COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 82, nº 3 (1992), p. 293-354

<http://www.numdam.org/item?id=CM_1992__82_3_293_0>

*Compositio Mathematica* **82**: 293–354, 1992.


# Root systems and Jacobi forms

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Received 8 March 1990; accepted 22 October 1991

## Introduction

Let $R$ be a (classical) root system, and let $Q$ denote its root lattice. For each elliptic curve $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ the Weyl group $W$ of $R$ acts on the abelian variety $A = Q \otimes_{\mathbb{Z}} E$. The quotient morphism $A \to A/W$ of this action plays an important role in understanding deformations of simply-elliptic surface singularities [Looijenga 2].

In fact it was for that purpose that the $W$-variety $A$ was first introduced in [Looijenga 1]. On $A$ there is an essentially unique minimal ample line bundle $\mathscr{L}$ on which $W$ acts too. The algebra of $W$-invariant sections of $\mathscr{L}$ and its tensor powers has been studied in loc. cit., [Bernshtein and Shvartsman], and [Saito 1, 2].

Later, Looijenga reconsidered the same situation in the more general setting of affine root systems [Looijenga 3]. In that context $A$ naturally appears as a family $A_H$ of abelian varieties, parametrized via $\tau$ by the complex upper half plane $H$. It was also noted by Looijenga that the natural action of the modular group $SL_2(\mathbb{Z})$ lifts to that family, as well as to the ample line bundle $\mathscr{L}_H$ on $A_H$. Some aspects of that action have been discussed in the work of [van Asch] and [Kac and Peterson].

The purpose of this paper is to study the invariant theory of $A_H$ and $\mathscr{L}_H$ with respect to the action of the modular group (or a subgroup $\Gamma$ of finite index) as well as of $W$. We form the quotients by $\Gamma$ and, as a first step, seek to extend the family

$$A_H/\Gamma \to H/\Gamma$$

over the cusps of $\Gamma$. This is achieved in a natural and quite explicit fashion, using a toroidal embedding technique from [Wirthmüller 2].

The invariants that we then study form a bi-graded algebra over the ring of modular forms, graded by weight (referring to behaviour with regard to $\Gamma$) and index (referring to the appropriate power of $\mathscr{L}$). We call these invariants Jacobi forms since in the special case when $R$ is of type $A_1$ they reduce to (weak) Jacobi forms in the sense of [Eichler and Zagier].

We succeed in determining the algebra of invariants for all types of root systems excluding $E_8$. The invariant algebra turns out to be a polynomial algebra over the ring of modular forms, with generators that do not depend on the particular choice of the group $\Gamma$.

The result has an application in singularity theory, to deformations of fat points in the plane with defining ideal

$$(x^2 - y^3, y^k) \quad \text{or} \quad (x^2 - y^3, xy^{k-1}) \quad (k \geqslant 3).$$

This complements the paper [Wirthmüller 1] and will appear elsewhere.

The author is grateful to the referee, who suggested various improvements to Section 1.

The paper is organized as follows:

*1. Toroidal embeddings and reflexive sheaves*
We describe a class of toroidal embeddings associated with certain properly discontinuous group actions on open cones. This class is sufficiently general for our purposes but on the other hand explicit enough to enable us to compute the cohomology of certain reflexive sheaves that arise naturally in the same context.

*2. A family of abelian varieties*
We introduce in detail the family $A_H \to H$ referred to at the beginning, as well as its compactification. We also describe the line bundle $\mathscr{L}_H$ and its extension as a reflexive sheaf. Using the results of Section 1 we then compute the cohomology of that extension.

*3. Jacobi forms*
We define the notion of Jacobi form and formulate our main result concerning the structure of the invariant algebra, in Theorem (3.6).

*4. Construction of Jacobi forms*
We first discuss some auxiliary results pertaining to the problem of extending sections of a sheaf that are given on some arrangement of divisors. These results facilitate an inductive construction of Jacobi forms, by picking a suitable sub root system of $R$ of smaller rank, and extending Jacobi forms associated with that sub root system and its conjugates under $W$.

*5. The individual root systems*
The construction prepared in Section 4 is carried out on a case-by-case basis, and thereby the proof of the main theorem is finally achieved.

## 1. Toroidal embeddings and reflexive sheaves

(1.1) Let $V$ be a finite dimensional real vector space, and let $\Lambda \subset V$ be a lattice with rank $\Lambda = \dim V$.

DEFINITION. A set $\Sigma$ consisting of cones in $V$ is called an *admissible fan* if the following hold:

(i) Each $\sigma \in \Sigma$ is a relatively open $\Lambda$-rational polyhedral cone; it does not contain any affine lines.
(ii) If $\sigma \in \Sigma$ then each face of $\bar{\sigma}$ belongs to $\Sigma$.
(iii) If $\sigma \in \Sigma$ and $\tau \in \Sigma$ then $\bar{\sigma} \cap \bar{\tau}$ is a union of faces of $\bar{\sigma}$.
(iv) The interior of

$$|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$$

is an open convex cone $I \subset V$, and $\sigma \subset \bar{I}$ for each $\sigma \in \Sigma$.
(v) If $C \subset V$ is any closed $\Lambda$-rational polyhedral cone with $C \subset |\Sigma|$ then $C$ meets but a finite number of $\sigma \in \Sigma$.

Note that if $\Sigma$ is an admissible fan then by (iv), each $\sigma \in \Sigma$ is either contained in $I$ or else is disjoint from it. By property (v) then $\{\sigma \cap I \mid \sigma \in \Sigma\}$ is a locally finite covering of $I$.

For any $\sigma \in \Sigma$ we use the symbol $\mathrm{St}(\sigma)$ to denote the star of $\sigma$ (with respect to $\Sigma$), i.e. the union of all $\tau \in \Sigma$ with $\sigma \subset \bar{\tau}$. While $\mathrm{St}(\sigma)$ need not be open in $|\Sigma|$ the intersection $\mathrm{St}(\sigma) \cap I$ is always open in $I$. We also record the following property of the star.

LEMMA. *For each $\sigma \in \Sigma$ and any $y \in I$ one has*

$$(y + \sigma) \cap \mathrm{St}(\sigma) \neq \varnothing$$

*Proof.* We choose a closed $\Lambda$-rational polyhedral cone $C' \subset I \cup \{0\}$ containing $y$ in its interior, and pick some point $z \in \sigma \cap \Lambda$. Then the convex cone spanned by $C'$ and $\{z\}$ is a closed $\Lambda$-rational polyhedral cone contained in $|\Sigma|$. By (v) it meets only a finite number of $\tau \in \Sigma$. Since the parametrized line segment

$$[0, 1] \ni t \mapsto (1-t)y + tz$$

maps into $C$ the point $(1-t)y + tz$ is contained in one and the same $\tau \in \Sigma$ for all

$t \in (0,1)$ sufficiently close to 1. We necessarily have $\tau \in \mathrm{St}(\sigma)$, and therefore

$$y + \frac{t}{1-t} z \in (y+\sigma) \cap \mathrm{St}(\sigma)$$

for those $t$. $\square$

Let us agree to call a subset $\Phi \subset \Sigma$ of an admissible fan *closed* if $\tau \in \Phi$ implies that all faces of $\bar{\tau}$ belong to $\Phi$. We put

$$|\Phi| = \bigcup_{\sigma \in \Phi} \sigma$$

and write

$$\Phi \cap I = \{\sigma \in \Phi \mid \sigma \subset I\}$$

by abuse of language.

Writing $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ we let for each such $\Phi$

$$V_{\mathbb{C}}/\Lambda = T(\Lambda) \hookrightarrow T(\Lambda, \Phi)$$

denote the torus embedding determined by $\Phi$ (i.e. by the collection $\{\bar{\sigma} \mid \sigma \in \Phi\}$, in the sense of [Kempf et al.] p. 24). We put

$$I(\Lambda) = (V + iI)/\Lambda,$$

and let

$$I(\Lambda, \Phi) \subset T(\Lambda, \Phi)$$

denote the open subset comprising all points that can be represented by vectors from $V + iI \subset V_{\mathbb{C}}$.

(1.2) Let $\Sigma$ be an admissible fan, and let again $I$ denote the interior of $|\Sigma|$.

LEMMA. *Let $\Gamma \subset GL(V)$ be a subgroup such that $\Lambda$ and $\Sigma$ are $\Gamma$-stable. Then $\Gamma$ has the Siegel property:*

*For any two cones $\sigma \in \Sigma \cap I$ and $\tau \in \Sigma$ one has $\sigma \cap \gamma\bar{\tau} = \varnothing$ for all but a finite number of $\gamma \in \Gamma$.*

*In particular $\Gamma$ acts on $\Sigma \cap I$ with finite isotropy groups $\Gamma_\sigma \subset \Gamma$.*

*Proof.* Let $\sigma \in \Sigma \cap I$; then $\mathrm{St}(\sigma)$ is the union of a finite number of $\tau \in \Sigma \cap I$. Thus the set of all indivisible lattice points that span a one-dimensional face in $\mathrm{St}(\sigma)$ is a finite subset of $\Lambda$ which contains a basis of $V$. Since the isotropy group $\Gamma_\sigma$ preserves this set $\Gamma_\sigma$ is finite.

Let now $\tau \in \Sigma$ be arbitrary. Whenever $\gamma\bar{\tau}$ meets $\sigma$ then $\sigma \subset \gamma\bar{\tau}$ and therefore $\gamma\tau \subset \mathrm{St}(\sigma)$. There are only finitely many cones with this property, and all are contained in $I$. The Siegel property follows since $\Gamma_\tau$ is finite. □

(1.3) PROPOSITION. *Assume that $\Sigma$ is admissible, and that $\Lambda$ and $\Sigma$ are $\Gamma$-stable. Then the (discrete) group $\Gamma$ acts properly on $I(\Lambda, \Sigma)$. In particular the topological quotient $I(\Lambda, \Sigma)/\Gamma$ is a locally compact Hausdorff space.*

*Proof.* Let $v, v' \in I(\Lambda, \Sigma)$; in order to prove properness we construct neighbourhoods $U$ of $v$ and $U'$ of $v'$ such that $U \cap \gamma U' = \varnothing$ for all but a finite number of $\gamma \in \Gamma$. The point $v$ belongs to the stratum $V_{\mathbb{C}}/(\Lambda + \mathbb{C}\sigma) \subset T(\Lambda, \Sigma)$ for some $\sigma \in \Sigma$, and is represented by $z = x + iy \in V + iI$, say. In view of Lemma (1.1) we may choose $y$ in $\mathrm{St}(\sigma) \cap I$. Let $\tau \in \Sigma \cap I$ be the cone containing $y$; then the set

$$V + i\,\mathrm{St}(\tau) \subset V + iI$$

represents an open neighbourhood $U$ of $v$ in $I(\Lambda, \Sigma)$. Similarly, a neighbourhood $U'$ of $v'$ is defined. The Siegel property then implies $U \cap \gamma U' = \varnothing$ for all but finitely many $\gamma \in \Gamma$ as required.

The last clause of the proposition holds since $I(\Lambda, \Sigma)$ is locally compact Hausdorff, and the action of $\Gamma$ is proper. □

COROLLARY. *Under the assumptions of the proposition $I(\Lambda, \Sigma)/\Gamma$ inherits a quotient analytic structure from $I(\Lambda, \Sigma)$, and in this way becomes a normal Cohen–Macaulay analytic variety.*

*Proof.* By [Kempf et al.] p. 52, toroidal singularities are normal and Cohen–Macaulay. Both properties descend to the quotient. □

(1.4). We shall construct certain reflexive analytic sheaves of rank one on the quotients $I(\Lambda, \Sigma)/\Gamma$. They arise from the following type of data.

DEFINITION. Let $\Sigma$ be an admissible fan. A *characteristic triple* $(\pi, \delta, \lambda)$ for $\Sigma$ consists of an epimorphism

$$\tilde{\Lambda} \xrightarrow{\pi} \Lambda$$

of lattices with kernel rank one, a preferred generator $\delta \in \ker \pi$, and, finally, a *characteristic section*

$$|\Sigma| \xrightarrow{\lambda} \tilde{V} := \mathbb{R} \otimes_{\mathbb{Z}} \tilde{\Lambda}.$$

The latter is by definition a section of $\pi\colon \tilde{V} \to V$ satisfying the two axioms:

(i) For each $\sigma \in \Sigma$ the map $\lambda \mid \bar{\sigma}$ is the restriction of a linear map from $\mathbb{R}\sigma$ to $\tilde{V}$, and
(ii) $\lambda(\sigma \cap \Lambda) \subset \tilde{\Lambda}$ for each $\sigma \in \Sigma$ with $\dim \sigma = 1$.

With each characteristic triple $(\pi, \delta, \lambda)$ we associate a sheaf $\mathscr{L}(\pi, \delta, \lambda)$ on $T(\Lambda, \Sigma)$ as follows. Let $I \subset V$ denote the interior of $|\Sigma|$ and put $\tilde{I} = \pi^{-1}(I) \subset \tilde{V}$. In $\tilde{V}$, consider the set of cones

$$\tilde{\Sigma} = \tilde{\Sigma}_- \cup \tilde{\Sigma}_0 \cup \tilde{\Sigma}_+$$

where

$$\tilde{\Sigma}_0 = \{\lambda(\sigma) \mid \sigma \in \Sigma\}$$
$$\tilde{\Sigma}_\pm = \{\lambda(\sigma) \pm \mathbb{R}_+\delta \mid \sigma \in \Sigma\}$$

($\mathbb{R}_+ = (0, \infty)$). This $\tilde{\Sigma}$ is an admissible fan with respect to $\tilde{\Lambda}$, and $\tilde{I}$ is the interior of $|\tilde{\Sigma}|$. In particular we have torus embeddings

$$T(\tilde{\Lambda}) \hookrightarrow T(\tilde{\Lambda}, \tilde{\Sigma}_0) \hookrightarrow T(\tilde{\Lambda}, \tilde{\Sigma}),$$

and the epimorphism $\pi$ induces morphisms of varieties

$$\begin{array}{ccc} I(\tilde{\Lambda}, \tilde{\Sigma}_0) & \hookrightarrow & T(\tilde{\Lambda}, \tilde{\Sigma}_0) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ I(\Lambda, \Sigma) & \hookrightarrow & T(\Lambda, \Sigma) \end{array}$$

also denoted $\pi$ by abuse of language.

NOTATION. Let $\Sigma$ be an admissible fan. Then for each $\tau \in \Sigma$ we put

$$\Sigma_\tau = \{\sigma \in \Sigma \mid \sigma \subset \bar{\tau}\},$$

and for each $d \in \mathbb{N}$,

$$\Sigma^d = \{\sigma \in \Sigma \mid \dim \sigma \leqslant d\};$$

clearly these are closed subsets of $\Sigma$.

LEMMA. *Let $(\pi, \delta, \lambda)$ be a characteristic triple for the admissible fan $\Sigma$, and let $\Phi \subset \Sigma$ be an arbitrary closed subset.*

*Then the inverse image of* $T(\Lambda, \Phi)$ *under* $T(\tilde{\Lambda}, \tilde{\Sigma}_0) \xrightarrow{\pi} T(\Lambda, \Sigma)$ *is* $T(\tilde{\Lambda}, \tilde{\Phi}_0)$ *where*

$$\tilde{\Phi}_0 = \{\tilde{\sigma} \in \tilde{\Sigma} \mid \pi\tilde{\sigma} \in \Phi\}.$$

*If the characteristic section* $\lambda$ *has the property*

$$\lambda(|\Phi| \cap \Lambda) \subset \tilde{\Lambda}$$

*then the restriction*

$$T(\tilde{\Lambda}, \tilde{\Phi}_0) \xrightarrow{\pi} T(\Lambda, \Phi)$$

*is an algebraic* $\mathbb{C}^*$*-principal bundle.*

*Proof.* The first statement is obvious. To prove the second, we choose for each $\sigma \in \Phi$ a linear extension $\lambda_\sigma$ of $\lambda \mid \bar{\sigma}$ that sends $\Lambda$ into $\tilde{\Lambda}$. Writing $\mathbb{C}^* = T(\mathbb{Z})$ we have an algebraic isomorphism

$$T(\Lambda, \Sigma_\sigma) \times T(\mathbb{Z}) \to T(\tilde{\Lambda}, \tilde{\Sigma}_{\lambda\sigma})$$
$$([z], [u]) \mapsto [\lambda_\sigma(z) + u\delta]$$

which is a trivialization of $T(\tilde{\Lambda}, \tilde{\Phi}_0) \xrightarrow{\pi} T(\Lambda, \Phi)$ over the affine open subset $T(\Lambda, \Sigma_\sigma)$. If $\sigma' \in \Phi$ is another cone the corresponding trivializations differ over

$$T(\Lambda, \Sigma_\sigma) \cap T(\Lambda, \Sigma_{\sigma'}) = T(\Lambda, \Sigma_\sigma \cap \Sigma_{\sigma'})$$

by an automorphism of $T(\tilde{\Lambda}, \tilde{\Sigma}_{\lambda\sigma} \cap \tilde{\Sigma}_{\lambda\sigma'})$ which is a translation on each fibre. □

The lemma shows in particular that

$$T(\tilde{\Lambda}, \tilde{\Sigma}_0^1) \xrightarrow{\pi} T(\Lambda, \Sigma^1)$$

always is a $\mathbb{C}^*$-bundle; we let $\mathscr{L}'$ be the sheaf of local sections of the associated line bundle and put

$$\mathscr{L}(\pi, \delta, \lambda) = j_* \mathscr{L}'$$

where $j\colon T(\Lambda, \Sigma^1) \hookrightarrow T(\Lambda, \Sigma)$ is the inclusion.

DEFINITION. We refer to $\mathscr{L}(\pi, \delta, \lambda)$ as the *reflexive sheaf associated to* the characteristic triple $(\pi, \delta, \lambda)$.

(1.5). We collect a few properties of the sheaves $\mathscr{L}(\pi, \lambda, \delta)$ which are immediate consequences of the way they are constructed.

(i) Let $(\pi, \delta, \lambda)$ be a characteristic triple. Any splitting of the exact sequence

$$0 \to \mathbb{Z}\delta \to \tilde{\Lambda} \to \Lambda \to 0$$

will identify the characteristic section $\lambda$ with a certain real-valued function $\mu$ on $|\Sigma|$. If $i\colon T(\Lambda) \hookrightarrow T(\Lambda, \Sigma)$ is the inclusion then $\mathscr{L}(\pi, \delta, \lambda)$ will appear as an $\mathcal{O}_{T(\Lambda,\Sigma)}$-submodule sheaf $\mathscr{M}$ of $i_*\mathcal{O}_{T(\Lambda)}$. The space of sections of that sheaf over $T(\Lambda, \Sigma_\sigma)$ is

$$H^0(T(\Lambda, \Sigma_\sigma); \mathscr{M}) = \bigoplus_{\substack{m\in\Lambda^\vee \\ m\geqslant\mu \text{ on } \sigma}} \mathbb{C}\cdot e^{2\pi i m} \tag{1.6}$$

where $\Lambda^\vee = \operatorname{Hom}(\Lambda, \mathbb{Z})$ denotes the lattice of characters of $T(\Lambda)$. In the terminology of [Kempf et al.], $\mathscr{M}$ is the complete coherent sheaf of $T(\Lambda)$-invariant fractional ideals characterized by the function $\operatorname{ord}\mathscr{M}$ which is the convex interpolation of $\mu$ on $|\Sigma^1|$, see loc. cit. p. 29. This fact may serve as a characterization of the class of sheaves that can be obtained from characteristic triples. It also proves that $\mathscr{L}(\pi, \delta, \lambda)$ is a reflexive sheaf of rank one indeed, as suggested by our terminology.

(ii) $\mathscr{L}(\pi, \delta, \lambda)$ restricts to an invertible sheaf over $T(\Lambda, \Phi)$ ($\Phi \subset \Sigma$ a closed subset) if and only if $\lambda(|\Phi| \cap \Lambda) \subset \tilde{\Lambda}$ holds.

(iii) Let us call two characteristic triples $(\pi, \delta, \lambda)$ and $(\pi', \delta', \lambda')$ isomorphic if there exists a linear isomorphism $\tilde{\Lambda} \xrightarrow{\varphi} \tilde{\Lambda}'$ such that $\pi' \circ \varphi = \pi$, $\varphi(\delta) = \delta'$, and $\varphi \circ \lambda = \lambda'$. Then isomorphic triples give rise to isomorphic sheaves on $T(\Lambda, \Sigma)$.

(iv) If $m$ is a non-zero integer then

$$\mathscr{L}\left(\tilde{\Lambda} + \frac{1}{m}\delta \xrightarrow{\pi} \Lambda, \frac{1}{m}\delta, \lambda\right)$$

is isomorphic to the reflexive hull of the $m$th tensor power $\mathscr{L}(\pi, \delta, \lambda)^m$ $(m > 0)$ respectively of $(\mathscr{L}(\pi, \delta, \lambda)^\vee)^{|m|}$ $(m < 0)$.

(v) The action of $T(\tilde{\Lambda})$ on itself by translation induces on action of $T(\tilde{\Lambda})$ on the bundle

$$T(\tilde{\Lambda}, \tilde{\Sigma}_0^1) \xrightarrow{\pi} T(\Lambda, \Sigma^1)$$

as a group of bundle equivalences, the action on the base being through the homomorphism $T(\tilde{\Lambda}) \to T(\Lambda)$. Therefore, for each $\sigma \in \Sigma$ the torus $T(\tilde{\Lambda})$ acts on the space of sections of $\mathscr{L}(\pi, \delta, \lambda)$ over $T(\Lambda, \Sigma_\sigma)$. Writing $\mathscr{L} = \mathscr{L}(\pi, \delta, \lambda)$ we have a decomposition

$$H^0(T(\Lambda, \Sigma_\sigma); \mathscr{L}) = \bigoplus_{l \in \tilde{\Lambda}^\vee} H^0(T(\Lambda, \Sigma_\sigma); \mathscr{L})^l$$

into $T(\tilde{\Lambda})$-weight spaces, indexed by the character lattice $\tilde{\Lambda}^\vee = \mathrm{Hom}(\tilde{\Lambda}, \mathbb{Z})$. We claim that

$$\dim H^0(T(\Lambda, \Sigma_\sigma); \mathscr{L})^l = 1 \quad \text{if } \langle \delta, l \rangle = 1 \text{ and } l \circ \lambda \leqslant 0 \text{ on } \sigma$$

while $H^0(T(\Lambda, \Sigma_\sigma); \mathscr{L})^l$ is trivial for all other $l \in \tilde{\Lambda}^\vee$. Indeed the kernel of the homomorphism $T(\tilde{\Lambda}) \to T(\Lambda)$ acts with weight 1 on all sections of $T(\tilde{\Lambda}) \to T(\Lambda)$, and for given $l \in \tilde{\Lambda}^\vee$ with $\langle \delta, l \rangle = 1$ the unique section of $\tilde{\Lambda} \xrightarrow{\pi} \Lambda$ with image $\ker l$ represents a generator $s_l \in H^0(T(\Lambda); \mathscr{L})^l$. This section $s_l$ is regular on $T(\Lambda, \Sigma_\sigma)$ if and only if $l \circ \lambda \leqslant 0$ holds on $\sigma$. The resulting decomposition

$$H^0(T(\Lambda, \Sigma_\sigma); \mathscr{L}) = \bigoplus_{\substack{l \in \tilde{\Lambda}^\vee \\ \langle \delta, l \rangle = 1 \\ l \circ \lambda \leqslant 0 \text{ on } \sigma}} \mathbb{C} \cdot s_l \tag{1.7}$$

is just (1.6) in an invariant guise.

For later use we record the following facts.

(1.8) PROPOSITION. *Let $(\pi, \delta, \lambda)$ be a characteristic triple.*

(a) *For each $\sigma \in \Sigma$ there exists an integer $m > 0$ such that the reflexive hull of $\mathscr{L}(\pi, \delta, \lambda)^m$ is invertible on $T(\Lambda, \Sigma_\sigma)$.*

(b) *$\mathscr{L}(\pi, \delta, \lambda)$ is a Cohen–Macaulay sheaf.*

*Proof.* (a) follows from (ii) and (iv) by choosing $m$ sufficiently divisible so that $\lambda(\bar{\sigma} \cap \Lambda)$ is contained in $\tilde{\Lambda} + (1/m)\mathbb{Z}\delta$.

Likewise, to prove the local property (b), we may work over $T(\Lambda, \Sigma_\sigma)$. If we determine $m$ as in (a) then

$$\lambda(\bar{\sigma} \cap m\Lambda) \subset m\tilde{\Lambda} + \mathbb{Z}\delta.$$

Thus the sheaf

$$\tilde{\mathscr{L}} := \mathscr{L}(m\tilde{\Lambda} + \mathbb{Z}\delta \xrightarrow{\pi} m\Lambda, \delta, \lambda)$$

is invertible on $T(m\Lambda, \Sigma_\sigma)$, in particular it is a Cohen–Macaulay sheaf there. If $q$ denotes the quotient morphism

$$T(m\Lambda, \Sigma_\sigma) \xrightarrow{q} T(\Lambda, \Sigma_\sigma)$$

corresponding to the exact sequence

$$0 \to \Lambda/m\Lambda \to T(\Lambda) \to T(m\Lambda) \to 0$$

then we have

$$\mathscr{L}(\pi, \delta, \lambda) = (q_* \tilde{\mathscr{L}})^{\Lambda/m\Lambda}$$

on $T(\Lambda, \Sigma_\sigma)$. The latter sheaf is a direct summand of a Cohen–Macaulay sheaf, and the assertion follows. □

(1.9) In this subsection we fix an admissible fan $\Sigma$ in $V$, and let $I$ denote the interior of $|\Sigma|$ as usual. For any closed subset $\Phi \subset \Sigma$ we define

$$\begin{aligned} I(\Lambda, \Phi)^{\cdot} &= I(\Lambda, \Phi)\backslash I(\Lambda, \Sigma\backslash(\Sigma \cap I)) \\ &= T(\Lambda, \Phi)\backslash T(\Lambda, \Sigma\backslash(\Sigma \cap I)). \end{aligned}$$

As the latter description shows $I(\Lambda, \Phi)^{\cdot}$ is an algebraic subscheme of $T(\Lambda, \Phi)$ rather than a mere analytic space. Note that $I(\Lambda, \Phi)^{\cdot}$ need not be of finite type though it always is so locally.

Let now $(\tilde{\Lambda} \xrightarrow{\pi} \Lambda, \delta, \lambda)$ be a characteristic triple for $\Sigma$. Following the procedure described in [Kempf et al.] p. 42, the (algebraic) cohomology groups $H^i(I(\Lambda, \Phi)^{\cdot}; \mathscr{L}(\pi, \delta, \lambda) \otimes \mathscr{O}_{I(\Lambda,\Phi)^{\cdot}})$ can be computed combinatorially. To this end we put, for each character $l \in \tilde{\Lambda}^\vee$ with $\langle \delta, l \rangle = 1$

$$Y_l(\Phi; \pi, \delta, \lambda) = \{y \in |\Phi| \cap I \mid \langle y, l \circ \lambda \rangle \leqslant 0\}$$
$$Y_l'(\Phi; \pi, \delta, \lambda) = \{y \in |\Phi| \cap I \mid \langle y, l \circ \lambda \rangle < 0\}.$$

(1.10) PROPOSITION. *For each $l \in \tilde{\Lambda}^\vee$ with $\langle \delta, l \rangle = 1$ there is a natural additive isomorphism*

$$\begin{aligned} &H^*(I(\Lambda, \Phi)^{\cdot}; \mathscr{L}(\pi, \delta, \lambda) \otimes \mathscr{O}_{I(\Lambda,\Phi)^{\cdot}})^l \\ &\qquad \simeq H^*(Y_l(\Phi; \pi, \delta, \lambda), Y_l'(\Phi; \pi, \delta, \lambda); \mathbb{C}). \end{aligned}$$

*Proof.* Almost verbatim that of loc. cit. p. 42. □

(1.11) We make the same assumptions as in the previous subsection.

DEFINITION. A subgroup $\Gamma \subset GL(V)$ is *admissible* (with respect to $\Sigma$) if

(i) $\Lambda$ and $\Sigma$ are $\Gamma$-stable, and
(ii) the orbit set $(\Sigma \cap I)/\Gamma$ is finite.

Our aim is to prove an analogue of (1.10) for the quotient of $I(\Lambda, \Sigma)^{\cdot}$ by an admissible group $\Gamma$. We first note:

(1.12) PROPOSITION. *The analytic subspace $I(\Lambda, \Sigma)^{\cdot}/\Gamma$ of $I(\Lambda, \Sigma)/\Gamma$ is compact.*

*Proof.* Let $\sigma \in \Sigma \cap I$. Since $\mathrm{St}(\sigma)$ is a neighbourhood of $\sigma$ in $I$ the closure of the stratum $V_{\mathbb{C}}/(\Lambda + \mathbb{C}\sigma)$ in $I(\Lambda, \Sigma)$ is a compact analytic space. As $\Gamma$ is assumed to act admissibly there are but a finite number of $\Gamma$-orbits in $\Sigma \cap I$, cf. (iii) of the definition in (1.2). Therefore $I(\Lambda, \Sigma)^{\cdot}/\Gamma$ is covered by a finite number of compact spaces, and the proposition follows. □

DEFINITION. A group $\Gamma \subset GL(\tilde{V})$ is said to act *admissibly* on the characteristic triple $(\pi, \delta, \lambda)$ if $\delta$ is a fixed point of $\Gamma$, and if the action of $\Gamma$ on $\tilde{V}$ is admissible with respect to $\tilde{\Sigma}$ and descends to an admissible action on $V$ that makes $|\Sigma| \xrightarrow{\lambda} \tilde{V}$ equivariant too.

If $\Gamma$ acts admissibly on $(\pi, \delta, \lambda)$, and if

$$I(\Lambda, \Sigma) \xrightarrow{q} I(\Lambda, \Sigma)/\Gamma$$

is the induced analytic quotient then $\mathscr{L}(\pi, \delta, \lambda)$ descends to a reflexive analytic sheaf

$$\mathscr{L}(\pi, \delta, \lambda)/\Gamma := (q_*\mathscr{L}(\pi, \delta, \lambda))^{\Gamma}$$

on $I(\Lambda, \Sigma)/\Gamma$. Since $q$ is locally a finite mapping this sheaf still is a Cohen–Macaulay sheaf as in (1.8).

Recall that for each closed $\Phi \subset \Sigma$ and each $l \in \tilde{\Lambda}^{\vee}$ with $\langle \delta, l \rangle = 1$ we have defined a topological pair

$$(Y_l(\Phi; \pi, \delta, \lambda),\ Y_l'(\Phi; \pi, \delta, \lambda)).$$

If $\Phi$ is $\Gamma$-stable then $\Gamma$ also acts on the disjoint sum

$$\sum_{\substack{l \in \tilde{\Lambda}^{\vee} \\ \langle \delta, l \rangle = 1}} (Y_l(\Phi; \pi, \delta, \lambda),\ Y_l'(\Phi; \pi, \delta, \lambda))$$

via

$$\Gamma \ni \gamma \colon (l, y) \mapsto (l \circ \gamma^{-1}, \gamma y).$$

This action is properly discontinuous with finite isotropy groups, and has a finite Hausdorff quotient

$$(Y(\Phi; \pi, \delta, \lambda),\ Y'(\Phi; \pi, \delta, \lambda)) = \left[\sum_l (Y_l(\Phi; \pi, \delta, \lambda),\ Y_l'(\Phi; \pi, \delta, \lambda))\right]\Big/\Gamma.$$

(1.13) THEOREM. *Let $\Sigma$ be an admissible fan in $V$, and let $(\pi, \delta, \lambda)$ be a* *characteristic triple for $\Sigma$, on which $\Gamma \subset GL(\tilde{V})$ acts admissibly. Then the (analytic) cohomology of $\mathscr{L}(\pi,\delta,\lambda)/\Gamma$ over $I(\Lambda,\Sigma)^{\cdot}/\Gamma$ can be computed via a canonical isomorphism*

$$H^*(I(\Lambda,\ \Sigma)^{\cdot}/\Gamma;\ \mathscr{L}(\pi,\ \delta,\ \lambda)/\Gamma \otimes \mathcal{O}_{I(\Lambda,\Sigma)^{\cdot}/\Gamma)} \simeq H^*(Y(\Sigma;\ \pi,\ \delta,\ \lambda),\ Y'(\Sigma;\ \pi,\ \delta,\ \lambda);\ \mathbb{C}).$$

*Proof.* We first prove the existence of a normal subgroup $\Gamma' \subset \Gamma$ of finite index, with the property:

(1.14) Whenever $\gamma \in \Gamma'$ and $\tau \in \Sigma \cap I$ are such that $\bar{\tau} \cap \gamma\bar{\tau} \cap I \neq \varnothing$ then $\gamma = 1$.

Let $F \subset \Lambda$ be the set of indivisible generators of one-dimensional cones in $\Sigma$, and put

$$F_2 = \{\{v,\ w\} \subset F \mid v \neq w, \text{ and } \{v,\ w\} \subset \overline{\mathrm{St}(\sigma)} \text{ for some } \sigma \in \Sigma^1 \cap I\}.$$

Since $\Gamma$ is admissible it acts on $F_2$ with a finite number of orbits, and we find a positive integer $m$ such that

$$v - w \notin m\Lambda \quad \text{for all } \{v,\ w\} \in F_2.$$

Then

$$\Gamma' := \{\gamma \in \Gamma \mid (\gamma - \mathrm{id})(\Lambda) \subset m\Lambda\}$$

is a normal subgroup of finite index in $\Gamma$ which satisfies (1.14). Indeed, let $\gamma \in \Gamma'$ and $\tau \in \Sigma \cap I$ be such that $\bar{\tau} \cap \gamma\bar{\tau} \cap I$ is non-empty. This intersection contains some $\sigma \in \Sigma^1 \cap I$, spanned by $v \in F$, say. We have

$$\gamma v \in \gamma\bar{\tau} \subset \overline{\mathrm{St}(\sigma)},$$

and by definition of $\Gamma'$ this implies $\gamma v = v$. Thus $\gamma$ maps $\overline{\mathrm{St}(\sigma)}$ into itself, and therefore must be the identity.

We now choose $\Gamma'$ so that (1.14) holds. For each $\tau \in \Sigma \cap I$ the union

$$\bigcup_{\gamma \in \Gamma'} I(\Lambda,\ \Sigma_{\gamma\tau})^{\cdot}$$

then is a disjoint one; therefore $I(\Lambda,\Sigma)^{\cdot}/\Gamma'$ is covered by the affine varieties

$$\left[\bigcup_{\gamma \in \Gamma'} I(\Lambda,\ \Sigma_{\gamma\tau})^{\cdot}\right] \Big/ \Gamma' = I(\Lambda,\ \Sigma_{\tau})^{\cdot},$$

with $\tau$ running through a system of representatives of $\Sigma \cap I \bmod \Gamma'$. Thus $I(\Lambda, \Sigma)^{\cdot}/\Gamma'$ itself has the structure of a complex algebraic variety, which is complete by (1.12). In view of the GAGA principle we may compute the cohomology groups in question as algebraic rather than analytic cohomology (for $\Gamma'$ in place of $\Gamma$).

Likewise (1.14) implies that for each $\tau \in \Sigma \cap I$ the union

$$\bigcup_{\gamma \in \Gamma'} (\gamma\bar{\tau} \cap I)$$

is disjoint, and that therefore the pair

$$(Y(\Sigma; \pi, \delta, \lambda), Y'(\Sigma; \pi, \delta, \lambda))$$

defined by $\Gamma'$ is covered by the pairs

$$\sum_{l} (Y_l(\Sigma_\tau; \pi, \delta, \lambda), Y_l'(\Sigma_\tau; \pi, \delta, \lambda))$$

($\tau \in (\Sigma \cap I)/\Gamma'$). The assertion of the theorem for the group $\Gamma'$ in place of $\Gamma$ now follows upon comparing the Čech complex of the affine covering of $I(\Lambda, \Sigma)^{\cdot}/\Gamma'$ (with coefficients in $\mathscr{L}(\pi, \delta, \lambda)/\Gamma' \otimes \mathcal{O}_{I(\Lambda,\Sigma)^{\cdot}/\Gamma'}$) to that of the covering of the pair $(Y(\Sigma; \pi, \delta, \lambda), Y'(\Sigma; \pi, \delta, \lambda))$ (with constant complex coefficients).

The isomorphism thus obtained is equivariant with respect to the finite group $\Gamma/\Gamma'$. The theorem therefore follows in general upon passing to the $\Gamma/\Gamma'$-fixed parts of the cohomology groups. □

(1.15) We keep using the notation from the previous subsections. There is a criterion of ampleness for the sheaves $\mathscr{L}(\pi, \delta, \lambda)/\Gamma$, as follows.

(1.16) THEOREM. *Let $\Sigma$ be an admissible fan in $V$, and let $(\pi, \delta, \lambda)$ be a characteristic triple for $\Sigma$. Further, let $\Gamma \subset GL(\tilde{V})$ act admissibly on the triple $(\pi, \delta, \lambda)$.*

*Then the sheaf $\mathscr{L}(\pi, \delta, \lambda)/\Gamma \otimes \mathcal{O}_{I(\Lambda,\Sigma)^{\cdot}/\Gamma}$ on $I(\Lambda, \Sigma)^{\cdot}/\Gamma$ is ample if and only if the characteristic section $\lambda$ is strictly convex on $\Sigma \cap I$ in the sense that for each $\sigma \in \Sigma \cap I$ there exists an $l \in \tilde{\Lambda}^\vee$ such that*

$$\langle \delta, l \rangle = 1$$

$l \circ \lambda \geqslant 0$ *everywhere on* $|\Sigma|$, *and*

$l \circ \lambda = 0$ *exactly on* $\bar{\sigma}$.

*Proof.* Since there are only finitely many $\Gamma$-orbits in $\Sigma \cap I$ Proposition (1.8) implies that for suitable $m > 0$ the reflexive hull of $\mathscr{L}(\pi, \delta, \lambda)^m$ is invertible. Replacing $(\tilde{\Lambda} \xrightarrow{\pi} \Lambda, \delta, \lambda)$ by $(\tilde{\Lambda} + (1/m)\mathbb{Z}\delta \xrightarrow{\pi} \Lambda, (1/m)\delta, \lambda)$ we thus may assume that $\mathscr{L} := \mathscr{L}(\pi, \delta, \lambda)$ itself is invertible.

First suppose that $\mathscr{L}/\Gamma \otimes \mathcal{O}_{I(\Lambda,\Sigma)^{\cdot}/\Gamma}$ is ample, and fix any $\sigma \in \Sigma \cap I$. The closure $S$ of the stratum $V_{\mathbb{C}}/(\Lambda + \mathbb{C}\sigma)$ in $I(\Lambda, \Sigma)^{\cdot}$ is a compact analytic variety, and the quotient mapping

$$S \xrightarrow{q} I(\Lambda, \Sigma)^{\cdot}/\Gamma$$

is a finite morphism. At the cost of raising $\mathscr{L}$ to a suitable tensor power we may assume that for each $s \in S$ the stalk $\mathscr{L}_s \otimes \mathcal{O}_{S,s}$ is isomorphic to $\mathcal{O}_{S,s}$ as a $\Gamma_s$-module. It then follows that

$$\mathscr{L} \otimes \mathcal{O}_S = q^*(\mathscr{L}/\Gamma \otimes \mathcal{O}_{I(\Lambda,\Sigma)^{\cdot}/\Gamma}),$$

which shows that $\mathscr{L} \otimes \mathcal{O}_S$ is an ample sheaf on $S$.

The variety $S$ is the torus embedding of $T(\Lambda/(\Lambda \cap \mathbb{R}\sigma))$ defined by projecting all members of $\Sigma$ that are contained in $\mathrm{St}(\sigma)$. Applying the ampleness criterion [Kempf et al.] p. 48, Theorem 13 we conclude that $\lambda$ is strictly convex at $\sigma$. This being true for all $\sigma \in \Sigma \cap I$ it easily follows that $\lambda$ is strictly convex globally, using the fact that each $\Lambda$-rational compact segment in $|\Sigma|$ meets only finitely many $\sigma \in \Sigma$.

Assume now that $\lambda$ is strictly convex on $\Sigma \cap I$. We choose a finite closed subset $\Phi \subset \Sigma$ sufficiently large so that $I(\Lambda, \Phi)^{\cdot}$ maps onto $I(\Lambda, \Sigma)^{\cdot}/\Gamma$. Arguing as in [Demazure 1] p. 568, Théorème 2 we find a positive integer $m$ and a finite number of characters $l \in \tilde{\Lambda}^{\vee}$ with $\langle \delta, l \rangle = m$ such that the corresponding sections in $H^0(I(\Lambda, \Sigma)^{\cdot}; \mathscr{L}^m \otimes \mathcal{O}_{I(\Lambda,\Sigma)^{\cdot}})$ define an embedding of $I(\Lambda, \Phi)^{\cdot}$ in a projective space. Enlarging $m$ we can arrange the choice of the characters $l$ so that all these sections have support in $I(\Lambda, \Phi)^{\cdot}$.

Let now $z$ and $z'$ be points in $I(\Lambda, \Phi)^{\cdot}$ with $\Gamma z \neq \Gamma z'$. By the preceding, we find an even larger $m$ and a section

$$s \in H^0(I(\Lambda, \Sigma)^{\cdot}; \mathscr{L}^m \otimes \mathcal{O}_{I(\Lambda,\Sigma)^{\cdot}})$$

that vanishes on all but one point of $\Gamma z \cup \Gamma z'$, and has support in $I(\Lambda, \Phi)^{\cdot}$. As the number of points any $\Gamma$-orbit can have in $I(\Lambda, \Phi)^{\cdot}$ is bounded, $m$ can be chosen uniformly for all choices of $z$ and $z'$ in $I(\Lambda, \Phi)^{\cdot}$. Since the action of $\Gamma$ on $(\pi, \delta, \lambda)$ is admissible the sum

$$\sum_{\gamma \in \Gamma} \gamma^* s$$

is locally finite and thus defines an invariant analytic section in

$$H^0(I(\Lambda, \Sigma)^{\cdot}; \mathscr{L}^m \otimes \mathscr{O}_{I(\Lambda,\Sigma)^{\cdot}})^\Gamma = H^0(I(\Lambda,\Sigma)^{\cdot}/\Gamma; \mathscr{L}^m/\Gamma \otimes \mathscr{O}_{I(\Lambda,\Sigma)^{\cdot}/\Gamma}).$$

By construction this section separates the orbits $\Gamma z$ and $\Gamma z'$.

Likewise one can construct invariant sections that separate tangent vectors. We omit the details. □

## 2. A family of abelian varieties

(2.1) Let $R$ be a reduced irreducible finite root system, and let

$$\tilde{R} := R^{\vee\wedge\vee}$$

be the dual of the affine root system obtained by completing $R^\vee$. A choice of a basis

$$\{\alpha_1, \ldots, \alpha_r\}$$

of $R$ also determines bases of $R^\vee$, $R^{\vee\wedge}$ and hence of $\tilde{R}$; we let $\alpha_0 \in \tilde{R}$ denote the extra base root. Note that $R$ is canonically embedded in $\tilde{R}$.

We think of $\tilde{R}$ as realized in a real vector space $\tilde{V}$ of (minimal) dimension $r+2$. Following [Looijenga 3] we introduce the Tits cone $\tilde{I}^+ \subset \tilde{V}$ of $\tilde{R}$; its interior $\tilde{I} \subset \tilde{V}$ determines the tube domain

$$\tilde{\Omega} = \tilde{V} + i\tilde{I} \subset \tilde{V}_{\mathbb{C}}.$$

Let $\tilde{Q} = \mathbb{Z}\tilde{R}$ respectively $Q = \mathbb{Z}R$ denote the root lattices of $\tilde{R}$ and $R$, and define $\delta \in \tilde{Q}$ as the smallest positive linear combination of the base roots that is orthogonal to $(\tilde{R})^\vee = R^{\vee\wedge}$. Then as the coefficient of $\alpha_0$ in $\delta$ is 1 the projection

$$\tilde{V} \xrightarrow{\pi} \tilde{V}/\mathbb{R}\delta =: V$$

will send the sublattice $Q \subset \tilde{Q}$ bijectively to $\tilde{Q}/\mathbb{Z}\delta$. In this way we will often identify these two lattices; in particular this allows us to think of $R$ as a root system in $V$. If we put

$$I^+ = \pi(\tilde{I}^+) \subset V, \qquad I = \pi(\tilde{I}), \qquad \Omega = \pi(\tilde{\Omega}) \subset V_{\mathbb{C}}$$

then

$$\tilde{I}^+ = \pi^{-1}(I^+), \qquad \tilde{I} = \pi^{-1}(I), \qquad \tilde{\Omega} = \pi^{-1}(\Omega).$$

(2.2) The root systems $R$ and $\tilde{R}$ generate Weyl groups

$$W \subset \tilde{W} \subset GL(\tilde{V}),$$

the first of which, $W$, preserves the subspace $\mathbb{R}Q \subset \tilde{V}$. We also have an extended Weyl group

$$W^\wedge = \tilde{Q} \rtimes \tilde{W} \subset \tilde{V} \rtimes GL(\tilde{V})$$

of $\tilde{R}$, which is a group of affine automorphisms of $\tilde{V}$. These groups also act on the tube domains $\Omega$ and $\tilde{\Omega}$, as is clear from the definitions.

The structure of $\tilde{W}$ and $W^\wedge$ can be understood as follows. Let $T \subset \tilde{W}$ be the kernel of the projection homomorphism

$$\tilde{W} \to GL(V)$$

(which is defined since $\delta$ is a fixed point of $\tilde{W}$). The obvious group homomorphisms fit into exact sequences

$$\begin{array}{ccccccccc} 1 \to & T & \to & \tilde{W} & \to & W & \to 1 \\ 1 \to & \tilde{Q} \rtimes T & \to & W^\wedge & \to & W & \to 1 \\ 0 \to & \mathbb{Z}\delta & \to & \tilde{Q} \rtimes T & \to & Q \times T & \to 1. \end{array} \tag{2.3}$$

In order to give an explicit description of these extensions we introduce the $W$-invariant scalar product $(?\,|\,?)$ on $Q$, normalized by the requirement

$$(\delta - \alpha_0 \,|\, \delta - \alpha_0) = 2$$

(note that $\delta - \alpha_0$ is a short root of $R$). We shall also think of $(?\,|\,?)$ as defined on the vector space $\mathbb{R}Q$, and sometimes even on $\tilde{Q}$ or $\mathbb{R}\tilde{Q}$ (with a radical spanned by $\delta$).

In order to fix coordinates on $\tilde{V}$ we choose a vector $\beta \in \tilde{V}$ which is orthogonal on $R^\vee$ and is normalized by

$$\langle \beta, \alpha_0^\vee \rangle = 1.$$

We then assign to each coordinate triple $(u, z, \tau) \in \mathbb{R} \times \mathbb{R}Q \times \mathbb{R}$ the point

$$-u\delta + z + \tau\beta \in \mathbb{R}\delta + \mathbb{R}Q + \mathbb{R}\beta = \tilde{V}.$$

Each element of $T \subset \tilde{W}$ can now be written

$$(u, z, \tau) \mapsto (u + (t \mid z) + \tfrac{1}{2}(t \mid t)\tau,\ z + \tau t,\ \tau)$$

for a uniquely determined $t \in Q$, and this sets up an isomorphism between $T$ and the additive group $Q$, cf. [Kac] p. 69 or [Looijenga 3] p. 28. This formula likewise describes the extensions (2.3); indeed the action of $W$ on $T$ by conjugation corresponds to the natural action of $W$ on the lattice $Q$, and $\tilde{Q} \rtimes T$ becomes a Heisenberg group with centre $\mathbb{Z}\delta$.

Note that the actions of $\tilde{W}$ on $\tilde{I}$, and of $W^\wedge$ on $\tilde{\Omega}$ are properly discontinuous with finite isotropy groups, and yield Hausdorff topological respectively analytic quotients. This follows directly from the explicit description of the actions though it also is part of the general theory of root systems [Looijenga 3].

(2.4) There is a natural action of the modular group $SL_2(\mathbb{Z})$ on $\tilde{V}_{\mathbb{C}}$ and $\tilde{\Omega}$, expressed by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u, z, \tau) \mapsto \left( u - \frac{c}{2} \frac{(z \mid z)}{c\tau + d}, \frac{1}{c\tau + d} z, \frac{a\tau + b}{c\tau + d} \right).$$

In fact this defines an action of a semi-direct product $W^\wedge \rtimes SL_2(\mathbb{Z})$, the latter being determined as follows: conjugation by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

acts trivially on $\mathbb{Z}\delta \times W \subset W^\wedge$, and sends $z \rtimes t \in Q \rtimes T \simeq Q \rtimes Q$ to

$$[(\tfrac{1}{2}ac(z \mid z) - bc(z \mid t) + \tfrac{1}{2}bd(t \mid t))\delta + az - bt] \rtimes (-cz + dt) \in \tilde{Q} \rtimes T.$$

Since $SL_2(\mathbb{Z})$ acts properly discontinuously, with finite isotropy groups, and with a Hausdorff analytic quotient on the $W^\wedge$-invariant coordinate $\tau$ varying in the upper half plane $H$ the same properties hold for the group $W^\wedge \rtimes SL_2(\mathbb{Z})$ with respect to its action on the tube domain $\tilde{\Omega}$.

We now fix a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ of finite index, and study the action of $\Gamma$ on the $\mathbb{C}^*$-principal bundle

$$\begin{array}{c} \mathbb{C}^* = \mathbb{C}\delta/\mathbb{Z}\delta \hookrightarrow \tilde{\Omega}/\tilde{Q} \rtimes T \\ \downarrow \\ \Omega/Q \rtimes T \end{array} \tag{2.5}$$

Note that $\tilde{\Omega} \subset \tilde{V}_{\mathbb{C}}$ and $\Omega \subset V_{\mathbb{C}}$ can be described as the inverse images of the upper half plane $H$ under the complex coordinate $\tau$ on $\tilde{V}_{\mathbb{C}}$ resp. $V_{\mathbb{C}}$, cf. [Kac], [Looijenga 3]. Adjoining the fibre $\mathbb{C}$ via

$$\begin{aligned} &\mathbb{C}\delta/\mathbb{Z}\delta = \mathbb{C}/\mathbb{Z} = \mathbb{C}^* \hookrightarrow \mathbb{C} \\ &u\delta \leftarrow\!\shortmid u \mapsto e^{2\pi i u} \end{aligned}$$

we pass to a line bundle on $\Omega/Q \rtimes T$. Let $\tilde{\mathscr{L}}$ denote the corresponding invertible sheaf. Dividing now by the action of $\Gamma$ we obtain an analytic quotient

$$\begin{array}{ccccc} \Omega/Q \rtimes T & \xrightarrow{q} & \Omega/(Q \rtimes T) \rtimes \Gamma & =: & A' \\ \downarrow{\scriptstyle\tau} & & \downarrow & & \downarrow{\scriptstyle p'} \\ H & \rightarrow & H/\Gamma & =: & C' \end{array}$$

together with the coherent analytic sheaf

$$\mathscr{L}' := (q_* \tilde{\mathscr{L}})^{\Gamma}$$

on $A'$. For any $\tau \in H$ the fibre $A'_{\Gamma\tau}$ of $p'$ over the orbit $\Gamma\tau \in C'$ is the quotient of the abelian variety $\Omega_\tau/Q \rtimes T$ by the isotropy group $\Gamma_\tau$ of $\tau$, with a non-reduced structure if this group contains elements other than $\pm 1$, i.e. if $\tau$ is an elliptic fixed point of $\Gamma$. Note that in case $\Gamma$ contains $-1$ the general fibre $A'_{\Gamma\tau}$ is not an abelian variety but its (singular) Kummer quotient.

The sheaf $\mathscr{L}'$ on $A'$ always is, by construction, a reflexive Cohen–Macaulay sheaf of rank one.

(2.6) Let $C$ be the compact curve obtained from $C'$ by adding the cusps of $\Gamma$. We wish to extend $A'$ and $\mathscr{L}'$ over $C$. Quite general compactifications of families of abelian varieties have been constructed by Namikawa, cf. [Namikawa]. The present situation is particularly simple as the abelian varieties in question are mere products of several copies of an elliptic curve, and only the modulus of that curve is allowed to vary. In fact quite an explicit compactification of $A'$ is readily at hand if a construction from [Wirthmüller 2] is used. Like Namikawa's it is of toroidal type and, based on the discussion of reflexive sheaves in Section 1, it also provides a natural extension of $\mathscr{L}'$.

We proceed to describe this compactification. As $SL_2(\mathbb{Z})$ acts transitively on the set of cusps of $\Gamma$ it suffices to deal with the standard cusp at $i\infty$. A typical punctured neighbourhood of this cusp is represented by the affine upper half plane

$$ic + H \subset \mathbb{C} \quad (c > 0),$$

and as is well known, for $c \geqslant 1$ each orbit of the isotropy group $\Gamma_{i\infty}$ in $ic + H$ is the intersection of $ic+H$ with a full $\Gamma$-orbit. If the cusp at $i\infty$ is a regular one we have

$$\begin{aligned}\Gamma_{i\infty} &= \left\{\begin{pmatrix}1 & ks\\ 0 & 1\end{pmatrix}\middle|\, k\in\mathbb{Z}\right\} \quad \text{if } -1\notin\Gamma\\ \Gamma_{i\infty} &= \{\pm 1\}\times\left\{\begin{pmatrix}1 & ks\\ 0 & 1\end{pmatrix}\middle|\, k\in\mathbb{Z}\right\} \quad \text{if } -1\in\Gamma\end{aligned} \tag{2.7}$$

for some positive integer $s$. Similarly, in the case of an irregular cusp we have

$$\Gamma_{i\infty} = \left\{\begin{pmatrix}(-1)^k & ks\\ 0 & (-1)^k\end{pmatrix}\middle|\, k\in\mathbb{Z}\right\}. \tag{2.8}$$

Note that the matrix

$$\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}$$

acts on $\tilde{V}$ as the translation by the vector $s\beta$.

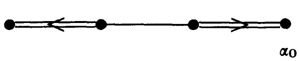
(2.9) The root basis $\{\alpha_0,\ldots,\alpha_r\}$ together with the vector $-s\beta$ constitutes a mixed root basis in the sense of [Wirthmüller 2] Section 9. Its Dynkin diagram $\mathscr{D}$ comprises the Dynkin diagram of $\tilde{R}$ as the full subdiagram $\mathscr{D}_{\text{black}}$ while $\mathscr{D}_{\text{white}}$ consists of just one vertex realized by the vector $-s\beta$. The remaining edges of $\mathscr{D}$ are determined by the values

$$\langle -s\beta, \alpha_0^\vee\rangle = -s$$
$$\langle s\beta, \alpha_i^\vee\rangle = 0 \quad \text{for } i=1,\ldots,r.$$

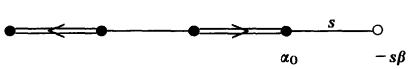
EXAMPLE. *If $R$ is of type $B_3$*



then the Dynkin diagram of $\tilde{R}$ is




and $\mathscr{D}$ is the mixed diagram



Following loc. cit. the mixed root basis gives rise to a toroidal embedding $\mathscr{X}(\mathscr{D})$ of $\tilde{\Omega}/\tilde{\Lambda}$ where $\tilde{\Lambda}$ denotes the lattice of rank $r+2$

$$\tilde{\Lambda} = \tilde{Q} + \mathbb{Z}s\beta.$$

Let $K \subset \tilde{V}$ be the convex cone spanned by the orbit $\tilde{W}\beta$; then $K \subset \tilde{I}^+$ since $\beta$ lies in the closed fundamental chamber $\bar{C}$ of $\tilde{R}$. The analytic space $\mathscr{X}(\mathscr{D})$ constructed in loc. cit. is, by definition, the union of a copy of the upper half plane $H$, and the open subset of the affine torus embedding determined by $\tilde{\Lambda}$ and $K$, comprising all points representable by vectors in $\tilde{\Omega}$.

(2.10) Rather than working with $\mathscr{X}(\mathscr{D})$ directly, we use the cone $K$ in order to construct a (non-affine) torus embedding of $T(\Lambda)$, where

$$\Lambda = Q + \mathbb{Z}s\beta \subset V.$$

To this end we need an admissible fan $\Sigma_K$ for $\Lambda$, which we take to be the set of projections of all proper faces of $K$ under the linear map $\tilde{V} \xrightarrow{\pi} V$. The following proposition will show, in particular, that $\Sigma_K$ is admissible indeed.

We let $\tilde{\Sigma}_K$ denote the set of proper faces of the closed cone

$$\bar{K} = K \cup \mathbb{R}_-\delta \quad (\mathbb{R}_- = (-\infty, 0)),$$

and put

$$K^\cdot = \bigcup_{\tilde{\sigma} \in \tilde{\Sigma}_K} \tilde{\sigma};$$

$K^\cdot$ is the topological boundary of $K$ in $\tilde{V}$. Thus

$$\Sigma_K = \{\pi\tilde{\sigma} \mid \tilde{\sigma} \in \tilde{\Sigma}_K\}$$

by definition.

(2.11) PROPOSITION. (a) $\pi(K) = I^+$, *and for each* $(z, \tau) \in I$ *the set*

$$\{u \in \mathbb{R} \mid (u, z, \tau) \in K\}$$

*is a closed ray in* $\mathbb{R}$, *bounded from below.*

(b) *The restriction of* $\pi\colon \tilde{V} \to V$

$$K^\cdot \cap \tilde{I} \xrightarrow{\pi} I$$

*is a homeomorphism.*

(c) *$K^{\cdot}\backslash\mathbb{R}_{-}\delta$ is the image of a section*

$$\lambda_K: I^+ \to \tilde{V}$$

*which is characteristic with respect to $\Sigma_K$ and $\tilde{\Lambda} \xrightarrow{\pi} \Lambda$.*

*Proof.* Clearly $\pi(K)$ is contained in $I^+$. On the other hand $\{(z,\tau)\in\pi(K)\mid\tau = 1\}$ is a non-empty convex set which is invariant under the lattice of translations $T \simeq Q$, whence this set is the full affine space $\{(z,\tau)\in V\mid\tau = 1\}$. This proves $\pi(K) = I^+$.

Let $\bar{C} \subset \tilde{V}$ denote the closed fundamental chamber of $\tilde{R}$ as before. Let $y\in\bar{C}\cap\tilde{I}$ be arbitrary. By [Looijenga 3] (2.4) the convex hull of the orbit $\tilde{W}y$ meets $\bar{C}$ exactly along the set

$$\bar{C}\cap(y + \bar{R}_{-}\{\alpha_0,\ldots,\alpha_r\})$$

where $\bar{\mathbb{R}}_{-} = (-\infty, 0]$. As the chambers induce a locally finite covering of the open Tits cone $\tilde{I}$ this implies that $K\cap\tilde{I}$ is closed in $\tilde{I}$ (put $y = \beta$). For $y\in\bar{C}\cap\tilde{I}\cap K$ it also implies

$$y + \bar{\mathbb{R}}_{-}\delta \subset \bar{C}\cap\tilde{I}\cap K,$$

and since $\bar{C}$ is a fundamental domain for the action of $\tilde{W}$ on $\tilde{I}^+$ we conclude

$$y + \mathbb{R}_{-}\delta \subset \tilde{I}\cap K \quad \text{for each } y\in\tilde{I}\cap K.$$

Given $(z,\tau)\in I$, the set $\{u\in\mathbb{R}\mid(u,z,\tau)\in K\}$ therefore equals $[t,\infty)$ for some $t\in\mathbb{R}$, or $t = -\infty$. In fact this last possibility is ruled out at once since $\tilde{W}\beta$, and hence $K$, is completely contained in the half space $\{(u,z,\tau)\in\tilde{V}\mid u \geqslant 0\}$. This completes the proof of (a).

We now know that for any $y\in I$ the set $\pi^{-1}(y)\cap K$ is a closed ray in $\tilde{V}$. This ray intersects $K^{\cdot}$ in its end point, which we define to be $\lambda_K(y)$. We complete the definition of $\lambda_K: I^+ \to \tilde{V}$ by putting $\lambda_K(0) = 0$. Then $\lambda_K\mid\bar{\sigma}$ is linear for each $\sigma\in\Sigma_K$ because $\pi$ is linear.

The fundamental chamber $\bar{C}$ meets but a finite number of cones from $\tilde{\Sigma}_K\cap\tilde{I}$. Since $\bar{C}$ has the form

$$\bar{C} = \pi^{-1}(\pi\bar{C})$$

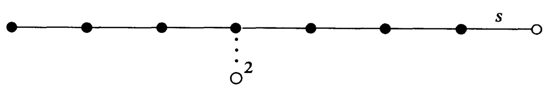
this means that $\pi\bar{C}$ meets but a finite number of cones from $\Sigma_K\cap I$, which in turn implies that $\Sigma_K\cap I$ is a locally finite covering of $I$. The section $I \xrightarrow{\lambda_K} K^{\cdot}\cap\tilde{I}$, being piecewise linear, thus is seen to be a continuous inverse to $K^{\cdot}\cap\tilde{I}\xrightarrow{\pi} I$.

In particular we now have proved (b), and in order to complete the proof of (c) it remains to show the integrality property

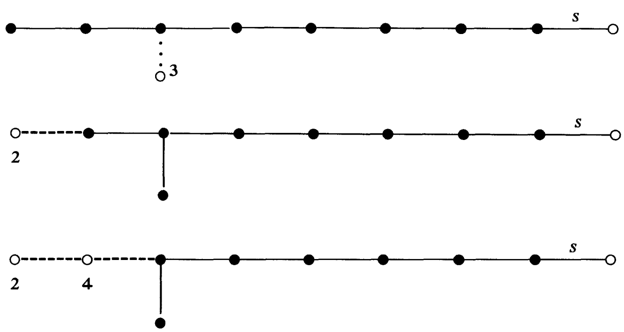
$$\lambda_K(\bar{\sigma} \cap \Lambda) \subset \tilde{\Lambda}$$

for each $\sigma \in \Sigma_K$ with $\dim \sigma = 1$. The image $\lambda_K(\sigma) \in \tilde{\Sigma}_K$ is a one-dimensional face of $K$, and in view of the classification of faces of $K$ given in [Wirthmüller 2] p. 230, Theorem 9.5 that face is $\tilde{W}$-conjugate to the ray $\mathbb{R}_+\beta \subset \tilde{V}$. We therefore may assume $\lambda_K(\sigma) = \mathbb{R}_+\beta$, and the assertion follows since $s \cdot \pi\beta \in V$ is a primitive vector in $\Lambda$. □

REMARK. It is not generally true that $\lambda_K(I^+ \cap \Lambda) \subset \tilde{\Lambda}$. To what extent this fails may be read off from the classification of faces of $K$. In fact, up to $\tilde{W}$-conjugacy these faces are classified by the subdiagrams of $\mathscr{D}$ in the sense of loc. cit. It is readily verified by inspection that the only $\sigma \in \Sigma_K$ with $\lambda_K(\bar{\sigma} \cap \Lambda) \not\subset \tilde{\Lambda}$ are those with $\lambda_K(\sigma)$ corresponding to one of the subdiagrams




($R$ of type $E_7$), or




($R$ of type $E_8$). The numbers marking the vertices that are not included in the subdiagram are their multiplicities in the greatest root $\delta - \alpha_0$ of $R$; in all cases the g.c.d. of these numbers is the smallest positive integer $m$ with $\lambda_K(\bar{\sigma} \cap \Lambda) \subset (1/m)\tilde{\Lambda}$.

(2.12) We let $\Sigma_k$ and $\lambda_K$ retain the meaning from the previous subsection.

LEMMA. *The group $T$ acts freely on $\Sigma_K \cap I$.*

*Proof.* Let $t \in T$ and $\sigma \in \Sigma_K \cap I$, and assume $t(\sigma) = \sigma$. As the coordinate $\tau: V \to \mathbb{R}$ is $T$-invariant $t$ must fix the relatively compact set

$$\sigma \cap \{(z, \tau) \in V \mid \tau = 1\}.$$

The barycentre of this set will then be a fixed point of $t$. But $t$ acts as a translation on the affine space $\{(z, \tau) \in V | \tau = 1\}$, and thus $t = 1$ follows. $\square$

The lemma shows in particular that the action of $T$ on $V$ is admissible with respect to $\Sigma_K$, and a fortiori $T$ acts admissibly on the characteristic triple $(\pi, \delta, \lambda_K)$. We therefore have an associated reflexive analytic sheaf $\mathscr{L}(\pi, \delta, \lambda_K)/T$ on $I(\Lambda, \Sigma_K)/T$. If the cusp at $i\infty$ is regular, and if $-1 \notin \Gamma$ then the open subset

$$\{[z, \tau] \,|\, \operatorname{Im} \tau > c\} \subset I(\Lambda, \Gamma_K)/T$$

naturally extends the restriction of $A' \xrightarrow{p'} C'$ over the punctured neighbourhood

$$\{\Gamma\tau \in C' \,|\, \operatorname{Im} \tau > c\}$$

of the cusp at $i\infty$. By construction the sheaf $\mathscr{L}(\pi, \delta, \lambda_K)/T$ extends $\mathscr{L}'$ in the same sense. In a slightly different way, this is also true if $-1 \in \Gamma$, or if the cusp is irregular. In the former case the extension is provided by the quotients of $I(\Lambda, \Sigma_K)/T$ and $\mathscr{L}(\pi, \delta, \lambda_K)/T$ by the involution

$$[z, \tau] \mapsto [-z, \tau].$$

Likewise, if the cusp is irregular one would begin the construction with $2s$ in place of $s$, and then form the quotients by

$$[z, \tau] \mapsto [-z, \tau + s].$$

Using the action of $\mathrm{SL}_2(\mathbb{Z})$ in order to handle all other cusps of $\Gamma$ we obtain a proper analytic morphism

$$A \xrightarrow{\ p\ } C$$

extending $p'$, as well as a reflexive rank one sheaf $\mathscr{L}$ on $A$ with $\mathscr{L} \,|\, A' = \mathscr{L}'$.

(2.13) We compute the cohomology of $\mathscr{L}$ and its powers along the fibres $A_c = p^{-1}(c)$, for all $c \in C$.

At first we look at the fibres of the projection

$$\Omega/Q \rtimes T \xrightarrow{\ \tau\ } H,$$

which is a smooth family of abelian varieties. For any fixed $\tau \in H$, we let

$$\tilde{\mathscr{L}}^m_\tau = \tilde{\mathscr{L}}^m \otimes_{\mathcal{O}_{H,\tau}} \mathbb{C}$$

be the invertible sheaf induced by $\tilde{\mathscr{L}}^m$ on the fibre $(\Omega/Q \rtimes T)_\tau$ over $\tau$. As is well known, for all exponents $m \neq 0$ one has

$$\dim H^*((\Omega/(Q \rtimes T)_\tau;\ \tilde{\mathscr{L}}^m_\tau) = |P/Q| \cdot m^r,$$

with all cohomology concentrated in degree 0 if $m > 0$, and in degree $r$ if $m < 0$. The lattice $P \subset \mathbb{Q}Q$ is the weight lattice of the finite root system $R$. For details see [Mumford] Section III.16 and [Looijenga 1] p. 19.

As to the cohomology of the trivial bundle we have, for each $\tau \in H$, a canonical isomorphism

$$H^*((\Omega/Q \rtimes T)_\tau;\ \mathcal{O}) = \Lambda^*\overline{\mathrm{Hom}_{\mathbb{C}}}(\mathbb{C}Q, \mathbb{C}) = \Lambda^*\mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{C})$$

For each $m \in \mathbb{Z}$ we let $\mathscr{L}^{(m)}$ denote the reflexive hull of the tensor power $\mathscr{L}^m$ on $A$. For its restriction to $A'$ we have the alternative description

$$\mathscr{L}^{(m)} = (q_*\tilde{\mathscr{L}}^m)^\Gamma$$

where $\Omega/Q \rtimes T \xrightarrow{q} A'$ is the quotient map. We now can determine the cohomology of $\mathscr{L}^{(m)}$ along the fibres of $A' \xrightarrow{p'} C'$.

(2.14) PROPOSITION. *Let $\tau \in H$, let $c \in C'$ be its image in $C'$, and put*

$$\mathscr{L}^{(m)}_c = \mathscr{L}^{(m)} \otimes_{\mathcal{O}_{C,c}} \mathbb{C}.$$

*Then*

$$H^*(A_c;\ \mathscr{L}^{(m)}_c) \simeq \begin{cases} H^*((\Omega/Q \rtimes T)_\tau;\ \tilde{\mathscr{L}}^m_\tau) & \text{if } -1 \notin \Gamma, \\ H^*((\Omega/Q \rtimes T)_\tau;\ \tilde{\mathscr{L}}^m_\tau)^{\{\pm 1\}} & \text{if } -1 \in \Gamma. \end{cases}$$

*Proof.* We let $\tilde{\tau} \subset H$ be the connected component of $\tau$ in the fibre of the quotient map $H \to C'$ over $c$. Thus $\tilde{\tau}$ is a point that carries a non-reduced structure in case $\tau$ is an elliptic fixed point of $\Gamma$. Since $\Omega/Q \rtimes T$ is smooth over $H$ and since the Betti numbers of the sheaves $\tilde{\mathscr{L}}^m_\tau$ do not depend on $\tau \in H$ the direct image sheaves $R^i\tau_*\tilde{\mathscr{L}}^m$ are locally free on $H$. In particular there exists, for each integer $i$, an isomorphism

$$H^i((\Omega/Q \rtimes T)_{\tilde{\tau}};\ \tilde{\mathscr{L}}^m_{\tilde{\tau}}) \simeq H^i((\Omega/Q \rtimes T)_\tau;\ \tilde{\mathscr{L}}^m_\tau) \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{\tau}}.$$

Though not canonical, such an isomorphism can be chosen equivariant with respect to the isotropy subgroup $\Gamma_\tau \subset \Gamma$. The Leray spectral sequence of

$$(\Omega/Q \rtimes T)_{\tilde{\tau}} \xrightarrow{q} A_c$$

now yields

$$\begin{aligned} H^i(A_c; \mathscr{L}_c^{(m)}) &= H^i(A_c; q_* \tilde{\mathscr{L}}^m \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^{\Gamma_\tau} \\ &= H^i(A_c; q_*(\tilde{\mathscr{L}}^m \otimes_{\mathcal{O}_{H,\tau}} \mathcal{O}_{\tilde{\tau}}))^{\Gamma_\tau} \\ &= H^i(A_c; q_* \tilde{\mathscr{L}}^m_{\tilde{\tau}})^{\Gamma_\tau} \\ &= H^i((\Omega/Q \rtimes T)_{\tilde{\tau}}; \tilde{\mathscr{L}}^m_{\tilde{\tau}})^{\Gamma_\tau} \\ &\simeq [H^i((\Omega/Q \rtimes T)_\tau; \tilde{\mathscr{L}}^m_\tau) \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{\tau}}]^{\Gamma_\tau}. \end{aligned}$$

The finite abelian group $\Gamma_\tau$ acts on $\mathcal{O}_{\tilde{\tau}}$ by the regular representation unless $-1 \in \Gamma$, in which case it acts via the regular representation of $\Gamma_\tau/\{\pm 1\}$. Therefore the last vector space is isomorphic to $H^i((\Omega/Q \rtimes T)_\tau; \tilde{\mathscr{L}}^m_\tau)$, respectively its even part $H^i((\Omega/Q \rtimes T)_\tau; \tilde{\mathscr{L}}^m_\tau)^{\{\pm 1\}}$. $\square$

(2.15) We complete the calculation of the previous subsection by computing the cohomology of $\mathscr{L}^{(m)}$ over a cusp of $\Gamma$. Again it suffices to consider the standard cusp at $i\infty$.

We first assume $-1 \notin \Gamma$. As before we put, for any $m \in \mathbb{Z}$

$$\mathscr{L}^{(m)}_{i\infty} = \mathscr{L}^{(m)} \otimes_{\mathcal{O}_{C,i\infty}} \mathbb{C}.$$

By Theorem (1.13), $H^*(A_{i\infty}; \mathscr{L}^{(m)}_{i\infty})$ is, for $m \neq 0$, the cohomology of the topological pair

$$(Y, Y') = \left( Y\left(\Sigma_K; \tilde{\Lambda} + \frac{1}{m}\delta \xrightarrow{\pi} \Lambda, \frac{1}{m}\delta, \lambda\right), Y'\left(\Sigma_K; \tilde{\Lambda} + \frac{1}{m}\delta \xrightarrow{\pi} \Lambda, \frac{1}{m}\delta, \lambda_K\right)\right).$$

Recall that by definition

$$(Y, Y') = \left[\sum_{\substack{l \in \tilde{\Lambda}^\vee \\ \langle \delta, l\rangle = m}} (Y_l, Y'_l)\right] \Big/ T,$$

where

$$\begin{aligned} Y_l &= \{y \in I \mid \langle y, l \circ \lambda_K \rangle \leqslant 0\}, \\ Y'_l &= \{y \in I \mid \langle y, l \circ \lambda_K \rangle < 0\}. \end{aligned}$$

Thus $(Y, Y')$ decomposes into a disjoint sum indexed by the $T$-orbits in the affine lattice $\{l \in \tilde{\Lambda}^\vee \mid \langle \delta, l \rangle = m\}$. Those $l$ with $l \circ \lambda_K$ positive everywhere on $I$ clearly make no contribution to $(Y, Y')$; they may be characterized by the property

$$l > 0 \text{ on } K \cap \tilde{I}.$$

If, on the other hand, $l$ has a zero in the interior of $K$ then the $T$-orbit of $l$ contributes an $r$-dimensional manifold with boundary to $Y$, and the interior of the same manifold to $Y'$. Thus from such $l$ there is still no contribution to the cohomology of $(Y, Y')$.

The remaining characters $l \in \tilde{\Lambda}^\vee$ (with $\langle \delta, l \rangle = m$) are those with $\ker l$ a supporting hyperplane of $\bar{K}$ that intersects $\bar{K}$ along the closure of some face from $\tilde{\Sigma}_K \cap \tilde{I}$. For $m > 0$ and any such $l$ the set $Y_l$ is the closure of a cone from $\Sigma_K$ while $Y_l'$ is empty. Thus the orbit gives a one-dimensional contribution to $H^0(Y, Y'; \mathbb{C})$. Likewise, for $m < 0$ there is a one-dimensional contribution from the orbit of $l$ to $H^r(Y, Y'; \mathbb{C})$.

It remains to count the number of relevant orbits in $\{l \in \tilde{\Lambda}^\vee \mid \langle \delta, l \rangle = m\}$. To this end we observe that any such $l$ is uniquely determined by its restriction to the sublattice $\Lambda \subset \tilde{\Lambda}$, and further restricts to a character on $Q \subset \Lambda$, i.e. a weight $l' \in P$ of the root system $R$. Conversely, given any weight $l' \in P$, there clearly is a unique rational extension $l: \tilde{\Lambda} \to \mathbb{Q}$ of $l'$ with $\langle \delta, l \rangle = m$ and $\ker l$ a supporting hyperplane of $\bar{K}$ intersecting $\bar{K}$ along the closure of a face from $\tilde{\Sigma}_K \cap \tilde{I}$. By the classification of faces of $\bar{K}$ [Wirthmüller 2] p. 230, Theorem 9.5, some $W$-conjugate of that face contains $\beta$; this implies that $l$ is integral on $\beta$, and therefore that $l \in \tilde{\Lambda}^\vee$.

We thus have established a bijective correspondence $l \leftrightarrow l'$; it turns out $T$-equivariant if we let $t \in T$ act on the lattice $P$ as the translation by $mt$. We therefore have exactly $|P/Q| \cdot m^r$ contributing orbits in $\{l \in \tilde{\Lambda}^\vee \mid \langle \delta, l \rangle = m\}$, and conclude

$$\dim H^*(A_{i\infty}; \mathscr{L}^{(m)}_{i\infty}) = |P/Q| \cdot m^r$$

for all $m \neq 0$, with all cohomology in $H^0$ for $m > 0$, in $H^r$ for $m < 0$.

In order to compute $H^*(A_{i\infty}; \mathcal{O}_{A_{i\infty}})$ we think of the trivial bundle as obtained from the characteristic triple $(\tilde{\Lambda} \xrightarrow{\pi} \Lambda, \delta, \lambda_0)$ with

$$\lambda_0(z, \tau) = (0, z, \tau) \in \tilde{V}.$$

The only contribution to the homology of $(Y, Y')$ comes from the character $l \in \tilde{\Lambda}^\vee$ (with $\langle \delta, l \rangle = 1$) that vanishes identically on $\Lambda$. Its contribution to $(Y, Y')$ is the pair $(I/T, \varnothing)$, and as $I/T$ is homotopy equivalent to a real torus of dimension $r$ we obtain

$$H^*(A_{i\infty}; \mathcal{O}_{A_{i\infty}}) \simeq \Lambda^* \mid \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{C}).$$

Finally, our calculations are summarised by

(2.16) THEOREM. *For each $m \in \mathbb{Z}$ and each $i \geqslant 0$ the function*

$$C \ni c \mapsto \dim_{\mathbb{C}} H^i(A_c; \mathscr{L}_c^{(m)})$$

*is constant on the curve $C$.*

*Proof.* This has been proved, with the exception of the irregular cusps and the case $-1 \in \Gamma$. Dividing by the extra involution takes care of these as well. □

COROLLARY. *For each $m \in \mathbb{Z}$ the direct image sheaves $R^i p_* \mathscr{L}^{(m)}$ are locally free sheaves on $C$, and for all points $c \in C$ one has*

$$H^i(A_c; \mathscr{L}_c^{(m)}) = R^i p_* \mathscr{L}^{(m)} \otimes_{\mathcal{O}_{C,c}} \mathbb{C}.$$

*Proof.* Being torsion free, $\mathscr{L}^{(m)}$ is flat over the curve $C$, and the theorem on cohomology and base change applies. □

## 3. Jacobi forms

(3.1) We introduce another reflexive rank one sheaf $\mathscr{M}$ on the variety $A$ constructed in the previous section. Recall from (2.12) that $A$ was obtained by gluing $A' = p^{-1}(C')$ with copies of $I(\Lambda, \Sigma_K)/T$, the latter divided by the involution $[z, \tau] \to [-z, \tau]$ if $-1 \in \Gamma$. We first define a sheaf $\mathscr{M}'$ on $A'$ in terms of the quotient map

$$\Omega/Q \rtimes T \xrightarrow{q} A':$$

a section of $\mathscr{M}'$ over an open subset $U' \subset A'$ is a holomorphic function $\varphi$ on $q^{-1}(U')$ that obeys the functional equation

$$\varphi\left(\frac{1}{c\tau + d} \cdot z, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d) \cdot \varphi(z, \tau) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

If $U'$ is also contained in $I(\Lambda, \Sigma_K)/T$ (respectively its quotient by the involution) then such $\varphi$ are just the holomorphic functions on $U'$. Therefore $\mathscr{M}'$ is canonically isomorphic to $\mathcal{O}_{A'}$ in some neighbourhood of $A \backslash A'$. We let $\mathscr{M}$ denote the sheaf on $A$ obtained from $\mathscr{M}'$ by gluing with $\mathcal{O}_A$ along such a neighbourhood.

For each integer $k$ we let $\mathscr{M}^{(k)}$ denote the reflexive hull of the tensor power $\mathscr{M}^k$.

For many choices of $\Gamma$ and $k$ the sheaf $\mathscr{M}^{(k)}$ is invertible, and in fact comes from a line bundle $\bar{\mathscr{M}}^{(k)}$ on $C$. A local section of $\bar{\mathscr{M}}^{(k)}$ over $C' = H/\Gamma$ is by definition a local function $\varphi$ on $H$ obeying the functional equation

$$\varphi\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \cdot \varphi(\tau) \quad \text{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

$\varphi$ is holomorphic at $i\infty$ if its Laurent expansion in $e^{2\pi i\tau}$ is a Taylor series; this also determines the notion of holomorphy at the remaining cusps. Thus the global sections of $\bar{\mathscr{M}}^{(k)}$ are just the modular forms of weight $k$; we write

$$M_k = H^0(C; \bar{\mathscr{M}}^{(k)}),$$

and

$$M_* = \bigoplus_{k=0}^{\infty} M_k$$

for the graded algebra of all modular forms.

We put $\bar{\mathscr{M}} = \bar{\mathscr{M}}^{(1)}$. Note that $\bar{\mathscr{M}}^{(k)}$ is an invertible sheaf unless $-1 \in \Gamma$ and $k$ is odd, in which case $\bar{\mathscr{M}}^{(k)}$ is the zero sheaf.

(3.2) PROPOSITION. *Each of the following conditions implies that the canonical homomorphism*

$$p^*\bar{\mathscr{M}}^{(k)} \to \mathscr{M}^{(k)}$$

*is an isomorphism:*

(i) $-1 \notin \Gamma$, *and* $\Gamma$ *has no fixed points on* $H$;
(ii) $k$ *is divisible by* 12.

*If* (i) *holds then* $\bar{\mathscr{M}}^{(k)} = \bar{\mathscr{M}}^k$ *while if* (ii) *holds then* $\bar{\mathscr{M}}^{(k)} = (\bar{\mathscr{M}}^{(12)})^{k/12}$.

*Proof.* Condition (i) means that $\Gamma$ acts freely on $H$ while (ii) implies that all isotropy groups $\Gamma_\tau$ $(\tau \in H)$ act trivially on the factor of automorphy $(c\tau + d)^k$. Either implies that $p^*\bar{\mathscr{M}}^{(k)} \to \mathscr{M}^{(k)}$ is surjective; the kernel, being a torsion subsheaf of $p^*\bar{\mathscr{M}}^{(k)}$, must then vanish. This proves the first assertion, while the second is straightforward. □

(3.3) We now let $\mathscr{I}_{km}$ denote the reflexive hull of the sheaf $\mathscr{M}^{(k)} \otimes \mathscr{L}^{(m)}$ on $A$. In view of Propositions (1.8)(a) and (3.2) the following statement makes sense.

(3.4) PROPOSITION. *For all positive integers k and m the sheaf* $\mathscr{I}_{km}$ *is ample.*

*Proof.* We may assume that $\mathscr{I}_{km} = p^*\bar{\mathscr{M}}^{(k)} \otimes \mathscr{L}^{(m)}$, and since $\bar{\mathscr{M}}^{(k)}$ is ample on $C$ it suffices to show that $\mathscr{L}^{(m)} \otimes \mathscr{O}_{A_c}$ is ample on each fibre $A_c$ of $A \xrightarrow{p} C$. As ampleness is not affected by taking finite quotients it suffices to see that

$\tilde{\mathscr{L}} \otimes_{\mathcal{O}_{H,\tau}} \mathbb{C}$ is ample on $(\Omega/Q \rtimes T)_\tau$ for each $\tau \in H$, and that

$$\mathscr{L}(\pi, \delta, \lambda_K)/T \otimes \mathcal{O}_{I(\Lambda,\Sigma_K)^{\cdot}/T}$$

is ample on $I(\Lambda, \Sigma_K)^{\cdot}/T$. The former is classical, cf. [Mumford], while the latter follows from Theorem (1.16). □

DEFINITION. The global sections of the sheaf $\mathscr{J}_{km}$ on $A$ are called *Jacobi forms* of *weight* $k$ and *index* $m$. We let

$$J_{km} = H^0(A; \mathscr{J}_{km})$$

be the complex vector space of such Jacobi forms, and put

$$J_{**} = \bigoplus_{\substack{k \in \mathbb{Z} \\ m \in \mathbb{Z}}} J_{km}.$$

Since for Jacobi forms of zero index one has

$$J_{k0} = H^0(C; \bar{\mathscr{M}}^{(k)}) = M_k \quad \text{for all } k,$$

$J_{**}$ is a bi-graded algebra over $M_*$. Note that while clearly $J_{km} = 0$ if $m < 0$ there may be non-trivial Jacobi forms of negative weight; as we shall see below this is in fact the case.

We are particularly interested in symmetric Jacobi forms $\varphi \in J_{km}^W$: those which are invariant with respect to the action of the finite Weyl group $W$ on $A$ and $\mathscr{J}_{km}$. The definition of Jacobi forms is easily rewritten in analytic terms. For the sake of simplicity we do this only for symmetric Jacobi forms $\varphi \in J_{km}^W$ in case $\Gamma = SL_2(\mathbb{Z})$ is the full modular group. Such forms correspond to holomorphic functions $\varphi: \Omega \to \mathbb{C}$ with the following properties:

(i) $\varphi(z + q, \tau) = \varphi(z, \tau)$
$\varphi(z + \tau q, \tau) = e^{-2\pi i m(q|z) - \pi i m(q|q)\tau} \cdot \varphi(z, \tau)$
for all $q \in Q$;

(ii) $\varphi\left(\frac{1}{c\tau + d} z, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \cdot e^{\pi i m(z|z)/(c\tau + d)} \cdot \varphi(z, \tau)$

(iii) $\varphi(wz, \tau) = \varphi(z, \tau)$ for all $w \in W$;

(iv) $\varphi(z, \tau)$ is a locally bounded function as $\operatorname{Im} \tau \to \infty$.

Indeed there is but one cusp to consider, and since all one-dimensional cones in the fan $\Sigma_K$ are $W$-conjugate to the projection of the ray $\mathbb{R}_+ \beta \subset \tilde{V}$ condition (iv) is sufficient, in the presence of (iii), to ensure holomorphy of $\varphi$ along $A_{i\infty}$.

Given the root system $R$, the family of abelian varieties $A \xrightarrow{p} C$ together with the sheaves $\mathcal{L}^{(m)}$ and $\mathcal{M}^{(k)}$ still depends on the choice of the subgroup $\Gamma \subset SL_2(\mathbb{Z})$. For the moment let us write, more precisely,

$$A_\Gamma \xrightarrow{p_\Gamma} C_\Gamma$$

and likewise $J_{km\Gamma}$ for the spaces of Jacobi forms. Clearly $J_{kmSL_2(\mathbb{Z})}$ may be identified with the subspace

$$(J_{km\Gamma})^{SL_2(\mathbb{Z})} \subset J_{km\Gamma}$$

of invariant forms. In particular any given Jacobi form $\varphi \in J_{kmSL_2(\mathbb{Z})}$ induces Jacobi forms

$$\varphi_\Gamma \in J_{km\Gamma}$$

for all subgroups $\Gamma \subset SL_2(\mathbb{Z})$ (of finite index). By abuse of language we say that the $\varphi_\Gamma$ obtained in this way do not depend on the choice of $\Gamma$.

(3.5) We proceed to formulate the main result of this paper. Let

$$(k(0), k(1), \ldots, k(r))$$

be an $(r+1)$-tuple of non-negative integers. For each multi-index $\alpha = (\alpha_0, \ldots, \alpha_r)$ (of non-negative integers) we then let

$$|\alpha| = \sum_{j=0}^{r} k(j)\alpha_j$$

be the weight of $\alpha$. The direct sum

$$\mathcal{S} = \bigoplus_{\alpha} \bar{\mathcal{M}}^{(|\alpha|)}$$

has the structure of a sheaf of $\mathcal{O}_C$-algebras in a natural way.

(3.6) THEOREM. *Let $R$ be a fixed classical (reduced and irreducible) root system of rank $r$, but not of type $E_8$. Then there exists a basis*

$$(\varphi_0, \varphi_1, \ldots, \varphi_r)$$

*of $W$-invariant Jacobi forms, in the following sense.*

(i) *Each*

$$\varphi_j \in J^W_{-k(j),m(j)}$$

*is a W-invariant Jacobi form of weight* $-k(j) \leqslant 0$ *and index* $m(j) > 0$, *and* $\varphi_j$ *does not depend on the choice of the subgroup* $\Gamma \subset SL_2(\mathbb{Z})$.

(ii) *The sheaf homomorphism*

$$\mathscr{S} = \bigoplus_{\alpha} \bar{\mathscr{M}}^{(|\alpha|)} \to \bigoplus_{m=0}^{\infty} p_* \mathscr{L}^{(m)W}$$

*which is multiplication by* $f_0^{\alpha_0} f_1^{\alpha_1} \cdot \cdots \cdot f_r^{\alpha_r}$ *on the summand with multi-index* $\alpha$ *is an isomorphism of* $\mathcal{O}_C$*-algebra sheaves.*

(iii) *For each* $c \in C$, *let* $|A_c|$ *denote the reduced fibre* $p^{-1}(c) \subset C$. *In case c is not a cusp we fix some* $\tau \in H$ *so that the restriction to* $|A_c|$ *of any Jacobi form* $\varphi \in J_{km}$ *is identified with a section*

$$\varphi_c \in H^0(A_c; \mathscr{L}^{(m)} \otimes \mathcal{O}_{|A_c|}).$$

*Then the algebra*

$$\bigoplus_{m=0}^{\infty} H^0(A_c; \mathscr{L}^{(m)} \otimes \mathcal{O}_{|A_c|}) = \mathbb{C}[(\varphi_0)_c, \ldots, (\varphi_r)_c]^{\Gamma_c}$$

*is the* $\Gamma_c$*-invariant part of the polynomial algebra in the indeterminates* $(\varphi_j)_c$, *where* $\Gamma_c = \Gamma_\tau$ *if c is not a cusp, and* $\Gamma_c = \Gamma \cap \{\pm 1\}$ *else.*

(iv) *The algebra of W-invariant Jacobi forms is the polynomial algebra*

$$J^W_{**} = M_*[\varphi_0, \ldots, \varphi_r]$$

*in the indeterminates* $\varphi_j$ *over the algebra of modular forms.*

REMARKS. The indices $m(j)$ of the basic Jacobi forms $(j = 0, \ldots, r)$ are the coefficients of $\delta$ written as a linear combination of the base roots $\alpha_0, \ldots, \alpha_r$ of $R$. This is already known from [Looijenga 1]. On the other hand, apart from $k(0) = 1$, the integers $k(j)$ turn out to be the degrees of the generators of the ring of invariant polynomials

$$\mathbb{C}[V]^W \subset \mathbb{C}[V],$$

i.e. the exponents of the Weyl group $W$ increased by 1. Some reason for this fact will be given in (5.22) below, but we do not know of a satisfactory a priori explanation.

Geometrically, (ii) implies that the family $A/W \to C$ is a bundle of weighted projective spaces (or their quotients by the action of $\{\pm 1\}$, if $-1 \in \Gamma$).

(3.7) We begin the proof of (3.6) with some preparations designed to eliminate case distinctions; the proof proper will be the subject of Sections 4 and 5.

We first note that for $s \geqslant 3$ the principal congruence subgroup

$$\Gamma(s) = \{g \in SL_2(\mathbb{Z}) \mid g \equiv 1 \ (\mathrm{mod}\, s)\}$$

acts freely on the upper half plane. Given the group $\Gamma$, we put

$$\tilde{\Gamma} = \Gamma \cap \Gamma(3) \quad \text{and} \quad \bar{\Gamma} = \Gamma/\tilde{\Gamma}.$$

In the sequel while using our standard notation we add a "$\sim$" symbol to indicate reference to $\tilde{\Gamma}$ rather than $\Gamma$. We have a commutative square

$$\begin{tikzcd} \tilde{A} \arrow[r, "q"] \arrow[d, "\tilde{p}"] & A \arrow[d, "p"] \\ \tilde{C} \arrow[r, "\bar{q}"] & C \end{tikzcd}$$

wherein $q$ and $\bar{q}$ are the quotient morphisms with respect to the action of the finite group $\bar{\Gamma}$. It follows from the definitions that for all integers $k$ and $m$ one has the identities

$$\bar{\mathscr{M}}^{(k)} = (\bar{q}_* \tilde{\bar{\mathscr{M}}}^{(k)})^{\bar{\Gamma}} = (\bar{q}_* \tilde{\bar{\mathscr{M}}}^{k})^{\bar{\Gamma}}$$

$$\mathscr{J}_{km} = (q_* \tilde{\mathscr{J}}_{km})^{\bar{\Gamma}} = (q_*(\tilde{\mathscr{M}}^{(k)} \otimes \tilde{\mathscr{L}}^{(m)}))^{\bar{\Gamma}} = (q_*(\tilde{p}^* \tilde{\bar{\mathscr{M}}}^{k} \otimes \tilde{\mathscr{L}}^{(m)}))^{\bar{\Gamma}}$$

while on the global level

$$M_k = H^0(\tilde{C}; \tilde{\bar{\mathscr{M}}}^{k})^{\bar{\Gamma}} = \tilde{M}_k^{\bar{\Gamma}}$$

$$J_{km} = H^0(\tilde{A}; \tilde{p}^* \tilde{\bar{\mathscr{M}}}^{k} \otimes \tilde{\mathscr{L}}^{(m)})^{\bar{\Gamma}} = \tilde{J}_{km}^{\bar{\Gamma}}.$$

We now assume that $\varphi_0, \varphi_1, \ldots, \varphi_r$ are Jacobi forms for which parts (i) and (ii) of Theorem (3.6) hold, with $\tilde{\Gamma}$ in place of $\Gamma$. We show that then necessarily parts (ii), (iii), and (iv) hold for $\Gamma$.

Indeed the isomorphism

$$\tilde{\mathscr{S}} = \bigoplus_{\alpha} \tilde{\bar{\mathscr{M}}}^{(|\alpha|)} \xrightarrow{\simeq} \bigoplus_{m=0}^{\infty} \tilde{p}_* \tilde{\mathscr{L}}^{(m)W}$$

yields, upon application of $\bar{q}_*$ and passing to $\bar{\Gamma}$-fixed spaces, an isomorphism

$$\mathscr{S} = \bigoplus_{\alpha} \bar{\mathscr{M}}^{(|\alpha|)} \xrightarrow{\simeq} \bigoplus_{m=0}^{\infty} (\bar{q}_* \tilde{p}_* \tilde{\mathscr{L}}^{(m)W})^{\bar{\Gamma}} = \bigoplus_{m=0}^{\infty} p_* \mathscr{L}^{(m)W},$$

which proves (ii) for the group. On the other hand (iii) for $\tilde{\Gamma}$ clearly follows from (ii) by restriction to the fibre while (iv) follows by taking global sections. Both properties then follow for $\Gamma$ too, by passing to $\Gamma_c$- and $\Gamma$-invariants, respectively.

For the proof of Theorem (3.6) it therefore suffices to construct Jacobi forms $\varphi_0, \varphi_1, \ldots, \varphi_r$ which, apart from (i), satisfy (ii) under the additional assumption that $\Gamma \subset \Gamma(3)$. In that case $\mathscr{S}$ is the symmetric algebra over the locally free sheaf of $\mathcal{O}_C$-modules

$$\bar{\mathscr{M}}^{k(0)} \oplus \bar{\mathscr{M}}^{k(1)} \oplus \cdots \oplus \bar{\mathscr{M}}^{k(r)}.$$

Since the sheaves $p_*\mathscr{L}^{(m)}$, and, a fortiori, the sheaves $p_*\mathscr{L}^{(m)W}$ are also locally free by (2.16) the isomorphy (ii) will in turn follow from (iii) which now reduces to the statement that for each $c \in C$

$$\bigoplus_{m=0}^{\infty} H^0(A_c; \mathscr{L}^{(m)} \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^W$$

is the polynomial algebra $\mathbb{C}[(\varphi_0)_c, \ldots, (\varphi_r)_c]$. Note that the truth of this statement is independent of the choice of $\Gamma \subset \Gamma(3)$.

(3.8) By the preceding the proof of Theorem (3.6) has, in particular, been reduced to the case of the group $\Gamma = \Gamma(3)$, to which we now stick. Denoting the corresponding family of abelian varieties by $A \xrightarrow{p} C$ again we outline the remaining steps to be done in order to complete the proof of the theorem.

(1) We shall construct symmetric Jacobi forms $\varphi_j$ on $A$ which furthermore are invariant under the group $\bar{\Gamma} = SL_2(\mathbb{Z})/\Gamma(3)$:

$$\varphi_j \in J^{W\bar{\Gamma}}_{-k(j),m(j)} \qquad (j = 0, 1, \ldots, r).$$

(2) We then show that for each $c \in C$

$$\bigoplus_{m=0}^{\infty} H^0(A_c; \mathscr{L}^{(m)} \otimes_{C,c} \mathbb{C})$$

is the polynomial algebra $\mathbb{C}[(\varphi_0)_c, \ldots, (\varphi_r)_c]$.

## 4. Construction of Jacobi forms

(4.1) We begin with a digression on arrangements of divisors. Throughout $A$ will denote an analytic Cohen-Macaulay variety, while $\mathscr{L}$ will be a coherent sheaf of Cohen-Macaulay $\mathcal{O}_A$-modules.

DEFINITION. A *type I arrangement* (*of divisors*) *on* $A$ is a finite set

$$\{Y_1, \dots, Y_n\}$$

of effective Weil divisors $Y_i$ on $A$ with the following properties.

(i) $Y := \sum_{i=1}^{n} Y_i$ is a Cartier divisor.

(ii) For $1 \leqslant i < j \leqslant n$ one always has

$$\dim Y_i \cap Y_j \leqslant \dim A - 2,$$

and for $1 \leqslant i < j < k \leqslant n$ one always has

$$\dim Y_i \cap Y_j \cap Y_k \leqslant \dim A - 3.$$

In the sequel we abbreviate

$$\mathscr{L}_i = \mathscr{L} \otimes \mathscr{O}_{Y_i}, \quad \mathscr{L}_{ij} = \mathscr{L} \otimes \mathscr{O}_{Y_i \cap Y_j},$$

etc.

(4.2) PROPOSITION. *Let* $\{Y_1, \dots, Y_n\}$ *be a type I arrangement on* $A$, *and put* $Y = \sum_{i=1}^{n} Y_i$. *Then the canonical sequence of restriction homomorphisms*

$$0 \to \mathscr{L} \otimes \mathscr{O}_Y \to \bigoplus_{1 \leqslant i \leqslant n} \mathscr{L}_i \to \bigoplus_{1 \leqslant i < j \leqslant n} \mathscr{L}_{ij}$$

*is exact, and therefore the corresponding global sequence*

$$0 \to H^0(Y; \mathscr{L} \otimes \mathscr{O}_Y) \to \bigoplus_i H^0(Y_i; \mathscr{L}_i) \to \bigoplus_{i<j} H^0(Y_i \cap Y_j; \mathscr{L}_{ij})$$

*also is exact.*

*Proof.* We put

$$U_1 = Y \backslash \bigcup_{i<j} (Y_i \cap Y_j)$$

$$U_2 = Y \backslash \bigcup_{i<j<k} (Y_i \cap Y_j \cap Y_k),$$

and let

$$U_1 \xrightarrow{e_1} Y \quad \text{and} \quad U_2 \xrightarrow{e_2} Y$$

denote the open inclusion maps. By assumption $Y\backslash U_1$ and $Y\backslash U_2$ are analytic subsets of codimension at least 1 and 2 in $Y$, respectively. Since $Y$ is a Cartier divisor on $A$ the sheaf $\mathscr{L} \otimes \mathcal{O}_Y$ is a Cohen-Macaulay sheaf on $Y$. By an extension theorem of Scheja, cf. [Siu and Trautmann] p. 36, Theorem (1.14), this implies that the restriction homomorphism

$$\mathscr{L} \otimes \mathcal{O}_Y \to e_{1*}(\mathscr{L} \otimes \mathcal{O}_Y \mid U_1)$$

is injective while

$$\mathscr{L} \otimes \mathcal{O}_Y \to e_{2*}(\mathscr{L} \otimes \mathcal{O}_Y \mid U_2)$$

even is an isomorphism. The former implies injectivity of

$$\mathscr{L} \otimes \mathcal{O}_Y \to \bigoplus_i \mathscr{L}_i$$

while the latter reduces the question of exactness at $\bigoplus_i \mathscr{L}_i$ to the special case when at any given point of $A$ no more than two of the $Y_i$ intersect. Under this extra assumption exactness at the term $\bigoplus_i \mathscr{L}_i$ is obvious. $\square$

(4.3) More generally, we now will allow up to three divisors to intersect along a codimension two subset of $A$. In that case though, we shall impose conditions on the arrangement as well as on $A$ and $\mathscr{L}$ that are much more restrictive in other ways.

For $A$ and $\mathscr{L}$ as above we put

$$B = \left\{ y \in A \,\middle|\, \begin{array}{l} y \text{ is a singular point of } A, \text{ or} \\ \mathscr{L}_y \text{ is not a free } \mathcal{O}_{A,y}\text{–module} \end{array} \right\}.$$

DEFINITION. A *type II arrangement on A* (*with respect to* $\mathscr{L}$) is a finite set

$$\{Y_1, \ldots, Y_n\}$$

of effective Weil divisors on $A$ with the following properties.

(i) $Y = \sum_{i=1}^{n} Y_i$ is a Cartier divisor.

(ii) For $1 \leqslant i < j \leqslant n$ one has

$$\dim Y_i \cap Y_j \cap B \leqslant \dim A - 3,$$

and the set

$$B_{ij} := \left\{ y \in Y_i \cap Y_j \backslash B \,\middle|\, \begin{array}{l} Y_i \text{ and } Y_j \text{ are not (smooth} \\ \text{and) transverse at } y \end{array} \right\}$$

also satisfies

$$\dim B_{ij} \leqslant \dim A - 3.$$

(iii) For $1 \leqslant i < j < k < l \leqslant n$ one always has

$$\dim Y_i \cap Y_j \cap Y_k \cap Y_l \leqslant \dim A - 3.$$

Given a type II arrangement $\{Y_1, \ldots, Y_n\}$ we further introduce the sets

$$V_{ij} = Y_i \cap Y_j \backslash \Big( \bigcup_{i \neq k \neq j} Y_k \cup B_{ij} \Big)$$

($1 \leqslant i < j \leqslant n$). For $1 \leqslant i < j < k \leqslant n$ the union of all irreducible components of dimension $\dim A - 2$ in the analytic set

$$Y_i \cap Y_j \cap Y_k \backslash (B_{jk} \cup B_{ik} \cup B_{ij})$$

will be denoted $V_{ijk}$. Both $V_{ij}$ and $V_{ijk}$ are constructible submanifolds of $A$ of dimension $\dim A - 2$.

(4.4) We let $\mathscr{K}$ denote the kernel of the restriction homomorphism

$$\bigoplus_i \mathscr{L}_i \to \bigoplus_{i<j} \mathscr{L}_{ij}$$

already considered in (4.2). We now shall construct a subsheaf $\mathscr{V} \subset \mathscr{K}$ which will turn out to comprise exactly those sections of the $\mathscr{L}_i$ that can be pieced together into a section of $\mathscr{L} \otimes \mathcal{O}_Y$.

Let us for the moment fix some point $y \in V_{ijk}$. Since $\mathscr{L}_y$ is a free $\mathcal{O}_{A,y}$-module we may pick some trivialization

$$\mathscr{L}_y \underset{\simeq}{\xrightarrow{t}} \mathcal{O}_{A,y} \oplus \cdots \oplus \mathcal{O}_{A,y}.$$

In view of the definition of the manifold $V_{ijk}$ we can find germs of vector fields

$$v_i \in (\mathscr{T} Y_i)_y, \qquad v_j \in (\mathscr{T} Y_j)_y, \quad \text{and} \quad v_k \in (\mathscr{T} Y_k)_y$$

such that

$$v_i + v_j + v_k = 0 \quad \text{in } (\mathscr{T} A)_y$$

while none of $v_i$, $v_j$ or $v_k$ is tangent to $V_{ijk}$ at $y$.

DEFINITION. Let $U \subset A$ be open. A section

$$s = (s_1, \ldots, s_n) \in \mathscr{K}(U) \subset \bigoplus_{i=1}^{n} \mathscr{L}_i(U)$$

belongs to $\mathscr{V}(U)$ if for all triples $(i, j, k)$ with $1 \leqslant i < j < k \leqslant n$ and all $y \in V_{ijk} \cap U$ the identity

$$\langle v_i, d(ts_i)\rangle + \langle v_j, d(ts_j)\rangle + \langle v_k, d(ts_k)\rangle = 0 \tag{4.5}$$

holds in $(\mathcal{O}_{A,y} \oplus \cdots \oplus \mathcal{O}_{A,y}) \otimes \mathcal{O}_{V_{ijk},y}$.

To justify this definition we prove:

LEMMA. *Given s, the validity of* (4.5) *does not depend on the choice of t nor on the choice of the vector fields* $v_i, v_j, v_k$.

*Proof.* As to the independence of $t$, it suffices to show that the left-hand side of (4.5) is an $\mathcal{O}_{A,y}$-linear function of $s$. Thus let $u \in \mathcal{O}_{A,y}$. Then

$$d(u \cdot ts_i) = u \cdot d(ts_i) + du \otimes ts_i$$

and since $s_i$, $s_j$ and $s_k$ restrict to one and the same germ in $\mathscr{L}_y \otimes \mathcal{O}_{V_{ijk},y}$ the contributions from the second terms add up to zero:

$$\begin{aligned}&\langle v_i, du \otimes ts_i\rangle + \langle v_j, du \otimes ts_j\rangle + \langle v_k, du \otimes ts_k\rangle\\ &\quad = \langle v_i + v_j + v_k, du \otimes ts_i\rangle = 0.\end{aligned}$$

Likewise, if $v_i'$, $v_j'$, $v_k'$ is a second choice of the triple of vector fields then the residue class of $v_i'$ in $\mathscr{T}V_i \otimes \mathcal{O}_{V_{ijk},y}$ satisfies

$$v_i' \equiv u \cdot v_i + w_i$$

for some $u \in \mathcal{O}_{V_{ijk},y}$ and $w_i \in (\mathscr{T}V_{ijk})_y$, and similarly

$$v_j' \equiv u \cdot v_j + w_j, \qquad v_k' \equiv u \cdot v_k + w_k,$$

with the same $u$. Summing up we obtain

$$\begin{aligned}&\langle v_i', d(ts_i)\rangle + \langle v_j', d(ts_j)\rangle + \langle v_k', d(ts_k)\rangle\\ &= u \cdot (\langle v_i, d(ts_i)\rangle + \langle v_j, d(ts_j)\rangle + \langle v_k, d(ts_k)\rangle) + \langle w_i + w_j + w_k, d(ts_i)\rangle.\end{aligned}$$

Since $w_i + w_j + w_k = 0$ the assertion follows. □

(4.6) PROPOSITION. *Let $\{Y_1, \ldots, Y_n\}$ be an arrangement of type II on A, and put $Y = \bigcup_{i=1}^n Y_i$. Then the restriction homomorphism*

$$\mathscr{L} \otimes \mathcal{O}_Y \to \bigoplus_{i=1}^{n} \mathscr{L}_i$$

*induces an isomorphism*

$$\mathscr{L} \otimes \mathcal{O}_Y \xrightarrow{\cong} \mathscr{V}.$$

*Proof.* If a local section $s = (s_1, \ldots, s_n)$ of $\bigoplus_i \mathscr{L}_i$ comes from a local section $\tilde{s}$ of $\mathscr{L} \otimes \mathcal{O}_Y$ then it certainly belongs to $\mathscr{K}$, and (4.5) just reflects the linearity of the differential of $t\tilde{s}$ at $y$. Thus $s$ is a local section of $\mathscr{V}$. On the other hand, for the same reason as in the proof of (4.2) the homomorphism

$$\mathscr{L} \otimes \mathcal{O}_Y \to \mathscr{V}$$

is an injection; the point in question is its surjectivity.

Surjectivity may be verified stalkwise, and is settled by Proposition (4.2) at all points $y \in A$ at which at most two of the divisors $Y_i$ $(1 \leqslant i \leqslant n)$ meet. Of the remaining points $y$ we need consider only those which belong to some $V_{ijk}$ $(1 \leqslant i < j < k \leqslant n)$, again by the extension argument used in the proof of (4.2).

Thus let $y \in V_{ijk}$, and let $s = (s_1, \ldots, s_n) \in \mathscr{V}_y$. We have to find an $\tilde{s} \in \mathscr{L}_y \otimes \mathcal{O}_{Y,y}$ simultaneously representing

$$s_i \in \mathscr{L}_{i,y}, \qquad s_j \in \mathscr{L}_{j,y}, \quad \text{and} \quad s_k \in \mathscr{L}_{k,y}.$$

Applying Proposition (4.2), at least we find a germ $\tilde{s}_{ij} \in \mathscr{L}_y \otimes \mathcal{O}_{Y,y}$ that represents $s_i$ and $s_j$. We let $s_{ij}$ be the image

$$s_{ij} \in \mathscr{L}_y \otimes \mathcal{O}_{Y_i \cup Y_j, y}$$

of $\tilde{s}_{ij}$.

In some neighbourhood of $y$ the intersection $(Y_i \cup Y_j) \cap Y_k$ coincides with the local divisor $2V_{ijk}$ on $Y_k$. Since $s \in \mathscr{V}_y$ by assumption, we have $s \in \mathscr{K}_y$, and (4.5) holds at $y$: these two facts together just mean that $s_{ij}$ and $s_k$ map to one and the same germ in $\mathscr{L}_y \otimes \mathcal{O}_{2V_{ijk},y}$. Another application of Proposition (4.2) now provides an $\tilde{s} \in \mathscr{L}_y \otimes \mathcal{O}_{Y,y}$ as required. $\square$

(4.7) We now assume a finite group $W$ of automorphism of $A$ is given, which also acts on the sheaf $\mathscr{L}$. Thus with each $w \in W$ there is associated an isomorphism

$$\rho_w : \mathscr{L} \xrightarrow{\cong} w^* \mathscr{L}$$

such that $\rho_1 = \mathrm{id}$, and if $v \in W$ is another element then the diagram

$$\begin{tikzcd} \mathscr{L} \arrow[r, "\rho_{vw}"] \arrow[dr, "\rho_w"'] & (vw)^*\mathscr{L} = w^*(v^*\mathscr{L}) \\ & w^*\mathscr{L} \arrow[u, "w^*\rho_v"'] \end{tikzcd}$$

commutes. We consider an arrangement $\{Y_1, \ldots, Y_n\}$ on $A$ (of type I or type II) with the property that $W$ permutes the divisors $Y_i$ transitively. Each $w \in W$, sending $Y_i$ to $Y_j$, say, then induces a linear isomorphism

$$w_* : H^0(Y_i;\ \mathscr{L}_i) \xrightarrow{\rho_w} H^0(Y_i;\ w^*\mathscr{L}_i) = H^0(Y_j;\ \mathscr{L}_j).$$

If $W_1 \subset W$ denotes the isotropy subgroup of $Y_1$ then projection to the first summand clearly gives an injective linear map

$$H^0\left(A;\ \bigoplus_i \mathscr{L}_i\right)^W \to H^0(Y_1;\ \mathscr{L}_1)^{W_1}.$$

Let again $\mathscr{K}$ be the kernel of

$$\bigoplus_i \mathscr{L}_i \to \bigoplus_{i<j} \mathscr{L}_{ij}.$$

(4.8) PROPOSITION. *Sections $s \in H^0(Y_1; \mathscr{L}_1)^{W_1}$ which belong to the image of $H^0(A; \mathscr{K})^W$ are characterized by their property that for all $i$ $(1 \leqslant i \leqslant n)$ and all $w \in W$ with $wY_1 = Y_i$ the sections*

$$s \in H^0(Y_1;\ \mathscr{L}_1)$$

*and*

$$w_* s \in H^0(Y_i;\ \mathscr{L}_i)$$

*represent the same section in $H^0(Y_1 \cap Y_i; \mathscr{L}_{ij})$.*

*Proof.* The condition clearly is necessary. On the other hand if it is satisfied, and if for $1 \leqslant i < j \leqslant n$ the automorphism $w_i \in W$ sends $Y_1$ to $Y_i$, and $w_j \in W$ sends $Y_1$ to $Y_j$, then the condition applied to $w := w_j^{-1} w_i$ shows that

$$w_{i*} s = w_{j*} w_* s \in H^0(Y_i;\ \mathscr{L}_i)$$

and

$$w_{j*}s \in H^0(Y_j; \mathscr{L}_j)$$

restrict to the same section in $H^0(Y_i \cap Y_j; \mathscr{L}_{ij})$. Then $s$ is the image of

$$(s, w_{2*}s, \ldots, w_{n*}s) \in H^0(A; \mathscr{K})^W. \qquad \square$$

For repeated use later on we record the following special case, which is particularly simple.

(4.9) PROPOSITION. *Let $\{Y_1, \ldots, Y_n\}$ be a type I arrangement on which $W$ acts. Assume that for each $i$ $(1 < i \leqslant n)$ there exists a $w_i \in W$ such that*

$$w_i Y_1 = Y_i, \qquad w_i \mid Y_1 \cap Y_i = \mathrm{id}_{Y_1 \cap Y_i},$$

*and such that the homomorphism*

$$\mathscr{L}_{1i} = \mathscr{L} \otimes \mathscr{O}_{Y_1 \cap Y_i} \xrightarrow{\rho w_i} w_i^* \mathscr{L} \otimes \mathscr{O}_{Y_1 \cap Y_i} = \mathscr{L}_{1i}$$

*induced by $\rho w_i$ is the identity. Then, with $Y = \Sigma_i Y_i$ as usual, the restriction homomorphism*

$$H^0(Y; \mathscr{L} \otimes \mathscr{O}_Y)^W \to H^0(Y_1; \mathscr{L}_1)^{W_1}$$

*is bijective.*

*Proof.* The condition in the previous proposition, which clearly does not depend on $w$ but rather its coset $wW_1$ in $W/W_1$, is satisfied with $w = w_1$, for all sections $s \in H^0(Y_1; \mathscr{L}_1)^{W_1}$. Therefore

$$H^0(A; \mathscr{K})^W \to H^0(Y_1; \mathscr{L}_1)^{W_1}$$

is bijective, and the statement now follows from Proposition (4.2). $\square$

(4.10) The infinite product

$$\omega(w, \tau) = (e^{\pi i w} - e^{-\pi i w}) \prod_{l=1}^{\infty} \frac{(1 - e^{2\pi i(l\tau + w)})(1 - e^{2\pi i(l\tau - w)})}{(1 - e^{2\pi i l\tau})^2}$$

converges locally uniformly for all $w \in \mathbb{C}$ and all $\tau \in H$, and thus defines a holomorphic function

$$\omega : \mathbb{C} \times H \to \mathbb{C}.$$

Clearly the limit

$$\lim_{\tau \to i\infty} \omega(w, \tau) = 2i \cdot \sin \pi w$$

exists, and is locally uniform in $w \in \mathbb{C}$. We refer to $\omega$ as the *fundamental Jacobi form* though it is not quite a Jacobi form in the technical sense. We list the functional equations satisfied by $\omega$; their verification is routine and therefore omitted.

$$\omega(-w, \tau) = -\omega(w, \tau)$$

$$\omega(w+1, \tau) = -\omega(w, \tau)$$

$$\omega(w+\tau, \tau) = -e^{-2\pi i w - \pi i \tau} \cdot \omega(w, \tau)$$

$$\omega(w, \tau+1) = \omega(w, \tau)$$

$$\omega(w/\tau, -1/\tau) = (1/\tau) \cdot e^{\pi i w^2/\tau} \cdot \omega(w, \tau).$$

We also note that the divisor of zeroes of $\omega$ is the 'lattice over $H$'

$$\{(w, \tau) \mid w \in \mathbb{Z} + \tau\mathbb{Z}\}.$$

(4.11) We return to the set-up of Section 3, and of (3.8) in particular. Let

$$\varphi \in J^{W\bar{\Gamma}}_{-K,M}$$

be an invariant Jacobi form of weight $-K \leqslant 0$ and index $M \geqslant 0$, and let $Y \subset A$ be the divisor of zeros of $\varphi$.

(4.12) PROPOSITION. *For each $k \geqslant 0$ the inclusion $Y \hookrightarrow A$ gives rise to an exact restriction sequence*

$$0 \to M_k^{\bar{\Gamma}} \xrightarrow{\varphi} J^{W\bar{\Gamma}}_{k-K,M} \to H^0(Y; \mathscr{J}_{k-K,M} \otimes \mathcal{O}_Y)^{W\bar{\Gamma}} \to 0.$$

*Proof.* We pass from the exact sequence

$$0 \to H^0(A; \mathscr{M}^k) \to H^0(A; \mathscr{J}_{k-K,M}) \to H^0(Y; \mathscr{J}_{k-K,M} \otimes \mathcal{O}_Y) \to H^1(A; \mathscr{M}^k)$$

to its $W\bar{\Gamma}$-fixed part (this is an exact functor), and only have to prove

$$H^1(A; \mathscr{M}^k)^{W\bar{\Gamma}} = 0.$$

To this end we consider the Leray sequence of $A \xrightarrow{p} C$, which comprises the sequence

$$0 \to H^1(C;\, p_* \mathscr{M}^k) \to H^1(A;\, \mathscr{M}^k) \to H^0(C;\, R^1 p_* \mathscr{M}^k) \to 0.$$

By the projection formula this reduces to

$$0 \to H^1(C;\, \bar{\mathscr{M}}^k) \to H^1(A;\, \mathscr{M}^k) \to H^0(C;\, \bar{\mathscr{M}}^k \otimes R^1 p_* \mathcal{O}_A) \to 0. \tag{4.13}$$

Passing to $W$-invariants we have to study

$$H^0(C;\, \bar{\mathscr{M}}^k \otimes R^1 p_* \mathcal{O}_A)^W = H^0(C;\, \bar{\mathscr{M}}^k \otimes (R^1 p_* \mathcal{O}_A)^W).$$

By the corollary to Theorem (2.16) we have for each $c \in C$

$$R^1 p_* \mathcal{O}_A \otimes_{\mathcal{O}_{C,c}} \mathbb{C} = H^1(A_c;\, \mathcal{O}_{A_c}).$$

If $c$ is not a cusp then this vector space identifies with

$$\overline{\mathrm{Hom}}_{\mathbb{C}}(\mathbb{C}Q,\, \mathbb{C}),$$

which is a non-trivial simple $W$-module, cf. (2.13). As the representation type of a finite group is locally constant the latter is still true if $c \in C$ is a cusp. In particular $H^1(A_c;\, \mathcal{O}_{A_c})^W = 0$ for all $c \in C$, and therefore

$$(R^1 p_* \mathcal{O}_A)^W = 0.$$

On the other hand $H^1(C; \mathscr{M}^k)$ vanishes for all even $k \geqslant 0$ by the classical theory of modular forms since $C = \overline{H/\Gamma(3)}$ is a rational curve, see e.g. [Gunning] p. 26. For $k$ odd we let

$$C \xrightarrow{q} C/\bar{\Gamma}$$

be the quotient morphism and note

$$H^1(C;\, \bar{\mathscr{M}}^k)^{\bar{\Gamma}} = H^1(C/\bar{\Gamma};\, q_* \bar{\mathscr{M}}^k)^{\bar{\Gamma}} = H^1(C/\bar{\Gamma};\, (q_* \bar{\mathscr{M}}^k)^{\bar{\Gamma}}).$$

Since $-1 \in \bar{\Gamma}$ the sheaf $\bar{\mathscr{M}}^k$ admits no $\bar{\Gamma}$-invariant local sections, and therefore $(q_* \bar{\mathscr{M}}^k)^{\bar{\Gamma}}$ is the zero sheaf. Thus

$$H^1(C;\, \bar{\mathscr{M}}^k)^{\bar{\Gamma}} = 0 \quad \text{for all } k \geqslant 0.$$

We now have proved that the $W\bar{\Gamma}$-invariant part of the sequence (4.13) is trivial, and the proposition follows. □

## 5. The individual root systems

(5.1) We now discuss the individual root systems $R$, and begin with the system of type $A_r$ $(r \geqslant 1)$, using its standard realization in

$$\mathbb{R}Q = \{z \in \mathbb{R}^{r+1} \mid z_0 + z_1 + \cdots + z_r = 0\},$$

see [Bourbaki] Planche I. We define

$$a_{r+1}(z, \tau) = \prod_{j=0}^{r} \omega(z_j, \tau).$$

It follows from the functional equations satisfied by the fundamental Jacobi form that $a_{r+1}$ has the properties listed in (3.3), and may therefore be identified with an invariant Jacobi form

$$a_{r+1} \in J^{W\bar{\Gamma}}_{-(r+1),1}.$$

For $r=1$ the divisor $Y$ of $a_2$ is represented by the set

$$\{(z, \tau) \in \mathbb{C}Q \times H \mid z_1 \in \mathbb{Z} + \tau\mathbb{Z}\},$$

counted with multiplicity 2 (since $z_0 = -z_1$). From Proposition (4.12) we read off the exact sequence

$$0 \to M_2^{\bar{\Gamma}} \to J_{01}^{W\bar{\Gamma}} \to H^0(Y; \mathscr{L} \otimes \mathscr{O}_Y)^{W\bar{\Gamma}} \to 0.$$

In this sequence the term $M_2^{\bar{\Gamma}}$ vanishes since there are no modular forms of weight 2 for the full modular group. On the other hand evaluation at $(0, \tau) \in \mathbb{C}Q \times H$ yields an isomorphism of the quotient term with

$$H^0(C; \mathscr{O}_C)^{\bar{\Gamma}} = \mathbb{C}.$$

This allows us to define a second Jacobi form

$$a_0 \in J_{01}^{W\bar{\Gamma}}$$

as the unique lift of the constant $1 \in \mathbb{C}$.

REMARK. Up to a constant factor $a_0$ and $a_2$ are the weak Jacobi forms $\tilde{\phi}_{0,1}$ and $\tilde{\phi}_{-2,1}$ of [Eichler and Zagier] p. 108. Indeed, making the identifications as before one easily verifies

$$\tilde{\phi}_{0,1} = 12a_0 \quad \text{and} \quad \tilde{\phi}_{-2,1} = -a_2.$$

Thus the quotient $a_0/a_2$, which is a meromorphic section of the sheaf $\mathscr{M}^{(2)}$ on $A$, is $1/4\pi^2$ times the Weierstraß $\wp$-function, cf. loc. cit. p. 39, Theorem 3.6.

We now turn to the case $r > 1$, and define further Jacobi forms $a_r$, $a_{r-1}, \ldots, a_2, a_0$ by induction on $r$. Let us specify the underlying root system $R$ by writing

$$A(R),\ J_{km}(R),\ W(R), \ldots$$

rather than just $A$, $J_{km}$, $W, \ldots$.

Let $Y \subset A(A_r)$ be the divisor of the Jacobi form

$$a_{r+1} \in J^{W(A_r)\bar{\Gamma}}_{-(r+1),1}$$

already constructed. $Y$ is a reduced Cartier divisor, and consists of $r+1$ irreducible components which are the $W(A_r)$-conjugates of the single divisor $Y_1 \subset A(A_r)$ represented by

$$\{(z, \tau) \in \mathbb{C}Q \times H \mid z_r \in \mathbb{Z} + \tau\mathbb{Z}\}.$$

The embedding

$$(z_0, z_1, \ldots, z_{r-1}, \tau) \to (z_0, z_1, \ldots, z_r, \tau)$$

identifies $Y_1$ with $A(A_{r-1})$, the sheaf $\mathscr{L}(A_r) \otimes \mathscr{O}_{Y_1}$ with $\mathscr{L}(A_{r-1})$, and the isotropy group $W(A_r)_{Y_1}$ with $W(A_{r-1})$.

Assuming $r > 2$ we are in the situation of Proposition (4.9): indeed the transposition $(1i) \in W(A_r)$ qualifies as the $w_i$ in the hypothesis of that proposition. Therefore for all $k \geqslant 0$

$$H^0(Y; \mathscr{J}_{k-r-1,1} \otimes \mathscr{O}_Y)^W = H^0(Y_1; \mathscr{J}_{k-r-1,1} \otimes \mathscr{O}_{Y_1})^{W_1} = J^{W(A_{r-1})}_{k-r-1,1}(A_{r-1}),$$

and the exact sequence of Proposition (4.12) comes down to

$$0 \to M^{\bar{\Gamma}}_k \xrightarrow{a_{r+1}} J^{W(A_r)\bar{\Gamma}}_{k-r-1,1}(A_r) \to J^{W(A_{r-1})\bar{\Gamma}}_{k-r-1,1}(A_{r-1}) \to 0.$$

Assuming inductively that Jacobi forms $a_r, a_{r-1}, \ldots, a_2, a_0$ with

$$a_j \in J_{-j,1}^{W(A_{r-1})\bar{\Gamma}}(A_{r-1}) \quad (j = r, r-1, \ldots, 2, 0)$$

have already been defined we now obtain the same number of new Jacobi forms on $A(A_r)$ by lifting the $a_j$. We fix such liftings arbitrarily and write them

$$a_j \in J_{-j,1}^{W(A_r)\bar{\Gamma}}(A_r)$$

again, by abuse of language.

This procedure still works in the case $r=2$ which is (mildly) exceptional since the three components of $Y$ do not form a type I arrangement. Nevertheless this arrangement still is of type II with respect to the sheaves $\mathscr{J}_{km}$. Let $\varphi \in J_{km}^{W(A_1)}(A_1)$ be a symmetric Jacobi form on $Y_1 = A(A_1)$, and let $\tilde{\varphi}$ be the corresponding $W(A_2)$-invariant section of the sheaf $\mathscr{K}$. Let $y$ be a point common to all components of $Y$. The choice of a $W_1$-equivariant isomorphism

$$(\mathscr{J}_{km})_y \xrightarrow[\simeq]{t} \mathcal{O}_{A(A_2),y}$$

makes each of the three components of $t(\tilde{\varphi}_y)$ a function germ which is even. Thus $\tilde{\varphi}$ satisfies (4.5) since the differentials in question must vanish at $y$. Therefore $\tilde{\varphi}$ is in fact a section of $\mathscr{V}$, which in turn comes from $H^0(Y; \mathscr{J}_{km} \otimes \mathcal{O}_Y)$ by Proposition (4.6). Applying this with $a_0$ and $a_2$ in place of $\varphi$ we now apply Proposition (4.12) and obtain liftings

$$a_0 \in J_{01}^{W(A_2)\bar{\Gamma}}(A_2) \quad \text{and} \quad a_2 \in J_{-2,1}^{W(A_2)\bar{\Gamma}}(A_2).$$

(5.2) We next consider $R$ of type $B_r$ ($r \geqslant 2$). The realization of [Bourbaki] Planche III is in $\mathbb{R}Q = \mathbb{R}^r$ with $Q = \mathbb{Z}^r$, and since the unit vectors are short roots the normalized invariant scalar product $(?\,|\,?)$ is twice the ordinary one on $\mathbb{R}^r$, compare (2.2).

For each $r \geqslant 2$ we introduce the Jacobi form

$$b_{2r} \in J_{-2r,1}^{W(B_r)\bar{\Gamma}}(B_r)$$

by the formula

$$b_{2r}(z, \tau) = \prod_{j=1}^{r} \omega(z_j, \tau)^2;$$

its divisor $Y \subset A(B_r)$ consists of $r$ components all $W(B_r)$-conjugate to $Y_1$ which is represented by the set

$$\{(z, \tau) \in \mathbb{C}^r \times H \mid z_r \in \mathbb{Z} + \tau\mathbb{Z}\}$$

and has multiplicity 2. We give these components the analytic structure which makes them the primary components of the Cartier divisor $Y$; then $Y$, having no embedded components, is the union of $Y_1$ and its $W(B_r)$-conjugates in the ideal-theoretic sense.

Since $W(B_r)$ contains the reflection at the hyperplane $\{z \in \mathbb{R}Q \mid z_r = 0\} \subset \mathbb{R}Q$ which represents $y_1$ the $W(B_r)_{Y_1}$ – invariant sections of $\mathscr{J}_{km}(B_r) \otimes \mathscr{O}_{Y_1}$ are simply the $W(B_{r-1})$ – invariant sections of $\mathscr{J}_{km}(B_r) \otimes \mathscr{O}_{|Y_1|}$ where $|Y_1|$ denotes the reduced space underlying $Y_1$. Since this space naturally identifies with $A(B_{r-1})$, and the sheaf $\mathscr{J}_{km}(B_r) \otimes \mathscr{O}_{|Y_1|}$ with $\mathscr{J}_{km}(B_{r-1})$ we obtain, using Propositions (4.9) and (4.12), an exact sequence

$$0 \to M_k^{\bar{\Gamma}} \xrightarrow{b_{2r}} J_{k-2r,1}^{W(B_r)\bar{\Gamma}}(B_r) \longrightarrow J_{k-2r,1}^{W(B_{r-1})\bar{\Gamma}}(B_{r-1}) \to 0$$

for each $k \geqslant 0$. Reading $B_1 = A_1$ we use these sequences inductively to lift the Jacobi forms $a_0, a_2, b_4, b_6, \ldots, b_{2r-2}$ from $A(B_{r-1})$ to $A(B_r)$. Again we fix such liftings

$$a_0 \in J_{01}^{W(B_r)\bar{\Gamma}}(B_r), \qquad a_2 \in J_{-2,1}^{W(B_r)\bar{\Gamma}}(B_r)$$

and

$$b_{2j} \in J_{-2j,1}^{W(B_r)\bar{\Gamma}}(B_r) \quad (2 \leqslant j < r)$$

arbitrarily, keeping the same symbols to denote them.

(5.3) The realization of the root system of type $D_r$ ($r \geqslant 3$) in [Bourbaki] Planche IV has the root lattice

$$Q = \left\{ z = (z_1, \ldots, z_r) \in \mathbb{Z}^r \,\middle|\, \sum_{i=1}^{r} z_i \equiv 0(2) \right\}.$$

For all $r$, the spaces of $W(D_r)$-invariant Jacobi forms $J_{km}^{W(D_r)}(D_r)$ split into even and odd parts

$$J_{km}^{W(D_r)}(D_r) = J_{km}^{W(D_r)+}(D_r) \oplus J_{km}^{W(D_r)-}(D_r)$$

according to parity with respect to the outer involution $\sigma$ of $R$ that changes the sign of $z_1$ (or of any other component of $z$).

For $r \geqslant 4$, we define the odd Jacobi form

$$d_r \in J_{-r,1}^{W(D_r)\bar{\Gamma}-}(D_r)$$

in terms of the fundamental Jacobi form $\omega$ by

$$d_r(z, \tau) = \prod_{j=1}^{r} \omega(z_j, \tau).$$

The divisor $Y$ of $d_r$ is reduced, and consists of the $r$ conjugates under $W(D_r)$ of the hypersurface $Y_1 \subset A(D_r)$ represented by

$$\{(z, \tau) \in \mathbb{C}^r \times H \mid z_r \in \mathbb{Z} + \tau\mathbb{Z}\}.$$

Thus $Y_1$ clearly identifies with $A(D_{r-1})$, and $\mathscr{J}_{km}(D_r) \otimes \mathscr{O}_{Y_1}$ with $\mathscr{J}_{km}(D_{r-1})$, but now the isotropy group $W(D_r)_{Y_1}$ includes the involution $\sigma$ as well as $W(D_{r-1}) \subset W(D_r)$. Applying (4.9) and (4.12) the usual way we obtain for each $k \geqslant 0$ an exact sequence

$$0 \to M_k^{\bar{\Gamma}} \xrightarrow{d_r} J_{k-r,1}^{W(D_r)\bar{\Gamma}}(D_r) \to J_{k-r,1}^{W(D_{r-1})\bar{\Gamma}+}(D_{r-1}) \to 0.$$

This sequence splits into

$$M_k^{\bar{\Gamma}} \xrightarrow{\cong} J_{k-r,1}^{W(D_r)\bar{\Gamma}-}(D_r)$$

and

$$J_{k-r,1}^{W(D_r)\bar{\Gamma}+}(D_r) \xrightarrow{\cong} J_{k-r,1}^{W(D_{r-1})\bar{\Gamma}+}(D_{r-1}).$$

In view of $D_3 = A_3$ the splitting uniquely defines three further Jacobi forms

$$a_j \in J_{-j,1}^{W(D_r)\bar{\Gamma}+}(D_r) \quad (j = 0, 2, 4)$$

for each $r$, by induction on $r \geqslant 4$.

(5.4) In this subsection $R$ is one of the root systems considered so far, i.e. of $A$, $B$ or $D$ type.

(5.5) PROPOSITION. *Fix any $c \in C$, and let*

$$\begin{array}{ll} a_0, a_2, a_3, \ldots, a_{r+1} & (\textit{case } A_r) \\ a_0, a_2, b_4, b_6, \ldots, b_{2r} & (\textit{case } B_r) \\ a_0, a_2, a_4, d_4 & (\textit{case } D_r) \end{array} \tag{5.6}$$

*be the Jacobi forms constructed above. Then the residue classes of these forms in the* $\mathbb{C}$*-algebra*

$$\bigoplus_{m=0}^{\infty} H^0(A_c;\, \mathscr{L}^{(m)} \otimes_{\mathcal{O}_{C,c}} \mathbb{C}) \simeq J_{**} \otimes_{M_*} \mathbb{C}$$

*are algebraically independent.*

*Proof.* Let $\psi$ stand for $a_{r+1}$, $b_{2r}$, and $d_4$, respectively, and assume an algebraic relation

$$\sum_{j=0}^{d} p_j \psi^j = 0 \tag{5.7}$$

holds in that algebra, with $p_j$ polynomial in the remaining forms from the list. Then $p_0$ vanishes along the divisor $Y_c \subset A_c$ of $\psi$, and by induction on $r$ we thus may assume $p_0$ is the zero polynomial. Since $\psi$ is not a zero divisor in $J_{**} \otimes_{M_*} \mathbb{C}$ we may divide (5.7) by $\psi$ and obtain an algebraic relation of strictly lower degree in $\psi$. The claim will follow after at most $d$ such steps. □

COROLLARY. *For each* $c \in C$ *the Jacobi forms* (5.6) *represent a basis of the complex vector space* $H^0(A_c;\, \mathscr{L} \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^W$.

*Proof.* Linear independence is trivially implied by the proposition, so it suffices to estimate the dimension of $H^0(A_c;\, \mathscr{L} \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^W$. This is done by the exact restriction sequence

$$\mathbb{C} \to H^0(A_c;\, \mathscr{L} \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^W \to H^0(Y_c;\, \mathscr{L} \otimes \mathcal{O}_{Y_c})^{W_1}$$

which in the three cases reads

$$\mathbb{C} \xrightarrow{a_{r+1}} H^0(A_c(A_r);\, \mathscr{L} \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^{W(A_r)} \longrightarrow H^0(A_c(A_{r-1});\, \mathscr{L} \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^{W(A_{r-1})}$$

$$\mathbb{C} \xrightarrow{b_{2r}} H^0(A_c(B_r);\, \mathscr{L} \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^{W(B_r)} \longrightarrow H^0(A_c(B_{r-1});\, \mathscr{L} \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^{W(B_{r-1})}$$

$$\mathbb{C} \xrightarrow{d_r} H^0(A_c(D_r);\, \mathscr{L} \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^{W(D_r)} \longrightarrow H^0(A_c(D_{r-1});\, \mathscr{L} \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^{W(D_{r-1})+}$$

respectively.

Clearly we have the further

COROLLARY. *The direct image sheaf* $p_* \mathscr{L}^W$ *on* $C$ *is*

$$p_* \mathscr{L}^W \simeq \mathcal{O} \oplus \mathscr{M}^2 \oplus \mathscr{M}^3 \oplus \cdots \oplus \mathscr{M}^{r+1}$$

$$p_* \mathscr{L}^W \simeq \mathcal{O} \oplus \mathscr{M}^2 \oplus \mathscr{M}^4 \oplus \cdots \oplus \mathscr{M}^{2r}$$

$$p_* \mathscr{L}^W \simeq \mathcal{O} \oplus \mathscr{M}^2 \oplus \mathscr{M}^4 \oplus \mathscr{M}^r,$$

*respectively.* □

We shall use this last corollary in order to construct Jacobi forms of index 2 on $A(D_r)$. First we prove in general:

(5.8) PROPOSITION. *Let* $\varphi \in J^{W\bar{\Gamma}}_{-K,M}$ *be a form of index* $M \geqslant 0$ *and weight* $-K \leqslant 0$, *and let* $Y \subset A$ *be its divisor of zeros. Let* $m > 0$ *be a positive integer, and assume that* $p_*\mathscr{L}^{(m)W}$ *is a sum of non-negative powers of the line bundle* $\bar{\mathscr{M}}$. *Then the restriction sequence*

$$0 \to J^W_{km} \xrightarrow{\varphi} J^W_{k-K,m+M} \to H^0(Y;\, \mathscr{J}_{k-K,m+M} \otimes \mathscr{O}_Y)^W \to 0$$

*is exact, for each* $k \geqslant 0$.

*Proof.* We show that $H^1(A; \mathscr{J}_{km})^W$ vanishes. In view of the projection formula the Leray sequence of $p$ yields

$$0 \to H^1(C;\, \bar{\mathscr{M}}^k \otimes p_*\mathscr{L}^{(m)}) \to H^1(A;\, \mathscr{J}_{km}) \to H^0(C;\, \bar{\mathscr{M}}^k \otimes R^1p_*\mathscr{L}^{(m)}) \to 0.$$

The $W$-invariant part of the first term is isomorphic to a direct sum

$$\bigoplus_\lambda H^1(C;\, \bar{\mathscr{M}}^{k_\lambda}) \quad \text{with all } k_\lambda \geqslant 0,$$

and therefore vanishes as $C$ is rational and $\bar{\mathscr{M}}$ has non-negative degree. On the other hand the quotient term of the sequence vanishes since $R^1p_*\mathscr{L}^{(m)}$ is the trivial sheaf. $\square$

Again specializing to the root system of type $D_r$ we have for each $k \geqslant 0$ an exact sequence

$$0 \longrightarrow J^{W(D_r)\bar{\Gamma}}_{k1}(D_r) \xrightarrow{d_r} J^{W(D_r)\bar{\Gamma}}_{k-r,2}(D_r) \longrightarrow J^{W(D_{r-1})\bar{\Gamma}+}_{k-r,2}(D_{r-1}) \longrightarrow 0$$

which splits into even and odd parts

$$0 \longrightarrow J^{W(D_r)\bar{\Gamma}+}_{k1}(D_r) \xrightarrow{d_r} J^{W(D_r)\bar{\Gamma}-}_{k-r,2}(D_r) \longrightarrow 0$$

$$0 \longrightarrow J^{W(D_r)\bar{\Gamma}-}_{k1}(D_r) \xrightarrow{d_r} J^{W(D_r)\bar{\Gamma}+}_{k-r,2}(D_r) \longrightarrow J^{W(D_{r-1})\bar{\Gamma}+}_{k-r,2}(D_{r-1}) \longrightarrow 0.$$

By induction on $r \geqslant 4$ the latter sequence allows to lift the Jacobi forms

$$a_3^2 \in J^{W(A_3)\bar{\Gamma}+}_{-6,2}(A_3)$$
$$d_4^2 \in J^{W(D_4)\bar{\Gamma}+}_{-8,2}(D_4), \ldots, d^2_{-(r-1)} \in J^{W(D_{r-1})\bar{\Gamma}+}_{-(2r-2),2}(D_{r-1})$$

to even Jacobi forms on $A(D_r)$. We fix such liftings and denote them

$$c_6 \in J^{W(D_r)\bar{\Gamma}+}_{-6,2}(D_r), \ldots, c_{2r-2} \in J^{W(D_r)\bar{\Gamma}+}_{-(2r-2),2}(D_r).$$

(5.9) PROPOSITION. *For each $c \in C$ the residue classes of $a_0$, $a_2$, $a_4$, $d_r$, $c_6$, $c_8, \ldots, c_{2r-2}$ in the $\mathbb{C}$-algebra*

$$\bigoplus_{m=0}^{\infty} H^0(A_c;\, \mathscr{L}^{(m)} \otimes_{\mathcal{O}_{C,c}} \mathbb{C}) \simeq J_{**}(D_r) \otimes_{M_*} \mathbb{C}$$

*are algebraically independent.*

*Proof.* This is the same argument that proved Proposition (5.5). □

(5.10) So far we have constructed, for each of the root systems of types $A_r$, $B_r$ or $D_r$, an $(r+1)$-tuple of symmetric Jacobi forms which for each $c \in C$ defines a weighted homogeneous embedding of the polynomial algebra

$$\mathbb{C}[T_0, \ldots, T_r] \hookrightarrow \bigoplus_{m=0}^{\infty} H^0(A_c;\, \mathscr{L}_c^{(m)})^W,$$

with $\mathscr{L}_c^{(m)} = \mathscr{L}^{(m)} \otimes_{\mathcal{O}_{C,c}} \mathbb{C}$. We must show that this embedding is surjective. To this end, again it suffices to estimate the dimensions of the $\mathbb{C}$-vector spaces $H^0(A_c;\, \mathscr{L}_c^{(m)})^W$, which is done in a straightforward way by induction on $r \geqslant 1$ and $m \geqslant 0$, using the exact sequences

$$H^0(A(A_r)_c;\, \mathscr{L}_c^{(m)})^{W(A_r)} \xrightarrow{a_{r+1}} H^0(A(A_r)_c;\, \mathscr{L}_c^{(m+1)})^{W(A_r)}$$

$$\longrightarrow H^0(A(A_{r-1})_c;\, \mathscr{L}_c^{(m+1)})^{W(A_{r-1})},$$

$$H^0(A(B_r)_c;\, \mathscr{L}_c^{(m)})^{W(B_r)} \xrightarrow{b_{2r}} H^0(A(B_r)_c;\, \mathscr{L}_c^{(m+1)})^{W(B_r)}$$

$$\longrightarrow H^0(A(B_{r-1})_c;\, \mathscr{L}_c^{(m+1)})^{W(B_{r-1})},$$

$$H^0(A(D_r)_c;\, \mathscr{L}_c^{(m)})^{W(D_r)} \xrightarrow{d_r} H^0(A(D_r)_c;\, \mathscr{L}_c^{(m+1)})^{W(D_r)}$$

$$\longrightarrow H^0(A(D_{r-1})_c;\, \mathscr{L}_c^{(m+1)})^{W(D_{r-1})+}.$$

This completes the proof of the assertions in (3.8) for the root systems $R$ of type $A_r$, $B_r$ and $D_r$.

(5.11) For the remaining root systems save $E_6$, $E_7$, $E_8$ the corresponding assertions follow easily from those already proven. Indeed, if $S$ is a root system of type $C_r$, $F_4$ or $G_2$ then we may identify the root lattices $Q(S)$ and $Q(R)$, with $R$

of type $D_r$, $D_4$, $A_2$ respectively. If this identification is made the Weyl group $W(S)$ becomes the full automorphism group $\mathrm{Aut}(R)$ in all cases but $C_4$ where $W(C_4)$ is generated by $W(D_4)$ and the involution $\sigma$ introduced in (5.3). We thus obtain the global invariants easily from the identity

$$J_{**}^{W(S)\bar{\Gamma}}(S) = J_{**}^{W(R)\bar{\Gamma}}(R)^{W(S)/W(R)},$$

while for each $c \in C$

$$\bigoplus_{m=0}^{\infty} H^0(A(S)_c;\ \mathscr{L}_c^{(m)})^{W(S)\bar{\Gamma}} = \left[\bigoplus_{m=0}^{\infty} H^0(A(R)_c;\ \mathscr{L}_c^{(m)})^{W(R)\bar{\Gamma}}\right]^{W(S)/W(R)}$$

Turning to the three particular cases we have

$$J_{**}^{W(C_r)\bar{\Gamma}}(C_r) = J_{**}^{W(D_r)\bar{\Gamma}+}(D_r) = M_*^{\bar{\Gamma}}[a_0, a_2, a_4, c_6, c_8, \ldots, c_{2r-2}, c_{2r}]$$

with $c_{2r} \in J_{-2r,2}^{W(B_r)\bar{\Gamma}}(B_r)$ corresponding to $a_3^2$ $(r=3)$ or to $d_r^2$ $(r>3)$. Similarly

$$J_{**}^{W(G_2)\bar{\Gamma}}(G_2) \subset J_{**}^{W(A_2)\bar{\Gamma}}(A_2)$$

is the subalgebra of Jacobi forms of even weight, i.e.

$$J_{**}^{W(G_2)\bar{\Gamma}}(G_2) = M_*^{\bar{\Gamma}}[a_0, a_2, c_6]$$

with $c_6 \in J_{-6,2}^{W(G_2)\bar{\Gamma}}(G_2)$ corresponding to $a_3^2$.

Finally, to settle the $F_4$ case, we have to determine the action of the group

$$W(F_4)/W(D_4) = \mathrm{Aut}(D_4)/W(D_4) \simeq \mathrm{Sym}(3)$$

on the algebra

$$J_{**}^{W(D_4)\bar{\Gamma}}(D_4) = M_*^{\bar{\Gamma}}[a_0, a_2, a_4, d_4, c_6].$$

We know that $a_0$, $a_2$, $a_4$ and $c_6$ are even (i.e. invariant) forms with respect to the involution $\sigma \in \mathrm{Aut}(D_4)$, while $d_4$ is odd. Since the divisor of zeros of $d_4$ clearly fails to be stable under the full group $\mathrm{Aut}(D_4)$ the line $\mathbb{C}d_4 \subset J_{-4,1}^{W(D_4)\bar{\Gamma}}(D_4)$ is not an invariant subspace of $J_{-4,1}^{W(D_4)\bar{\Gamma}}(D_4)$, and this implies that

$$J_{-4,1}^{W(D_4)\bar{\Gamma}}(D_4) = \mathbb{C}a_4 \oplus \mathbb{C}d_4$$

is a simple Sym(3)-module. On the other hand Sym(3) acts trivially on

$$J_{-2,1}^{W(D_4)\bar{\Gamma}}(D_4) = \mathbb{C}a_2,$$

and the decompositions

$$J_{01}^{W(D_4)\bar{\Gamma}}(D_4) = \mathbb{C}a_0 \oplus M_4^{\bar{\Gamma}} \cdot J_{-4,1}^{W(D_4)\bar{\Gamma}}(D_4)$$
$$J_{-6,2}^{W(D_4)\bar{\Gamma}}(D_4) = \mathbb{C}c_6 \oplus a_2 \cdot J_{-4,1}^{W(D_4)\bar{\Gamma}}(D_4)$$

show that $a_0$ and $c_6$ can be made Sym(3)-invariants by adding suitable linear combinations of $a_4$ and $d_4$. Since the algebra of invariants of the representation of Sym(3) as the dihedral group on $\mathbb{C}a_4 \oplus \mathbb{C}d_4$ is $\mathbb{C}[a_4^2 + d_4^2, a_4 d_4^2]$ we conclude

$$J_{**}^{W(F_4)\bar{\Gamma}}(F_4) = M_*^{\bar{\Gamma}}[a_0, a_2, c_6, f_8, f_{12}]$$

with $f_8 \in J_{-8,2}^{W(F_4)\bar{\Gamma}}(F_4)$ and $f_{12} \in J_{-12,3}^{W(F_4)\bar{\Gamma}}(F_4)$ corresponding to $a_4^2 + d_4^2$ and $a_4 d_4^2$ respectively.

(5.12) We turn to the root systems of type $E_6$ and $E_7$. Rather than with Bourbaki's representation, we prefer to work with the realization of these root systems in the Picard group of a del Pezzo surface. For $5 \leqslant r \leqslant 8$ we thus let $\mathrm{Pic}(X) \simeq \mathbb{Z}^{r+1}$ denote the Picard lattice of a smooth del Pezzo surface $X$ of degree $9-r$. We let $(?\,|\,?)$ be minus the intersection from on $\mathrm{Pic}(X)$. If $\kappa \in \mathrm{Pic}(X)$ denotes the canonical class of $X$ then the orthogonal complement

$$Q(R) = \{x \in \mathrm{Pic}(X) \mid (x \mid \kappa) = 0\}$$

is the root lattice of a root system

$$R = \{x \in Q(R) \mid (x \mid x) = 2\},$$

which is of type $D_5, E_6, E_7, E_8$ respectively. The Weyl group $W(R)$ identifies with the group of isometries of $\mathrm{Pic}(X)$ fixing $\kappa$. These and other details concerning the Picard group of a del Pezzo surface are conveniently found in [Demazure 2].

The finitely many exceptional classes $j \in \mathrm{Pic}(X)$, i.e. those with $(j \mid j) = (j \mid \kappa) = 1$, are each represented by a (unique) exceptional curve on $X$ which we shall refer to as a *line* on $X$. In the case $r = 6$, to which we now turn, $X$ is a smooth cubic in $\mathbb{P}^3$, and the lines are the famous 27 lines on $X$ in the ordinary sense of the word.

If $j \in \mathrm{Pic}(X)$ is one such line the orthogonal complement

$$Q_j = \{z \in Q \mid (z \mid j) = 0\}$$

is spanned by the set

$$R_j = R \cap Q_j,$$

which is a root system of type $D_5$. Both the isotropy group of $j$ and the normalizer of $R$ in $W(R)$ coincide with the Weyl group $W(R_j) \subset W(R)$.

The set of pairs $(i, j)$ of exceptional classes consists of two $W(E_6)$-orbits: Firstly, there are pairs $(i, j)$ such that the corresponding lines are skew, i.e. $(i \mid j) = 0$. In this case one has $i - j \in R$, and the reflection associated to $i - j$ swaps $i$ and $j$, hence $Q_i$ and $Q_j$, while it restricts to the identity on $Q_i \cap Q_j$. The second $W(E_6)$-orbit comprises pairs $(i, j)$ such that the corresponding lines on $X$ meet. In that case there is a unique further exceptional class $k \in \mathrm{Pic}(X)$ such that the lines corresponding to $i$, $j$, and $k$ form a triangle on $X$:

$$(j \mid k) = (i \mid k) = (i \mid j) = -1.$$

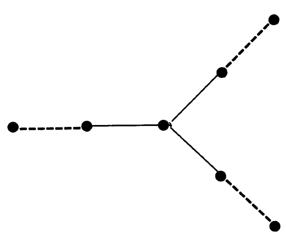
$i - j$ is the sum of two orthogonal roots $\alpha$ and $\alpha'$. The product $w \in W(E_6)$ of the associated reflections swaps $Q_i$ and $Q_j$, and leaves $Q_k$ invariant. The root system

$$R_{ij} = R \cap Q_i \cap Q_j$$

is of type $D_4$ and generates the lattice

$$Q_{ij} = Q_j \cap Q_k = Q_i \cap Q_k = Q_i \cap Q_j.$$

The inclusions of $Q_{ij}$ in $Q_i$, $Q_j$, and $Q_k$ realise the three possible extensions of the Dynkin diagram $D_4$:



If suitable identifications are made the restriction of $w$ to $Q_{ij}$ becomes the unique diagram automorphism of $D_4$ that extends to $D_5$ under the inclusion $R_{ij} \hookrightarrow R_k$.

(5.13) Using the fundamental Jacobi form $\omega$ we define a function

$$e_{27}(z, \tau) = \prod_j \omega((z \mid j), \tau);$$

the product is taken over all exceptional classes $j \in \mathrm{Pic}(X)$. A straightforward verification shows that

$$e_{27} \in J_{-27,6}^{W(E_6)\bar{\Gamma}}(E_6)$$

is a symmetric Jacobi form.

We let

$$Y = \sum_j Y_j$$

be the divisor of zeros of $e_{27}$ where again $j$ runs through the set of exceptional classes in $\mathrm{Pic}(X)$. We arbitrarily single out three of those classes $j$ that define a triangle on $X$, and denote them $j = 1, 2, 3$. We also fix an element $w_j \in W(E_6)$ for each such $j$, with

$$w_1 = 1 \in W(E_6) \quad \text{and} \quad w_j(j) = 1 \in \mathrm{Pic}(X).$$

The inclusions

$$Q_{12} \subset Q_1 \subset Q$$

considered in (5.12) then define canonical embeddings

$$A(D_4) \hookrightarrow A(D_5) \simeq Y_1 \hookrightarrow A(E_6).$$

We must determine which symmetric Jacobi forms

$$\varphi_1 \in J_{**}^{W(D_5)\bar{\Gamma}}(D_5)$$

have symmetric extensions over $Y$. After the choice of the $w_j$ each $\varphi_1 \in J_{**}^{W(D_5)\bar{\Gamma}}(D_5)$ determines a section

$$(\varphi_j) \in \bigoplus_j H^0(Y_j; \mathscr{J}_{**}(E_6) \otimes \mathcal{O}_{Y_j}),$$

and as a necessary condition for extendability we note:

(5.14) PROPOSITION. *Let $\mathscr{K}$ be the kernel of the sheaf homomorphism*

$$\bigoplus_j \mathscr{J}_{**}(E_6) \otimes \mathcal{O}_{Y_j} \to \bigoplus_{i \neq j} \mathscr{J}_{**}(E_6) \otimes \mathcal{O}_{Y_i \cap Y_j}$$

*as in* (4.4). *Then $(\varphi_j)$ is a section of $\mathscr{K}$ if and only if the residue class*

$$\overline{\varphi_1} \in J_{**}^{W(D_4)\bar{\Gamma}}(D_4)$$

*of $\varphi_1$ is invariant under the full automorphism group* $\mathrm{Aut}(D_4) = W(F_4)$.

*Proof.* This follows from Propositions (4.8) and (4.9), in view of the discussion in (5.12). □

We restate this proposition in terms of the already known structure of the algebras of Jacobi forms concerned. We put

$$R = M_*^{\bar{\Gamma}}[a_0, a_2, a_4, d_5, c_6, c_8] = J_{**}^{W(D_5)\bar{\Gamma}}(D_5),$$
$$R_0 = M_*^{\bar{\Gamma}}[a_0, a_2, a_4, c_6, c_8] \subset R;$$

thus $R_0$ projects isomorphically onto

$$\bar{R} = R/d_5R.$$

We further introduce the subalgebra

$$\bar{S} = M_*^{\bar{\Gamma}}[a_0, a_2, c_6, f_8, f_{12}] = J_{**}^{W(F_4)\bar{\Gamma}}(F_4) = \bar{R}^{W(F_4)};$$

recall from (5.11) that

$$f_8 = a_4^2 + d_4^2 = a_4^2 + c_8$$
$$f_{12} = a_4 d_4^2 = a_4 c_8.$$

The exact diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "d_5"] & R \arrow[r, "\pi"] & \bar{R} \arrow[r] & 0 \\ 0 \arrow[r] & R \arrow[r] \arrow[u, equal] & S \arrow[r] \arrow[u] & \bar{S} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

defines the subalgebra

$$S = \pi^{-1}(\bar{S}) \subset R,$$

and what (5.14) assures is that the Jacobi forms $\varphi_1 \in S$ are exactly those with $(\varphi_j)$ a secton of $\mathscr{K}$. We next have to determine those $\varphi_1 \in S$ with $(\varphi_j)$ in fact a section of the subsheaf $\mathscr{V} \subset \mathscr{K}$ introduced in (4.4). Since triple intersections $Y_i \cap Y_j \cap Y_k$ have codimension three in $A(E_6)$ unless $i, j, k$ form a triangle on $X$ the quotient sheaf $\mathscr{K}/\mathscr{V}$ is supported on $Y_{12} = Y_1 \cap Y_2$ and its $W(E_6)$-conjugates.

(5.15) The algebra $\bar{S} \subset \bar{R}$ is the isomorphic image under $R \xrightarrow{\pi} \bar{R}$ of

$$S_0 = M_*^{\bar{\Gamma}}[a_0, a_2, c_6, f_8, f_{12}] \subset R_0.$$

In view of the fact that $R_0$ is the free $S_0$-module with basis $\{1, a_4, a_4^2\}$ we may decompose $S$ as an $S_0$-module, as follows:

$$S = S_0 \oplus S_0 d_5 \oplus S_0 a_4 d_5 \oplus S_0 a_4^2 d_5 \oplus R d_5^2.$$

The subalgebra $S_0 \subset J_{**}^{W(D_5)\bar{\Gamma}}(D_5)$ is even with respect to the reflection of $\mathbb{R}Q_1$ at the hyperplane $\mathbb{R}Q_{12}$; in particular for each $\varphi_1 \in S_0$ the differential $d\varphi_1$ is everywhere zero along $\mathbb{C}Q_{12}$. Therefore $(\varphi_j)$ is a section of $\mathscr{V}$ for all $\varphi_1 \in S_0$, and, for the same reason, for all $\varphi_1 \in Rd_5^2$.

We shall now show that $(\varphi_j)$ is a section of $\mathscr{V}$ if $\varphi_1 = d_5$ or $\varphi_1 = a_4 d_5$, but not if $\varphi_1 = a_4^2 d_5$.

Let $t \in W(E_6)$ be an element of order 3 that induces a cyclic permutation on the set of exceptional classes $\{1, 2, 3\} \subset \mathrm{Pic}(X)$. Then for each vector $v_1 \in \mathbb{C}Q_1$ the sum

$$v_1 + tv_1 + t^2 v_1 \in V_{\mathbb{C}}(E_6)$$

belongs to $\mathbb{C}Q_{12}$. The condition (4.5) on $\varphi_1$ that $(\varphi_j)$ be a section of $\mathscr{V}$ therefore reads

$$\langle v_1, d_x\varphi_1\rangle + \langle tv_1, d_{tx}(\varphi_1 \circ t^{-1})\rangle + \langle t^2 v_1, d_{t^2x}(\varphi_1 \circ t^{-2})\rangle = 0$$

for all $x \in \mathbb{C}Q_{12}$ and all $v_1 \in \mathbb{C}Q_1$. The left hand side reduces to

$$\begin{aligned}&\langle v_1, d_x\varphi_1\rangle + \langle v_1, d_{tx}\varphi_1\rangle + \langle v_1, d_{t^2x}\varphi_1\rangle\\ &\quad = \langle v_1, d_x\varphi_1 + d_{tx}\varphi_1 + d_{t^2x}\varphi_1\rangle.\end{aligned}$$

If $\varphi_1$ is a multiple of $d_5$, say

$$\varphi_1 = \psi \cdot d_5$$

then the derivatives of $\varphi_1$ normal to $\mathbb{C}Q_{12}$ are proportional to the values of the function $\psi \cdot d_4$. The condition for $(\varphi_j)$ to be a section of $\mathscr{V}$ thus comes down to

$$(\psi d_4)(x) + (\psi d_4)(tx) + (\psi d_4)(t^2x) = 0 \quad \text{for all } x \in \mathbb{C}Q_{12}.$$

This in turn means that in the decomposition of $J_{**}^{W(D_4)\bar{\Gamma}}(D_4)$ into $t$-weight spaces, the Jacobi form $\psi d_4$ has no component in the $t$-fixed part.

We have to check whether this last condition holds with $\psi = 1$, $a_4$, and $a_4^2$, respectively. The subgroup of $W(E_6)$ spanned by the elements $t$ and $w$ is the symmetric group Sym(3), and we know from (5.11) that the Sym(3)-module generated by the Jacobi form $d_4$ is the plane

$$\mathbb{C}a_4 \oplus \mathbb{C}d_4 \subset J_{**}^{W(D_4)\bar{\Gamma}}(D_4);$$

this is the unique simple Sym(3)-module of dimension 2. If $V_0$, $V_+$, $V_-$ denote the simple representations of the cyclic group $\langle t \rangle$ ($V_0$ trivial) then

$$\mathbb{C}a_4 \oplus \mathbb{C}d_4 \simeq V_+ \oplus V_-$$

as $\langle t \rangle$-modules. Therefore $\varphi_1 = d_5$ does determine a section $(\varphi_j)$ of $\mathscr{V}$.

Turning to $\psi = a_4$, we note that $a_4 d_4$ is contained in the Sym(3)-module

$$\mathbb{C}a_4^2 \oplus \mathbb{C}a_4 d_4 \oplus \mathbb{C}d_4^2.$$

Its simple components are

$$\mathbb{C}(a_4^2 + d_4^2) \quad \text{and} \quad \mathbb{C}(a_4^2 - d_4^2) \oplus \mathbb{C}a_4 d_4,$$

and as $a_4 d_4$ belongs to the latter it has no $\langle t \rangle$-fixed part. Thus $\varphi_1 = a_4 d_5$ also determines a section $(\varphi_j)$ of $\mathscr{V}$.

Finally, to study the case $\psi = a_4^2$ we decompose the Sym(3)-module

$$\mathbb{C}a_4^3 \oplus \mathbb{C}a_4^2 d_4 \oplus \mathbb{C}a_4 d_4^2 \oplus \mathbb{C}d_4^3:$$

it is the direct sum of a two-dimensional and two one-dimensional representations. Since $a_4^2 + d_4^2$ is an invariant the former must be the simple submodule

$$\mathbb{C}(a_4^2 + d_4^2)a_4 \oplus \mathbb{C}(a_4^2 + d_4^2)d_4.$$

The element $\psi d_4 = a_4^2 d_4$ does not belong to this submodule, and therefore has a non-zero component in the subspace of $\langle t \rangle$-fixed elements. This proves that $a_4^2 d_5$ does not lead to a section of $\mathscr{V}$.

In view of Proposition (4.6) we now have proved:

(5.16) PROPOSITION. *Restriction from $Y$ to $Y_1$ identifies*

$$\bigoplus_{k,m} H^0(Y; \mathscr{I}_{km} \otimes \mathscr{O}_Y)^{W(E_6)\bar{\Gamma}}$$

*with the subalgebra*

$$S_0 \oplus S_0 d_5 \oplus S_0 a_4 d_5 \oplus R d_5^2 \subset R = J_{**}^{W(D_5)\bar{\Gamma}}(D_5). \qquad \square$$

REMARK. The same reasoning would have shown the analogous result for each fibre of the projection $Y \xrightarrow{p} C$.

The structure of this subalgebra is given by

(5.17) PROPOSITION. *The algebra*

$$S_0 \oplus S_0 d_5 \oplus S_0 a_4 d_5 \oplus R d_5^2$$

*is generated over* $M_*^{\bar{\Gamma}}$ *by the Jacobi forms*

$$a_0,\ a_2,\ c_6,\ d_5,\ e_9, f_8, \text{ and } f_{12}$$

*with*

$$e_9 = a_4 d_5$$

*and*

$$f_8 = a_4^2 + c_8$$

$$f_{12} = a_4 c_8$$

*as above. These generators are subject to the single generating relation*

$$e_9^3 - d_5^2 e_9 f_8 + d_5^3 f_{12} = 0$$

*in* $J_{-27,6}$.

*Proof.* Let $S' \subset R$ be the subalgebra generated by the elements listed. Then clearly

$$S_0 \oplus S_0 d_5 \oplus S_0 a_4 d_5 \subset S'.$$

On the other hand

$$R d_5^2 = R_0[d_5] \cdot d_5^2,$$

and since $R_0$ is spanned, as an $S_0$-module, by $1, a_4$, and $a_4^2$ it follows that $S'$ also contains $Rd_5^2$. This proves that the elements from the list generate. The ideal of relations is principal because $a_0$, $a_2$, $c_6$, $d_5$, $f_8$, and $f_{12}$ are algebraically independent. The second part of the proposition now follows easily. □

(5.18) We have the following proposition, which complements (4.12) and (5.8):

(5.19) Let $\varphi \in J_{-K,M}^{W\bar{\Gamma}}$ be a form of index $M \geqslant 0$ and weight $-K \leqslant 0$, and let

$Y \subset A$ be its divisor of zeros. Assume that the rank $r$ of the underlying root system $R$ is at least 2. Then for all $m < M$, and any $k \in \mathbb{Z}$ restriction from $A$ to $Y$ gives an isomorphism

$$J_{km} \simeq H^0(Y; \mathscr{J}_{km} \otimes \mathcal{O}_Y).$$

*Proof.* The exact sequence

$$\begin{aligned} 0 &\to H^1(C; \mathscr{M}^{k+K} \otimes p_* \mathscr{L}^{(m-M)}) \to H^1(A; \mathscr{J}_{k+K,m-M}) \\ &\to H^0(C; \mathscr{M}^{k+K} \otimes Rp_*^1 \mathscr{L}^{(m-M)}) \to 0 \end{aligned}$$

is trivial. □

Since $a_0$, $a_2$, $c_6$, $d_5$, $e_9$, $f_8$, and $f_{12}$ all have index at most 3 we may use (5.19) in order to obtain (unique) liftings of these forms in $J_{**}^{W(E_6)\bar{\Gamma}}(E_6)$. In this latter algebra we have a relation

$$e_9^3 - d_5^2 e_9 f_8 + d_5^3 f_{12} = \lambda \cdot e_{27}$$

where $\lambda$ is a Jacobi form of weight and index 0, i.e. a constant.

(5.20) PROPOSITION. *$\lambda \neq 0$, and for each $c \in C$ the forms $a_0$, $a_2$, $c_6$, $d_5$, $e_9$, $f_8$, $f_{12}$ have algebraically independent residue classes in*

$$\bigoplus_{m=0}^{\infty} H^0(A_c; \mathscr{L}^{(m)} \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^W = J_{**}(E_6)^W \otimes_{M_*} \mathbb{C}.$$

*Proof.* Let

$$J \subset J_{**}(E_6)^W \otimes_{M_*} \mathbb{C}$$

be the graded subalgebra generated by those classes. It is proved in [Looijenga 1] (4.2) that for generic $c \in C$, the algebra $J_{**}^W \otimes_{M_*} \mathbb{C}$ is a polynomial algebra on homogeneous generators, their degrees being the indices of $a_0$, $a_2$, $c_6$, $d_5$, $e_9$, $f_8$, $f_{12}$ (i.e. 1, 1, 2, 1, 2, 2, 3). Since by (5.19), $J$ contains at least $J_{*m}^W \otimes_{M_*} \mathbb{C}$ for $m < 6$ the classes of $a_0$, $a_2$, $c_6$, $d_5$, $e_9$, $f_8$, $f_{12}$ also generate the polynomial algebra $J_{**}^W \otimes_{M_*} \mathbb{C}$ ($c \in C$ generic). In particular we conclude $\lambda \neq 0$.

We now let $c \in C$ be arbitrary. In view of the structure of

$$\bigoplus_{m=0}^{\infty} H^0(Y_c; \mathscr{L}^{(m)} \otimes \mathcal{O}_{Y_c}),$$

see (5.17), every relation between $a_0$, $a_2$, $c_6$, $d_5$, $e_9$, $f_8$, $f_{12}$ in $J$ is divisible by

$e_9^3 - d_5^2 e_9 f_8 + d_5^3 f_{12}$, which is not a zero divisor in $J$ because $\lambda \neq 0$. From this it follows by induction that there are no non-trivial relations at all. □

Estimating the dimensions of the graded pieces as in (5.4) we now obtain that in fact

$$J = J_{**}(E_6)^W \otimes_{M_*} \mathbb{C}$$

for each $c \in C$, and that therefore the symmetric Jacobi forms

$$a_0 \in J_{01}^W,\ a_2 \in J_{-2,1}^W,\ d_5 \in J_{-5,1}^W,$$

$$c_6 \in J_{-6,2}^W,\ e_9 \in J_{-9,2}^W,\ f_8 \in J_{-8,2}^W,$$

$$f_{12} \in J_{-12,3}^W$$

have all the properties stipulated in (3.8). This completes the proof of Theorem (3.6) for $R$ of type $E_6$.

(5.21) We finally treat the case of $R$ of type $E_7$. The 56 "lines" on the del Pezzo surface of degree 2 occur in 28 pairs which are invariant under the symmetry $-1 \in W(E_7)$. We arbitrarily pick one representative $j$ from each pair and form the product function

$$e_{28}(z, \tau) = \prod_j \omega((z \mid j), \tau).$$

Again this is seen to be an invariant Jacobi form

$$e_{28} \in J_{-28,6}^{W(E_7)\bar{\Gamma}}(E_7).$$

Let $Y \subset A(E_7)$ be its divisor of zeros; then the components of $Y$ are isomorphic to $A(E_6)$, and form an arrangement of type I on $A(E_7)$. Since $-1 \in W(E_7)$ stabilises each component only Jacobi forms of even degree can be extended from $A(E_6)$ to $A(E_7)$. The Weyl group $W(E_7)$ acts doubly transitively on the set of components of $Y$, and their intersections give rise to no further obstruction to extending Jacobi forms symmetrically. We therefore have

$$\bigoplus_{k,m} H^0(Y; \mathscr{J}_{km} \otimes \mathcal{O}_Y)^{W(E_7)\bar{\Gamma}} = \bigoplus_{\substack{k,m \\ k \text{ even}}} J_{km}^{W(E_6)\bar{\Gamma}}(E_6).$$

If we put

$$e_{10} = d_5^2,\ e_{14} = d_5 e_9,\ e_{18} = e_9^2$$

the right hand side comes down as the algebra

$$M_*^{\bar{\Gamma}}[a_0, a_2, c_6, e_{10}, e_{14}, e_{18}, f_8, f_{12}]$$

with generating relation

$$e_{10} \cdot e_{18} = e_{14}^2.$$

Exactly the same reasoning as in the $E_6$ case allows to lift the generators to Jacobi forms on $A(E_7)$:

$$a_0 \in J_{01}^W, \quad a_2 \in J_{-2,1}^W,$$
$$c_6 \in J_{-6,2}^W, \quad e_{10} \in J_{-10,2}, \quad f_8 \in J_{-8,2},$$
$$e_{14} \in J_{-14,3}^W, \quad f_{12} \in J_{-12,3},$$
$$e_{18} \in J_{-18,4}^W.$$

It also follows as before that these forms are in fact algebraically independent and generating along each fibre of $A \xrightarrow{p} C$.

This completes the proof of Theorem (3.6). □

(5.22) REMARKS. The inductive procedure we have followed in order to construct $W$-invariant Jacobi forms would also serve to construct basic invariants for the linear action of $W$ on the vector space $V_{\mathbb{C}}$ (excluding the root system of type $E_8$). To some extent this explains why the weights of the basic symmetric Jacobi forms correspond to the degrees occurring in the linear invariant theory of the Weyl group.

We do not know to what extent Theorem (3.6) holds for $R$ of type $E_8$. Since the root lattice of $E_8$ is unimodular the Riemann theta function associated with that lattice is a symmetric Jacobi form

$$\theta \in J_{4,1}^{W(E_8)\bar{\Gamma}}(E_8)$$

which is seen to span the vector space

$$H^0(A_c; \mathscr{L} \otimes_{\mathcal{O}_{C,c}} \mathbb{C})^W$$

for each $c \in C$.

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