-\gamma_i}}.$$

The Hirzebruch-Riemann-Roch formula gives

$$\chi(X, [\Omega^{\cdot}_{X/\mathbf{C}} \otimes \mathscr{F}]) = r \int_X \prod_i \gamma_i.$$

Hence $\chi(X, [\Omega^{\cdot}_{X/\mathbf{C}} \otimes \mathscr{F}])$ is proportional to the rank of $\mathscr{F}$. Taking $\mathscr{F} = \mathscr{O}_X$, we see that the constant of proportionality must be $\chi(X, [\Omega^{\cdot}_{X/\mathbf{C}}])$. □

4.6. *A formula for* $\rho_{D,n}$

For each irreducible component Z_i of Z let Z_i' denote the union of the irreducible components of Z distinct from Z_i and set

m_i =: the multiplicity with which Z_i occurs in D,

χ_i =: the Euler characteristic of $Z_i \backslash Z_i'$,

$\chi_i(Y)$ =: the Euler characteristic of $Y \cap (Z_i \backslash Z_i')$.

The key technical result of the paper is

PROPOSITION 4.6.1. *For all coherent sheaves $\mathscr{E}$ on X supported in Z equipped with an integrable connection ∇ regular singular along Z, the relation*

$$\rho_{D,n}(\mathscr{E}) = \prod_i \left(\frac{(-\operatorname{Res}_i \omega)^{\delta_i}}{P_i(\mathscr{E}; m_i)} \right)^{\chi_i - \chi_i(Y)}$$

holds asymptotically in D and *n, where for each irreducible component Z_i of Z, δ_i is the degree of the characteristic polynomial $P_i(\mathscr{E}; t)$ of $\mathscr{E}$* along *Z_i.*

Proof. By the existence of virtuous filtrations (Prop. 3.6.1) and the compatibility of $\rho_{D,n}$ with exact sequences (Prop. 4.4.2) on the one hand, and the compatibility of $P_i(\mathscr{E}, t)$ with exact sequences (§3.8) on the other, we may assume that $\mathscr{E}$ is pure and annihilated by the defining ideal of an irreducible component Z_i of Z. In view of Prop. 4.4.3, we only have to prove that

$$r'(G_{D,n}(\mathscr{E})) = -(\chi_i - \chi_i(Y)) r_i(\mathscr{E})$$

holds asymptotically in D and n, where $r_i(\mathscr{E})$ is the dimension over the function field of Z_i of the stalk of $\mathscr{E}$ at the generic point of Z_i. By the ampleness of Y,

$$r'(G_{D,n}(\mathscr{E})) = \sum_p p(-1)^p \chi(X, [\mathscr{G}^p_{D,n}(\mathscr{E})])$$

asymptotically in D and n. Denoting the union of the irreducible components of Z distinct from Z_i by Z'_i, we have a relation

$$[\mathscr{G}^p_{D,n}(\mathscr{E})] = \sum_{j=p,p-1} [\Omega^j_{Z_i/k}(\log((Z'_i \cup Y) \cap Z_i)) \otimes \mathscr{E}(D + nY)]$$

in the Grothendieck group of X by virtue of the existence of the residual filtration (§3.2.3). Therefore

$$r'(G_{D,n}(\mathscr{E})) = -\chi(Z_i, [\Omega^{\cdot}_{Z_i/k}(\log((Z'_i \cup Y) \cap Z_i)) \otimes \mathscr{E}(D + nY)])$$

asymptotically in D and n. By Prop. 4.5.1, the right-hand side is of the desired form. □

4.7. *Proof of the theorem*

Now B induces a horizontal homomorphism $\mathscr{E}' \to \iota_*\iota^*\mathscr{E}$ factoring through $\mathscr{E}(D)$ for a suitable effective Weil divisor D of X supported in Z, where $\iota: U \to X$ denotes the inclusion. If the theorem holds for $\mathscr{E}'$, $\mathscr{E}(D)$ and the B-induced horizontal isomorphism $\iota^*\mathscr{E}' \xrightarrow{\sim} \iota^*\mathscr{E}(D)$ and also for $\mathscr{E}$, $\mathscr{E}(D)$ and the inclusion-induced horizontal isomorphism $\iota^*\mathscr{E} \to \iota^*\mathscr{E}(D)$ then it must hold for B itself. Thus we may assume without loss of generality that B is the restriction of a horizontal monomorphism $\mathscr{E}' \to \mathscr{E}$ which we again denote by B. After a little diagram-chasing, one finds that

$$\frac{\det(Q(B)) \circ \varepsilon_{D,n}(\mathscr{E}')}{\varepsilon_{D,n}(\mathscr{E}) \circ \det(P_n(B))} = \frac{\det(G_{D,n}(B), \partial_\nabla)}{\det(G_{D,n}(B), \partial_\omega)}$$

asymptotically in D and n. Since Y is ample, the sequence

$$0 \to G_{D,n}(\mathscr{E}') \xrightarrow{G_{D,n}(B)} G_{D,n}(\mathscr{E}) \to G_{D,n}(\operatorname{coker}(B)) \to 0$$

is exact asymptotically in D and n. It follows (§2.4.2) that

$$\rho_{D,n}(\operatorname{coker}(B)) = \frac{\det((G_{D,n}(B), \partial_\nabla)}{\det((G_{D,n}(B), \partial_\omega)}$$

asymptotically in D and n. By Prop. 3.9.1, for each irreducible component Z_i of

Z, the exponents of $\mathscr{E}$ and $\mathscr{E}'$ along Z_i can be indexed so that $\ell_{ij} =: a'_{ij} - a_{ij}$ is a nonnegative integer and

$$P_i(\mathrm{coker}(B); t) = \prod_j \prod_{\lambda=0}^{\ell_{ij}-1} (t - a_{ij} - \lambda).$$

By Prop. 4.6.1,

$$\rho_{D,n}(\mathrm{coker}(B)) = \prod_i \left(\prod_j \prod_{\lambda=0}^{\ell_{ij}-1} \frac{\mathrm{Res}_i \omega}{a_{ij} - m_i + \lambda} \right)^{\chi_i - \chi_i(Y)}$$

With this the proof of Thm. 4.3.1 is complete. □

5. A semilinear variant of the main result

5.1. *Notation and setting*

5.1.1. As in the preceding section, let k be an algebraically closed field of characteristic 0. But now fix an automorphism τ of k the fixed field k_0 of which is again algebraically closed. Given a k-vectorspace V, let V^τ denote the tensor product $k \otimes_{k_0} V$ modulo the k_0-subspace generated by all expressions of the form

$$x(\tau y) \otimes v - x \otimes yv \quad (x, y \in k;\ v \in V).$$

A k-linear map (isomorphism) $B: V^\tau \to V$ will be called a τ-*linear* endomorphism (automorphism) of V. Given such, we define $\det(B \mid V)$ to be the determinant of any matrix B_{ij} representing B with respect to a k-basis $\{v_i\}$ of V in the sense that

$$B(1 \otimes v_j) = \sum_i B_{ij} v_i.$$

Such a determinant is well defined up to a factor in $(k^\times)^{\tau - 1}$. Given $x, y \in k^\times$ we write $x \equiv y$ if $x/y = z^{\tau - 1}$ for some $z \in k^\times$.

5.1.2. Fix a smooth projective variety X_0/k_0, an effective divisor Z_0/k_0 of X_0 whose irreducible components Z_{0i} are smooth and cross normally. Let U_0 be the complement of Z_0 in X_0 and assume that U_0 is affine. Let ω_0 be a nonvanishing global section of $\Gamma(X_0, \Omega^1_{X_0/k_0}(\log Z_0))$ with nonzero residues $\mathrm{Res}_i \omega_0 \in k_0$ along each irreducible component Z_{0i} of Z_0. Let Y_0 be a hyperplane section of X_0 in sufficiently general position so as to be smooth, cross Z_0 normally and avoid the zeroes of ω_0. Under these assumptions the pullback of ω_0 to Y_0 again has isolated zeroes. We denote the base-change to k of each of the preceding objects by the corresponding symbol without the subscript 0, e.g.

$$X =: X_0 \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k).$$

More generally, given any locally closed subscheme W_0 of X_0, W denotes the corresponding locally closed subscheme of X. Given a quasi-coherent sheaf $\mathscr{F}$ on X set

$$\mathscr{F}^\tau =: (\mathrm{id}_{X_0} \times \mathrm{Spec}(\tau))^* \mathscr{F}.$$

Note that

$$\Gamma(X, \mathscr{F}^\tau) = \Gamma(X, \mathscr{F})^\tau.$$

5.1.3. Let $\mathscr{E}$ be a locally free coherent sheaf on X equipped with a k-linear integrable connection ∇ regular singular along Z. Note that $\mathscr{E}^\tau$ is equipped by transport of structure with an integrable connection ∇^τ. Let

$$B: (\mathscr{E}^\tau, \nabla^\tau)|_U \tilde{\to} (\mathscr{E}, \nabla)|_U$$

be a horizontal isomorphism. Given a smooth closed subscheme W_0 of U_0, B induces τ-linear automorphism of the hypercohomology group $H^*_{DR}(W, \mathscr{E})$ and therefore we may define a determinant

$$\varepsilon(W_0, \mathscr{E}) \equiv: \prod_i \det(B \mid H^i_{DR}(W, \mathscr{E}))^{(-1)^i}$$

well defined up to a factor in $(k^\times)^{\tau - 1}$.

5.2. *Statement of the variant*

Let $\mathscr{E}$ be a locally free coherent sheaf on X equipped with k-linear integrable connection ∇ regular singular along Z and an isomorphism

$$B: (\mathscr{E}^\tau. \nabla^\tau)|_U \tilde{\to} (\mathscr{E}, \nabla)|_U.$$

The invariant $\varepsilon(U_0, \mathscr{E})$ has the following description: For each irreducible component Z_i of Z let χ_i denote the Euler characteristic of the complement in Z_i of the union of the irreducible components of Z distinct from Z_i and let $\{a_{ij}\}$ be the collection of exponents of $\mathscr{E}$ along Z_i. Then $\{\tau a_{ij}\}$ is the collection of exponents of $\mathscr{E}^\tau$ along Z_i. By Prop. 3.9.1 a re-indexing $\{a'_{ij}\}$ of $\{\tau a_{ij}\}$ exists such that $a'_{ij} - a_{ij} \in \mathbf{Z}$ for all indices j.

THEOREM 5.2.1. *For all sufficiently large positive integers* N,

$$\varepsilon(U_0, \mathscr{E}) \equiv \left(\prod_{u_0} \varepsilon(u_0, \mathscr{E})^{\mathrm{ord}_u \omega}\right)^{(-1)^{\dim(U)}} \prod_i \left(\prod_j (\mathrm{Res}_i(\omega))^{a'_{ij} - a_{ij}} \frac{\Gamma(a_{ij} - N)}{\Gamma(a'_{ij} - N)}\right)^{\chi_i},$$

where u_0 *ranges over the closed points of* U_0.

Evidently a necessary condition for the theorem to hold is that for all N sufficiently large and positive, the right-hand side of the asserted formula is independent of N up to factors in $(k^\times)^{\tau-1}$. Equivalently, it is necessary that

$$1 \equiv \prod_i \left(\prod_j \frac{a_{ij} - N}{a'_{ij} - N}\right)^{\chi_i}$$

for all N sufficiently large and positive. But the latter is clearly the case, because for each i the collection of exponents $\{a'_{ij}\}$ of $\mathscr{E}^\tau$ along Z_i coincides, up to re-indexing, with $\{\tau a_{ij}\}$.

5.3. *Proof of the variant*

5.3.1. Let W_0 be a closed subscheme of U_0 such that the pullback $\mu_0^* \omega_0$ has isolated zeroes on W_0, where $\mu_0 \colon W_0 \to X_0$ denotes the inclusion. (The only two cases we have in mind are $W_0 = U_0$ and $W_0 = Y_0$.) It follows (§3.2.2) that $\mathscr{H}^*(\Omega^{\cdot}_{W/k}, \partial_{\mu^*\omega})$ is a graded coherent sheaf concentrated in dimension equal to $\dim(W)$ and supported on a finite set of closed points of W. Now B induces a τ-linear automorphism of $H^*_\omega(W, \mathscr{E}(nY))$ and therefore we may form a determinant

$$\varepsilon_n(W_0, \mathscr{E}, \omega_0) \equiv: \prod_i \det(B \mid H^i_\omega(W, \mathscr{E}(nY)))^{(-1)^i}$$

well defined up to a factor in $(k^\times)^{\tau-1}$. A spectral sequence argument gives

$$H^*_\omega(W, \mathscr{E}(nY)) = \Gamma(W, \mathscr{H}^*(\Omega^{\cdot}_{W/k}, \partial_{\mu^*\omega}) \otimes \mu^*\mathscr{E}(nY)),$$

whence the formula

$$\varepsilon_n(W_0, \mathscr{E}, \omega_0) \equiv \left(\prod_{w_0} \varepsilon(w_0, \mathscr{E})^{\mathrm{ord}_w \mu^* \omega}\right)^{(-1)^{\dim(W)}}$$

where w_0 runs over the closed points of W_0. In particular, it follows that $\varepsilon_n(W_0, \mathscr{E}, \omega_0)$ is independent of n.

5.3.2. In order to prove the theorem it will be enough to show that, for all sufficiently large positive integers N, the formula

$$\frac{\varepsilon(U_0, \mathscr{E})}{\varepsilon_n(U_0, \mathscr{E}, \omega_0)} \equiv \prod_i \left(\prod_j (\mathrm{Res}_i(\omega))^{a'_{ij} - a_{ij}} \frac{\Gamma(a_{ij} - N)}{\Gamma(a'_{ij} - N)} \right)^{\chi_i}.$$

holds for some (hence all) n. By induction on dimension, the formula

$$\frac{\varepsilon(U_0 \cap Y_0, \mathscr{E})}{\varepsilon_n(U_0 \cap Y_0, \mathscr{E}, \omega_0)} \equiv \prod_i \left(\prod_j (\mathrm{Res}_i(\omega))^{a'_{ij} - a_{ij}} \frac{\Gamma(a_{ij} - N)}{\Gamma(a'_{ij} - N)} \right)^{\chi_i(Y)}$$

holds for all N sufficiently large and positive and all n, where, as in Thm. 4.3.1, we denote by $\chi_i(Y)$ the Euler characteristic of the intersection of Y with the complement in Z_i of the union of the irreducible components of Z distinct from Z_i.

5.3.3. Now B induces τ-linear automorphisms of the graded k-vectorspaces $P_n(\mathscr{E})$ and $Q(\mathscr{E})$. Hence we can form determinants

$$\varepsilon_n(U_0, Y_0, \mathscr{E}, \omega_0) \equiv: \prod_i \det(B \mid P_n^i(\mathscr{E}))^{(-1)^i}$$

$$\varepsilon(U_0, Y_0, \mathscr{E}) \equiv: \prod_i \det(B \mid Q^i(\mathscr{E}))^{(-1)^i}$$

well defined up to factors in $(k^\times)^{\tau - 1}$. By definition

$$\frac{\varepsilon(U_0, Y_0, \mathscr{E})}{\varepsilon_n(U_0, Y_0, \mathscr{E}, \omega_0)} \equiv \frac{\det(Q(B)) \circ \varepsilon_{D,n}(\mathscr{E}^\tau)}{\varepsilon_{D,n}(\mathscr{E}) \circ \det(P_n(B))}$$

asymptotically in Weil divisors D of X supported in Z and integers n. Therefore, by Thm. 4.3.1,

$$\frac{\varepsilon(U_0, Y_0, \mathscr{E})}{\varepsilon_n(U_0, Y_0, \mathscr{E}, \omega_0)} \equiv \prod_i \left(\prod_j (-\mathrm{Res}_i(\omega))^{a'_{ij} - a_{ij}} \frac{\Gamma(a_{ij} - m_i)}{\Gamma(a'_{ij} - m_i)} \right)^{\chi_i - \chi_i(Y)},$$

asymptotically in D and n, where m_i denotes the multiplicity with which the irreducible component Z_i of Z occurs in D.

5.3.4. By considering the natural long exact sequences of section 4.2, we get a relation

$$\frac{\varepsilon(U_0, \mathscr{E})}{\varepsilon_n(U_0, \mathscr{E}, \omega_0)} \equiv \left(\frac{\varepsilon(U_0, Y_0, \mathscr{E})}{\varepsilon_n(U_0, Y_0, \mathscr{E}, \omega_0)} \right) \left(\frac{\varepsilon(Y_0 \cap U_0, \mathscr{E})}{\varepsilon_n(Y_0 \cap U_0, \mathscr{E}, \omega_0)} \right).$$

With this, the proof of Thm. 5.2.1 is complete. □

5.4. *Proof of the theorem stated in the introduction*

We continue to work in the setting of section 5.1. We specialize however, taking k to be an algebraic closure of the field $\mathbf{C}(s)$ of rational functions in a variable s defined over $\mathbf{C}$ and take τ be a $\mathbf{C}$-linear automorphism of k such that $\tau s = s + 1$. The fixed field k_0 of τ is determined by

LEMMA 5.4.1. *If* $y \in k^\times$ *satisfies* $y^{\tau - 1} \in \mathbf{C}(s)^\times$, *then* $y \in \mathbf{C}(s)^\times$. *In particular*, $k_0 = \mathbf{C}$.

Proof. Such an element y generates a finite algebraic extension K of $\mathbf{C}(s)$ stable under τ, and consequently the set of places of $\mathbf{C}(s)/\mathbf{C}$ ramified in K is a finite τ-stable set. This can happen only if $K/\mathbf{C}(s)$ ramifies at $s = \infty$ only. But then no places of $\mathbf{C}(s)/\mathbf{C}$ can ramify in K at all, hence $K = \mathbf{C}(s)$. It follows that $k_0 \subseteq \mathbf{C}(s)$, but since clearly $k_0 \cap \mathbf{C}(s) = \mathbf{C}$, necessarily $k_0 = \mathbf{C}$. □

Now let f_0 be a meromorphic function on X_0 defined and nowhere vanishing on U_0 and take $\mathscr{E}$ to be a copy of the structure sheaf of X equipped with the unique integrable connection ∇ regular singular along Z such that

$$\partial_\nabla(\eta \otimes e) = d\eta \otimes e + \left(s \frac{df}{f} \wedge \eta\right) \otimes e,$$

and equipped with the horizontal isomorphism

$$B =: (e^\tau \mapsto fe) : (\mathscr{E}^\tau, \nabla^\tau)|_U \xrightarrow{\sim} (\mathscr{E}, \nabla)|_U.$$

Then

$$\varepsilon(U_0, \mathscr{E}) \equiv \varepsilon(U_0, f_0),$$

where the latter is as defined in the introduction. (The symbols $X_0, U_0, \ldots$, here correspond to the symbols $X, U, \ldots$, in the introduction.) Now $\varepsilon(U_0, f_0)$ is defined up to a factor in $(\mathbf{C}(s)^\times)^{\tau - 1}$, but in view of Lemma 5.4.1 it will suffice merely to determine $\varepsilon(U_0, f_0)$ up to a factor in $(k^\times)^{\tau - 1}$.

Note that the exponent of $\mathscr{E}$ (resp. $\mathscr{E}^\tau$) along Z_i is $m_i s$ (resp. $m_i(s + 1)$), where m_i is the order of vanishing of f along Z_i. By Thm. 5.2.1,

$$\varepsilon(U_0, \mathscr{E}) \equiv \left(\prod_{u_0} f_0(u_0)^{\mathrm{ord}_{u_0}\omega}\right)^{(-1)^{\dim(U)}} \prod_i \left((\mathrm{Res}_i\, \omega_0)^{m_i} \frac{\Gamma(m_i s - N)}{\Gamma(m_i(s + 1) - N)}\right)^{\chi_i}$$

for all sufficiently large positive integers N. But the right-hand side, for N ranging over the integers, is independent of N modulo coboundaries; taking $N = 0$ we get the desired formula. The proof of Thm. 1.1 is complete. □

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Note added in proof

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