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## Uniqueness of Whittaker model for some supercuspidal representations of the metaplectic group

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The metaplectic extension by  $\{+1, -1\}$  of either  $SL(2)$  or  $GL(2)$  over a local or global field has been a center of interest for the last twenty-five years. Two points of view developed first simultaneously: the study of automorphic forms of half-integral weight on  $SL(2, \mathbb{Q})$  or  $GL(2, \mathbb{Q})$ , culminating with the Shimura correspondence, and the study of Weil (or oscillator) representations, for which I cannot help citing Weil's paper [15]. Far from giving an historical account of those subjects I only want to stress now how papers by Gelbart and Piatetskii-Shapiro [8] and Flicker [6] emphasized the crucial part played in the theory by the Weil representation, granting that:

- The local components of an automorphic square-integrable representation of an adelic metaplectic group over  $GL(2)$  that is determined by only one Fourier coefficient are all Weil representations ([8]).
- Locally, odd Weil representations are the only supercuspidal representations of the two-fold metaplectic group over  $GL(2)$  that map, in the local Shimura correspondence, onto special representations of  $GL(2)$  ([6]).

Let now  $F$  be a local non archimedean field with  $n$   $n$ -th roots of unity ( $n$  prime to the residual characteristic of  $F$ ), and let  $\tilde{G}$  be the  $n$ -fold metaplectic extension of  $GL(2, F)$ . All genuine irreducible admissible representations of  $\tilde{G}$  are known ([6]) and the supercuspidal ones can all be obtained by induction from an open compact mod center subgroup ([3]). When  $n$  is odd no peculiar phenomenon occurs: the Shimura correspondence maps supercuspidal representations of  $\tilde{G}$  onto supercuspidal representations of  $GL(2, F)$ , the construction by induction of supercuspidal representations of  $\tilde{G}$  follows closely the construction previously made for  $GL(2, F)$ , and the Whittaker spaces of those representations all have the same dimension  $n$ . When  $n$  is even however this smoothness disappears as follows:

1. There are supercuspidal representations of  $\tilde{G}$  that map, in the Shimura correspondence, onto special representations of  $GL(2, F)$ .
2. The general pattern of construction by induction from open compact mod

center subgroups is broken for some induced representations, that split into a direct sum of a finite number of supercuspidal representations.

3. Most supercuspidal representations of  $\tilde{G}$  have an  $n$ -dimensional Whittaker space, yet for a finite number of them (up to the central character), the Whittaker space has dimension  $n/2$ .

Not surprisingly, those three peculiar behaviours occur for exactly the same supercuspidal representations, obtained, up to twisting, as follows: take a cuspidal representation of the finite group  $GL(2, \mathbf{k}_F)$  associated to a regular character of trivial square of the quadratic extension of  $\mathbf{k}_F$ , pull it back to the canonical homomorphic image of  $GL(2, \mathfrak{O}_F)$  in  $\tilde{G}$ , twist it by a compatible genuine character of the center of  $\tilde{G}$  and induce the resulting representation to  $\tilde{G}$ ; the induced representation splits into two inequivalent supercuspidal representations. The set of supercuspidal representations obtained in this way is exactly the set of genuine supercuspidal representations of  $\tilde{G}$  that map, in the Shimura correspondence, onto special representations of  $GL(2, F)$ , and is exactly the set of genuine supercuspidal representations of  $\tilde{G}$  the Whittaker space of which has dimension  $n/2$ ; if  $n = 2$  this set is exactly the set of odd Weil representations. In view of the three peculiarities above, we can expect to find some interesting supercuspidal representations of the metaplectic groups over  $GL(r, F)$ ,  $r \geq 2$ , related in some sense to the odd Weil representation if we are dealing with an  $r$ -fold covering, while studying the construction by induction of supercuspidal representations of this group and giving special attention to the cases when the induced representations split most and to the dimensions of the Whittaker spaces of the representations under study; so far we deliberately put aside, as important as they may be, the global point of view and the point of view of the Shimura correspondence.

In a previous paper ([4]) we initiated the program described above by studying the process of induction of representations pulled back from representations of  $GL(r, \mathbf{k}_F)$ ; we will here study the, say, “unramified series” of supercuspidal representations of  $\tilde{G}$ , namely the representations obtained by induction and, possibly, splitting, from a *very cuspidal* (see [5]) representation of  $GL(r, \mathfrak{O}_F)$  (pulled back to the canonical homomorphic image of this subgroup in  $\tilde{G}$ ) twisted by a compatible genuine character of the center of  $\tilde{G}$ . The first part of the paper will set up the structure of the metaplectic group and the necessary notations. The second part will detail the process of induction and give an interpretation in terms of the Langlands correspondence for  $GL(r)$  of the cases when the induced representation splits. The last part will describe the Whittaker spaces of the representations just constructed and show that the minimal dimension for the Whittaker space occurs exactly for the factors of an induced representation in case of maximal reducibility.

Note that for an  $r$ -fold metaplectic group over  $GL(r, F)$  we get, among the

representations constructed, supercuspidal representations with a one-dimensional Whittaker space, that is, with a unique Whittaker model. Those representations, associated to some cuspidal representations of  $GL(r, \mathbf{k}_F)$ , enjoy some of the properties expected from a generalized odd Weil representation: they have a similar construction and a unique Whittaker model, as is shown here, and we will describe elsewhere an interesting model for them that generalizes the original Weil construction; it would be interesting to determine their behaviour in the Shimura correspondence as described in [7] and to investigate their possible fitting into exceptional automorphic forms for the global metaplectic group. For  $n = r = 3$  they do fit into an exceptional automorphic form: the cubic analogue of the cuspidal theta series constructed by Patterson and Piatetski-Shapiro in [13]; indeed the computation of their Gamma factor as defined in [13] shows that they are equivalent to the supercuspidal representations with unique Whittaker model the existence of which is proved in [13] through a converse theorem.

**1. The general metaplectic group**

1.1. Let  $F$  be a local non archimedean field with  $\mathfrak{O}$  as ring of integers,  $\mathfrak{P}$  as maximal ideal,  $\mathbf{k}$  as residual field, with  $q$  elements,  $p$  as residual characteristic and  $\varpi$  as a uniformizing element; let  $r \geq 2$  be an integer, let  $n \geq 1$  be another integer such that  $|n|_F = 1$  and  $\Xi = \{\xi \in F^\times / \xi^n = 1\}$  has cardinality  $n$ .

Let  $G = GL_r(F)$ . We will work on the general metaplectic group  $\tilde{G}$ , that is, the central extension of  $G$  by  $\Xi$  described<sup>1</sup> in [11]:

$$1 \rightarrow \Xi \xrightarrow{i} \tilde{G} \xrightarrow{p} G \rightarrow 1$$

We fix as in [11] the section  $s$  and the associated 2-cocycle  $\sigma^{(c)}$  (here  $c$  is an integer mod  $n$ ); let  $A$  be the subgroup of diagonal matrices in  $G$  and  $a = \text{diag}[a_1, \dots, a_r]$ ,  $b = \text{diag}[b_1, \dots, b_r]$  be two elements of  $A$ , then:

$$\sigma^{(c)}(a, b) = \left( \prod_{1 \leq i < j \leq r} (a_i, b_j) \right) (\det a, \det b)^c$$

with  $(, )$  denoting the  $n$ -th Hilbert symbol over  $F^\times$ .

1.2 A general element in  $\tilde{G}$  will be denoted by  $\tilde{g}$ ; we agree on  $p(\tilde{g}) = g$ , namely  $\tilde{g}$  is any pullback of  $g \in G$  to  $\tilde{G}$ . For any subgroup  $H$  of  $G$  its inverse image  $p^{-1}(H)$

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<sup>1</sup> All the details and properties we give here without proof can be found in Kazhdan and Patterson's paper [11], "Metaplectic Forms".

in  $\tilde{G}$  will be denoted by  $\tilde{H}$ . We call a representation  $(\pi, V)$  of  $\tilde{H}$  *genuine* if the kernel of the extension acts through an injective character  $\varepsilon$  from  $\Xi$  into the group of complex  $n$ -roots of unity:  $\pi \circ i(\xi) = \varepsilon(\xi)I_V$  for  $\xi \in \Xi$ . We fix once and for all such an  $\varepsilon$ , through which  $\Xi$  will be supposed to act in any genuine representation considered, unless otherwise mentioned.

1.3 Let  $N$  be the subgroup of unipotent uppertriangular matrices in  $G$ . One has for all  $g, h$  in  $G$  and  $n, m$  in  $N$ , the property  $\sigma^{(e)}(ng, hm) = \sigma^{(e)}(g, h)$ , hence the section  $s$  restricted to  $N$  is a homomorphism from  $N$  into  $\tilde{G}$ ; now:

LEMMA 1. *Let  $H$  be a subgroup of  $G$ ; assume the map  $h \mapsto h^n$  from  $H$  to itself is onto. Then a homomorphic section of  $H$  into  $\tilde{G}$  is unique.*

*Proof.* Two homomorphic sections from  $H$  to  $\tilde{G}$  can only differ through a morphism  $\alpha$  from  $H$  into  $\Xi$ ; it satisfies  $\alpha(h^n) = \alpha(h)^n = 1$  for all  $h$  in  $H$ , hence is trivial.

For any subgroup  $H$  of  $G$  such that there exist a unique homomorphic section of itself into  $\tilde{G}$  we can unambiguously denote by  $H$  the image of this section in  $\tilde{G}$ . Since the assumption in the lemma holds for any unipotent radical of a parabolic subgroup in  $G$ , the section  $s$  is the unique homomorphic section of  $N$  in  $\tilde{G}$  and we will still denote  $s(N)$  by  $N$ .

Let  $T$  be a subgroup of a group  $M$ ; we call  $N_M(T)$  the normalizer of  $T$  in  $M$ . Note that for any  $\tilde{g}$  in  $\tilde{G}$  the inner automorphism of  $\tilde{G}$  defined by  $\tilde{g}$  only depends on its image  $g$  in  $G$  – the extension is central – and for any subgroup  $H$  of  $G$  the equality  $N_{\tilde{G}}(\tilde{H}) = \widetilde{N_G(H)}$  holds. Now:

LEMMA 2. *Let  $H$  be a subgroup of  $G$  such that there exist a unique homomorphic section  $s_H$  of  $H$  into  $\tilde{G}$ ; then the equality  $N_{\tilde{G}}(H) = \widetilde{N_G(H)}$  holds.*

*Proof.* The inclusion  $N_{\tilde{G}}(H) \subseteq \widetilde{N_G(H)}$  is clear. Conversely let  $g \in N_G(H)$ ; the map  $h \mapsto \tilde{g}s_H(g^{-1}hg)\tilde{g}^{-1}$  is a homomorphic section of  $H$  into  $\tilde{G}$ ; we get from uniqueness  $\tilde{g}s_H(g^{-1}hg)\tilde{g}^{-1} = s_H(h)$ , hence  $\tilde{g}$  normalizes  $H$  in  $\tilde{G}$ .

COROLLARY 1. *The normalizer of  $N$  in  $\tilde{G}$  is  $\tilde{A}N$ .*

1.4. We still call  $\det \circ p$  from  $\tilde{G}$  to  $F^\times$ ; hence  $\det \tilde{g} = \det g$ .

We denote by  $\overset{\circ}{\tilde{G}}$  the kernel of the morphism  $\tilde{G} \xrightarrow{\det} F^\times$ , namely  $p^{-1}(\text{SL}_r(F))$ .

We denote by  $\overset{\sim}{\tilde{G}}$  the kernel of the morphism  $\tilde{G} \xrightarrow{\det} F^\times/F^{\times n}$ .

For any subgroup  $H$  of  $G$  we use the notations  $\overset{\circ}{\tilde{H}} = \tilde{H} \cap \overset{\circ}{\tilde{G}}$  and  $\overset{\sim}{\tilde{H}} = \tilde{H} \cap \overset{\sim}{\tilde{G}}$ .

1.5. Since we are dealing with a central extension the commutator  $[\tilde{g}, \tilde{h}] = \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}$  of two elements  $\tilde{g}, \tilde{h} \in \tilde{G}$  depends only on their projections  $g$  and  $h$ ; sometimes we will just denote it by  $[g, h]$ .

The center  $Z$  of  $G$  is the subgroup of scalar matrices in  $G$ . Let  $\zeta$  be the injection of  $F^\times$  into  $\tilde{Z}$  defined by  $\zeta(\lambda) = s(\lambda I_r)$  for  $\lambda$  in  $F^\times$ . The want of commutativity between  $\tilde{Z}$  and  $\tilde{G}$  can be measured:

LEMMA 3. For  $\lambda \in F^\times$  and  $\tilde{g} \in \tilde{G}$  we have:

$$[\zeta(\lambda), \tilde{g}] = \zeta(\lambda)\tilde{g}\zeta(\lambda)^{-1}\tilde{g}^{-1} = (\lambda, \det g)^{r-1+2rc}$$

The center of  $\tilde{G}$  is  $\tilde{Z}^{n'} = p^{-1}(\{\lambda I_r, \lambda^{r-1+2rc} \in F^{\times n'}\})$ , with  $n' = \frac{n}{(n, r-1+2rc)}$ .

*Proof.* The commutator  $[\zeta(\lambda), \tilde{g}]$  certainly belongs to  $i(\Xi)$ , central subgroup in  $\tilde{G}$ ; it follows that the map  $g \mapsto [\zeta(\lambda), \tilde{g}]$  is, for any  $\lambda$ , a morphism from  $G$  to  $i(\Xi)$  hence factors through the determinant. It is now enough to compute it for a diagonal  $g$ , which is straightforward.

So  $\tilde{Z}$  is generally not central in  $\tilde{G}$ ; however  $\tilde{Z}$  is the commutant of  ${}^n\tilde{G}$  in  $\tilde{G}$ .

1.6. Let  $K = \text{GL}_r(\mathfrak{O})$  be the standard maximal compact subgroup of  $G$ . Kazhdan and Patterson have shown in [11] that under the assumption  $|n|_F = 1$  there exists a canonical continuous homomorphic section  $\kappa: K \mapsto \tilde{G}$  with the following characteristic properties (here  $W$  is the subgroup of permutation matrices in  $G$ , contained in  $K$ ):

$$\kappa|_{A \cap K} = s|_{A \cap K} \tag{1}$$

$$\kappa|_W = s|_W \tag{2}$$

$$\kappa|_{N \cap K} = s|_{N \cap K} \tag{3}$$

The image  $\kappa(K)$  of  $K$  in  $\tilde{G}$  through  $\kappa$  will be denoted by  $K^\star$ . The existence of the section  $\kappa$  plays a crucial part in the construction of “an unramified series” of supercuspidal representations of  $\tilde{G}$ , as we will now explain.

## 2. A construction of supercuspidal representations

2.1. Let  $H$  be a closed subgroup of a locally compact totally discontinuous group  $G$ ; let  $\rho$  be a smooth irreducible representation of  $H$  in a complex vector space  $V$ . The (compactly) induced representation of  $\rho$  to  $G$  is the representation  $\text{ind}_H^G \rho$  of  $G$  through right translations in the space  $\mathcal{C}_c(G, H, \rho)$  of functions from  $G$  into  $V$  that are smooth, compactly supported mod  $H$ , and satisfy  $f(hg) = \rho(h)f(g)$  for all  $h$  in  $H$  and  $g$  in  $G$ . It is a smooth representation. Dropping the assumption of compact support mod  $H$  leads to the non-compactly induced representation of  $\rho$  to  $G$ .

Let  $\rho$  be a smooth irreducible representation of  $K$  and  $\rho^\star$  the associated smooth irreducible representation of  $K^\star$ , defined by  $\rho^\star \circ \kappa = \rho$ ; let  $\chi$  be a genuine character of the center  $\tilde{Z}^{n'}$  of  $\tilde{G}$  such that  $\chi$  and the central character of  $\rho^\star$  agree on  $\tilde{Z}^{n'} \cap K^\star$ . Then the tensor product  $\chi \otimes \rho^\star$  is a smooth irreducible

genuine representation of  $\widetilde{Z}^n K^\star$ . We want to investigate its (compactly) induced representation to  $\widetilde{G}$ ,  $\pi(\rho, \chi) = \text{ind}_{\widetilde{Z}^n K^\star}^{\widetilde{G}} \chi \otimes \rho^\star$ , a smooth genuine representation of  $\widetilde{G}$ , in the case when  $\rho$  is *very cuspidal* in the sense of Carayol ([5, Définition 4.1]). The next four sections will be devoted to prove the following:

**THEOREM 1.** *Let  $\rho$  be a very cuspidal irreducible representation of  $K$  and  $\chi$  be a genuine character of  $\widetilde{Z}^n$  agreeing on  $\widetilde{Z}^n \cap K^\star$  with the central character of  $\rho^\star$ . The subgroup of all  $z$  in  $\widetilde{Z}$  the adjoint action of which fix the equivalence class of  $\chi \otimes \rho^\star$  has the form  $\widetilde{Z}^{n/k} \kappa(Z \cap K)$  where  $k$  is an integer dividing  $d = \text{g.c.d.}(n, r)$ . The induced representation  $\pi(\rho, \chi)$  of  $\chi \otimes \rho^\star$  to  $\widetilde{G}$  is a direct sum of  $k$  inequivalent genuine irreducible supercuspidal representations of  $\widetilde{G}$ , induced from the  $k$  inequivalent extensions to  $\widetilde{Z}^{n/k} K^\star$  of  $\chi \otimes \rho^\star$ .*

**REMARKS:**

1. If  $n$  is prime to  $r$  the representation  $\pi(\rho, \chi)$  is always irreducible.
2. After proving this theorem we will be able to give an interpretation of the integer  $k$  associated to  $\rho$  through Moy's results on the Langlands correspondence for  $G$  (see [12]); for that purpose we will have to assume that  $F$  has characteristic 0 and  $r$  is prime to  $p$ .

2.2. According to [5, Proposition 1.5] the induced representation  $\text{ind}_{ZK}^G \nu \otimes \rho$  (where  $\nu$  is any appropriate character of  $Z$ ) is an irreducible supercuspidal representation of  $G$  provided that the following condition be satisfied<sup>2</sup> for any  $g \in G$ ,  $g \notin ZK$ :

$$\text{The representations } \rho_{|K \cap gKg^{-1}} \text{ and } {}^g \rho_{|K \cap gKg^{-1}} \text{ are disjoint.} \quad (4)$$

An analogous criterion of irreducibility, hence supercuspidality, for  $\pi(\rho, \chi)$  is (see [5] or [3, 1.4.3]) that the following condition be satisfied for all  $\tilde{g} \in \widetilde{G}$ ,  $\tilde{g} \notin \widetilde{Z}^n K^\star$ :

$$\begin{aligned} &\text{The representations } \chi \otimes \rho_{|\widetilde{Z}^n K^\star \cap \tilde{g}\widetilde{Z}^n K^\star \tilde{g}^{-1}} \text{ and} \\ &{}^g(\chi \otimes \rho^\star)_{|\widetilde{Z}^n K^\star \cap \tilde{g}\widetilde{Z}^n K^\star \tilde{g}^{-1}} \text{ are disjoint.} \end{aligned} \quad (5)$$

The study of (5) requires a close look at the subgroups  $\widetilde{Z}^n K^\star \cap \tilde{g}\widetilde{Z}^n K^\star \tilde{g}^{-1}$ .

2.3. Denote by  $k \mapsto \mathbf{P}(k)$  the projection morphism from  $K$  onto  $\text{GL}_r(\mathbf{k})$ —obtained through reducing the entries mod  $\mathfrak{P}$ —, the kernel of which is  $K_1 = I_r + M_r(\mathfrak{P})$ . Since  $n$  is prime to the residual characteristic of  $F$  the subgroup  $1 + \mathfrak{P}$  lies in the kernel of the  $n$ -th Hilbert symbol and we will freely regard  $(\ , \ )$  as defined on  $\mathbf{k}^\times$ .

For any  $g$  in  $G$  we define a homomorphic section  ${}^g \kappa$  of  $gKg^{-1}$  into  $\widetilde{G}$  through

<sup>2</sup>The conjugate representation  ${}^g \rho$  is the representation of  $gKg^{-1}$  defined by  ${}^g \rho(gkg^{-1}) = \rho(k)$  for  $k \in K$ .

${}^g\kappa(gkg^{-1}) = \tilde{g}\kappa(k)\tilde{g}^{-1}$  for  $k \in K$ . We want to compare  ${}^g\kappa$  and  $\kappa$  on their common domain, hence we put  $i(\xi(g, k)) = {}^g\kappa(k)\kappa(k)^{-1}$  for  $k \in K \cap gKg^{-1}$ ; we have:

LEMMA 4. 1. *The map  $k \mapsto \xi(g, k)$  is a continuous morphism from  $K \cap gKg^{-1}$  into  $\Xi$ , trivial on the intersection with  $K_1$ .*

2. *Let  $h, l$  be in  $K$ ; then  $\xi(hgl, hkh^{-1}) = \xi(g, k)$  for any  $k \in K \cap gKg^{-1}$ .*

3. *Let  $a = \text{diag}[a_1, \dots, a_r]$  be in  $A$  and assume  $\text{val } a_i \leq \text{val } a_{i+1}$  for  $1 \leq i < r$ ; then  $\mathbf{P}(K \cap aKa^{-1})$  is a standard parabolic subgroup in  $\text{GL}_r(\mathbf{k})$  with Levi subgroup  $\prod_{j=1}^s \text{GL}_{r_j}(\mathbf{k})$ . Let  $k_1, \dots, k_s$  be the corresponding diagonal blocks of  $\mathbf{P}(k)$  for a  $k$  in  $K \cap aKa^{-1}$ ; then*

$$\xi(a, k) = \left( \prod_{j=1}^s (a_{r_1+\dots+r_j}, \det k_j)^{-1} \right) (\det a, \det k)^{1+2c}$$

*Proof.* 1. Because  ${}^g\kappa$  and  $\kappa$  are continuous morphisms and  $i(\Xi)$  is central in  $\tilde{G}$ , the given map is a continuous morphism; because  $K_1$  is a pro- $p$ -group and  $n$ , the order of  $\Xi$ , is prime to  $p$ , it is trivial on  $K_1$ .

2. Follows from a straightforward computation, since  $\kappa$  is a morphism and  $i(\Xi)$  is central.

3. One can check that  $\mathbf{P}(K \cap aKa^{-1})$  is the standard upper-block-triangular parabolic subgroup with the given Levi subgroup where  $\prod_{j=1}^s \text{GL}_{r_j}(F)$  is the centralizer of  $a$  in  $G$ . Let  $n$  belong to  $N \cap K \cap aKa^{-1} = N \cap K$ ; then  $\kappa(n) = s(n)$  from 1.6 and  ${}^a\kappa(n) = s(a)\kappa(a^{-1}na)s(a)^{-1} = s(a)s(a^{-1}na)s(a)^{-1} = s(n)$  from 1.3. Hence  $\xi(a, n) = 1$ . It follows that the morphism  $k \mapsto \xi(a, k)$  viewed as a morphism from  $\mathbf{P}(K \cap aKa^{-1})$  into  $\Xi$  factors through its Levi subgroup, and is trivial on  $\mathbf{P}(N \cap K) \cap \prod_{j=1}^s \text{GL}_{r_j}(\mathbf{k})$  and on any of its conjugates in this Levi subgroup. Henceforth it factors on each  $\text{GL}_{r_j}$  through the determinant and the given formula is easily checked on diagonal elements, using 1.6.

2.4. The representation  ${}^g(\chi \otimes \rho^\star)$  appearing in condition 5 is the representation  $\chi \otimes {}^g\rho^\star$  of  $\tilde{g}\widehat{Z}^n K^\star \tilde{g}^{-1} = \widehat{Z}^{n^g} \kappa(gKg^{-1})$  and is disjoint from  $\chi \otimes \rho^\star$  on their common domain if and only if  $\rho^\star$  and  $\varepsilon \otimes {}^g\rho^\star$  are disjoint on  $K^\star \cap \widehat{gKg^{-1}}$ . Let  $k$  belong to  $K \cap gKg^{-1}$ ; then  $\kappa(k) = i(\xi(g, k))^{-1}{}^g\kappa(k)$  and  $\rho^\star(\kappa(k)) = \rho(k)$  while

$$\begin{aligned} \varepsilon \otimes {}^g\rho^\star(\kappa(k)) &= \varepsilon(\xi(g, k))^{-1}{}^g\rho^\star({}^g\kappa(k)) \\ &= \varepsilon(\xi(g, k))^{-1}{}^g\rho^\star(\tilde{g}\kappa(g^{-1}kg)\tilde{g}^{-1}) \\ &= \varepsilon(\xi(g, k))^{-1}\rho^\star(\kappa(g^{-1}kg)) \\ &= \varepsilon(\xi(g, k))^{-1}{}^g\rho(k) \end{aligned}$$

We can now restate condition 5 as follows:

The representations  $k \mapsto \rho(k)$  and  $k \mapsto \varepsilon(\xi(g, k))^{-1}{}^g\rho(k)$  of  $K \cap gKg^{-1}$  are disjoint.

(6)



Whenever  $\rho$  is very cuspidal, condition 6 is satisfied for any  $\tilde{g}$  in  $\tilde{G} - \widetilde{ZK}$ . Indeed, the proof of Théorème 4.2 in [5] shows that:

- either  $\rho$  and  ${}^g\rho$  are disjoint on  $K_1 \cap gK_1g^{-1}$ ; since we know from the above lemma that  $\varepsilon(\xi(g, k))$  is trivial on that group the assertion follows.
- or  $\rho$  factors through  $\mathbf{P}$  to a cuspidal representation of  $GL_r(\mathbf{k})$ ; in this case part 2 of the above lemma shows that we need only to consider the double coset  $KgK$  and can then assume that  $g = a$  is a diagonal element satisfying the conditions of part 3 of the lemma. The proof in [5] exhibits a standard unipotent subgroup  $U$  of  $G$  such that  $\rho$  and  ${}^a\rho$  be disjoint on  $U \cap K$ . Since we know from part 3 in the above lemma that  $\varepsilon(\xi(g, k))$  is trivial on that group the assertion follows.

2.5. We are left with condition 6 for  $\tilde{g} \in \widetilde{ZK}$ ,  $\tilde{g} \notin \widetilde{Z}^n K^\star$ , equivalently for  $\tilde{g} = \zeta(\lambda)$  with  $\lambda \in F^\times$ ,  $\lambda \notin F^{\times n'}$ . Now  ${}^g\rho = \rho$  and  $\varepsilon(\xi(g, k)) = [\zeta(\lambda), \kappa(k)] = (\lambda, \det k)^{r-1+2rc}$  from lemma 3. We can as well take  $\lambda = \varpi^i$ , the integer  $i$  running from 1 to  $n' - 1$ . We denote by  $\omega$  the character  $x \mapsto \varepsilon((\varpi, x))^{r-1+2rc}$  of  $F^\times$  and by  $\omega_\star$  its restriction to  $\mathfrak{D}^\times$ . We have to examine whether  $\rho$  and  ${}^3\omega_\star \otimes \rho$  are disjoint or equivalent on  $K$ .

LEMMA 5. 1. *The characters  $\omega$  and  $\omega_\star$  have the same order  $n'$ .*

2. *The irreducible representations  $\omega_\star^i \otimes \rho$  and  $\rho$  are disjoint whenever  $i$  is not a multiple of  $\frac{n'}{(n', r)} = \frac{n'}{(n, r)}$ .*

*Proof.* 1. Since  $n$  is prime to  $p$  we can use the formula in [14, Proposition 8, p. 217] to compute:

$$\omega(x) = \varepsilon((-1)^{\text{val } x} \varpi^{\text{val } x/x})^{(r-1+2rc)(q-1/n)} = \varepsilon((-1)^{\text{val } x} \varpi^{\text{val } x/x})^{q-1/n'}$$

where the unit under  $\varepsilon$  is regarded as an element of  $k^\times$  (remember that  $n$  divides  $q - 1$ ). The order of this character certainly divides  $n' = (n, r - 1 + 2rc)$ ; furthermore on a unit  $x$  we get  $\omega(x) = \varepsilon(x^{-(q-1/n)})$  with order exactly  $n'$ , hence the first assertion.

2. An obvious necessary condition for equivalence is the agreement of central characters:  $\omega^{ir}$  should be trivial on  $\mathfrak{D}^\times$ , hence  $i$  should be a multiple of

$$\frac{n'}{(n', r)} = \frac{n'}{(n, r)}.$$

Part 2 of the lemma shows that the subgroup of those  $i$  in  $\mathbb{Z}$  such that the representations  $\omega_\star^i \otimes \rho$  and  $\rho$  be equivalent is generated by some  $n'/k$  where  $k$  divides  $d$ . Let  $A$  be an operator intertwining  $\chi \otimes \rho^\star$  and its conjugate under  $\zeta(\varpi^{n'/k})$ ; then  $A^k$  is a scalar operator since  $\zeta(\varpi^{n'/k})^k$  acts as a scalar in  $\chi \otimes \rho^\star$  and

<sup>3</sup> Here the tensor product stands for twisting by the given character of the determinant.

we can choose  $A$  such that  $A^k = \chi \otimes \rho^*(\zeta(\varpi^{n'/k})^k)$ . There are exactly  $k$  such choices for  $A$ , each of whom determines an extension of  $\chi \otimes \rho^*$  to  $\widetilde{Z}^{n'/k} K^*$  by letting  $\zeta(\varpi^{n'/k})$  act through  $A$ . Those  $k$  extensions, say  $\sigma_j(\rho, \chi)$ ,  $j$  running from 1 to  $k$ , are inequivalent since the only operators intertwining  $\chi \otimes \rho^*$  with itself are the scalar operators. The induced representation of  $\chi \otimes \rho^*$  to  $\widetilde{Z}^{n'/k} K^*$  is the direct sum of the  $\sigma_j(\rho, \chi)$ ,  $1 \leq j \leq k$ , and  $\pi(\rho, \chi)$  is the direct sum of the  $\text{ind}_{\widetilde{Z}^{n'/k} K^*}^{\widetilde{G}} \sigma_j(\rho, \chi)$ ,  $1 \leq j \leq k$ ; each of those is irreducible from condition 6 for  $\tilde{g}$  in  $\widetilde{G}$ ,  $\tilde{g}$  not in  $\widetilde{Z}^{n'/k} K^*$ , and they are inequivalent from [5, Proposition 1.5, (2)]. The theorem is now proved.

2.6. In order to get a clear understanding of the way the integer  $k$  in the above theorem is related to  $\rho$  we will now make use of the tame Langlands correspondence established in [12], which forces us to make the hypotheses that  $F$  has characteristic 0 and  $r$  is prime to  $p$ . We call  $\mathcal{L}$  the bijection constructed in [12] between the set of equivalence classes of irreducible supercuspidal representations of  $G = \text{GL}_r(F)$  and the set of equivalence classes of irreducible representations of the Weil group  $W_F$  of  $F$  of degree  $r$ . Theorem 2.2.2 in [12] states that for any irreducible supercuspidal representation  $R$  of  $G$  there exists an extension  $E$  of degree  $r$  of  $F$  and an admissible character  $\theta$  of  $E^\times$  such that  $\mathcal{L}(R)$  is the induced representation from  $W_E$  to  $W_F$  of  $\theta$  (that is, of the character of  $W_E$  associated to  $\theta$  via class field theory) and that such a pair  $(E, \theta)$  is determined up to conjugacy by an element of  $W_F$ . The fundamental claim here is the following:

**PROPOSITION 1.** *Let  $\rho$  be a very cuspidal representation of  $K$  and  $\nu$  be any character of  $Z$  agreeing on  $Z \cap K$  with the central character of  $\rho$ . Let  $\mathcal{L}(\text{ind}_{ZK}^G \nu \otimes \rho) = \text{ind}_{W_E}^{W_F} \theta$  where  $\theta$  is an admissible character of  $E^\times$ . Then  $E$  is an unramified extension of  $F$ .*

*Proof.* 1. We fix an unramified character  $\mu$  of  $F^\times$  of order  $r$ . Since  $\nu \otimes \rho \otimes \mu \circ \det$  is equal to  $\nu \otimes \rho$  on  $ZK$ , the representations  $R$  and  $R \otimes \mu$  are equivalent. Since the bijection  $\mathcal{L}$  commutes with twisting by characters of  $F^\times$  ([12, Corollary 4.2.4]) we must have  $\mu \otimes \mathcal{L}(R) \simeq \mathcal{L}(R)$ . On the other hand induction also commutes with twisting, hence  $\mu \otimes \mathcal{L}(R)$  is induced from the admissible character  $\theta\mu \circ N_{E/F}$  of  $E^\times$ . Equivalence with  $\mathcal{L}(R)$  implies that  $\theta\mu \circ N_{E/F}$  is conjugate to  $\theta$ .

2. Let  $\sigma$  be an  $F$ -automorphism of  $E$  such that  $\theta(\sigma(x)) = \theta(x)\mu(N_{E/F}(x))$  for all  $x$  in  $E^\times$ , and let  $L$  be the subfield of  $E$  consisting of the fixed points of  $\sigma$ . Then  $E$  is a cyclic extension of  $L$  and Hilbert's Theorem 90 asserts that the image of the map  $x \mapsto \sigma(x)/x$ , for  $x$  in  $E^\times$ , is  $\text{Ker } N_{E/L}$ , the kernel of the norm map. Given a uniformizing element  $\varpi_L$  of  $L$ , we can pick (see [10], with the tameness of  $E - r$  is prime to  $p$  - in mind) a uniformizing element  $\varpi_E$  of  $E$  such that  $\varpi_E^e = \varpi_L \xi$  where  $e$  is the ramification index of  $E$  over  $L$  and  $\xi$  is a root of unity in  $E^\times$  of order prime to  $p$ . The group  $\text{Ker } N_{E/L}$  is generated by the  $\sigma(x)/x$  for  $x$  in  $\mathfrak{O}_E^\times$  and  $\sigma(\varpi_E)/\varpi_E$ ; since  $(\sigma(\varpi_E)/\varpi_E)^e = \sigma(\xi)/\xi$  is a root of unity of order prime to  $p$ , the

element  $\sigma(\varpi_E)/\varpi_E$  itself is a root of unity of order  $k$  prime to  $p$ . I claim that the  $\sigma(x)/x$  for  $x$  in  $\mathfrak{D}_E^\times$  generate a subgroup of  $\text{Ker } N_{E/L}$  containing  $\text{Ker } N_{E/L} \cap (1 + \mathfrak{P}_E)$ . Indeed let  $1 + z$  belong to  $\text{Ker } N_{E/L} \cap (1 + \mathfrak{P}_E)$ . There is a unique  $1 + t$  in  $1 + \mathfrak{P}_E$  such that  $(1 + t)^k = 1 + z$ , because  $k$  is prime to  $p$ . Its norm  $N_{E/L}(1 + t)$  belongs to  $1 + \mathfrak{P}_L$  and is a  $k$ -root of unity, hence equals 1, so  $1 + t = (\sigma(\varpi_E)/\varpi_E)^j \sigma(y)/y$  for some  $j$  in  $\mathbb{Z}$ ,  $y$  in  $\mathfrak{D}_E^\times$ . Now  $1 + z = (1 + t)^k = (\sigma(\varpi_E)/\varpi_E)^{jk} \sigma(y^k)/y^k = \sigma(x)/x$  where  $x = y^k$  belongs to  $\mathfrak{D}_E^\times$  as desired.

Since  $\mu$  is unramified, for all  $x$  in  $\mathfrak{D}_E^\times$  we have  $\theta(\sigma(x)) = \theta(x)$ , so that  $\theta$  is trivial on  $\text{Ker } N_{E/L} \cap (1 + \mathfrak{P}_E)$  and its restriction to  $1 + \mathfrak{P}_E$  factors through  $N_{E/L}$ . By definition of an admissible character, this implies that  $E$  is an unramified extension of  $L$ .

3. Let  $f$  be the residual degree of  $E$  over  $F$ . We can now pick  $\varpi_E = \varpi_L$ , hence  $\sigma(\varpi_E) = \varpi_E$ . We get  $\mu \circ N_{E/F}(\varpi_E) = \theta(\sigma(\varpi_E))/\theta(\varpi_E) = 1$ . Since  $\mu$  is unramified this amounts to  $\mu(\varpi_F) = 1$ . But  $\mu$  has order  $r$ ; we conclude  $f = r$  as announced.

2.7. We are now in a position to give the interpretation of the integer  $k$  arising in Theorem 1, in terms of the Langlands correspondence:

**THEOREM 2.** *Let  $\rho$  be a very cuspidal representation of  $K$ ; pick a character  $\nu$  of  $Z$  agreeing on  $Z \cap K$  with the central character of  $\rho$  and let  $E$  be the unramified extension of degree  $r$  of  $F$  and  $\theta$  be an admissible character of  $E^\times$  such that  $\mathcal{L}(\text{ind}_{ZK}^G \nu \otimes \rho) = \text{ind}_{W_E}^{W_F} \theta$ . The positive integers  $s$  such that  $\rho$  is equivalent to  $\omega_\star^s \otimes \rho$  are the multiples of  $n'/k$  with  $k$  a divisor of  $d = \text{g.c.d.}(n, r)$ . The integer  $k$  is the degree of  $E$  over the smallest subextension  $L_0$  of  $E$  over  $F$  such that  $\theta^n$  factors through the norm map from  $E$  to  $L_0$ .*

*Proof.* 1. Since the bijection  $\mathcal{L}$  commutes with twisting by characters of  $F^\times$  ([12, Corollary 4.2.4]) and induction also commutes with twisting, from  $\rho \simeq \omega_\star^s \otimes \rho$  we deduce that  $\theta$  is conjugate to  $\theta \omega^s \circ N_{E/F}$ . Let  $\sigma$  in  $\text{Gal}(E/F)$  be such that  $\theta \omega^s \circ N_{E/F} = \theta \circ \sigma$  and let  $L$  be the fixed field of  $\sigma$ . I claim that the degree  $l$  of  $E$  over  $L$  is the order of  $\omega^s$ , which divides  $n'$  from Lemma 5, as well as the order of the restriction of  $\theta$  to  $\text{Ker } N_{E/L}$  (from this will follow that  $l$  also divides  $r$  hence  $d$  and,  $l$  being prime to  $p$ , that the restriction of  $\theta$  to  $1 + \mathfrak{P}_E$  factors through  $N_{E/L}$ ). Indeed we have  $\theta \circ \sigma^j = \theta(\omega^s \circ N_{E/F})^j$  for all  $j$ , while  $\theta$  is regular, so the order of  $\sigma$  equals the order of  $\omega^s \circ N_{E/F}$ , itself the order of  $\omega^s$  from Lemma 5,  $E$  being unramified. Now for all  $x$  in  $E^\times$  we have  $\theta(\sigma(x)/x) = \omega^s \circ N_{E/F}(x)$ , hence  $\theta$  restricted to  $\text{Ker } N_{E/L}$  has order  $l$ .

This proves that for any such integer  $s$  the quotient  $n'/\text{g.c.d.}(n', s) = l$  divides  $d$ , so the smallest such  $s$  is some  $n'/k$  where  $k$  divides  $d$ , and that  $\theta^l$ , hence  $\theta^n$ , factors through  $N_{E/L}$  where  $l$  is the degree of  $E$  over  $L$ .

2. Let  $L_0$  be the smallest subextension of  $E$  such that  $\theta^n$  factor through  $N_{E/L_0}$ ; from 1, the degree  $l_0$  of  $E$  over  $L_0$  is a multiple of  $k$ . We have to show that  $l_0 = k$ .

Let  $\sigma$  generate  $\text{Gal } E/L_0$  and let  $t$  be the order of  $\theta$  restricted to  $\text{Ker } N_{E/L_0}$ ; then  $t$  divides  $n$  and we can use the  $n$ -th Hilbert symbol  $(\ , \ )_{n,E}$  on  $E^*$  and find an  $a$  in  $E^*$  such that for all  $x$  in  $E^*$ :

$$\theta(\sigma(x)/x) = (a^{n/t}, x)_{n,E}$$

Note that  $\varpi_F$  is a uniformizing element for  $E$ , so the character  $x \mapsto \theta(\sigma(x)/x)$  has order  $t$  on  $\mathfrak{D}_E^\times$  itself; but for  $x$  in  $\mathfrak{D}_E^\times : (a^{n/t}, x)_{n,E} = (x^{-n \text{val } a/t})^{(q_E - 1)/n}$ , from [14, Proposition 8, p. 217]. We deduce that  $\text{val } a$  must be prime to  $t$ . Now for  $x$  in  $L_0^\times$  we must have  $(a^{n/t}, x)_{n,E} = 1$  or  $((N_{E/L_0} a)^{n/t}, x)_{n,L_0} = 1$ , hence  $N_{E/L_0} a^{n/t} \in L_0^{\times n}$ , or, since  $F$  has  $n$   $n$ -th roots of 1,  $N_{E/L_0} a \in L_0^{\times t}$ . Write  $a = \varpi_F^i u$  with  $u \in \mathfrak{D}_E^\times$ ; we get that  $il_0$  must be a multiple of  $t$  while  $i$  is prime to  $t$ , so  $l_0$  is a multiple of  $t$ ; on the other hand  $N_{E/L_0}(u)$  must belong to  $\mathfrak{D}_{L_0}^{\times t}$  so  $u$  must belong to  $\mathfrak{D}_E^{\times t}$ . We get:

$$\theta(\sigma(x)/x) = (\varpi^{in/t}, x)_{n,E} = (\varpi^{in/t}, N_{E/F} x)_{n,F}$$

We just let  $\theta(\sigma(x)/x)$  go through the norm  $N_{E/F}$ ; now we can iterate and find that:

$$\theta(\sigma^t(x)) = \theta(x)(\varpi^{in/t}, N_{E/F} x)_{n,F}^t = \theta(x)$$

But  $\theta$  is regular, so this in turn implies  $\sigma^t = 1$  hence  $t = l_0$ . Also we have  $n/t = (n, r - 1 + 2rc)n'/t$  and picking  $j$  such that  $in/t \equiv ij(r - 1 + 2rc)n'/t$  modulo  $n$  we can write:

$$\theta(\sigma(x)) = \theta(x)(\varpi^{ijn'/t}, N_{E/F} x)_{n,F}^{r-1+2rc} = \theta(x)(\omega \circ N_{E/F}(x))^{ijn'/t}$$

Now  $ij$  being prime to  $t$  we deduce that  $n'/t$  is a multiple of  $n'/k$ , hence  $t$  divides  $k$ , while  $t = l_0$  is a multiple of  $k$ ; so  $l_0 = k$  and the theorem is proved.

2.8. We have noticed already, just after Theorem 1, that when  $n$  is prime to  $r$  the representation  $\pi(\rho, \chi)$  is always irreducible. On the other hand the cases of greatest interest are those with  $n = r$  as will be illustrated in part 3. Note anyhow that whenever  $r$  is a prime (different from  $p$ ) and  $F$  has characteristic 0 the situation is very clear cut: if  $n$  is not prime to  $r$  then  $d = r$  and the integer  $k$  in Theorems 1 and 2 is either 1 or  $r$ . Now the case when  $k = r$  ( $r$  a prime or not) has been completely described in a previous work; indeed Theorem 2 and its proof show that if  $k = r$  the restriction to  $1 + \mathfrak{P}_E$  of the character  $\theta$  factors through  $N_{E/F}$  so that up to twisting  $\theta$  has conductor 1 and  $\rho$  comes through  $\mathbf{P}$  from a cuspidal representation of  $\text{GL}_r(\mathbf{k})$ : this is precisely the situation examined in [4].

### 3. Whittaker spaces

3.1. We shall now discuss the Whittaker spaces of the representations constructed in part 2, starting with the definitions we need.

We fix a non trivial continuous character  $\psi$  of the additive group  $F$ . Any smooth character of  $N$  has the following form:

$$n = \begin{bmatrix} 1 & n_{12} & \cdots & \cdots & n_{1r} \\ 0 & 1 & n_{23} & \cdots & n_{2r} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & n_{r-1r} \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix} \mapsto e(n)$$

$$e(n) = \psi(\alpha_1 n_{12} + \alpha_2 n_{23} + \cdots + \alpha_{r-1} n_{r-1r}), \quad \alpha_i \in F.$$

Such a character  $e$  of  $N$  is *non degenerate* if each  $\alpha_i$  belongs to  $F^\times$ , in other words if its fixator in  $A$  is  $Z$ . From now on, we fix such a non degenerate character  $e$  of  $N$ .

Let  $(\pi, V)$  be a smooth representation of  $\tilde{G}$  and  $V^*$  be the algebraic dual of  $V$ ; the *Whittaker space* of  $\pi$  with respect to  $e$  is:

$$Wh(\pi, e) = \{L \in V^*/L(\pi(n)v) = e(n)L(v), \forall v \in V, n \in N\}$$

Any nonzero  $L$  in  $Wh(\pi, e)$  is called a *Whittaker form* on  $V$ . Note that  $Wh(\pi, e)$  is the dual space of  $V_e = V/V(e)$  (the Jacquet module of  $V$ ) where  $V(e)$  is the subspace of  $V$  generated by all vectors  $\pi(n)v - e(n)v$  with  $v \in V$  and  $n \in N$ . The subgroup of those elements in the normalizer  $\tilde{A}N$  of  $N$  that fix  $e$ , namely  $\tilde{Z}N$ , acts naturally on the three spaces  $V(e)$ ,  $V_e$  and  $Wh(\pi, e)$ .

Let  $L$  be a Whittaker form on  $V$ ; define for each  $v$  in  $V$  a function  $\mathcal{L}(v)$  on  $\tilde{G}$  by  $\mathcal{L}(v)(g) = L(\pi(g)v)$  for  $g \in \tilde{G}$ . These functions belong to the space  $\mathcal{C}(\tilde{G}, N, e)$  of *Whittaker functions* on  $\tilde{G}$ , i.e. of smooth functions from  $\tilde{G}$  into  $\mathbb{C}$  satisfying  $f(ng) = e(n)f(g)$  for all  $n$  in  $N$ ,  $g$  in  $\tilde{G}$ , and the map  $\mathcal{L}: v \mapsto \mathcal{L}(v)$  intertwines  $\pi$  and the representation of  $\tilde{G}$  in  $\mathcal{C}(\tilde{G}, N, e)$  through right translations. The map  $L \mapsto \mathcal{L}$  provides us with an isomorphism:

$$Wh(\pi, e) \xrightarrow{\sim} \text{Hom}_{\tilde{G}}(\pi, \mathcal{C}(\tilde{G}, N, e))$$

If  $\pi$  is irreducible and its Whittaker space is non trivial, a map  $\mathcal{L}$  as above is injective; its image, a subspace of  $\mathcal{C}(\tilde{G}, N, e)$  on which  $\pi$  acts through right translations, is called a *Whittaker model* for  $(\pi, V)$ . We will always assume that  $\pi$  is genuine, hence work with Whittaker functions in  $\mathcal{C}(\tilde{G}, \tilde{N}, e \otimes \varepsilon)$ .

3.2. When working with the group  $G$  it has been known since [9] that an irreducible supercuspidal representation of  $G$  has a *unique* Whittaker model, in other words has a one-dimensional Whittaker space. This is not true any more for the metaplectic group; however we know from [11] that the Whittaker space of a smooth representation  $(\pi, V)$  of  $\tilde{G}$ :

- is finite-dimensional if  $(\pi, V)$  is irreducible;
- is non trivial if  $(\pi, V)$  is supercuspidal.

Unicity of Whittaker models for supercuspidal irreducible representations of  $G$  stands as a remarkable result with remarkable consequences among which is global multiplicity one for automorphic forms on  $GL(r)$ . Likewise, finding the genuine irreducible representations of  $\tilde{G}$ , if any, that have a one-dimensional Whittaker space can be expected to have, in some cases at least, interesting global applications, as happened in [11] where Kazhdan and Patterson constructed in the case when  $n = r$  a representation, quotient of a particular reducible principal series representation of  $\tilde{G}$ , having a unique Whittaker model, and could then fit it into an automorphic form on the global metaplectic group with remarkable Fourier coefficients. We will not reach such results here; we merely wish to describe the Whittaker spaces of the supercuspidal representations constructed in part 2. The first step in this direction is the following:

**PROPOSITION 2.** *Let  $\sigma$  be a smooth genuine irreducible representation of  $\widetilde{ZK}$  in a complex vector space  $W$  such that the induced representation  $\text{ind}_{\widetilde{ZK}}^{\tilde{G}} \sigma = \pi$ , acting in  $V = \mathcal{C}_c(\tilde{G}, \widetilde{ZK}, \sigma)$ , is a genuine irreducible supercuspidal representation of  $\tilde{G}$ .*

*Let  $E(\sigma, e)$  be the space of smooth functions  $\Phi$  from  $\tilde{A}$  into  $W$ , compactly supported mod  $\tilde{A} \cap \widetilde{ZK}$ , satisfying:*

$$\Phi(a_0 a) = \sigma(a_0) \Phi(a) \quad (a \in \tilde{A}, a_0 \in \tilde{A} \cap \widetilde{ZK}) \quad (7)$$

$$\sigma(n) \Phi(a) = e(a^{-1} n a) \Phi(a) \quad (a \in \tilde{A}, n \in N \cap K^*) \quad (8)$$

*For any  $f$  in  $V$  define a function  $\phi(f)$  from  $\tilde{G}$  into  $W$  by:*

$$\phi(f)(g) = \int_N \overline{e(n)} f(gn) \, dn \quad (g \in \tilde{G})$$

(i) *For any  $a$  in  $\tilde{A}$ , the dimension of the subspace of  $E(\sigma, e)$  made of the functions with support in  $(\tilde{A} \cap \widetilde{ZK})a$  is equal to the multiplicity of the character  ${}^a e: {}^a e(n) = e(a^{-1} n a)$ , of  $N \cap K^*$ , in the representation  $\sigma$ .*

(ii) *The morphism  $\phi$  factors through the Jacquet module  $V_e$  of  $(\pi, V)$  and determines, by restriction of the functions  $\phi(f)$  to  $\tilde{A}$ , an isomorphism still denoted by  $\phi$  from  $V_e$  into  $E(\sigma, e)$  that intertwines the natural action of  $\tilde{Z}$  on  $V_e$  and the action of  $\tilde{Z}$  on  $E(\sigma, e)$  through right translations.*

*Proof.* (i) Condition (8) on  $\Phi$  in  $E(\sigma, e)$  means that  $\Phi(a)$  belongs to the subspace  $W^{ae}$  of  $W$  on which  $N \cap K^*$  acts through  ${}^ae$ ; conversely if  $w$  belongs to  $W^{ae}$  there is a unique function  $\Phi_{w,a}$  in  $E(\sigma, e)$  with support in  $(\tilde{A} \cap \tilde{ZK})a$  satisfying  $\Phi_{w,a}(a) = w$ : it is defined through (7).

(ii) The integral defining  $\phi(f)(g)$  converges: the function  $f$  is compactly supported mod  $\tilde{ZK}$  hence the function  $n \mapsto f(gn)$  is compactly supported on  $N$  for any  $g$ . For any  $n$  in  $N$ ,  $k$  in  $\tilde{ZK}$  and  $a$  in  $\tilde{A}$  we have  $\phi(f)(kan) = e(n)\sigma(k)\phi(f)(a)$  hence  $\phi(f)$  is completely determined by its restriction to  $\tilde{A}$  which itself belongs to  $E(\sigma, e)$  since the first condition is obviously satisfied and the second arises from the following, for  $n$  in  $N \cap K^*$ :

$$\sigma(n)\phi(f)(a) = \phi(f)(na) = \phi(f)(aa^{-1}na) = e(a^{-1}na)\phi(f)(a)$$

Now  $N$  is exhausted by its compact subgroups: we can pick an increasing sequence  $(N_j)$ ,  $j$  running through all integers, of compact subgroups of  $N$  such that  $N = \bigcup_j N_j$  and we can define operators in  $V$  by  $\phi_j(f) = \int_{N_j} \overline{e(n)}\pi(n)f \, dn$ ; we have:

1. For every  $f$  in  $V$  the sequence of functions  $\phi_j(f)$  simply converges to  $\phi(f)$ ; actually, for any  $g$  in  $\tilde{G}$  there is an integer  $k$  depending on  $g$  such that  $\phi(f)(g) = \phi_j(f)(g)$  if  $j \geq k$ . Note that  $\phi(f)$  does not belong to  $V$ .
2. The kernels of the  $\phi_j$ 's form an increasing sequence with union  $V(e)$  (see for instance [1, Lemma 2.33]).

We want to show that  $V(e)$  is the kernel of  $\phi$  itself. Clearly the kernel of  $\phi$  contains the kernel of  $\phi_j$  for all  $j$ , hence  $V(e)$ . Conversely let  $f$  belong to the kernel of  $\phi$ ; from  $\phi(f)(g) = 0$  for all  $g$  in  $\tilde{G}$  and property 1 we deduce that the decreasing sequence of compact mod the center subsets of  $\tilde{G}$  formed by the supports of the functions  $\phi_j(f)$  has an empty intersection, so we can find a finite collection of them with an empty intersection and consequently we can find an integer  $k$  such that the support of  $\phi_k(f)$  be empty. This just means that  $f$  belongs to the kernel of  $\phi_k$  hence, from property 2, to  $V(e)$ . Injectivity is proven.

Since  $E(\sigma, e)$  is the direct sum of the subspaces studied in (i) it is enough to prove that those subspaces belong to the image of  $\phi$ . Let  $w$  belong to  $W^{ae}$  and  $\Phi_{w,a}$  be the function defined in the proof of (i). Let  $f$  be the function on  $\tilde{G}$  defined by  $f(ka) = \sigma(k)w$  if  $k \in \tilde{ZK}$ ,  $f(g) = 0$  if  $g \notin \tilde{ZK}a$ . It is easy to check that  $\phi(f)$  is, up to a non-zero constant, equal to  $\Phi_{w,a}$ . Surjectivity is proven.

That  $\phi$  commutes with the action of  $\tilde{Z}$  is easily checked.

For  $1 \leq i < r$ , define the following subgroups of  $N$ :

$$U_i = \{u \in N / u_{jk} = 0 \text{ if } j \neq k \text{ and } (j, k) \neq (i, i + 1)\}$$

We say that a character  $e$  of an open subgroup  $N'$  of  $N$  is a *non degenerate character of  $N'$*  if it is non trivial on  $U_i \cap N'$  for each  $i$ ,  $1 \leq i < r$ . With this definition we can state:

**COROLLARY 2.** *Let  $\rho$  be a smooth irreducible representation of  $K$  such that, for a suitable character  $\tau$  of  $Z$ , the representation  $\text{ind}_{ZK}^G \tau \otimes \rho$  is irreducible. Let  $e$  be a non degenerate character of  $N$ . Then:*

- *the multiplicity of the restriction of  $e$  to  $N \cap K$  in  $\rho$  is 0 or 1;*
- *the set of elements  $a$  in  $A$  for which the multiplicity of the character  ${}^a e$  of  $N \cap K$  in  $\rho$  is 1 is exactly one left coset of  $A \cap ZK$  in  $A$ ;*
- *pick  $\alpha$  in  $A$  such that the multiplicity of the character  ${}^\alpha e$  of  $N \cap K$  in  $\rho$  be 1; then  ${}^\alpha e$  is a non degenerate character of  $N \cap K$ .*

*Proof.* The induced representation is irreducible, hence supercuspidal, so it has a unique Whittaker model, or, by previous remarks, a one-dimensional Jacquet module relative to  $e$ . Applying the previous proposition to  $\sigma = \tau \otimes \rho$  yields  $\dim E(\sigma, e) = 1$ , so that the multiplicity of  ${}^a e$  in  $\sigma$  is 0 for all but one  $A \cap ZK$ -left-coset, say the coset of  $\alpha$ , for which it is 1. Furthermore, assume  ${}^\alpha e$  is trivial on some subgroup  $U_i \cap K$ ; then  $\alpha = \text{diag}[\alpha_1, \dots, \alpha_r]$  satisfies  $\text{val}(\alpha_{i+1}/\alpha_i) \geq \text{cond}(\psi)$  and for  $a_i = \text{diag}[1, \dots, 1, \varpi, \dots, \varpi]$  (the first  $\varpi$  in the  $i + 1$ -th entry) we get  ${}^{a_i} {}^\alpha e = {}^\alpha e$  on  $N \cap K$ , hence the multiplicity of the character  ${}^{a_i} {}^\alpha e$  of  $N \cap K$  in  $\rho$  is also 1, which is impossible since  $a_i \alpha$  does not belong to the left coset  $(A \cap ZK)\alpha$ .

3.3. Let us assume that  $\sigma$  is obtained as in part 2; we are then ready to describe completely the Whittaker space of the representation induced from  $\sigma$ :

**THEOREM 3.** *Let  $\rho, \chi$  and  $k$  be as defined in Theorem 1, let  $\rho_i$ , for  $1 \leq i \leq k$ , be the  $k$  inequivalent extensions of  $\rho^* \otimes \chi$  to  $\widehat{Z}^{n/k} K^*$ , let  $\sigma_i$  be the induced representation of  $\rho_i$  to  $\widehat{Z} K^*$  and  $\pi_i(\rho, \chi)$  be the induced representation of  $\sigma_i$  to  $\widetilde{G}$ . Then the dimension of the Whittaker space  $\text{Wh}(\pi_i(\rho, \chi), e)$  of  $\pi_i(\rho, \chi)$  is  $n'/k$  and a basis for it is given by the following Whittaker forms:*

$$\phi_j(f) = \int_N \overline{e(n)} \lambda(f(\zeta(\varpi^j) s(\alpha) n)) \, dn \quad (0 \leq j < n'/k),$$

where:

- *the element  $\alpha$  of  $A$  is such that the character  ${}^\alpha e$  has multiplicity one in  $\rho$ ;*
- *let  $v$  be a non zero vector in the space of  $\rho$  on which  $N \cap K$  acts through  ${}^\alpha e$ ; then  $\lambda$  must be a linear form on the space of  $\sigma_i$  such that  $\lambda(v) \neq 0$  and  $\lambda(\sigma_i(\zeta(\varpi^j) w)) = 0$  for any  $w$  in the space of  $\rho$  and  $j$  between 1 and  $n'/k - 1$ .*



The Whittaker space  $Wh(\pi_i(\rho, \chi), e)$  has length  $d/k$  as a  $\tilde{Z}$ -module; it is an irreducible  $\tilde{Z}$ -module if and only if  $k = d = (n, r)$ .

*Proof.* (1) From Proposition 2, the dimension of  $Wh(\pi_i, e)$  is the sum of the multiplicities of the characters  ${}^a e$  in  $\sigma_i$ ,  $a$  running through  $(\tilde{A} \cap \tilde{ZK}) \backslash \tilde{A}$ . The restriction to  $N \cap K^\star$  of  $\sigma_i$  is equivalent to the direct sum of  $n'/k$  copies of the restriction to  $N \cap K^\star$  of  $\rho^\star$ , hence the multiplicity of  ${}^a e$  in  $\sigma_i$  is  $n'/k$  times the multiplicity of  ${}^a e$  in  $\rho$ . From Corollary 2 we get that the dimension of  $Wh(\pi_i, e)$  is  $n'/k$  as announced.

(2) From Corollary 2 we can find  $\alpha$  in  $A$ , the  $A \cap ZK$ -left coset of which is uniquely determined by  $\rho$ , such that  ${}^a e$  has multiplicity one in  $\rho$ . Let  $v$  be a non zero vector in the space of  $\rho$  on which  $N \cap K$  acts through  ${}^a e$ ; it is uniquely determined up to a scalar. The isomorphism  $\phi$  constructed in Proposition 2 maps the Jacquet module  $V_{i,e}$  of  $\pi_i$  onto the space of functions on  $(\tilde{A} \cap \tilde{ZK}) \backslash \tilde{A}$  satisfying (7) and (8). Evaluating those functions at  $s(\alpha)$  is again an isomorphism, transforming the action of  $\tilde{Z}$  on  $V_{i,e}$  into its action  $\sigma_i$  on the subspace  $W_i({}^a e)$  of vectors in the space of  $\sigma_i$  on which  $N \cap K^\star$  acts through  ${}^a e$ , twisted by the character  $z \mapsto \varepsilon(\det \alpha, z)$ , itself equivalent to the induced representation to  $\tilde{Z}$  of the character  $z \mapsto \varepsilon(\det \alpha, z) \chi_i(z)$  of  $\tilde{Z}^{n'/k} \cdot \tilde{Z} \cap K_\star$  corresponding to the action of  $\tilde{Z}^{n'/k} \cdot \tilde{Z} \cap K_\star$  on  $v$ . Therefore the Jacquet module  $V_{i,e}$  is isomorphic, as a  $\tilde{Z}$ -module, to  $\text{ind}_{\tilde{Z}^{n'/k} \cdot \tilde{Z} \cap K_\star}^{\tilde{Z}} \chi_i \otimes \varepsilon(\det \alpha, \cdot)$ .

(3) Fix a non trivial linear form  $\lambda$  on the space  $W_\rho$  of  $\rho$  such that  $\lambda(v) \neq 0$  and extend it to a linear form on the space of  $\sigma_i$  by letting it be 0 on the subspaces  $\sigma_i(\mathfrak{w}^j)(W_\rho)$  for  $1 \leq j < n'/k$ ; then the forms  $\lambda \circ \sigma_i(\zeta(\mathfrak{w}^j))$  for  $0 \leq j < n'/k$  make up a basis for the dual space of  $W_i({}^a e)$  and by composition we get that  $f \mapsto \lambda \circ \sigma_i(\zeta(\mathfrak{w}^j))(\phi(f))$  make up a basis for  $Wh(\pi_i(\rho, \chi), e)$ .

(4) The computations in section 1.5 show that  $\tilde{Z}$  is an Heisenberg group with center  $\tilde{Z} \cap {}^n \tilde{G} = {}^{n/d} \tilde{Z}$ ; furthermore  ${}^{n'/d} \tilde{Z} \cdot \tilde{Z} \cap K_\star$  is a maximal commutative subgroup of index  $n'/d$  in  $\tilde{Z}$  so that irreducible genuine representations of  $\tilde{Z}$  are determined by their central character and have dimension  $n'/d$ . Here  $k$  divides  $d$  and the Whittaker space is a genuine representation of  $\tilde{Z}$  of dimension  $n'/k$ , hence length  $d/k$  as announced.

3.4. The time has come to draw some conclusions and ask some questions.

(1) It turns out that the set of dimensions of the Whittaker spaces of the genuine irreducible supercuspidal representations constructed in part 2 is included in (and likely equal to, see Theorem 2) the set of integers  $\{n'/k, k \text{ a divisor of } d = g.c.d.(n, r)\}$ . Is it the set of dimensions of the Whittaker spaces of all genuine irreducible supercuspidal representations of  $\tilde{G}$ ?

(2) Whenever  $n$  is prime to  $r$ , the Whittaker spaces of the representations constructed in part 2 all have the same dimension  $n'$  and are irreducible  $\tilde{Z}$ -modules. Does the Whittaker space of a genuine irreducible supercuspidal representation of  $\tilde{G}$  always have dimension  $n'$  in this case?

(3) Assume as before that  $n$  is prime to  $r$  and assume further that  $n' = 1$ , i.e. that  $n$  divides  $(n, r - 1 + 2rc)$ : we get a series of supercuspidal representations of  $\tilde{G}$  with one-dimensional Whittaker spaces, i.e. unique Whittaker model. This situation arises for instance when  $n$  divides  $r + 1$  and  $c = -1$ ; note that the case when  $n = r + 1$  and  $c = -1$  is, apart from the case  $n = r$ , the only case for which the exceptional representation of Kazhdan and Patterson [11] has a unique Whittaker model.

(4) We now turn to the case when  $d = r$ : then the Whittaker spaces of minimal dimension in our series are obtained for  $k = r$  and are irreducible  $\tilde{Z}$ -modules of dimension  $n'/r$  — under the assumptions of 2.6, the corresponding “minimal” representations of  $\tilde{G}$  come from cuspidal representations of  $GL_r(\mathbf{k})$ , see 2.8 and [4]. This minimal dimension is 1 if and only if  $n' = r$ , i.e. if  $n$  is the product of  $r$  by a divisor of  $r - 1 + 2rc$ .

(5) Assume  $n = r$  and  $F$  has characteristic 0: Theorems 2 and 3 imply that the only representations among those constructed in part 2 that have a one-dimensional Whittaker model come, after twisting, inducing and splitting, from cuspidal representation of  $GL_r(\mathbf{k})$  associated to regular characters  $\theta$  of the extension of degree  $r$  of  $\mathbf{k}$  such that  $\theta^n$  factors through the norm over  $\mathbf{k}$ ; up to twisting by a character of the determinant, we get a finite number of such representations in the series. This is a nice enough situation and we can wonder whether the set of genuine irreducible supercuspidal representations having a one-dimensional Whittaker space is, up to the central character, a *finite* set, and whether we have obtained this whole set in our series. For  $n = r = 2$ , it is indeed the case (see [3]) and the representations that make up this set are the odd Weil representations.

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