LUCY MOSER-JAUSLIN

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# The Chow rings of smooth complete SL(2)-embeddings

LUCY MOSER-JAUSLIN*

*Section de Mathématiques, Université de Genève, rue du Lievre 2-4, Case Postale 240, 1211 Genève 24, Switzerland*

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## Introduction

An SL(2)-embedding is a three-dimensional algebraic SL(2)-variety over **C** having an open orbit equivariantly isomorphic to SL(2). The smooth complete SL(2)-embeddings have been classified in a combinatorial way by assigning them "diagrams" which contain information about the local rings of the orbits (see [LV] and [$MJ_3$]). In this paper we will describe how to calculate the Chow ring of such a variety directly from its diagram. The Chow ring will be shown to be isomorphic to the cohomology ring.

The determination of the Chow ring is an important aspect in the study of the geometrical properties of an embedding. Thus it is useful to be able to see it directly from its diagram. In some ways, the theory of SL(2)-embeddings is similar to that of torus embeddings (see [KKMS], [Dan], and [Oda]). In the case of a torus, each embedding corresponds to a "fan" from which many geometrical properties can be studied. The Chow ring, for example, was calculated by Danilov and Jurkiewicz (see [Dan] and [$Jur_1$]). For the case of SL(2)-embeddings, some questions become easier (since we are only interested in three-dimensional varieties) and others become more complicated. For example, like for torus embeddings, we will see that the Chow ring is generated by stable divisors. However, unlike the toric case, the relations between these generators are not at all apparent. For this reason we introduce an additional generator, which is the closure of a certain Borel subgroup $B$ of SL(2).

In section 1, we give a brief review of the theory of SL(2)-embeddings. In section 2, we prove some results concerning how the generators of the Chow ring intersect. We show, for example, that the stable divisors intersect transversely and that they are smooth surfaces, which are determined very simply by the diagram. Then in section 3, we calculate explicitly the Chow ring. This is done as follows: we first prove that we do indeed have a set of generators; then we give a set of relations which can be described geometrically, and finally we

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*Present address: Lucy Moser-Jauslin c/o Moser, 49 Greifenseestr, CH-8603 Schwerzenbach, Switzerland (Partially supported by the Swiss National Science Foundation, Grant No. 8220-025974).

prove using information about the dimensions of the groups of classes of cycles in the Chow ring that we have found enough relations. (For this last part, we use the fact that the Chow ring coincides with the cohomology ring, and we can therefore use Poincaré duality.) In section 4 we calculate the Chow ring for two examples.

In section 5, we deduce the canonical divisor from the diagram of an embedding. Then in section 6, we study the cone of effective one-cycles. We show that it is a finite polyhedral cone generated by curves stable by the action of a Borel subgroup of SL(2), though not necessarily stable by SL(2). (This is similar to what happens for torus embeddings; there the generators are all stable by the torus action [Reid].) Having a finite polyhedral cone of effective one-cycles is a very useful property. For example, it means that Nakai's criterion for a divisor to be ample can be simplified to the statement that a divisor $D$ is ample if and only if $D \cdot C > 0$ for all effective one-cycles $C$. This is not true for general varieties (see [Har]). Using this simplified Nakai's criterion, we find a necessary and sufficient condition for a smooth complete SL(2)-embedding to be projective. In section 7, we look at some specific examples of the cones. Finally, in section 8, we use the knowledge of the Chow rings to solve the following problem posed by V. L. Popov: Find the degree of a closed three-dimensional orbit of an affine irreducible representation of SL(2).

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## 1. A review of SL(2)-embeddings

An SL(2)-*embedding* is an algebraic variety $X$ over **C** endowed with an action of SL(2) and an equivariant open immersion $i\colon \mathrm{SL}(2) \hookrightarrow X$. Thus $X$ is a three-dimensional variety with an open orbit which we identify to SL(2) using the immersion $i$. (We consider the embedding with the base point given by the image of the identity element under $i$.) Thus for a subgroup $H$ of SL(2), we can talk about the closure of $H$ in $X$. The set of smooth complete SL(2)-embeddings have been classified in [$MJ_3$]. To each such embedding one associates a "diagram" which characterizes the local rings of the orbits. Throughout this paper, when not specified, all varieties are considered to be smooth.

Throughout this paper, we will be interested in SL(2)-stable and $B$-stable subvarieties of embeddings, where $B$ is a Borel subgroup of SL(2). When not specified, the word "stable" means "SL(2)-stable," and "invariant" means "invariant under the action of SL(2)."

First we recall some important facts about the classification. To each stable irreducible divisor of an embedding corresponds an invariant geometric valuation ring with the same quotient field as $\mathbf{C}[\mathrm{SL}(2)]$, the ring of regular functions on SL(2). The set of these valuations are found as follows. Choose a Borel subgroup $B$ of SL(2). Then the set of eigenvectors of $B$ in $\mathbf{C}[\mathrm{SL}(2)]$ is the set of homogeneous polynomials in two variables, which we call $z$ and $w$. An invariant valuation is determined by its values on this set, and therefore on the set $\{az + bw\}_{(a:b)\in\mathbf{P}^1}$. It is normalized such that it takes the value $-1$ for all but possibly one of these elements, and $r \geqslant -1$ on the remaining one. There is one valuation which corresponds to a divisor comprised of an infinite number of one-dimensional orbits; it takes the value $-1$ on all the elements above. We denote this valuation by $v(\ , -1)$. All the others correspond to divisors which contain an open orbit. Let $D$ be a $B$-stable divisor of SL(2) corresponding to the function $a_0z + b_0w$, $(a_0:b_0)\in\mathbf{P}^1$. Then one finds for each $r\in\mathbf{Q}\cap(-1,1]$ an invariant valuation denoted $v(D,r)$ with

$$v(D, r)(az + bw) = \begin{cases} r & \text{if } (a:b) = (a_0:b_0) \\ -1 & \text{otherwise.} \end{cases}$$

That there are no other valuations of stable divisors is easy to see using elementary properties of valuations. One way to show the existence of these valuations is by using limits of "curves." This process is described in [LV, §4]. A "curve" in SL(2) is an element $\lambda\in\mathrm{SL}(2,\mathbf{C}((t)))$. Such a $\lambda$ induces a map

$$i_\lambda\colon \mathbf{C}[\mathrm{SL}(2)] \xrightarrow{\text{comult}} \mathbf{C}[\mathrm{SL}(2)] \otimes \mathbf{C}[\mathrm{SL}(2)] \xrightarrow{1\otimes\lambda} \mathbf{C}[\mathrm{SL}(2)] \otimes \mathbf{C}((t))$$

$$\hookrightarrow \mathbf{C}(\mathrm{SL}(2))((t)).$$

To $\lambda$ we associate the valuation $v_\lambda = v_t \circ i_\lambda$, where $v_t$ gives the order of $t$. One can show that $v_\lambda$ is the valuation of some stable divisor containing a 2-dimensional orbit, and conversely, all such valuations are obtained in this way. One can also see that it is enough to restrict the study to curves in $B$. Consider $\mathbf{C}(B) \subset \mathbf{C}(\mathrm{SL}(2))$ by using the projection of the big cell $U^- \times B \to B$, where $U^-$ is the unipotent radical of a Borel subgroup distinct from $B$. Then the valuation is determined by its restriction to $\mathbf{C}(B)$. (Geometrically, this is the valuation ring of the intersection of the divisor with $\bar{B}$, the closure of $B$, considered as a divisor of $\bar{B}$.)

One knows that $X$ has no fixed point, since it is smooth (see [MJ$_3$]). Thus all remaining closed orbits are of dimension one (isomorphic to $\mathbf{P}^1$). *They are determined by the set of B-stable divisors containing them.* We identify the $B$-stable divisors of SL(2) with $\mathbf{P}^1$ (from the notation given earlier, they are the divisors given by the zeroes of a function of the form $az + bw$, $(a:b)\in\mathbf{P}^1$); their closures in $X$ are clearly $B$-stable divisors. Thus the set of possible $B$-stable divisors is given by $\mathbf{P}^1 \cup \{\mathrm{SL}(2)\text{-stable divisors}\}$. There are several types of orbits. For each type we give (i) the set of valuations of SL(2)-stable divisors containing it; (ii) the $B$-stable divisors in $\mathbf{P}^1$ whose closures contain it; and (iii) the set of geometric invariant valuations which dominate the orbit. This last part is used to see what part of the diagram refers to the orbit (see the description below). The valuations are given in the form $v(D, r)$ where $D \in \mathbf{P}^1$, and $r = p/q$ with $p$ and $q$ relatively prime and $q > 0$. Given two distinct valuations $v = v(D, r)$ and $v' = v(D', r')$, we say that $v'$ lies "above" $v$ if $D = D'$ and $r' > r$; otherwise we say it lies below $v$.

*Type AB:*

(i) $v(D, r_1)$ and $v(D, r_2)$, with $D \in \mathbf{P}^1$ and $-1 \leqslant r_1 < r_2 \leqslant 1$ and $q_1 p_2 - q_2 p_1 = 1$ (this last condition is needed for smoothness of the orbit in the embedding);
(ii) no elements of $\mathbf{P}^1$;
(iii) $\{$valuations lying above $v(D, r_1)$ and below $v(D, r_2)\}$;

*Type $B_+$:*

(i) $v(D, r)$ with $D \in \mathbf{P}^1$ and $r = 0$ or $-1$;
(ii) $D$;
(iii) valuations lying above $v(D, r)$;

*Type $B_-$:*

(i) $v(D, r)$ with $D \in \mathbf{P}^1$ and $r = 1/q$, $q \geqslant 2$;
(ii) $\mathbf{P}^1 \backslash D$;
(iii) valuations lying above $v(D, r)$;

*Type $A_1$:*

(i) $v(D, r)$ with $D \in \mathbf{P}^1$ and $r = -1/q$, $q \geqslant 2$;
(ii) $\mathbf{P}^1 \backslash D$;
(iii) valuations lying below $v(D, r)$;

*Type $A_2$:*

(i) $v(D_1, r_1)$, $v(D_2, r_2)$ where $D_1 \neq D_2$ and either $r_1 = 1$ and $r_2 = (q-1)/q$, $q \geqslant 1$, or $r_1 = r_2 = 0$;
(ii) $\mathbf{P}^1 \backslash \{D_1, D_2\}$;
(iii) valuations lying below $v(D_1, r_1)$ and $v(D_2, r_2)$.

In this paper we sometimes identify an orbit by giving its type and the valuations in (i). For example, we say an orbit is of type $B_+$ "with $v(D,0)$" or simply "with $r=0$" if it is contained in a stable divisor with valuation $v(D,0)$. Note that the valuations of (iii) determine the type. This is because we only consider smooth embeddings.

The diagram of an embedding $X$ is given as follows. First we draw a diagram of all the invariant valuations (see Fig. 1a). This will be the "skeleton" of the diagram. For each stable divisor we mark the corresponding valuations (see Fig. 1b). Each connected component of the skeleton minus the marks corresponds to a one-dimensional orbit whose boundaries in the diagram correspond to the SL(2)-stable divisors containing it. The valuations in this part of the diagram are those which dominate the local ring of the orbit. For orbits of type $B_+$ and $B_-$, we distinguish the two by labeling the orbits with either $+$ or $-$ (see Fig. 1c). As said before, the type is in fact determined by the situation of the orbit in the diagram (coming from the fact that we are only looking at smooth orbits), and it would therefore not be necessary to label the $B_-$ and $B_+$ orbits. However, for clarity, we make the distinction.

An important divisor which we will use in the following sections is the closure of $B$. We will want to calculate how this divisor intersects with the stable cycles. What follows will be of use in this direction.

Given a $B$-embedding $S$, we can construct a special SL(2)-embedding $\mathrm{SL}(2)\times_B S = \mathrm{SL}(2)\times S/\sim$, where $(g,s)\sim(gb^{-1},bs)$ for $g\in\mathrm{SL}(2)$, $b\in B$, and $s\in S$.

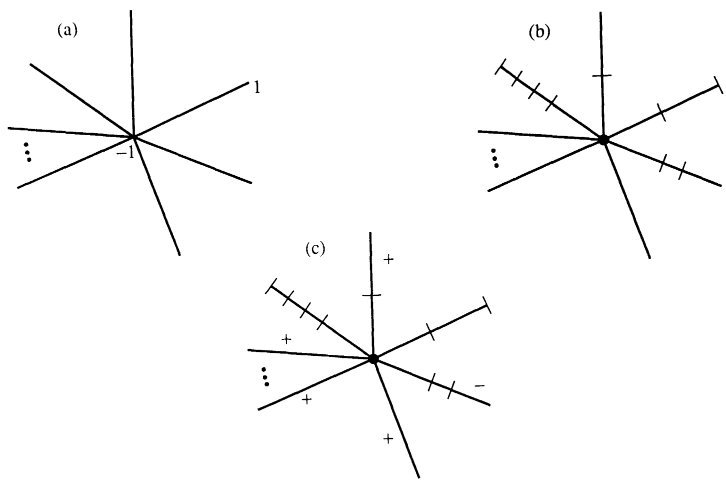


*Fig.* 1. The diagram of an embedding. (a) gives the "skeleton" diagram of the geometric stable valuations. There is a "ray" for each $D\in\mathbf{P}^1$. To each $r\in(-1,1]\cap\mathbf{Q}$ there corresponds a valuation $v(D,r)$ on the ray of $D$. The center point corresponds to the valuation $v(\ ,-1)$. In (b) we mark the valuations of stable divisors of an embedding. (c) is the diagram of an embedding. It has an infinite number of orbits: the open orbit, 9 orbits of dimension 2, 9 orbits of type AB, 1 of type $B_-$, and an infinite number of type $B_+$.

The group SL(2) acts by left multiplication. One has the locally trivial equivariant fibre bundle structure $\mathrm{SL}(2) \times_B S \to \mathrm{SL}(2)/B$ with fibre $S$. The orbits of $\mathrm{SL}(2) \times_B S$ are clearly obtained by taking the SL(2)-orbits of the $B$-orbits in $S$, when considering $S$ as the closure of $B$ in the embedding $\mathrm{SL}(2) \times_B S$.

LEMMA 1.1. (see ($\mathrm{MJ}_1$]) *A normal* SL(2)*-embedding is of the form* $\mathrm{SL}(2) \times_B S$ *if and only if no orbit is contained in the closure of* $B$.

*Proof.* First we show that every orbit intersects the closure of $B$. In other words, we show that $\mathrm{SL}(2) \cdot \bar{B} = X$. Consider the dominant morphism

$$\psi\colon \mathrm{SL}(2) \times_B \bar{B} \to X$$
$$(s, x) \mapsto sx.$$

Since $\mathrm{SL}(2)/B$ is complete, this map is proper (see e.g. [Kr] III.2.5, Satz 2), and therefore surjective.

I claim that if no orbit of $X$ is entirely contained in the closure of $B$, then $\psi$ is an isomorphism. We know it is surjective and birational, since it induces an isomorphism on the open orbit. Thus by Zariski's Main Theorem it suffices to prove that the fibres of $\psi$ are finite (since $X$ is normal).

Let $z$ be in $\bar{B}$. Now $\dim \mathrm{SL}(2)z \geqslant \dim Bz + 1$. This means that $\dim B + 1 - \dim B_z \leqslant \dim \mathrm{SL}(2) - \dim \mathrm{SL}(2)_z$, where $G_z$ means the isotropy subgroup of $z$ in $G$. Now since $B$ has codimension one in SL(2), we have that $\dim \mathrm{SL}(2)_z \leqslant \dim B_z$, and since $B$ is in SL(2), we have equality. Thus $\mathrm{SL}(2)z \cap \bar{B}$ has a finite number of $B$-orbits, and $\mathrm{SL}(2)z/B_z$ is finite. This implies that the fibres of $\psi$ are finite, and thus $\psi$ is an isomorphism. □

Note that $B$ itself is a $B$-stable irreducible divisor of SL(2); thus from the information given above about the types of orbits one can tell immediately if the orbit is in the closure of $B$. In the diagram of $X$ there is one ray which we call the "special ray" for which the valuations are of the form $v(B, r)$. Let $Z$ be an orbit of an embedding. Then $Z$ is in the closure of $B$ if and only if $Z$ is one of the following types:

(a) Type $\mathrm{B}_+$ where $D$ is the "special ray";
(b) Type $\mathrm{B}_-$ where $D$ is not the "special ray";
(c) Type $\mathrm{A}_1$ where $D$ is not the "special ray";
(d) Type $\mathrm{A}_2$ where neither $D_1$ nor $D_2$ are the "special ray."

(See Fig. 2.)

As we have described it, the diagram depends on the choice of the Borel subgroup $B$. One might ask how the diagram changes when one chooses another Borel subgroup, say $B' = sBs^{-1}$ with $s \in \mathrm{SL}(2)$. I claim that the only change is that the "rays" of the diagram are permuted. This is because $D$ is a $B$-stable

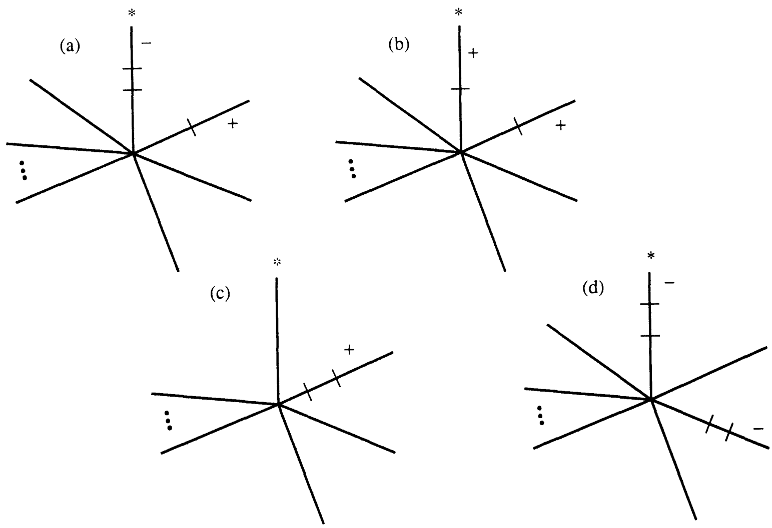


*Fig. 2.* (a) Gives an example of an embedding isomorphic to $SL(2) \times_B \bar{B}$. We mark the special ray with a star. (b), (c), and (d) are examples not of this form.

divisor of SL(2) if and only if $sD$ is a $B'$-stable divisor, and we have $v(D, r) = v(sD, r)$, since they are both stable by SL(2). Thus in the classification using $B$, the ray with $D = Bs^{-1}$ becomes the "special ray" using $B'$. Throughout this paper we will consider several different Borel subgroups of SL(2). In order to avoid confusion in the notation, instead of changing the Borel subgroup used for the classification, when we say simply $D$ is the "special divisor" for $B'$, we mean that $D = Bs^{-1}$ where $B' = sBs^{-1}$. Thus we need not ever refer directly to the Borel subgroup used for the classification.

## 2. Some preliminary results

In the next section we will describe the Chow ring of a smooth SL(2)-embedding $X$ using the irreducible stable divisors of $X$ and the closure of a certain Borel subgroup as generators. In order to calculate the Chow ring, we must know the intersections of these divisors. In this section we prove some important results concerning this. First of all we calculate explicitly the local rings of the orbits. Using this information we can describe just how these generators intersect.

First we give some notation. Let $B_1$ and $B_2$ be two distinct Borel subgroups of SL(2) with $B_2 = s^{-1}B_1 s$, $s \in SL(2)$. Choose coordinates of SL(2) such that the coordinate ring $\mathbf{C}[SL(2)] = \mathbf{C}[x, y, z, w]/(xw - yz - 1)$, where the equations of $B_1$ and $B_2$ are $z = 0$ and $y = 0$ respectively, and $x = s^{-1}z$ and $w = sy$ (e.g. $x$, $y$, $z$, and $w$ are the matrix coordinates, $B_1$ is the subgroup of upper triangular matrices, $B_2$ of lower triangular matrices, and $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$). Let $D_i$ be a divisor with "special ray" $B_i$, $i = 1, 2$. (That is, $D_i = Bs_i$ where $B$ is the Borel subgroup used for the classification, and $s_i$ is chosen such that $B_i = s_i^{-1}Bs_i$.)

PROPOSITION 2.1. *Let $Z$ be a stable subvariety of a smooth* SL(2)*-embedding. Suppose that $Z$ is not contained in the closure of $B_1$. Then the local ring of $Z$ is given by the localization of the ring $A$ in the ideal $\not{p}$ where $A$ and $\not{p}$ are given as follows:*

(1) *If $Z$ is of dimension two with valuation $v(D_1, p/q)$ where $p$ and $q$ are relatively prime integers and $q > 0$, then*

$$A = \mathbf{C}\left[\frac{x}{z}, z^q w^p, z^k w^m\right] \qquad \not{p} = (z^k \omega^m)$$

*where $pk - qm = 1$;*

(2) *If $Z$ is an orbit of type* $\mathrm{B}_+$ *with $v(D_2, 0)$, then*

$$A = \mathbf{C}\left[\frac{x}{z}, w, \frac{1}{z}\right] \qquad \not{p} = \left(w, \frac{1}{z}\right);$$

*If $Z$ is an orbit of type* $\mathrm{B}_+$ *with $v( \ , -1)$ contained on the closure of $B_2$, then*

$$A = \mathbf{C}\left[\frac{x}{z}, \frac{w}{z}, \frac{1}{z}\right] \qquad \not{p} = \left(\frac{w}{z}, \frac{1}{z}\right);$$

(3) *If $Z$ is an orbit of type* $\mathrm{B}_-$ *with $v\left(D_1, \frac{1}{q}\right)$, then*

$$A = \mathbf{C}\left[\frac{x}{z}, z^q w, z\right] \qquad \not{p} = (z^q w, z);$$

(4) *If $Z$ is an orbit of type $AB$ with $v\left(D_1, \frac{p_1}{q_1}\right)$ and $v\left(D_1, \frac{p_2}{q_2}\right)$ with $\frac{p_1}{q_1} < \frac{p_2}{q_2}$ and $p_1 q_2 - q_1 p_2 = 1$, then*

$$A = \mathbf{C}\left[\frac{x}{z}, z^{q_1} w^{p_1}, \frac{1}{z^{q_2} w^{p_2}}\right] \qquad \not{p} = \left(z^{q_1} w^{p_1}, \frac{1}{z^{q_2} w^{p_2}}\right);$$

(5) *If $Z$ is an orbit of type* $\mathrm{A}_1$ *with* $v\left(D_1, \dfrac{-1}{q}\right)$, *then*

$$A = \mathbf{C}\left[\frac{x}{z}, \frac{w}{z^q}, \frac{1}{z}\right] \qquad \not{p} = \left(\frac{w}{z^q}, \frac{1}{z}\right);$$

(6) *If $Z$ is an orbit of type* $\mathrm{A}_2$ *with* $v(D_1, 0)$ *and* $v(D_2, 0)$, *then*

$$A = \mathbf{C}\left[\frac{x}{z}, \frac{1}{z}, \frac{1}{w}\right] \qquad \not{p} = \left(\frac{1}{z}, \frac{1}{w}\right);$$

*If $Z$ is an orbit of type* $\mathrm{A}_2$ *with* $v(D_1, 1)$ *and* $v\left(D_2, \dfrac{q-1}{q}\right)$, *then*

$$A = \mathbf{C}\left[\frac{x}{z}, \frac{1}{zw}, \frac{1}{z^{q-1}w^q}\right] \qquad \mathscr{P} = \left(\frac{1}{zw}, \frac{1}{z^{q-1}w^q}\right).$$

*Also in cases* (2)–(6) *the localization of $A$ in the ideal* $\not{p} + \left(\dfrac{x}{z}\right)$ *is the local ring of the point $z_0$ of* $Z \cong \mathrm{SL}(2)/B_1$ *with isotropy group $B_2$.*

*Proof.* First note that in each case, $A$ is a $\mathbf{C}$-algebra with quotient field $\mathbf{C}(\mathrm{SL}(2))$, the field of rational functions on SL(2). Now (1) is easily proven by remarking that $A_{\not{p}}$ is a valuation ring dominated by the valuation ring of $Z$; thus we have equality.

For the other cases, we use a construction of an embedding from [LV]. Consider an embedding as the set of local rings of its closed points. The action of SL(2) on $\mathbf{C}(\mathrm{SL}(2))$ induces an action of its Lie algebra $\mathfrak{sl}_2$ by derivations. Given a finitely generated $\mathbf{C}$-algebra $A$ with quotient field $\mathbf{C}(\mathrm{SL}(2))$ which is stable under this action of $\mathfrak{sl}_2$, one can construct an embedding $\mathrm{SL}(2) \cdot X_A$, where $X_A$ is the set of localizations of $A$ in its maximal ideals (see §1.6 of [LV]). Then $X_A$ is an affine open subvariety of $\mathrm{SL}(2) \cdot X_A$ which intersects all the orbits; in particular, $\mathrm{SL}(2) \cdot X_A$ is smooth if and only if $A$ is a regular ring. Now $\mathfrak{sl}_2$ is generated by the derivations

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w},\ x\frac{\partial}{\partial z} + y\frac{\partial}{\partial w} \quad \text{and} \quad z\frac{\partial}{\partial x} + w\frac{\partial}{\partial y}.$$

For this proposition, it is easily checked that in all the cases, $A$ is stable by $\mathfrak{sl}_2$, hence we can do the construction above. Also, $A$ is regular, therefore the embedding is always smooth. Note that in each case $A$ and $\not{p}$ are stable by $B_1$.

One checks in a straightforward manner that the localization of $A$ in $\not{p}$ is also stable by the action of $s$ and $B_2$, and hence it is stable by SL(2). Therefore it is the local ring of a stable subvariety of a smooth embedding of dimension 1. Since smooth embeddings do not have fixed points, this subvariety is an orbit. It remains to check that this orbit is indeed $Z$. To do this, it is enough to find the invariant valuations which dominate the given ring. If, for example, we are in case (2) with $v(D_2, 0)$, then for $v = v(D, r)$ to dominate $A_{\not{p}}$ it is necessary that $v(w) > 0$; thus we have $D = D_2$ and $r > 0$. Conversely, if this is the case, then the valuation ring of $v$ is known using (1), and we see that it dominates the ring. The other cases are treated similarly.

As for the last remark, one needs only to check it in the embedding $\mathrm{SL}(2)\cdot X_A$, where it is clearly the case. □

REMARK. In fact we know more. In all cases except where $Z$ is of type $\mathrm{B}_+$ with $r = -1$, the point $z_0$ is in the closure of all the orbits of $\mathrm{SL}(2)\cdot X_A$; thus $X_A$ is completely contained in any embedding $X$ containing $Z$. This gives some $B_1$-stable charts of $X$ of the form $\mathbf{A}^3_{\mathbf{C}}$. We will not use this remark in what follows.

COROLLARY 2.2. *The irreducible stable divisors of a smooth embedding are smooth rational surfaces, and they intersect transversely.*

*Proof.* The questions of smoothness and transversality need only to be verified at the one-dimensional orbits, where one can check the local rings from the proposition. (For smoothness, for example, one simply checks that the local rings of the one-dimensional orbits *in the residue fields of the irreducible stable divisors* are regular.) Also an irreducible stable divisor is rational since the residue field of its local ring is of the form $\mathbf{C}\left(\dfrac{x}{z}, z^q w^p\right)$ from (1) of the proposition. □

In fact, we will show that the irreducible stable divisors of a smooth complete embedding are rational ruled surfaces, and we can determine explicitly which one given its valuation ring; this will be done in the next proposition. First let us review some basic general facts about smooth rational ruled surfaces. (For a reference, see [Beau] or [Saf].) For $n \geqslant 0$, we denote by $\mathbf{F}_n$ the surface given by $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$. For example, $\mathbf{F}_0$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, and $\mathbf{F}_1$ is isomorphic to the blow up of $\mathbf{P}^2$ in a point. Each smooth rational ruled surface is isomorphic to $\mathbf{F}_n$ for some $n \geqslant 0$. If $n \geqslant 1$, there is a unique morphism $\pi_n\colon \mathbf{F}_n \to \mathbf{P}^1$, and $\mathbf{F}_n$ has a section $e_n$ corresponding to the "section at infinity" of $\mathcal{O}_{\mathbf{P}^1}(n)$ with self-intersection $-n$; it is the only curve of $\mathbf{F}_n$ with strictly negative self-intersection. Let $f_n$ be a fibre of $\pi_n$, and $d_n$ be a section of $\mathcal{O}_{\mathbf{P}^1}(n)$ considered as a section of $\pi_n$ (that is, $d_n$ is a section which does not intersect $e_n$). Then we have that $\mathrm{Pic}(\mathbf{F}_n)$ is generated by $[e_n]$ and $[f_n]$, where the brackets indicate the linear equivalence class, and $[d_n] = [e_n] + n[f_n]$.

From now on $X$ will always denote a complete smooth SL(2)-embedding. Let

$Y$ be an irreducible stable divisor in $X$. In the diagram of $X$, $Y$ corresponds to a valuation which we denote by $v(D, r)$. Let $Z$ be a stable subvariety which intersects $Y$ but does not equal $Y$. We say that $Z$ lies "above" $Y$ if the invariant valuations which dominate the local ring of $Z$ are of the form $v(D, r')$ with $r' > r$. Otherwise we say that $Z$ lies "below" $Y$.

Whenever we write $r = p/q$, we choose $p$ and $q$ to be relatively prime integers with $q > 0$.

In the next proposition we will describe the stable irreducible divisors as surfaces and give notation for some of the curves in an embedding.

PROPOSITION 2.3. *Suppose $Y$ is an irreducible stable divisor of a smooth complete* SL(2)*-embedding $X$ associated to the valuation $v(D, r)$ with* $-1 \leqslant r \leqslant 1$.

(i) *If* $-1 < r < 1$, *then $Y$ is equivariantly isomorphic to $\mathbf{F}_{p+q}$ where $r = p/q$. The curve $e$ with self-intersection $-(p + q)$ corresponds to the stable curve which lies "above" $Y$, and the stable curve which lies "below" corresponds to a section $d$ of $\pi\colon \mathbf{F}_{p+q} \to \mathbf{P}^1$ which does not intersect $e$. A fibre $f$ of $\pi$ corresponds to the intersection of $Y$ with the closure of any Borel subgroup containing neither $e$ nor $d$.*

(ii) *If $r = -1$, then $Y$ is equivariantly isomorphic to* $\mathrm{SL}(2)/B \times \mathbf{P}^1$ *where $B$ is a Borel subgroup of* SL(2) *and* SL(2) *acts trivially on* $\mathbf{P}^1$.

(iii) *If $r = 1$, then $Y$ is equivariantly isomorphic to* $\mathrm{SL}(2)/B \times \mathrm{SL}(2)/B$ *for a Borel subgroup $B$ of* SL(2).

NOTATION.

(1) If we are in Case (ii) of Proposition 2.3, then we call $[e]$ the class of an orbit. The class of a fibre of the projection to $\mathrm{SL}(2)/B$ we call $[f]$.

(2) If we are in Case (iii) of Proposition 2.3, then we denote by $c$ the fibre obtained by intersection $Y$ with the closure of the Borel subgroup whose "special ray" is $D$, and $c'$ a fibre of the other ruling. The stable curve that lies "below" $Y$ is the diagonal, whose rational class is $[c] + [c']$.

*Proof.* First, if the complement to the open orbit in $X$ is irreducible, then the proposition holds (see proof of Lemma 3.3). Otherwise it is easily checked that there exists a Borel subgroup $B$ whose closure does not contain any orbits in $Y$. Thus by Lemma 1.1, $Y$ is contained in an open neighborhood of the form $\mathrm{SL}(2) \times_B S$ where $S$ is a smooth open subvariety of the closure of $B$ in $X$ (simply take away all closed orbits in the closure of $B$). Thus we have that $Y$ is equivariantly isomorphic to $\mathrm{SL}(2) \times_B C$, where $C$ is a complete curve is $S$. Since $Y$ is smooth and rational, we have that $C$ is isomorphic to $\mathbf{P}^1$, and thus $Y$ is ruled and isomorphic to $\mathbf{F}_n$ for some $n \geqslant 0$. To find $n$ we must check the action of $B$ on $C \cong \mathbf{P}^1$. If it acts by the character $\chi^n$, where $\chi$ is a generator of the characters of $B$, then $Y$ is isomorphic to $\mathbf{F}_n$. In this case, $Y$ has three orbits: $e_n$, another closed orbit $d_n$, and the rest (unless $n = 0$, in which case there are an infinite number of orbits). If $B$ acts in "the standard way" on $\mathbf{P}^1$, i.e. there is only one fixed point, then we have $Y \cong \mathrm{SL}(2)/B \times \mathrm{SL}(2)/B \cong \mathbf{P}^1 \times \mathbf{P}^1$. There are two orbits: the diagonal and its complement. To find which action we have, we must check if $Y$ has an open orbit, and if so, how SL(2) acts on this orbit.

First of all if $r = -1$, then $Y$ has an infinite number of orbits, and thus $Y \cong \mathrm{SL}(2)/B \times \mathbf{P}^1 \cong \mathbf{F}_0$. This proves (ii).

Now if $r > -1$, then $Y$ has an open orbit, and it is isomorphic to $\mathbf{F}_n$ where $n$ is determined by the action of SL(2) on this orbit; in particular, we see that $n$ depends only on the valuation ring of $Y$, and not on the closed orbits.

The divisor $Y$ has two orbits if and only if $r = 1$. This gives (iii).

For the remaining cases, $B$ acts on $C$ by a character $\chi^n$, and we must determine $n$. In the notation of Proposition 2.1, we can suppose that $B = B_1$ and $D = D_1$ (since $n$ only depends on the valuation ring, we can suppose that for this choice $\bar{B}$ does not contain any orbit of $Y$) and $B_2$ is another Borel subgroup. Then since $C$ is the intersection of $Y$ with the closure of $B$, the residue field of $Y$ is $\mathbf{C}(z^q w^p)$. Now $B$ acts on the function $z^q w^p$ modulo the prime ideal of $C$ by the character $\chi^{p+q}$, where $\chi$ generates the group of characters; thus $Y$ is isomorphic to $\mathbf{F}_{p+q}$.

It remains to check which of the stable sections has self-intersection $-n$ and which has $n$. We will show that $Y'$, the vector bundle over $\mathbf{P}^1$ obtained by taking the curve lying "below" $Y$ away from $Y$, is $\mathcal{O}(-n)$ (its only section is the zero section). We will cover $Y'$ with two affine charts. Let $z_0$ be the point of the 1-dimensional orbit of $Y'$ with isotropy group $B_2$; from Proposition 2.1 we have that the local ring of $z_0$ in $Y'$ is obtained by localizing the ring $A' = \mathbf{C}[\bar{x}/\bar{z}, \bar{z}^q \bar{w}^p]$ at the origin where the bar indicates the class of the function in the residue field of $Y$ (thus $\bar{f} = 0$ if and only if $v(D, r)$ is strictly positive on $f$). In fact I claim that Spec $A'$ is included in $Y'$; this is because Spec $A'$ has two orbits under the action of $B_1$ both of which contain $z_0$ in their closures, and the rational map from Spec $A'$ to $Y'$ must be $B_1$-equivariant. Similarly Spec $A''$ where $A'' = sA' = \mathbf{C}[\bar{z}/\bar{x}, \bar{x}^q \bar{y}^p]$ is contained in $Y'$. These two charts cover $Y'$, and the map to $\mathrm{SL}(2)/B \cong \mathbf{P}^1$ is given by the projection to the first coordinate. Now we have that $\bar{x}^q \bar{y}^p = (\bar{x}/\bar{z})^{q+p} \bar{z}^q \bar{w}^p$ since $y/w = x/z - 1/(zw)$ in $\mathbf{C}(\mathrm{SL}(2))$, and $1/(zw)$ is in the ideal of $Y$. This shows that $Y'$ is isomorphic to $\mathcal{O}(-n)$. Thus the zero section of $Y'$, which is the stable curve lying "above" $Y$, has self-intersection $-n$ in $Y$.

As for the fibres, from Lemma 1.1 we know that $Y$ is fibred over $\mathrm{SL}(2)/B' \cong \mathrm{SL}(2)/B$ for any Borel subgroup $B'$ whose closure contains neither $e$ nor $d$, and thus $Y \cap \bar{B}'$ is a fibre of $Y$. □

REMARK. Another way to check which stable curve of $Y \cong \mathbf{F}_n$ has negative self-intersection is as follows. Denote by $T$ the torus $B_1 \cap B_2$. Pick a point $x \in Y$ in the open orbit in a fibre not fixed by $T$. Using the local rings, one can see that both $\lim_{t \to 0} tx$ and $\lim_{t \to \infty} tx$ for $t \in T$ are in the curve lying "below" $Y$. Thus $\overline{Tx}$ is an irreducible curve which intersects strictly positively with the curve lying "below" $Y$, and it is disjoint from the curve lying "above." Using the structure of $\operatorname{Pic} \mathbf{F}_n$, this proves the result.

We will be interested in the intersections of $\bar{B}$ with itself and with the stable irreducible divisors for special cases of a Borel subgroup $B$. Firstly, if $X$ is of the form $\mathrm{SL}(2) \times_B \bar{B}$, then this is easy, since $\bar{B}$ is a fibre. Thus in this case, $[\bar{B}]^2 = 0$, and $[\bar{B}][Y] = \{\text{a fibre of } Y\}$, where $Y$ is an irreducible stable divisor. For the general case, we must work a little harder.

PROPOSITION 2.4. *Let $B$ be a Borel subgroup of* SL(2) *such that the closure of $B$ in $X$ contains no orbit of type $\mathrm{A}_\alpha$ or of type $\mathrm{B}_+$ with $r = -1$. Denote by $e_1, \ldots, e_k$ the orbits in the closure of $B$. (Note that they are all of type $\mathrm{B}_+$ or $B_-$.) Choose $Y_1, \ldots, Y_k$ such that $e_i$ is in $Y_i$. We call $f_i$ a fibre of $Y_i$, and $Y_i$ corresponds to the valuation $v(D_i, r_i)$ with $r_i = p_i/q_i$ with $p_i$ and $q_i$ relatively prime and $q_i > 0$. Then*

(i) $[\bar{B}]^2 = \sum_{i=1}^k a_i[e_i]$, *where $a_i = 1$ if $e_i$ is of type $\mathrm{B}_+$ and $a_i = q_i - 1$ if $e_i$ is of type $\mathrm{B}_-$;*
(ii) $[\bar{B}][Y_i] = [e_i] + [f_i]$, $i = 1, \ldots, k$;
(iii) $[\bar{B}][f_i] = [\cdot]$, *the class of a point of $X$ in the group $A^3(X) \cong \mathbf{Z}$, $i = 1, \ldots, k$.*

*Proof.* The method we use is the following. For $s \in \mathrm{SL}(2)\backslash B$, we have that $s\bar{B}$ is rationally equivalent to $\bar{B}$, and we show that set theoretically we have that $\bar{B} \cap s\bar{B} = \bigcup_{i=1}^k e_i$ and $\bar{B} \cap Y_i = e_i \cup f_i$, $i = 1, \ldots, k$ where $f_i$ is a fibre of $Y_i \to \mathbf{P}^1$. Also we show that $s\bar{B} \cap f_i = e_i \cap f_i$. Thus we need only to check the multiplicities of the intersections. This can be done using the local ring of the point $e_i \cap f_i$ for each $i$, which we know from Proposition 2.1.

To see that $\bar{B} \cap s\bar{B} = \bigcup_{i=1}^k e_i$ and $\bar{B} \cap Y_i = e_i \cup f_i$, $i = 1, \ldots, k$, we first look at the intersections away from the one-dimensional orbits. Let $X' = X\backslash\{\text{1-dimensional orbits}\}$. We know from Lemma 1.1 that $X'$ is fibred over $\mathrm{SL}(2)/B$, thus we have that $\bar{B} \cap s\bar{B} \cap X' = \varnothing$ and $\bar{B} \cap Y_i \cap X' = f_i \cap X'$. Now by definition the $e_i$'s are exactly those 1-dimensional orbits contained in $\bar{B} \cap s\bar{B}$, and $e_i$ is the unique 1-dimensional orbit which is in $\bar{B} \cap Y_i$. Finally, $s\bar{B} \cap f_i = s\bar{B} \cap \bar{B} \cap f_i = \bigcup_{j=1}^k (e_j \cap f_i) = f_i \cap e_i$ since $Y_i$ and $Y_j$ do not intersect for $i$ and $j$ distinct and $i, j = 1, \ldots, k$.

Fix an $i$ between 1 and $k$. We apply Proposition 2.1 where $B_2 = B$, $Z = e_i$, and $B_1 = sBs^{-1}$ is a Borel subgroup whose closure does not contain $e_i$. Note that $f_i \cap e_i$ is the point of $e_i$ with isotropy group $B$, since $f_i$ is stable by the action of $B$; thus $f_i \cap e_i = z_0$ from Proposition 2.1, and we have an explicit representation of its local ring.

If $e_i$ is of type $\mathrm{B}_+$ (with $r_i = 0$), the local ring of $z_0$ is obtained by localizing the ring $\mathbf{C}\left[\frac{x}{z}, w, \frac{1}{z}\right]$ in the maximal ideal $\left(\frac{x}{z}, w, \frac{1}{z}\right)$. The equations in this ring for the subvarieties needed are as follows:

$$\begin{array}{llll} \bar{B}\colon & y = \left(\dfrac{x}{z}\right) w - \dfrac{1}{z} & e_i\colon & \left(w, \dfrac{1}{z}\right) \\[2ex] s\bar{B}\colon & w & f_i\colon & \left(\dfrac{x}{z}, \dfrac{1}{z}\right). \\[2ex] Y_i\colon & \dfrac{1}{z} & & \end{array}$$

(For example, the function given for $\bar{B}$ is certainly zero on $\bar{B}$, and since it generates a prime ideal, it defines $\bar{B}$.) The results in the proposition are easily verified.

If $e_i$ is of type $\mathrm{B}_-$ (with $r_i = 1/q_i$), the local ring of $z_0$ is the localization of $\mathbf{C}\left[\dfrac{x}{z}, z^{q_i}w, z\right]$ in the maximal ideal $\left(\dfrac{x}{z}, z^{q_i}w, \dfrac{1}{z}\right)$, and the equations of the subvarieties are as follows:

$$\begin{array}{llll} \bar{B}\colon & z^{q_i}y = \left(\dfrac{x}{z}\right)(z^{q_i}w) - (z)^{q_i-1} & e_i\colon & (z^{q_i}w, z) \\[2ex] \bar{s}B\colon & z^{q_i}w & f_i\colon & \left(\dfrac{x}{z}, z\right). \\[2ex] Y_i\colon & z & & \end{array}$$

The results given in the proposition are easily verified. □

We prove one more useful proposition using similar methods.

PROPOSITION 2.5. *Let $Y$ and $Y'$ be stable divisors of $X$ which intersect. Suppose also that $B$ is a Borel subgroup whose closure does not contain any orbit of type $A_2$. Then*

(i) $[Y][Y'][\bar{B}] = [\cdot]$;

(ii) *If $Y$ corresponds to a valuation $v(D, 1)$, then, using the notation of Proposition 2.3, we have*

$$[Y][\bar{B}] = \begin{cases} [c] & \text{if } B \text{ has "special ray" } D \\ [c'] & \text{if not.} \end{cases}$$

*Proof.* First note that $[Y][Y']$ is the class of a section of $Y$ counted with multiplicity one, since $Y$ and $Y'$ intersect transversely, and it is an orbit of type AB or $\mathrm{A}_2$. By the choice of $B$, this orbit is not contained in the closure of $B$, thus it has an open SL(2)-stable neighborhood $V$ which is fibred over SL(2)/$B$, by Lemma 1.1. This proves (i).

If $Y$ is as in (ii), then $V$ contains $Y$, since $Y$ has only two orbits. Thus $[Y][\bar{B}]$ is the class of a fibre of one of the two rulings of $Y$ counted with the multiplicity 1. By definition, if $B$ has "special ray" $D$, then $Y \cap \bar{B}$ is $c$, so $[Y][\bar{B}] = [c]$.

On the other hand if $B$ is another Borel subgroup, we must check that the intersection $Y \cap \bar{B}$ is not in the class of $c$; consequently it is in the class of $c'$. To do this, we use the same method as in the previous proposition: we check intersections in a neighborhood. We apply Proposition 2.1 where $B_1 = B$, $B_2$ has "special ray" $D$ and $Z$ is the closed orbit in $Y$. The local ring of $z_0$ is the localization of $\mathbf{C}\left[\frac{x}{z}, \frac{1}{zw}, z^{m+1}w^m\right]$ in the ideal $\left(\frac{x}{z}, \frac{1}{zw}, z^{m+1}w^m\right)$ for a specific choice of $m$. In this ring the ideal for $c$ is given by $\left(z^{m+1}w^m, \frac{y}{w} = \frac{x}{z} - \frac{1}{zw}\right)$, and the ideal for $\gamma$, the closed orbit of $Y$, is $\left(\frac{1}{zw}, z^{m+1}w^m\right)$. Now the equation for $s^{-1}\bar{B}$ is $\frac{x}{z}$, thus $s^{-1}\bar{B}$ passes through $z_0 = c \cap \gamma c$, but it does not contain $c$. Since we know that $Y \cap \bar{B}$ is a fibre of one of the rulings it must be $c'$. □

REMARK. The last part of Proposition 2.5 can also be proven in another way. Let $B'$ be the Borel subgroup whose special ray is $D$. Using the notation of the proposition, one first shows that $\bar{B} \cap \bar{B}' \cap \gamma = \varnothing$, so that $\bar{B}$ does not contain $\bar{B}' \cap Y = c$. Then one shows that $\bar{B}$ intersects $c$ by using the limits of the curve $\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Since $\{\lambda(t) \mid t \in \mathbf{C}^*\}$ is the intersection of $B$ and $B'$, its limits are in $\bar{B} \cap \bar{B}'$, and one finds that it has a limit in $Y$, using the map $i_\lambda$ described in section 1. Thus $\bar{B} \cap Y$ must be a fibre of the other ruling. One advantage to this proof is that one sees how one uses that the valuation of $Y$ is at the "end" of a ray ($r = 1$): the limits of $\lambda(t)$ are on orbits whose local rings are dominated by valuations of this type. But since we have not developed the ideas of curve limits here, and since we already know the local rings from Proposition 2.1, we use this latter method.

## 3. The Chow Rings

In this section we will calculate explicitly the Chow ring with rational coefficients of a smooth complete SL(2, C)-embedding $X$. (The only difference between choosing rational rather than integer coefficients is that torsion is ignored. In fact we will show in Lemma 3.2 that there is no torsion.) This turns out to be the same as the cohomology ring with coefficients in $\mathbf{Q}$. First we show that the Chow ring is generated over $\mathbf{Q}$ by the classes of irreducible stable divisors (Lemmas 3.3 and 3.4). The relations in terms of these generators is quite complicated; so we add another generator, which is the closure of a certain Borel subgroup. Then the relations can all be understood in terms of the geometry explained in section 2.

We will achieve our goal by first choosing an appropriate set of generators. Then we divide the problem into four cases:

(n = 1) $X$ has exactly one irreducible stable divisor.

(F) There exists a Borel subgroup $B$ whose closure does not contain any orbit of $X$;

(P) in the diagram of $X$ all "rays" end in an orbit of type $\mathrm{B}_+$ or type $\mathrm{A}_\alpha$, and $X$ has at least two irreducible stable divisors;

(NP) there are at least two orbits of type $\mathrm{B}_-$ in the diagram of $X$.

These four cases are mutually exclusive and cover all the possibilities. The Case (n = 1) has only one special embedding in it (all its orbits are of type $\mathrm{B}_+$ with $v(\ ,-1)$), which we treat separately in Lemma 3.3. If $X$ has exactly one orbit of type $\mathrm{B}_-$, then it is in Case (F) with $B$ chosen such that the type $\mathrm{B}_-$ orbit is on the "special ray." If $X$ has no type $\mathrm{B}_-$ orbit but has a divisor with valuation $v(D, 1)$, then it is in Case (F) with $B$ chosen such that $D$ is the "special ray." Otherwise we are in one of the other two cases. For each case we find a set of relations which generate the relations of the Chow ring. These relations are found using the results of section 2.

The notation (n = 1) stands for "one irreducible stable divisor," (F) stands for "fibred over SL(2)/$B$" (from Lemma 1.1) and (P) and (NP) stand for "projective" and "non-projective," respectively. In Proposition 6.4 it will be shown that the varieties of Cases (n = 1), (F) and (P) are projective, while those of Case (NP) are not.

Let $X$ be a variety of dimension $n$. We denote by $A^i(X) = A_{n-i}(X) = \{(n-i)-\text{dimensional algebraic cycles}\}/\text{rational equivalence}$. Then $A^*(X)$ forms a ring, where multiplication corresponds to intersection of cycles; this is called the *Chow ring* (see [Ful]). We will calculate the ring $A^*_{\mathbf{Q}}(X) = A^*(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ where $X$ is a smooth complete SL(2)-embedding. If $Z$ is a cycle, we denote its class in $A^*_{\mathbf{Q}}(X)$ as $[Z]$.

LEMMA 3.1. *For* SL(2) *one has* $A_0(\mathrm{SL}(2)) = A_1(\mathrm{SL}(2)) = A_2(\mathrm{SL}(2)) = 0$.

*Proof.* We have the exact sequence

$$A_i(B) \to A_i(\mathrm{SL}(2)) \to A_i(\mathrm{SL}(2)\backslash B) \to 0$$

where $B$ is a Borel subgroup of SL(2) [Ful, p. 21]. Now as varieties, we have $B \cong \mathbf{C} \times \mathbf{C}^*$ and $\mathrm{SL}(2)\backslash B \cong \mathbf{C}^2 \times \mathbf{C}^*$ using the Bruhat decomposition. For $i = 0$ or 1 we have that $A_i(B) = A_i(\mathrm{SL}(2)\backslash B) = 0$, thus $A_i(\mathrm{SL}(2)) = 0$. For $i = 2$, we have $A_2(\mathrm{SL}(2)\backslash B) = 0$ and the cycle $[B]$ in $A_2(\mathrm{SL}(2))$ is equivalent to 0, thus the first map is zero; so $A_2(\mathrm{SL}(2)) = 0$. $\square$

Let $X$ be a smooth complete SL(2)-embedding. We denote by $Y_1, \ldots, Y_n$ the distinct irreducible components of the complement to the open orbit.

LEMMA 3.2. *Let $X$ be a smooth complete* SL(2)*-embedding. Then*

(i) *there is an isomorphism $A_*(X) \xrightarrow{\sim} H_{2*}(X; \mathbf{Z})$; also these groups are torsion-free;*
(ii) $A^0(X) \cong A^3(X) \cong \mathbf{Z}$;
(iii) *$A^1(X)$ is freely generated by the classes of $Y_1, \ldots, Y_n$;*
(iv) rank$(A^2(X)) = n$, *and $A^2(X)$ is generated by curves in the complement to the open orbit.*

*Proof.* For any variety $X$ there is a natural map $A_*(X) \to H_{2*}(X, \mathbf{Z})$ given by taking the homology class of a cycle [Ful, p. 373]. We will show that in our case, this map is an isomorphism. First of all, if $X$ is projective, then we can use the Bialynicki-Birula decomposition [BB] (using the action of a maximal torus $T$ of SL(2) on $X$) to see that this is indeed the case and that these groups are torsion-free (the components of the fixed points for $T$ are either points or $\mathbf{P}^1$ if $X$ has an infinite number of orbits; to see this, note that if $Y$ is a divisor with an infinite number of one-dimensional orbits, then it is isomorphic to $\mathrm{SL}(2)/B \times \mathbf{P}^1$ from Proposition 2.3). Now I claim that this holds even if $X$ is not projective. Note that by blowing up a finite number of times along stable curves ($\cong \mathbf{P}^1$), the result is a projective embedding. Thus we need only check what happens to the two groups by blowing up. Now on the stable curve $Z$ to be blown up and on the exceptional divisor $\tilde{Z}$ the groups $H_{2i}$ and $A_i$ are isomorphic. Also for the two groups one has split exact sequences which yield the following commutative diagram [Ful, pp. 115] and [G-H, p. 605]:

$$\begin{tikzcd}
0 \arrow[r] & A_i(Z) \arrow[r] \arrow[d, "\wr"] & A_i(\tilde{Z}) \oplus A_i(X) \arrow[r] \arrow[d] & A_i(\tilde{X}) \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & H_{2i}(Z) \arrow[r] & H_{2i}(\tilde{Z}) \oplus H_{2i}(X) \arrow[r] & H_{2i}(\tilde{X}) \arrow[r] & 0
\end{tikzcd}$$

where $\tilde{X}$ is the blow up of $X$ in $Z$. Thus if $A_i(\tilde{X})$ is isomorphic to $H_{2i}(\tilde{X})$, then we have $A_i(X) \cong H_{2i}(X)$.

Thus (i) is true, and we know that the ranks of $A_2(X) = A^1(X)$ and $A_1(X) = A^2(X)$ are the same by Poincaré duality.

Now we use the exact sequence

$$A_i\left(\bigcup_{j=1}^{n} Y_j\right) \to A_i(X) \to A_i(\mathrm{SL}(2)) \to 0.$$

By choosing $i = 1$ or 2 we see that the corresponding groups are generated by cycles in the complement to the open orbit. Thus we are left to show that there are no relations in $A_2(X) = A^1(X)$ between the classes of $Y_1, \ldots, Y_n$. Suppose $f$ were a nonzero rational function on $X$ such that the support of div($f$) were entirely in the complement of the open orbit. Then $f$ regarded as an element of the coordinate ring of SL(2) would be a unit; thus $f$ would be constant on SL(2) and therefore constant on $X$, which gives a contradiction. □

REMARK. The variety $X$ has no homology of odd dimensions (by the same proof as (i) of the previous lemma). Thus the map which sends an algebraic cycle to its cohomology class induces an isomorphism between the Chow ring and the cohomology ring.

LEMMA 3.3. *Let $X$ be the smooth complete* SL(2)*-embedding with one irreducible stable divisor $Y$. Then we have*

$$\mathbf{Z}[y, e]/(y^2 - 2e, e^2) \cong A^*(X)$$

*where $y$ is mapped to $[Y]$, $e$ is mapped to the class of a closed orbit in $X$, and $ye$ is mapped to $[\cdot]$, the class of a point. In particular we have $\mathbf{Q}[y]/(y^4) \cong A^*_{\mathbf{Q}}(X)$.*

*Proof.* All the orbits of $X$ are of type $\mathrm{B}_+$ with $r = -1$ and the divisor $Y$ corresponds to the vauation $v(\ ,-1)$. In this case, $X$ is a quadric in $\mathbf{P}^4$ given by the equation $z_{11}z_{22} - z_{12}z_{21} - z_0^2 = 0$. Now $Y$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ and is given by the equation $z_0 = 0$. Let $B$ be the Borel subgroup whose closure in $X$ is given by the equation $z_{21} = 0$. Then $[\bar{B}] = [Y]$, and one can easily check that $[\bar{B}][Y] = [e] + [f]$, where $[e]$ and $[f]$ are the two generators of Pic $Y$, given in Proposition 2.3(ii). Also, by considering the equivalences of divisors in $\bar{B}$, we see that $[e] = [f]$. Thus we have that $[Y]^2 = [\bar{B}][Y] = 2e = 2f$. Similarly, $[Y]^3 = 2[\cdot]$. Now from Lemma 3.2, we know that $[Y]$ generates $A^1(X)$ and $[e] = [f]$ generates $A^2(X)$. □

LEMMA 3.4. *For $X$ with $n \geqslant 2$ the ring $A^*(X)$ is generated over $\mathbf{Z}$ by the classes of $Y_1, \ldots, Y_n$.*

*Proof.* First of all by Lemma 3.2, $A^1(X)$ is generated by $[Y_1], \ldots, [Y_n]$, and $A^2(X)$ is generated by the classes of curves in $\bigcup_{i=1}^n Y_i$. We must express the class of a section and a fibre of each $Y_i$ by a polynomial in $[Y_1], \ldots, [Y_n]$. Now we know, either using [Bor] or simply by checking the diagrams, that $\bigcup_{i=1}^n Y_i$ is connected. Hence for each $Y_i$ there is a $Y_j \neq Y_i$ such that the intersection $Y_i \cap Y_j$ is not empty. Thus as is noted in the first line of the proof of Proposition 2.5, $[Y_i][Y_j]$ is the class of a section of $Y_i$. Also as noted in the proof of Proposition 2.3, there exists a Borel subgroup $B_i$ whose closure contains no orbit of $Y_i$. Now $[\bar{B}_i] \in A^1(X)$ can be expressed as a linear combination of $[Y_1], \ldots, [Y_n]$. We have that $[\bar{B}_i][Y_i]$ is the class of a fibre of $Y_i$ (see remark before Proposition 2.4). As for $A^3(X)$, from Proposition 2.5 we have that $[\bar{B}_i][Y_i][Y_j] = [\cdot]$. □

Thus to describe the Chow ring we need only to find the relations between the stable divisors. These relations, however, are not at all intuitive. So instead we add another divisor to the set of generators. This gives us "room to move" subvarieties to equivalent subvarieties, and the relations become understandable in terms of the geometry of the embedding described in the previous section. For example, in Lemma 3.3 to find the Chow ring when $n = 1$ we used that $Y_1$ is rationally equivalent to the closure of a Borel subgroup to find the self-intersection of $Y_1$. We use this idea in all the cases. The extra generator will be the class of the closure of a Borel subgroup $B$ of SL(2) where $B$ is chosen as follows: If $X$ is in Case

(n = 1) then $B$ is any Borel subgroup (its closure contains one orbit of type $B_+$ with $v( \ , -1)$. From the proof of Lemma 3.3, for all choices of $B$ the class of its closure is equivalent to the class of $Y_1$);

(F) then $B$ contains no orbit of $X$ in its closure;

(P) then the closure of $B$ contains one orbit of type $B_+$ with $r = 0$ (it is the only orbit in the closure of $B$);

(NP) then one orbit of type $B_-$ is not contained in the closure of $B$ (the closure of $B$ contains all the other type $B_-$ orbits).

In particular, $B$ is always chosen such that its closure contains as few orbits as possible and never any orbits of type $A_\alpha$, $\alpha = 1, 2$.

We will describe the kernel of the map

$$\varphi: \mathbf{Q}[v, y_1, \ldots, y_n] \to A^*_{\mathbf{Q}}(X)$$

given by $y_i \to [Y_i]$, $(i = 1, \ldots, n)$ and $v \to [\bar{B}]$ where $B$ is chosen as above. We list some polynomials which will be used in Theorem 3.5 to generate the kernel of this map.

We denote by $v(D_i, r_i)$ the valuation associated to $Y_i$, and $r_i = p_i/q_i$ with $p_i$ and $q_i$ relatively prime integers with $q_i > 0$.

First we order the $Y_i$'s such that $Y_1, \ldots, Y_s$ have $D_i = D$, the "special ray" of $B$, and $r_1 > r_2 > \cdots > r_s \geqslant -1$. We call $Y_i$ and $Y_j$ neighbors if they are distinct but intersect. Choose $Y_{s+1}$ to be a neighbor below $Y_s$. (Note that by the choice of $B$, we always have $s \geqslant 1$.)

Consider the following homogeneous polynomials:

$$v + \sum_{i=1}^{s} p_i y_i - \sum_{i=s+1}^{n} q_i y_i; \tag{1}$$

$$y_i y_j \quad \text{if } i \neq j \text{ and } Y_j \text{ are not neighbors;} \tag{2}$$

$$y_i y_j + (p_i + q_i) y_i v - y_i y_k \quad \text{if } Y_j \text{ (resp. } Y_k\text{) is a neighbor above (resp. below) } Y_i. \tag{3}$$

If $y_i$ does not appear in (1), then we will need another generator of the kernel.

This polynomial depends on where the divisor $Y_i$ is in relation to the others. We have that $y_i$ does not appear in (1) if and only if the valuation of $Y_i$ is $v(D, 0)$, i.e. $r_i = 0$. We consider 4 cases depending on whether there do or do not exist divisors "above" and "below" $Y_i$ on the same "ray."

$$vy_s - y_s^2 - q_{s-1}y_sy_{s-1} + p_{s+1}y_sy_{s+1} \quad \text{if } i = s > 1 \tag{4}$$

$$vy_i - y_i^2 - q_{i-1}y_iy_{i-1} - q_{i+1}y_iy_{i+1} \quad \text{if } s > i > 1 \tag{4'}$$

$$vy_1 - v^2 - y_1^2 + p_2y_1y_2 \quad \text{if } s = i = 1 \tag{4''}$$

$$vy_1 - v^2 - y_1^2 - q_2y_1y_2 \quad \text{if } s > i = 1 \tag{4'''}$$

Also if $n = 2$ we add an extra polynomial:

$$y_1y_2^2 - (p_1 + q_1)vy_1y_2 \quad \text{if } n = 2. \tag{5}$$

(In fact, polynomial (5) is always in the kernel, but it is only needed as a generator if $n = 2$.)

CASE (F): If $\bar{B}$ does not contain any orbit of $X$, then add the polynomial

$$v^2. \tag{$6_F$}$$

CASE (P): If $n \geq 2$ and all "rays" end in orbits of type $\mathrm{B}_+$ or type $\mathrm{A}_\alpha$, then add the polynomials

$$\begin{aligned} &v^2y_i \quad (i = 2, \ldots, n) \\ &vy_1 - y_1y_2. \end{aligned} \tag{$6_P$}$$

CASE (NP): If there are at least two orbits of type $\mathrm{B}_-$, then choose $Y'_1, \ldots, Y'_m$ to be the divisors other than $Y_1$ which contain an orbit of type $\mathrm{B}_-$, and choose $Y''_1, \ldots, Y''_m$ as their neighbors, and consider also the polynomials

$$\begin{aligned} &v^2y_i \quad \text{if } Y_i \neq Y'_j \quad (j = 1, \ldots, m) \\ &v^2y'_j - (1 - q'_j)vy_1y_2 \quad (j = 1, \ldots, m) \\ &v^2 + \sum_{j=1}^{m} \frac{q'_j - 1}{q'_j}\, y'_j(y''_j - (q'_j + 1)v). \end{aligned} \tag{$6_{NP}$}$$

THEOREM 3.5. *Let $I$ be the ideal generated by equations* (1)–(5) *and* ($6_i$) *for the ith case above* ($i = F$, $P$, *or* $NP$) *and all homogeneous polynomials of degree* 4. *Then $\varphi$ induces an isomorphism*

$$\mathbf{Q}[v, y_1, \ldots, y_n]/I \cong A_{\mathbf{Q}}^*(X).$$

*Proof.* First we must check that $I$ is in the kernel of $\varphi: \mathbf{Q}[v, y_1, \ldots, y_n] \to A^*_{\mathbf{Q}}(X)$. The kernel of $\varphi$ contains (1), because it is the principal divisor $\mathrm{div}(f)$ where $f$ is chosen in the coordinate ring of SL(2) such that it generates the ideal of $B$. If $Y_i$ and $Y_j$ are not neighbors, they do not intersect; thus (2) is in the kernel. (3) comes from Proposition 2.3(i), since with $i, j$ and $k$ chosen as in (3), we have that $[Y_i][Y_j] = [e_i]$, $[Y_i][Y_k] = [d_i]$, and $[Y_i][\bar{B}] = [f_i]$ (for the last equality note that $e_i$ and $d_i$ are of type AB or $\mathrm{A}_2$, and thus by the choice of $B$, they are not in the closure of $B$). As for equations (4)–(4‴), let $B' \neq B$ be a Borel subgroup which contains neither $e_i$ nor $d_i$ in its closure. Then there is a relation in $A^*_{\mathbf{Q}}(X)$

$$[\bar{B}'] - [Y_i] + \sum_{j \neq i} *[Y_j] = 0$$

which comes from $\mathrm{div}(f')$, where $f'$ is a function in the ring of regular functions of SL(2), which generates the ideal of $B'$. We have that $[\bar{B}'][Y_i] = [f_i]$ and either $[\bar{B}][Y_i]$ is $[f_i]$ or, if $i = 1$, we have that $[\bar{B}][Y_i] = [f_i] + [e_i]$ and $[\bar{B}]^2 = [e_i]$ using Proposition 2.4. We multiply the equation above by $[Y_i]$ and substitute either $[\bar{B}][Y_i]$ or $[\bar{B}][Y_i] - [\bar{B}]^2$ for $[\bar{B}'][Y_i]$ to see that (4)–(4‴) are in the kernel. For (5) we have

$$[Y_1][Y_2]^2 = \begin{cases} [d_1][Y_2] = ([e_1] + (p_1 + q_1)[f_1])[Y_2] & \\ \qquad = (p_1 + q_1)[\bar{B}][Y_1][Y_2] & \text{if } r_1 \neq 1; \\ ([c_1] + [c'_1])[Y_2] = 2[\bar{B}][Y_1][Y_2] & \text{if } r_1 = 1 \end{cases}$$

using Propositions 2.3 and 2.5.

In Case (F), we have that $X \cong \mathrm{SL}(2) \times_B \bar{B}$. Therefore $[\bar{B}]$ is rationally equivalent to $[s\bar{B}]$, and $\bar{B}$ and $s\bar{B}$ do not intersect for any $s \in \mathrm{SL}(2) \backslash B$, thus $[\bar{B}]^2 = [\bar{B}][s\bar{B}] = 0$. In Case (P) the first polynomial of $(6_P)$ is clearly in the kernel because for $i = 2, \ldots, n$ the divisor $Y_i$ is in an open neighborhood fibred over $\mathrm{SL}(2)/B$ as in Case (F), so $[\bar{B}]^2[Y_i] = 0$. The second one comes from Propositions 2.3 and 2.4 (remember that $r_1 = 0$ in this case). For Case (NP) the first part of $(6_{NP})$ is clearly in the kernel as in Case (P). For the second one, we have $[\bar{B}]^2[Y'_j] = [\bar{B}]([e'_j] + [f'_j]) = [\bar{B}]([d'_j] - q'_j[f'_j])$ using Propositions 2.3 and 2.4, and by Propositions 2.4(iii) and 2.5(i) this is equal to $(1 - q'_j)[\bar{B}][Y_1][Y_2]$. The third also comes for Propositions 2.3 and 2.4:

We have

$$[\bar{B}]^2 = \sum_{j=1}^{m} (q'_j - 1)[e'_j];$$

$$[Y'_j][\bar{B}] = [e'_j] + [f'_j];$$

$$[Y'_j][Y''_j] = [d'_j] = [e'_j] + (q'_j + 1)[f'_j].$$

Together these yield ($6_{NP}$).

Now we check that $I$ generates all the relations.

For the Case (n = 1), by Lemma 3.3, $I$ is generated by $v - y_1$ (which is (1)) and $y_1^4$. So now we can assume that $n \geqslant 2$ and we are in one of the other 3 cases.

That equation (1) generates the degree one relations is clear from Lemma 3.2 (one simply checks dimensions).

As for the degree 2 equations, we proceed as follows. First we show that using the equations above, we can "move" all second degree monomials into a linear sum of $y_1 y_2$, $vy_i$, $(i = 1, \ldots, n)$ using the generators of $I$. Then we show that these $n + 1$ elements are linearly dependent modulo $I$, thus they generate an $n$-dimensional vector space, which by Lemma 3.2 is $A^2_{\mathbf{Q}}(X)$.

For degree 3, we show that any third degree monomial can be "moved" to a multiple of $vy_1y_2 = [\cdot]$, the class of a point in $X$.

We give a list of the monomials with the relations of $I$ needed to "move" them:

*Degree* 2 *monomials:*

$$\begin{array}{rl}
y_i y_j \quad i \neq j\text{:} & \text{use (2) and (3);} \\
y_i^2 \quad \text{if } y_i \text{ appears in (1):} & \text{use } y_i \cdot (1); \\
v^2\text{:} & \text{use } v \cdot (1); \\
y_i^2 \quad \text{if } y_i \text{ is not in (1):} & \text{use } (4) - (4''').
\end{array}$$

The relation satisfied by $vy_i$ $(i = 1, \ldots, n)$ and $y_1 y_2$ is given by (6). In Case (F), use $v \cdot (1) - v^2$. In Case (P), use $vy_1 - y_1 y_2$. In Case (NP), solve for $v^2$ in $v \cdot (1)$ and the third part of ($6_{NP}$) (this clearly does give a non-trivial relation, because $vy_1$ does not occur in ($6_{NP}$), but it does occur in $v \cdot (1)$).

*Degree* 3 *monomials:*

Using the result for monomials of degree 2, we see it suffices to check the monomials $y_1^2 y_2$ and $v^2 y_i$ $(i = 1, \ldots, n)$.

$$\begin{array}{rl}
v^2 y_i\text{:} & \text{use (6);} \\
y_1^2 y_2 \quad \text{if } n \geqslant 3\text{:} & \text{use } y_1 \cdot (3); \\
y_1^2 y_2 \quad \text{if } n = 2\text{:} & \text{use (5).}
\end{array}$$

($y_1^2 y_2 = y_2(* \, y_1 y_2 + \sum * \, vy_i)$. The only term which still has to be calculated is $y_1 y_2^2$, for which we use (5).)

This finishes the proof of the theorem. □

In fact, using Theorem 3.5 we can calculate the Chow ring with coefficients in **Z**. If $n = 1$, we know the Chow ring from Lemma 3.3. For $n \geqslant 2$, from Lemma 3.4 we know that $\varphi$ induces a surjective morphism over the integers. Since the groups $A^i(X)$ have no torsion (Lemma 3.2), we get an isomorphism between

$A^*(X)$ and $\mathbf{Z}[v, y_1, \ldots, y_n]/(I \cap \mathbf{Z}[v, y_1, \ldots, y_n])$. However, the relations given generate $I$ over $\mathbf{Q}$, but not necessarily over $\mathbf{Z}$.

Note that though at first view the relations given in this theorem might look complicated, they can be understood using simple geometrical properties of the irreducible stable subvarieties.

Another comment is that in special cases, one can often avoid using some of the relations given in the theorem. For example, sometimes there are several possible choices for the Borel subgroup. In this case, by using several different possibilities, we find new relations which can replace ones such as (4)–(4‴) in the theorem. This method will be used in some of the examples of the following sections. Although the idea works well in specific calculations, it is not appropriate for the general theorem, because there are too many separate cases to study. For this reason, we use only one Borel subgroup in Theorem 3.5.

I do not claim that the given set of generators of the relations of the Chow ring are in any way minimal. In fact, one can easily see that in many specific examples, many of the relations given are not needed (see e.g. the first example of the next section). But as mentioned in the previous comment, to make the theory work in general without dividing into too many cases, we must add some generators which are superfluous in some special cases.

## 4. Two examples

For our first example, let $X_q$, $q \geqslant 2$ be an embedding with 6 orbits: the open orbit, 2 two-dimensional orbits with valuations $v(D, 0)$ and $v(D, -1/q)$, and an orbit of type $\mathrm{B}_+$, one of type AB, and one of type $\mathrm{A}_1$ (see Figure 3a). We call $Y_1$ (resp. $Y_2$) the stable divisor with valuation $v(D, 0)$ (resp. $v(D, -1/q)$). From Theorem 3.5, the Chow ring $A^*_{\mathbf{Q}}(X_q)$ is isomorphic to $\mathbf{Q}[v, y_1, y_2]/I$ where $I$ is generated by:

$v - y_2$;

$vy_1 - v^2 - y_1^2 - qy_1y_2$;

$y_1y_2^2 - vy_1y_2$;

$v^2y_2$;

$vy_1 - y_1y_2$

all homogeneous polynomials of degree four.

Clearly, the third and fifth relations are superfluous. We find

$$A^*_{\mathbf{Q}}(X_q) \cong \mathbf{Q}[y_1, y_2]/((1 - q)y_1y_2 - y_2^2 - y_1^2, y_2^3, \text{homogeneous polynomials of degree four})$$

where $[\cdot]$, the class of a point in $A^3(X_q)$, maps to $y_1 y_2^2$. We can calculate for example that $[Y_2]^3 = 0$ and $[Y_1]^3 = (q^2 - 2q)[\cdot]$.

In the next example, there are four two-dimensional orbits whose closures are: $Y_1$ with valuation $v(D, 1/n)$, $Y_2$ with valuation $v(D, 0)$, $Y'_1$ with valuation $v(D', 1/m)$ and $Y'_2$ with $v(D', 0)$ with $n$, $m \geqslant 2$ (see Figure 3b). Here the Chow ring with rational coefficients is isomorphic to $\mathbf{Q}[v, y_1, y_2, y'_1, y'_2]/I$ where $I$ is generated by

$$v + y_1 - y'_2 - my'_1;$$

$$y_1y'_1,\ y_1y'_2,\ y_2y'_1;$$

$$y_1y_2 - y_2y'_2 + vy_2,\ y'_1y'_2 - y_2y'_2 + vy'_2;$$

$$vy_2 - y_2^2 - ny_1y_2;$$

$$v^2y_1,\ v^2y_2,\ v^2y'_2;$$

$$v^2y'_1 - (1 - m)vy_1y_2;$$

$$v^2 + \frac{m-1}{m}\, y'_1(y'_2 - (m+1)v);$$

all homogeneous polynomials of degree four.

Here we can calculate $[Y_1]^3 = [Y'_1]^3 = 0$, $[Y_2]^3 = -(n^2 + 2n)[\cdot]$, $[Y'_2]^3 = -(m^2 + 2m)[\cdot]$ and $[\bar{B}]^3 = (m - m^2)[\cdot]$.

## 5. Canonical Divisors

In this section we will calculate the canonical divisor of a smooth complete SL(2)-embedding in terms of the irreducible stable divisors $Y_1, \ldots, Y_n$ using the diagram. The result is as follows.

PROPOSITION 5.1. *Let $X$ be a smooth* SL(2)*-embedding with irreducible stable*

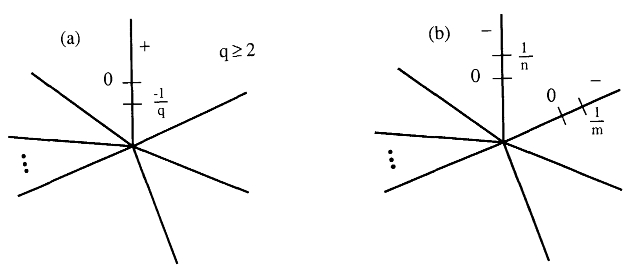


*Fig. 3.* Diagrams for the examples of section 4.

*divisors $Y_1, \ldots, Y_n$ where $Y_i$ corresponds to the valuation $v(D_i, r_i)$, and $r_i = p_i/q_i$ with $p_i$ and $q_i$ relatively prime and $q_i > 0$. Then the canonical divisor $K_X$ is given by*

$$K_X = \sum_{i=1}^{n} (p_i - q_i - 1)[Y_i].$$

*Proof.* We choose a three-form $\theta$ on SL(2) whose divisor is trivial, and then we extend it to $X$ and check the multiplicity of each stable divisor in the divisor of $\theta$. As usual, we choose coordinates of $\mathbf{C}[\mathrm{SL}(2)] = \mathbf{C}[x, y, z, w]/(xw - yz - 1)$. Given a stable divisor $Y$ with valuation $v = v(D, p/q)$ with $v(z) = p/q$ and $v(w) = -1$ we know from Proposition 2.1 that the valuation ring of $v$ is given by $\mathbf{C}[(x/z, z^q w^p, z^k w^m]_{(z^k w^m)}$ where $m$ and $k$ are chosen such that $pk - qm = 1$. Let $U$ be equal to $\mathrm{Spec}(\mathbf{C}[x/z, z^q w^p, z^k w^m])$. It has a divisor $Y'$ with the same valuation ring as $Y$. The multiplicity of $Y$ in the divisor of $\theta$ on $X$ is the same as that of $Y'$ for $\theta$ on $U$.

Let $\theta$ be the form given by $x\,dx \wedge dy \wedge dz$ in $\mathrm{SL}(2)\backslash\{x = 0\}$. It is easily checked that on $\mathrm{SL}(2)\backslash\{z = 0\}$ we have $\theta = -z\,dx \wedge dz \wedge dw$. Thus on SL(2) the divisor of $\theta$ is trivial. Now we check $\theta$ on $U$. Let $a = x/z$, $b = z^q w^p$, and $c = z^k w^m$. One finds $\theta = b^{-1-m+k} c^{-1-q+p} da \wedge db \wedge dc$. Thus the multiplicity of $Y$ is $p - q - 1$. □

## 6. The cone of effective one-cycles

In general, if $X$ is an algebraic normal variety, we can define a vector space $N_1(X)$ by

$$N_1(X) = (\{1\text{-cycles}\}/\text{numerical equivalence}) \otimes_{\mathbf{Z}} \mathbf{R}.$$

DEFINITION. The *cone of effective one-cycles of $X$* is the convex cone in $N_1(X)$ generated by all effective one-cycles. It is denoted by $NE(X)$, and its closure by $\overline{NE}(X)$. If $X$ is projective, then by a criterion of Kleiman [Kle] or [Har], the cone $\overline{NE}(X)$ contains no lines. In this case, a ray $R_+$ in $\overline{NE}(X)$ is called *bad extremal* if it is extremal in the usual sense (that is, if $C + C' \in R_+$ with $C$ and $C' \in NE(X)$, then $C$ and $C'$ are in $R_+$) and we have $C \in R_+$ with $K_X \cdot C \geqslant 0$. It is called *good extremal* if it is extremal in the usual sense, and it contains a 1-cycle whose intersection with the canonical divisor is strictly negative. A curve is called *extremal* (good or bad) if it generates an extremal ray of $\overline{NE}(X)$.

In general for a projective variety $X$, the cone $NE(X)$ can be very complicated. It is not always closed, and its closure does not have to be finitely generated. A theorem by S. Mori [Mor] proves that a part of this cone is polyhedral and locally finitely generated, and the extremal rays in this part are generated by rational curves. (It is the part which has negative intersection with the canonical divisor.) For a smooth projective variety of dimension 3, Mori shows that these good extremal rays correspond to "contractions" which in some sense generalize the notion of blowing down exceptional curves for smooth surfaces. It is, however, more complicated. For example he must allow for certain singularities to occur.

In our situation (where $X$ is a smooth SL(2)-embedding) these results become much simpler, much as in the case of torus embeddings (see [Reid]). In this section, we describe what happens for SL(2)-embeddings.

In section 3, we calculated the 1-cycles modulo rational equivalence. In this case, this is the same as numerical equivalence. To see this, first note that as shown in Lemma 3.2(i), a 1-cycle is rationally equivalent to 0 if and only if it is homologically equivalent to 0. Let $Z$ be an algebraic 1-cycle. If it is homologically equivalent to 0, then it is clearly numerically equivalent to 0, since intersections depend only on the homology class. Conversely, if $Z$ is not homologically equivalent to 0, then by Poincaré duality, there is a homological 4-cycle $Z'$ which intersects $Z$ positively (remember that there is no torsion in the homology groups). Again by Lemma 3.2(i), $Z'$ corresponds to an algebraic 2-cycle; thus $Z$ is not numerically equivalent to 0. Thus we have all the information needed to calculate $NE(X)$.

First we will prove that the cone of effective 1-cycles is a closed finite polyhedral cone generated by rational curves. This will simplify matters tremendously, and we will discuss some of the consequences afterwards.

From now on, to simplify notation we denote by $v$ the class of the closure of $B$, and by $y_i$ the class of $Y_i$, and we do not distinguish between curves and their classes.

The following lemma was formulated and proven by Th. Vust with the help of M. Brion and M. Reid.

LEMMA 6.1. *Let $G$ be a connected linear algebraic group and $H$ an algebraic subgroup such that the homogeneous space $G/H$ is one-dimensional but not complete (i.e. it is not isomorphic to $\mathbf{P}^1$), and let $f: X \to X'$ be a $G$-equivariant morphism between complete $G$-varieties. Suppose that $C$ is a closed subvariety of $X$ stable by $H$ and contained in a fibre of $f$. Then $C$ is rationally equivalent to an effective cycle which is stable by $G$ and which is contained in a fibre of $f$.*

*Proof.* (The idea of this proof comes from [Reid].) Denote by $p_1$ and $p_2$ the projections on the first and second coordinates of $\mathbf{P}^1 \times X$. We will find a closed subvariety $Y$ of $\mathbf{P}^1 \times X$ with the following property: there exists two point $x_1$, $x_0 \in \mathbf{P}^1$ with $p_2(\pi^{-1}(x_1)) = C$ and $p_2(\pi^{-1}(x_0)) = C_0$ is stable by the action of $G$ where $\pi$ is the restriction of $p_1$ to $Y$. Therefore $C$ and $C_0$ are rationally equivalent.

First we show that the image $G \times_H C$ of the morphism

$$G \times C \to G/H \times X$$
$$(s, c) \mapsto (sH, sc)$$

is closed. Note that $G \times_H C$ is $G$-stable where $G$ acts diagonally on $G/H \times X$. Let $Z$ be the closure of $G \times_H X$, and suppose $z \in Z - G \times_H C$. Since $Z$ is two-dimensional, its intersection with $F = H/H \times X$ is purely one-dimensional, and by equivariance, we can assume that $z \in F \cap Z$. Thus there is a curve $C'$ in $F \cap Z$ where $C'$ is not contained in $G \times_H C$. In other words, $C' - G \times_H C$ is one-dimensional, and thus $G(C' - G \times_H C)$ is two-dimensional, and it is contained in $Z - G \times_H C$. This is a contradiction to the existence of $z$.

Now since $G/H$ is one-dimensional, there is an equivariant open immersion $i: G/H \to \mathbf{P}^1$. Consider the equivariant morphism

$$G \times_H C \hookrightarrow G/H \times X \overset{i \times id}{\hookrightarrow} \mathbf{P}^1 \times X$$
$$(s, c) \mapsto (sH, sc)$$

where $G$ acts diagonally on $G/H \times X$ and on $\mathbf{P}^1 \times X$. Denote by $Y$ the closure of the image. Now since $G/H$ is not complete, there exists a point $x_0 \in \mathbf{P}^1$ fixed by $G$. Let $x_1$ be the image of $H/H$ in $\mathbf{P}^1$. Since $G \times_H C$ is closed in $G/H \times X$, we have that $p_2(\pi^{-1}x_1)) = C$, and since the maps above are equivariant, we have that $C_0 = p_2(\pi^{-1}(x_0))$ is $G$-stable.

Moreover, if $C \subset f^{-1}(x')$, then consider the morphism $G/H \to X'$ given by $gH \to gx'$. It extends to a morphism $\varphi: \mathbf{P}^1 \to X'$. Now the diagram

$$\begin{tikzcd} Y \arrow[r, "p_2"] \arrow[d, "\pi"] & X \arrow[d, "f"] \\ \mathbf{P}^1 \arrow[r, "\varphi"] & X' \end{tikzcd}$$

is commutative (it is enough to check commutativity on the image of $G \times_H C$). Thus $C_0$ is contained in the fibre of $\varphi(x_0)$. $\square$

PROPOSITION 6.2. *The cone $NE(X)$ is polyhedral generated by the curves $f_i$ or $c_i$ and $c_i'$ (when $Y_i$ is a stable divisor isomorphic to* SL(2)/$B \times$ SL(2)/$B$) ($i = 1, \ldots, n$), *and the* SL(2)*-orbits of types* $\mathrm{B}_+$ *with* $r = 0$ *and types* $\mathrm{B}_-$. (*See Proposition* 2.3 *and notation following it for the explanation of these curves.*)

*Proof.* First of all, by using the lemma twice, we see that all curves are equivalent to effective one-cycles which are stable by a Borel subgroup $B$ of SL(2) (first use $G/H = U/\{e\}$ where $U$ is the unipotent radical of $B$, and then

$G/H = B/U$). In particular, the cone $NE(X)$ is generated by curves in the complement to the open orbit.

Now the cycles outside of the open orbit are generated by the curves given above, since the effective 1-cycles of $Y_i$ are generated by $e_i$ and $f_i$ or $c_i$ and $c_i'$, and if $e_i$ is not of type $\mathrm{B}_+$ or $\mathrm{B}_-$, then it is equal to $d_j$ for some $j$. If $n \geqslant 2$, then any orbit of type $\mathrm{B}_+$ *with* $r = -1$ is also equivalent to $d_j$ for some $j$ (see Proposition 2.3(iii)), and if $n = 1$, then it is equivalent to a fibre (see Lemma 3.3). □

Thus the possible extremal rays are those curves listed in the proposition. For each embedding we can calculate which ones of these are in fact extremal, and then using Proposition 5.1, we can find which are good and bad extremal. In Proposition 6.5, we will give some general partial results. First let us show a nice consequence of this proposition.

COROLLARY 6.3. *A divisor $D$ of $X$ is ample if and only if $D$ intersects strictly positively with all the curves listed in Proposition* 6.2.

*Proof.* We know from the Proposition that $NE(X)$ is closed. Thus by a result of Kleiman, Nakai's Criterion can be simplified to the statement that a divisor is ample if and only if it intersects all effective 1-cycles strictly positively (see [Kle] and also [Har, p. 42]). □

Now we can prove

PROPOSITION 6.4. *Let $X$ be a smooth complete* SL(2)*-embedding. Then $X$ is projective if and only if it contains at most one orbit of type* $\mathrm{B}_-$.

*Proof.* First of all, if it contains two orbits of type $\mathrm{B}_-$, we will use the knowledge about the Chow ring to find an effective 1-cycle which is equivalent to 0. We fix some notation: let $Y_1, \dots, Y_s$ have valuations $v(D, r_i)$ with $r_1 > \cdots > r_s$ with $e_1$ of type $\mathrm{B}_-$. Let $Y_t$ be another divisor with $e_t$ of type $\mathrm{B}_-$. Denote by $D'$ the divisor such that $Y_t$ has valuation $v(D', r_t)$, and choose the ordering such that $Y_{s+1}, \dots, Y_t$ are the divisors having valuations with $D_i = D'$, $i = s+1, \dots, t$. Let $Y_{t+1}, \dots, Y_k$ be the other divisors with $e_i$ of type $\mathrm{B}_-$. We have by (1) of section 3 and Proposition 2.4 that

$$v^2 + \sum_{i=1}^{s} p_i y_i v - \sum_{i=s+1}^{n} q_i y_i v$$
$$= \sum_{j=t}^{k} (q_j - 1)e_j + \sum_{i=1}^{s} p_i f_i - \sum_{j=s+1}^{n} q_j f_j - \sum_{j=t}^{k} q_j e_j = 0;$$

thus

$$\sum_{i=1}^{s} p_i f_i = \sum_{i=s+1}^{n} q_i f_i + \sum_{j=t}^{k} e_j. \tag{1}$$

Now exchange the role of $Y_t$ and $Y_1$. By symmetry we find

$$\sum_{i=s+1}^{t} p_i f_i = e_1 + \sum_{i=1}^{s} q_i f_i + \sum_{i=t+1}^{n} q_i f_i + \sum_{j=t+1}^{k} e_j. \tag{2}$$

Solving (1) and (2) together, we get

$$0 = e_1 + e_t + 2 \sum_{j=t+1}^{k} e_j + \sum_{i=1}^{t} (q_i - p_i) f_i + 2 \sum_{i=t+1}^{n} q_i f_i,$$

and since $q_i - p_i$ is always non-negative, this gives an effective cycle equivalent to 0.

Now the embedding in Case (n = 1) of section 3 is clearly projective (see Lemma 3.3 for a complete description).

If $X$ is in Case (F) of section 3, that is, it is isomorphic to $\mathrm{SL}(2) \times_B \bar{B}$, then it is projective. To see this, first note that since $\bar{B}$ is a smooth (algebraic) surface, it is projective. Now there are two ways to proceed. Firstly, we use that $\bar{B}$ can be embedded equivariantly in $\mathbf{P}^n$, where $B$ acts linearly on $\mathbf{P}^n$. Now $G \times_B \mathbf{P}^n$ is a projective bundle over $\mathbf{P}^1$. By a theorem of Kodaira, any bundle with fibre isomorphic to projective space over a projective variety is projective (see [Hir], Theorem 18.3.1 and [Kod], Theorem 8). A second method goes as follows: We use that $\mathrm{SL}(2) \times_B \bar{B}$ is a locally trivial equivariant fibre bundle, whose fibre is a projective variety and whose base space is homogeneous. Therefore we can apply the Moišezon Criterion: Given $G$ a connected algebraic group, if a $G$-variety $X$ contains an open quasi-projective set which intersects all orbits in $X$, then $X$ is quasi-projective (see [Moi], p. 43).

We are left with $X$ of Case (P) of section 3, that is that all rays end in either orbits of type $\mathrm{B}_+$ or of type $\mathrm{A}_\alpha$ and $n \geqslant 2$. In [$\mathrm{MJ}_3$], a list of all minimal smooth SL(2)-embeddings is given. (An embedding is *minimal* if it is not obtained by blowing up another smooth embedding.) One finds that any embedding in this case is obtained by blowing up one of the embeddings with diagram in Figure 4.

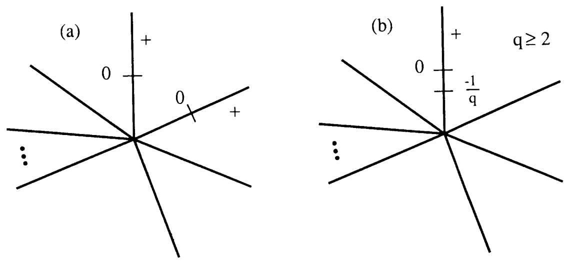


*Fig. 4.* The diagrams of minimal models where all the rays end in orbits of types $\mathrm{B}_+$ or $\mathrm{A}_\alpha$.

Thus we need only show that those in the figure are projective. For this we will find an ample divisor. For the embedding with the diagram in Figure 4a, one can easily show using Corollary 6.3 that $Y_1 + Y_2$ is ample. For Figure 4b, the ample divisor we find is $Y_1 + (q+1)Y_2$. In order to show how to use the results of the previous sections, we give the details for the case of the diagram in Figure 4b (this is the first example from section 4). Let $Y_1$ have valuation $v(D, 0)$, $Y_2$ have valuation $v(D, -1/q)$ and $B$ be the Borel subgroup with "special ray" $D$. Then by the relation (1) of section 3, we have $v - y_2 = 0$; thus $v^2 = vy_2$. Now $vy_2 = f_2$, and by Proposition 2.4, $v^2 = e_1$. We know that the cone $NE(X)$ is generated by $e_1 = f_2$ and $f_1$. We have that the intersection

$$(y_1 + (q+1)y_2)e_1 = y_1 f_2 + (q+1)y_2 e_1 = 1 > 0.$$

As for the intersection with $f_1$, we have $(y_1 + (q+1)y_2)f_1 = y_1 f_1 + q + 1$. Now $y_1 f_1 = y_1(d_1 - e_1) = y_1(e_2 - f_2) = y_1(d_2 - (q-1)f_2 - f_2) = -q$. Thus $(y_1 + (q+1)y_2)f_1 = 1 > 0$. For Figure 4a, the reasoning is similar, but a bit easier. □

REMARKS. (1) Those minimal embeddings of the form $\mathrm{SL}(2) \times_B S$ can be described further: we have that $S$ is a minimal embedding of $B$, and they are classified in [$\mathrm{MJ}_2$]. Thus we have that an embedding of this form is of the type $\mathrm{SL}(2) \times_B \mathbf{F}_n$, $n \neq 1$, or $\mathrm{SL}(2) \times_B \mathbf{P}^2$. The diagrams of these types can be determined using the action of $B$ on $S$ (see [$\mathrm{MJ}_1$] for a complete list).

(2) One can in fact show that each embedding of Figure 4 is the total space of an equivariant $\mathbf{P}^1$-bundle over $\mathbf{P}^2$, where $\mathbf{P}^2 = \mathbf{P}(R_0 \oplus R_1)$ and $R_i$ is the $(i+1)$-dimensional irreducible representation of SL(2). In fact, the one with the diagram of Figure 4a is isomorphic to the flag variety SL(3)/{a Borel subgroup}. Thus one can use the same argument as when $X$ is fibred over $\mathrm{SL}(2)/B$ to show that it is projective.

(3) For those with more than one type $\mathrm{B}_-$ orbits, there are several other ways to show that they are not projective. One way is by using the action of certain maximal torus and showing that the Bialynicki-Birula decomposition is not filtrable, because the variety contains a "cycle" using this torus (see [BB] and [$\mathrm{Jur}_2$] for a reference of these terms).

PROPOSITION 6.5. *Let $X$ be a smooth projective* SL(2)*-embedding. Then*

(i) *Stable curves of type* $\mathrm{B}_-$ *are always bad extremal;*
(ii) *Stable curves of type* $\mathrm{B}_+$ *always have negative intersection with the canonical divisor, but they are extremal if and only if either $r = -1$ and $n = 1$ or $r = 0$ and $n = 2$.*

*Proof.* (i) First we will show that a type $\mathrm{B}_-$ orbit $e_i$ is extremal. Choose a Borel subgroup $B'$ whose closure contains the orbit $e_i$ and at most one other orbit; this second orbit should be of type $\mathrm{B}_+$ or $\mathrm{A}_1$ (to see that this is possible see the description of the different types in section 1). We denote the class of $\bar{B}'$ by $w$. I claim that the intersection $e_i w$ is strictly negative, and the intersection of $w$ with any other 1-dimensional orbit is nonnegative. Once this is proven, suppose we have $ae_i = \sum b_j \gamma_j$, where $a$ and the $b_j$'s are positive numbers and the $\gamma_j$'s are irreducible curves. Since $X$ is projective, we can assume that none of the $\gamma_j$'s are equal to $e_i$. Now one of the $\gamma_j$'s, say $\gamma_1$ has a strictly negative intersection with $w$ and therefore it is in $\bar{B}'$. But this is also true for $s\bar{B}'$ for all $s \in \mathrm{SL}(2)$; for $s \in \mathrm{SL}(2)\backslash B'$ we have that $\bar{B} \cap s\bar{B}$ is a union of 1-dimensional orbits, so $\gamma_1$ is an orbit, and thus $\gamma_1 = e_i$; this gives a contradiction.

It remains to prove the claim. For the first part, note that since $\bar{B}'$ contains $e_i$, its "special ray" is not the one containing $e_i$; Thus $w - q_i y_i + \cdots = 0$, that is, $e_i w = q_i e_i y_i$. If $B$ is chosen such that its "special ray" does contain $e_i$ (that is $\bar{B}$ does not contain $e_i$), then $v + y_i + \cdots = 0$, and $e_i y_i = -e_i v < 0$, so $e_i w < 0$. Now let $\gamma$ be another 1-dimensional orbit; if it is not in the closure of $B'$, it clearly has a non-negative intersection with $w$. Now if $\gamma = e_j$ is of type $\mathrm{B}_+$ and in the closure of $B'$, then we know that $w + p_j y_j + \cdots = 0$, thus $we_j = -p_j y_j e_j$. If $r_j = 0$, then $p_j = 0$, and the claim is true. If $r_j = -1$, the $p_j = -1$ and $y_j e_j = ve_j = 1$, where $v$ is the class of the closure of the Borel subgroup $B$ which does not contain $e_j$. On the other hand, if $\gamma = d_j$ is of type $\mathrm{A}_1$, Then we have $w - q_j y_j + \cdots = 0$, thus $wd_j = q_j y_j d_j$. Also, for $v$ as before, we have $v - y_j + \cdots = 0$, since $p_j = -1$. Thus $y_j d_j = vd_j = 1$.

To check it is bad extremal we must check the intersection with the canonical divisor $K_X$ whose class is $(1 - q_i - 1)y_i + \cdots$. We have $K_X e_i = -q_i y_i e_i > 0$ by the calculations above.

(ii) If $n = 1$, the result is clear, because we have $K_X = -3Y_1$, and $y_1 e_1 = ve_1 = 1$.

For all $n$ one can easily check that the intersection with the canonical divisor is always negative in this case. It remains to check when such a curve is extremal.

If $n$ is not 1, then an orbit of type $\mathrm{B}_+$ with $r = -1$ is not extremal, because it is equivalent to $d_k$ for some $k$ with $k = 1, \ldots, n$.

If $n > 2$ then we will show that $e_i$ is not extremal, where $e_i$ is an orbit of type $\mathrm{B}_+$ with $r_i = 0$. If $X$ has no orbit of type $B_-$, we choose $B$ such that $\bar{B}$ contains $e_i$ (that is, the "special ray" of $B$ contains $e_i$), and we denote its class by $v$. Then $v^2 = e_i$ by Proposition 2.4. By the relation (1) of section 3, we have

$$v = -\sum p_k y_k + \sum q_j y_j$$

where all the $p_k$'s that appear in the equation are negative. By multiplying the equation by $v$, we see that $e_i$ is a positive sum with $n - 1$ terms. Thus if $e_i$ were extremal, then $e_i$ and all these terms would be in the same ray of $NE(X)$. We will show that this is impossible. Denote $e_i$ by $e_1$, and by $y_2$ and $y_3$ the classes of two intersecting stable divisors distinct from $y_1$ (where $Y_1$ contains $e_1$). Then we have

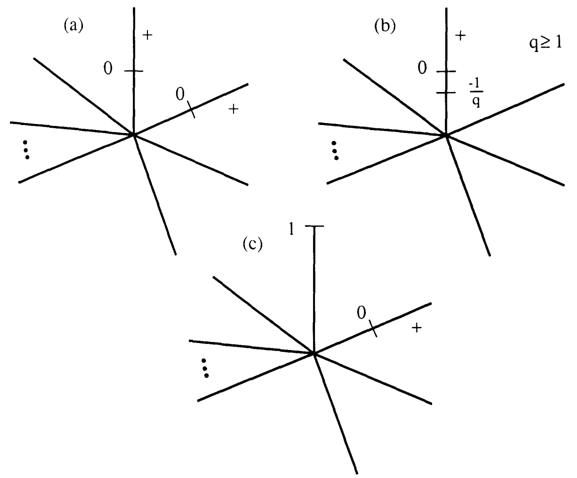


*Fig. 5.* The possibilities with $n = 2$ and a type $\mathrm{B}_+$ orbit with $r = 0$.

$e_1 y_3 = 0$ and $vy_2 y_3 > 0$; thus $e_1$ and $vy_2$ are not in the same ray.

If $X$ does contain an orbit $e_k$ of type $\mathrm{B}_-$, we still choose $B$ such that its closure contains $e_i$ (even though this is not the choice of $B$ in section 3), and this time, by Proposition 2.4, we have $v^2 = e_i + (q_k - 1)e_k$ and $vy_k = f_k + e_k$. Again the equation above multiplied by $v$ shows that $e_i$ is not extremal.

Now we check all the possibilities for $n = 2$ with an orbit of type $\mathrm{B}_+$ with $r = 0$. There are three cases shown in Fig. 5. Since $n = 2$, we know that $NE(X)$ is generated by two elements among those of Proposition 6.2. One checks easily that in the three cases of Figure 5, there are two equivalence classes of curves among the curves of Proposition 6.2, thus they are all extremal. For example, in the first case, we have $Y_1$ with valuation $v(D_1, 0)$ and $Y_2$ with valuation $v(D_2, 0)$ and $v$ is the class of the closure of the Borel subgroup containing $e_1$. Then by equation (1) of section 3, we have $v = y_2$; thus $e_1 = v^2 = vy_2 = f_2$. Similarly we have $e_2 = f_1$. The classes of curves from Proposition 6.2 are $e_1 = f_2$ and $e_2 = f_1$. The other two cases are treated similarly (see also Example 1 in section 7). □

REMARK. Note that there is a "contraction" of an orbit of type $\mathrm{B}_-$, namely we send the orbit to a fixed point (see [$\mathrm{MJ}_1$] or [$\mathrm{MJ}_3$]). However the image has a "bad" singularity, and this does not count as a contraction in Mori's sense. In fact by [Nak] we know that the image of a Mori contraction is always smooth in the case of smooth SL(2)-embeddings. In general, from trying out many examples, it seems to me that a curve $C$ of a smooth projective embedding $X$ is extremal if and only if the following is true: there exists an equivariant morphism $f: X \to \tilde{X}$ where $\tilde{X}$ is a normal projective SL(2)-variety, such that a curve $C'$ of $X$ is contracted to a point by $f$ if and only if $C'$ and $C$ are in the same ray in $NE(X)$. Then the image is smooth if and only if the curve is good extremal. As of now I have no proof of this.

## 7. Some examples of the cone of effective one-cycles

In this section we will calculate the cone of effective one-cycles with their extremal rays for three examples. In order to do this, we first determine the Chow rings.

EXAMPLE 1 Diagram of Figure 6a:

Here we have $n = 2$, $Y_1$ with valuation $v(D_1, 1)$ and $Y_2$ with valuation $v(D_2, 0)$. There are two one-dimensional orbits: one of type $A_2$ and one of type $B_+$. From Lemma 1.1, we see that we are in Case (F) of section 3, that is $X$ is isomorphic to $SL(2) \times_B \bar{B}$, where $B$ is chosen to have "special ray" $D_1$. Remember that $v$ is the class of the closure of $B$. We have that the Chow ring $A^*_{\mathbf{Q}}(X)$ is isomorphic to $\mathbf{Q}[v, y_1, y_2]$, where $v$, $y_1$ and $y_2$ satisfy the relations

$$\begin{aligned} v + y_1 - y_2 &= 0 \\ y_1 y_2^2 - 2v y_1 y_2 &= 0 \\ v^2 &= 0 \end{aligned}$$

all homogeneous polynomials of degree $4 = 0$.

Now let us check the cone $NE(X)$. We know from Proposition 6.2 that the generators are among $c_1 = vy_1$, $c_1' = y_1y_2 - vy_1$, $f_2 = vy_2$ and $e_2 = y_1y_2 - vy_2$ (see Proposition 2.3). Thus

$$f_2 = v(v + y_1) = vy_1 = c_1$$

and

$$e_2 = y_1y_2 - f_2 = y_1y_2 - c_1 = c_1'.$$

Now we know that $NE(X)$ is generated by two elements, because it is a cone which over the real numbers generates $A_1(X) \otimes \mathbf{R}$, which is of dimension $n = 2$. Thus all the curves above are extremal.

We check the intersections with $K_X$. We know already from Proposition 6.5 that $e_2 = c_1'$ is good extremal. Now $K_X = -y_1 - 2y_2$. Thus

$$K_X f_2 = -y_1 f_2 - 2y_2 c_1 = -3 < 0.$$

Thus $f_2 = c_1$ is also good extremal.

In fact one can show (using the method mentioned in the Remark (1) after Proposition 6.4) that this embedding is isomorphic to

$$\mathbf{P}(R_0 \oplus R_1) \times SL(2)/B \cong \mathbf{P}^2 \times \mathbf{P}^1.$$

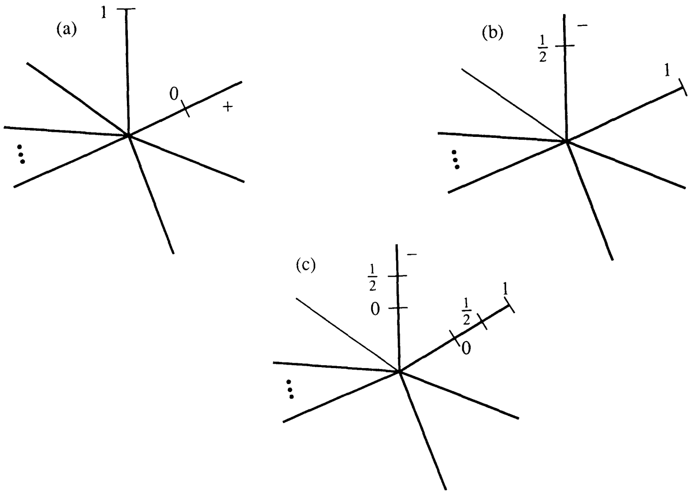


*Fig. 6.* Examples for section 7.

The contractions corresponding to the two good extremal rays are the projections to $\mathbf{P}^1$ and $\mathbf{P}^2$.

EXAMPLE 2 Diagram of Figure 6b:

We have $n = 2$ and $Y_1$ with valuation $v(D_1, 1/2)$ and $Y_2$ with valuation $v(D_2, 1)$. The two one-dimensional orbits are of type $A_2$ and $B_-$. We choose $B$ with "special ray" $D_1$, and as before $X$ is isomorphic to $SL(2) \times_B \bar{B}$. The Chow ring is given by $\mathbf{Q}[v, y_1, y_2]$ where the generators satisfy the relations

$$\begin{aligned} v + y_1 - y_2 &= 0 \\ y_1y_2^2 - 3vy_1y_2 &= 0 \\ v^2 &= 0 \end{aligned}$$

all homogeneous polynomials of degree $4 = 0$.

The generators of $NE(X)$ are among $f_1 = vy_1$, $e_1 = d_1 - 3f_1 = y_1y_2 - 3vy_1$, $c_2' = vy_2$, and $c_2 = y_1y_2 - vy_2$ (see Propositions 2.3 and 2.5(ii)). We already know from Proposition 6.5 that $e_1$ is bad extremal. For the others we have

$$c_2' = vy_2 = vy_1 = f_1$$

and

$$c_2 = y_1y_2 - v(v + y_1) = e_1 + 2f_1.$$

Again the cone is generated by exactly two classes, thus we have that $c_2$ is not extremal and $c_2' = f_1$ and $e_1$ are extremal.

The canonical divisor is given by $-2y_1 - y_2$. Thus

$$K_X f_1 = -2y_1c_2' - y_2f_1 = -3 < 0.$$

So this curve is good extremal.

One can show that this embedding is isomorphic to $SL(2) \times_B \mathbf{P}^2$ where $B$ acts on $\mathbf{P}^2$ by:

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot (z_0 : z_1 : z_2) = (a^{-2}z_0 : az_1 + bz_2 : a^{-1}z_2).$$

The one good extremal ray corresponds to the contraction to $SL(2)/B$.

EXAMPLE 3 Diagram of Figure 6c:

In this case we have $n = 5$. The valuations correspond to the stable divisors as follows:

| | |
|---|---|
| $Y_1$: $v(D_1, \frac{1}{2})$ | $Y_3$: $v(D_2, 0)$ |
| $Y_2$: $v(D_1, 0)$ | $Y_4$: $v(D_2, \frac{1}{2})$ |
| | $Y_5$: $v(D_2, 1)$. |

There are 3 orbits of type AB, 1 of type $B_-$, and 1 of type $A_2$. The Chow ring is given by $\mathbf{Q}[v, y_1, y_2, y_3, y_4, y_5]$ where the generators satisfy the following relations:

$$v + y_1 - y_3 - 2y_4 - y_5 = 0 \tag{1}$$
$$y_i y_j = 0 \quad \text{if } j \neq i-1, i, i+1 \tag{2}$$
$$y_1y_2 + y_2v - y_2y_3 = 0 \tag{3}$$
$$y_3y_4 + y_3v - y_2y_3 = 0 \tag{4}$$
$$y_4y_5 + 3y_4v - y_3y_4 = 0 \tag{5}$$
$$vy_2 - y_2^2 - 2y_1y_2 = 0 \tag{6}$$
$$v^2 = 0 \tag{7}$$

all homogeneous polynomials of degree 4 = 0

As for the cone $NE(X)$, we know it is generated by $e_1$, $f_1$, $f_2$, $f_3$, $f_4$, $c_5$ and $c_5'$. Also since $n = 5$, it needs at least 5 generators. We will find two of these curves which are not extremal, and then we can conclude that all the rest are extremal. First of all from (1) and (7) we have $f_1 = vy_1 = vy_3 + 2vy_4 + vy_5 = f_3 + 2f_4 + c_5'$ is not extremal (if it were, then we would have that $f_1$, $f_3$, $f_4$ and $c_5'$ would all be in the same ray, and we could not have five distinct rays generating $NE(X)$). One can also see that $c_5 = f_2 + 2f_3 + 3f_4 + 2c_5' + e_1$, and thus it is not extremal.

Thus the extremal curves are given by $e_1$, $f_2$, $f_3$, $f_4$, and $c'_5$.

We already know from Proposition 6.5 that $e_1$ is bad extremal. One can calculate the other intersections with $K_X = -2y_1 - 2y_2 - 2y_3 - 2y_4 - y_5$ to see that the only good extremal curve is $f_4$. Its contraction corresponds to the blow down of the divisor $Y_4$ in $X$ to another smooth embedding (see [$MJ_3$]).

In Propositions 6.1 and 6.2 of [$MJ_3$] we find all the equivariant morphisms between SL(2)-embeddings, and we show that among smooth varieties, they always correspond to a composition of blowing downs. In particular, one sees that one can blow down the divisor $Y_5$ to obtain a smooth embedding with another orbit of type $B_-$. However, this new embedding is not projective by Proposition 6.4, thus this is not a contraction in the sense of Mori. The resulting variety is very similar to the non-projective PGL(2)-embedding described in [LMJV] which can be quotiented to obtain an analytic variety which is not a scheme (Artin-Moišezon space).

## 8. An application

In this section we will use the calculation of the Chow rings from section 3 to resolve a special case of a problem posed by V. L. Popov.

Let $G$ be a connected algebraic reductive group, $V$ a finite dimensional rational $G$-module and $v$ an element of $V$. Suppose that the orbit $Gv$ is closed and that the isotropy group $G_v$ is finite.

PROBLEM (V. L. Popov) Calculate $|Gv \cap A_1 \cap \cdots A_d| = \deg(Gv)$ where the $A_i$'s are affine hyperplanes in general position in $V$ and $d$ is the dimension of $G$.

This problem was resolved by Th. Vust for the case $G = \mathrm{SL}(2)$ and $V = R_n$ is the irreducible SL(2)-module of dimension $n + 1$. We present here his solution. (For the case where all roots of $v$ are simple, the degree of $Gv$ can also be calculated by a different method, which is independent of the results of this paper. This will be discussed at the end of this section.)

Consider the inclusion $V \subset \mathbf{P}(V \oplus 1)$ and the closure $\overline{Gv}$ of $Gv$ in $\mathbf{P}(V \oplus 1)$. Since $Gv$ is of dimension three, if $H_1$, $H_2$ and $H_3$ are hyperplanes in $\mathbf{P}(V \oplus 1)$ in general position, then $H_1 \cap H_2 \cap H_3$ does not intersect $\overline{Gv}$ at infinity, and the divisors $H_i \cap \overline{Gv}$ of $\overline{Gv}$ intersect transversely. Thus the number we want to calculate is $|Gv \cap H_1 \cap H_2 \cap H_3| = [\overline{Gv} \cap H_1]^3$ in the Chow ring of $\overline{Gv}$.

The variety $\overline{Gv}$ is not smooth and is not necessarily an embedding of SL(2). However, we can find a smooth complete SL(2)-embedding which dominates it. Consider the rational map

$$\mathbf{P}(M_2(\mathbf{C}) \oplus 1) \dashrightarrow \mathbf{P}(V \oplus 1)$$

$$(s:t) \mapsto (sv:t^n)$$

where $M_2(\mathbf{C})$ is the set of two-by-two matrices. This restricts to a rational map

$$\varphi: X_{-1} \dashrightarrow \overline{Gv}$$

where $X_{-1}$ is the smooth complete SL(2)-embedding in which the complement to the open orbit is irreducible (the embedding from Lemma 3.3). Now $v = v_1^{n_1} \cdots v_p^{n_p}$ where $v_1, \dots, v_p$ are distinct linear forms in two variables and $\sum_{i=1}^{p} n_i = n$. We know that $n - 2n_i \geqslant 0$ for $i = 1, \dots, p$ since the orbit of $v$ is closed. The rational map $\varphi$ is undefined at exactly $p$ orbits: they are where $t = 0$ and $sv_i = 0$, $i = 1, \dots, p$. Denote by $D_i$ the $B$-stable divisor of SL(2) such that the closure of $D_i$ contains the orbit $\{(s:t) \in X_{-1} \mid t = 0 \text{ and } sv_i = 0\}$. (Here $B$ is the arbitrary Borel subgroup used for the classification of the SL(2)-embeddings from section 1.)

Now we resolve the indeterminacy of $\varphi$. First we blow up the $p$ orbits of $X_{-1}$ where $\varphi$ is not defined to obtain a new embedding $X'_p$. From section 6 of [MJ$_3$] we can find the diagram of $X'_p$. It has stable irreducible divisors with valuations $v( \ , -1)$ and $v(D_i, 0)$ $i = 1, \dots, p$. The rational map $\varphi$ induces a rational map from $X'_p$ to $\overline{Gv}$. By studying the local coordinates of the blow-up, it can be seen that this map is still not regular. We blow up $p$ more orbits to obtain a variety $X_p$ whose irreducible stable divisors have the following valuations:

$W$: $v( \ , -1)$

$Z_i$: $v(D_i, 0)$ $\qquad i = 1, \dots, p$

$Y_i$: $v(D_i, 1)$

(see Figure 7). The induced rational map $\varphi_p: X_p \to \overline{Gv}$ is in fact regular. Moreover, using the direct calculations of the local coordinates of the blow-ups, one can show that the divisor

$$\Lambda = nW + \sum_{i=1}^{p} (n - n_i)Z_i + \sum_{i=1}^{p} (n - 2n_i)Y_i$$

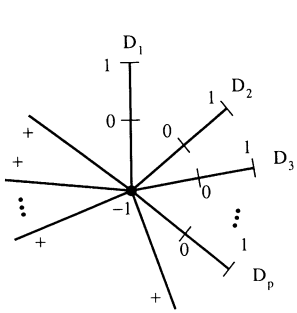


*Fig. 7.* Diagram for section 8.

is in the linear system induced by $\varphi_p$ (i.e. $\Lambda$ is the pullback of the hyperplane at infinity in $\overline{Gv}$). Now since $\varphi_p$ is proper we see that the number we are looking for is simply

$$\frac{[\Lambda]^3}{|G_v|}$$

where $G_v$ is the isotropy group of $v$. Using the results from section 3, we can calculate that in $A^3(X_p) \cong \mathbf{Z}[\cdot] \cong \mathbf{Z}$ we have that

$$\begin{array}{lll} Y_i^2 Z_i = -1 & Y_i^3 = 0 & \\ Y_i Z_i^2 = 2 & Z_i^3 = -4 & (i = 1, \dots, p) \\ Z_i^2 W = 0 & W^3 = -2(p-1) & \\ Z_i W^2 = 1 & & \end{array}$$

and all the other monomials of degree three are 0. Thus one can calculate that

$$[\Lambda]^3 = -2(p-1)n^3 - 4\sum_{i=1}^{p}(n-n_i)^3 + 3n^2\sum_{i=1}^{p}(n-n_i) + 3n\sum_{i=1}^{p}(n-n_i)(n-2n_i).$$

Now suppose that all roots of $v$ are simple (i.e. $n_i = 1$ for all $i$ and $p = n$). Then

$$[\Lambda]^3 = 2n(n-1)(n-2).$$

Also, if we assume $v$ is in general position (i.e. the isotropy group of $v$ is as small as possible), then the order of $G_v$ is

3 if $n = 3$

8 if $n = 4$

1 if $n \geqslant 5$ and odd

2 if $n \geqslant 6$ and even.

Therefore we have

$$|\mathrm{SL}(2)v \cap A_1 \cap A_2 \cap A_3| = \begin{cases} 4 & \text{if } n = 3 \\ 6 & \text{if } n = 4 \\ 2n(n-1)(n-2) & \text{if } n \text{ is odd and } \geqslant 5 \\ n(n-1)(n-2) & \text{if } n \text{ is even and } \geqslant 6 \end{cases}$$

where the $A_i$'s are hyperplanes in $R_n$ in general position.

REMARK. One could also calculate the Chow ring of $X_p$ directly using the behavior of the Chow ring under blow ups.

If all roots of $v$ are simple, then the degree of $Gv$ can also be deduced from a calculation of Enriques and Fano [E-F]. They find the degree of the closure $X(v)$ of $G[v]$ in the projective space $\mathbf{P}(R_n)$ to be $2n(n-1)(n-2)/|G_{[v]}|$, where $G_{[v]}$ is the isotropy group of the line $[v]$. A proof of their result can be found in [M-U], Proposition 1.10, but this presentation is incomplete. They find the intersection of $G[v]$ with three specific hyperplanes, but they do not show that this intersection is transversal, i.e. that the hyperplanes are in general position. This missing step can be proven by studying the tangent space of $G[v]$ at $[v]$ and showing it is complementary to the intersection of the three hyperplanes. To obtain the degree of $Gv \subset R_n$, consider the finite morphism $f: \overline{Gv} \to X(v)$, where $\overline{Gv}$ is the closure of $Gv$ in $\mathbf{P}(R_n \oplus 1)$, and the map is given by projection away from the origin. It is of degree $d = |G_v/G_{[v]}|$. Note that given a hyperplane section $D$ in $X(v)$, the pullback $f^*D$ is a hyperplane section of $\overline{Gv}$. (In other words, one can find a hyperplane in $R_n$ which goes through the origin and intersects $Gv$ transversely. This can either be checked directly, or, using the fact that the map $\mathbf{P}(R_n \oplus 1) - \{\text{origin}\} \to \mathbf{P}(R_n)$ is flat, it is a consequence of the pull-back formula for flat morphisms (see [Ful], p. 34).) Thus $\deg(Gv) = [f^*D]^3$, and $\deg(X(v)) = [D]^3$. By the projection formula ([Ful], p. 34), we find $\deg(Gv) = d \cdot \deg(X(v))$.

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