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A metric characterization of manifolds with boundary

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Introduction

This paper characterizes topological spaces admitting the structure of a smooth manifold with boundary as precisely those finite dimensional spaces admitting a (metrically) complete inner metric of bounded curvature. The boundary points are characterized in terms of geodesic completeness; in consequence, the Hopf–Rinow Theorem is completely generalized to the class of topological manifolds (without boundary) with inner metric of bounded curvature. Since an inner metric can be derived from a Riemannian metric, an inner metric of bounded curvature is easy to obtain for a smooth manifold with boundary. Most of this paper is devoted to proving the converse statement, by studying the local geometric and topological properties which result from both upper and lower metric curvature bounds.

An *inner metric space* (X, d) is a metric space X with distance d such that for all $x, y \in X$, $d(x, y)$ is the infimum of the lengths of curves α joining x and y in X. Inner metric spaces appear naturally in the study of Gromov–Hausdorff ([G]) convergence of Riemannian manifolds (e.g., [F], [FY], [GLP], [GP1], [GP2], [GPW], [GW], [P]): A limit of Riemannian spaces inherits an inner metric structure and, depending on the nature of the spaces converging to it, various other geometric properties. These properties and their topological implications are treated here abstractly, supporting the point of view that much of what is true for limits of Riemannian spaces is directly a result of the geometry they possess, not the (presumably) more special fact that they are limits.

Such an approach can be fruitful. The main result of [N] is that if X is an inner metric space which (1) is geodesically complete, (2) has curvature locally bounded below, and (3) has curvature locally bounded above, then X has the structure of a smooth manifold with $c^{1,\alpha}$ Riemannian metric. Using simple 'metric' arguments, on can show (cf. [P2]) that limits of Riemannian manifolds in the class treated by [GW] and [P] satisfy (1), (2), and (3), thus obtaining a 'metric' proof of the Convergence Theorem. More generally one can argue that limits of Riemannian manifolds with diameters bounded above, and curvature and injectivity radius bounded below, are smooth manifolds with at least *continuous* Riemannian metric ([P1], [P2]).

In larger precompact classes of Riemannian manifolds (e.g., with a lower bound on sectional curvatuare and upper bound on diameter, but either without a lower volume bound, to allow 'collapsing,' or without an upper curvature bound ([GP1], [GPW])), limit spaces generally only satisfy (2). Of particular interest are limits in the latter class, which are known to be generalized manifolds ([GPW]), and conjectured to be manifolds. We have treated two less general cases: In the present paper we show that spaces satisfying (2) and (3) are manifolds with boundary, and in [P1] we show that those satisfying (1) and (2) are manifolds. Excluding the case in [N], there exist non-manifolds satisfying any other subset of the above three conditions.

The failure of geodesic completeness is treated in this paper by introducing the notion of 'geodesic terminal' to mean a point at which some geodesic 'stops' (a 'geodesic' is an arclength parameterized curve which is locally distance minimizing). Such points do not, of course, exist in the Riemannian case, but in limits of Riemannian spaces geodesic terminals can even occur in the interior of a manifold with inner metric (2.5). For the remainder of this paper, the single word 'complete' will refer to metric completeness (as distinct from geodesic completeness).

The main results of this paper are given below. For the definition of 'independent', see the beginning of Chapter 3.

THEOREM A. *Let X be a complete, locally compact inner metric space of locally bounded curvature (above and below). Then the following are equivalent:*

(a) *X is finite dimensional,*
(b) *there are at most finitely many independent elements in the space of directions at some point in X,*
(c) *the set $\mathscr{T}$ of geodesic teminals is nowhere dense, and*
(d) *X is a manifold with boundary and $\partial X = \mathscr{T}$.*

Furthermore, if $\dim X = n < \infty$ then at each point in X the maximum number of independent elements in the space of directions is n.

COROLLARY B. *Let M be a topological manifold and d be an inner metric on M of locally bounded curvature. Then the following are equivalent:*

(a) *(M, d) is complete,*
(b) *(M, d) is geodesically complete,*
(c) *there exists a point $p \in M$ such that each geodesic starting at p is defined on all of $\mathbb{R}^+$, and*
(d) *every closed, bounded subset of M is compact.*

COROLLARY C. *A locally compact, infinite dimensional inner metric space of locally bounded curvature has a dense set of geodesic terminals.*

The 'normal' coordinates used to prove Theorem A are not in general smooth at the boundary. However, the boundary can be 'smoothed:'

THEOREM D. *A topological space X admits the structure of a smooth manifold with boundary if an only if X possesses a complete metric of locally bounded curvature.*

1. Metric geometry

'Bounded curvature' has different meanings in the literature; accordingly, we sketch here the basic theory used in this paper (simplified and revised from the more general treatment in [R]), and establish our notation. The only new concepts in Chapter 1 are those of 'geodesic terminal' and 'comparison radius', defined at the end.

Throughout this paper, curves will always be assumed to be parameterized proportional to arclength. (X, d) will always denote an inner metric space. If α is a curve in X from x to y such that $l(\alpha) = d(x, y)$, where $l(\alpha)$ denotes the length of α, then α is called a *minimal curve*. A *geodesic* is a curve γ which is locally minimal; specifically, if γ is defined on an interval I, then for every $t \in I$ there exists an interval $J = [t - \delta, t + \delta]$, $\delta > 0$, such that $\gamma|_{J \cap I}$ is a minimal curve. In this paper, γ_{xy} will always denote a geodesic from x to y. Under the assumptions of metric completeness and local compactness, every pair of points in X can be joined by at least one minimal curve, and every closed and bounded subset of X is compact.

In all curvature discussions in this paper, the value of $K^{-1/2}$ will be taken to be ∞ if $K \leqslant 0$. A *triple* $(a; b, c)$ in X is a set of three points $a, b, c \in X$ such that $a \neq b$ and $a \neq c$. For any K, let S_K denote the (complete) dimension two, simply connected Riemannian space form of constant curvature K. If $(a; b, c$ is a triple such that $d(a, b) + d(b, c) + d(c, a) < 2\pi/\sqrt{K}$, then there is a uniquely determined (up to congruence in S_K) triangle $T_K(a; b, c)$ in S_K having sides of length $d(a, b)$, $d(b, c)$, and $d(c, a)$. Let $\alpha_K(a; b, c)$ denote the angle corresponding to a in $T_K(a; b, c)$.

DEFINITION 1.1. An open set U in X is said to be a *region of curvature* $\leqslant K$ (resp. $\geqslant K$) if for every triple $(a: b, c)$ in U,

(a) $(a; b, c)$ has a representative in S_K, and
(b) if $a \neq b$ and $a \neq c$ and γ_{ab}, γ_{ac} are minimal curves, then the distance between any points x on γ_{ab} and y on γ_{ac} is $\leqslant$ (resp. $\geqslant$) the distance between the corresponding points x' and y' in $T_K(a; b, c)$.

THEOREM 1.2. *Let U be a region of curvature $\geqslant K$ in X. Then*

(a) *If γ_{ab} and γ_{ac} lie in U, then for any number $\tilde{K}$,* $\lim_{s,t \to 0} \alpha_{\tilde{K}}(a; \gamma_{ab}(s), \gamma_{ac}(t))$

exists, and is independent of both $\tilde{K}$ *and the parameterizations of the curves; this number is called the angle between* γ_{ab} *and* γ_{ac}*, denoted* $\alpha(\gamma_{ab}, \gamma_{ac})$.

(b) *The triangle inequality holds for angles.*

(c) *If* γ_{ab} *and c lie in U, then for all x on* γ_{ab} *strictly between a and b,* $\alpha(\gamma_{xa}, \gamma_{xc}) + \alpha(\gamma_{xc}, \gamma_{xb}) = \pi$.

(d) *If* γ_{ab} *and c lie in U, x is on* γ_{ab} *strictly between a and b, and* $\mathrm{d}(c, x) = \mathrm{d}(c, \gamma_{ab})$, *then* $\alpha(\gamma_{xa}, \gamma_{xc}) = \alpha(\gamma_{xc}, \gamma_{xb}) = \pi/2$.

In a region of curvature bounded above, conditions (a) and (b) still hold, but (c) and (d) fail in general (2.3).

LEMMA 1.3. *Let* γ_{ab}, γ_{ac} *be minimal curves in a region of curvature bounded below. Then for any positive constants* c_1, c_2,

$$\lim_{t \to 0} \mathrm{d}(\gamma_{ab}(c_1 t), \gamma_{ac}(c_2(t))/t = (c_1^2 + c_2^2 - 2 \cdot c_1 \cdot c_2 \cdot \cos \alpha(\gamma_{ab}, \gamma_{ac}))^{1/2}.$$

A space X is said to have *curvature locally bounded below* (resp. *above*) if each $x \in X$ is contained in a region of curvature bounded below (resp. above) by some number K possibly dependent on x. X is said simply to have *locally bounded curvature* if X has curvature locally bounded above and below. If X has curvature locally bounded below, two geodesics have angle 0 if and only if they coincide on their maximal domain of definition. The angle is therefore a bona fide metric on the space S_p of all unit geodesics of maximal domain starting at a point $p \in X$, called the *space of directions* at p.

A point $x \in X$ is called a *branch point* if there exist distinct points a, b, c different from x and minimal curves γ_{ab}, γ_{ac} such that x lies on both γ_{ab} and γ_{ac}, the two curves coincide between a and x, and $\mathrm{d}(a, x) = \mathrm{d}(b, x)$. At the branch point x, the geodesic γ_{ax} 'branches' to form two distinct geodesics γ_{ab} and γ_{ac}. A region of curvature $\geqslant K$ contains no branch points.

A subset A of X is called *strictly convex* if every pair of points in A is joined by a unique minimal curve, and that curve lies entirely in A. If $r < \pi/2\sqrt{K}$, and $B(x, r)$ is compact and contained in a region of curvature $\leqslant K$, then $B(x, r/2)$ is strictly convex. Hence, in a locally compact space of curvature locally bounded above, every point is contained in a strictly convex neighborhood.

In a region U of curvature $\leqslant K$ (resp. $\geqslant K$), the following equivalent conditions hold, whenever the given geodesics exist in U:

A1. *If* $(a; b, c)$ *is a triple in U such that* $b \neq c$, *then* $\alpha(\gamma_{ab}, \gamma_{ac}) \leqslant$ *(resp* $\geqslant$*)* $\alpha_K(a; b, c)$.

A2. *If* $(a; b, c)$ *is a triple such that* $b \neq c$, *and ABC denotes the uniquely determined triangle in* S_K *with* $AB = \mathrm{d}(a, b)$, $AC = \mathrm{d}(a, c)$, *angle* $\alpha(\gamma_{ab}, \gamma_{ac})$ *at A, and side BC of minimal length, then* $\mathrm{d}(b, c) \geqslant$ *(resp.* $\leqslant$*) BC.*

An inner metric space X is geodesically complete if every geodesic in X has a unit parameterization defined on all of $\mathbb{R}$.

DEFINITION 1.4. A point $x \in X$ is called the *terminal* of a geodesic γ_{ax} if γ_{ax} cannot be extended beyond x (as a geodesic). More generally, a point is called a *geodesic terminal* if it is the terminal of some geodesic.

In the metrically complete case, geodesic completeness is equivalent to the absence of geodesic terminals; it is therefore both simple and convenient to have a notion of geodesic completeness for an arbitrary open set:

DEFINITION 1.5. If X is a complete inner metric space, an open set $U \subseteq X$ is said to be *geodesically complete* if U has no geodesic terminals.

DEFINITION 1.6. If x lies in a region of curvature $\leqslant K$, the *upper comparison radius* for K at x is defined to be

$$c^K(x) = \sup\{r: B(x, r) \text{ is a region of curvature } \leqslant K\}.$$

If x is not contained a region of curvature $\leqslant K$, then $c^K(x)$ is defined to be 0. A point x is called a *singularity* if $c^K(x) = 0$ for all K.

The inequality $c^K(x) \geqslant c^K(y) - \mathrm{d}(x, y)$ holds for all $x, y \in X$, and shows that c^K is either everywhere infinite or a continuous map from X into the non-negative reals; if c^K is positive on X, X is said to have *curvature* $\leqslant K$. Finally, $c^K(X)$ will denote $\inf_{x \in X} c^K(x)$, the upper comparison radius of X.

By reversing the inequalities in the above definitions, one can similarly define, for any K, the lower comparison radius $c_K(x)$ (with $c_K(X) = \inf_{x \in X} c_K(x)$), and curvature $\geqslant K$ for the whole space X.

2. Examples

EXAMPLE 2.1. If M is a Riemannian manifold, then the distance induced by the Riemannian metric is by definition an inner metric. If the sectional curvature k on M satisfies $k \leqslant U$, then $c^U(x) > 0$ for all x; if $k \geqslant L$, then by Toponogov's Theorem, $c_L(M) = \infty$ ([K]). Toponogov's Theorem can be generalized to geodesically complete inner metric spaces of curvature locally $\geqslant k$ ([P1]), but it is not known if there exist more general inner metric spaces having curvature $\geqslant L$ and $c_L(X) < \infty$. Such spaces could not, for example, be constructed as limits of spaces for which global comparisons hold ([P2]).

EXAMPLE 2.2. If X is an arbitrary metric space and $Y \subset X$ is finitely path connected (every $x, y \in Y$ can be joined by a rectifiable curve in Y), the induced inner metric $\mathrm{d}_I(x, y)$ on Y is the infimum of the lengths (in the metric of X) of all curves connecting x and y in Y. If 'close' points in the original metric can be connected by 'short' curves in Y, the induced inner metric is topologically equivalent to the usual induced metric, but in general, $\mathrm{d}_I(x, y) \geqslant \mathrm{d}(x, y)$. In particular, every finitely path connected metric space has an inner metric. If N is a Riemannian submanifold of a Riemannian manifold M, the induced inner metric is simply the usual distance associated with the induced Riemannian metric on N.

EXAMPLE 2.3. In $\mathbb{R}^3$, let X be the union of the (x, y)-plane with the z-axis, and let d be the induced inner metric from $\mathbb{R}^3$. Then X is geodesically complete, $c^0(X) = \infty$ (so X is strictly convex), and $c_0(x) > 0$ except at the origin (which is a branch point). This example illustrates that the local euclidean structure of an otherwise 'nice' space is easily destroyed by a single point at which curvature is not bounded below. Note that the angle between the z-axis and any geodesic in the plane through the origin is π.

EXAMPLE 2.4. Let X be an n-sphere ($n \geqslant 2$), with an open 'cap' sliced off along some latitudinal 'circle' above the equator. Then the boundary circle becomes a 'new' geodesic in the induced inner metric, and every point on it is a branch point and a geodesic terminal. The lower curvature bound of the original sphere has been destroyed. The original upper curvature bound also no longer holds.

If the slice is made along or below the equator, the remaining closed disk is strictly convex and retains the constant curvature of the sphere, and points on the equator all become geodesic terminals.

EXAMPLE 2.5. Let X_i be a flat cone with the apex smoothly rounded off at some positive distance ε_i from the end, with $\varepsilon_i \to 0$, and the wide end smoothly capped (to make the space a compact Riemannian manifold). Then the limit X of these spaces is a cone (with the wide end rounded off), which has curvature bounded above and below by 0 around, but excluding, the apex. The apex is a singularity and a geodesic terminal, since it is always shorter to pass around the cone than through the apex. Corollary B shows that such isolated, 'interior' terminals do not occur when the curvature is locally bounded.

EXAMPLE 2.6. Let X be the convex hull of infinitely many independent vectors $\{v_1, v_2, \ldots\}$ in $\mathbb{R}^\infty$, where $\|v_i\| = 2^{-1}$ (if the v_i are orthogonal, X is simply the Hilbert Cube). X is compact, and since X is a convex subset of $\mathbb{R}^\infty$, the induced inner metric on X is flat (i.e., of curvature bounded above and below by 0). The 'faces' of this infinite parallelpiped form a dense set of geodesic terminals. If the angle between v_i and the span of $\{v_1, \ldots, v_{i-1}\}$ tends to 0, then X will have a compact, but infinite dimensional space of directions at the origin. This shows that condition (b) of Theorem A cannot be weakened to 'the space of directions at one point in X is precompact'.

3. Finite dimensional spaces of locally bounded curvature

If X is a space of curvature locally bounded below and there exists a point $x \in X$ with at most two directions, it is easy to show that X is homeomorphic to an interval or a circle. Some of the lemmas below fail for this trivial case, and to avoid special exceptions in the statements, the direction space at each point will be assumed, when necessary, to have at least three elements.

DEFINITION 3.1. Let X be a space of curvature locally bounded below. The *tangent space* T_p at a point $p \in X$ is the metric space obtained from $S_p \times \mathbb{R}^+$ by identifying all points of the form $(\gamma, 0)$ (and denoting the resulting point by 0) with the following metric, where the class of (γ, t) in the identification space is denoted $t \cdot \gamma$:

$$\delta(t \cdot \gamma,\, s \cdot \beta) = (t^2 + s^2 - 2\mathrm{st} \cdot \cos \alpha(\gamma, \beta))^{1/2}$$

For each $\gamma \in S_p$, let $T(\gamma) = \sup\{s\colon \gamma(s) \text{ is defined}\}$; that is, γ terminates at $\gamma(T(\gamma))$ if $T(\gamma) < \infty$, and has no terminal if $T(\gamma) = \infty$. T will be called the *terminal map* on S_p. The *exponential map* is defined by $\exp_p(s \cdot \gamma) = \gamma(s)$, for all $s \leqslant T(\gamma)$. Exp_p is, by A2 (resp. A1), continuous (resp. open) on the intersection of its domain of definition with any $B(0, r)$ such that $\exp_p(B(0, r))$ is contained in a region of curvature $\geqslant K$ (resp. $\leqslant K$). Furthermore, $\exp_p$ is a radial isometry, and preserves the angle between radial geodesics (i.e., starting at p). If X is complete and has locally bounded curvature, then any sequence of geodesics whose directions are Cauchy and whose lengths have a positive lower bound has a limit which is again a geodesic; in this case $\exp_p$ is continuous on its domain of definition, and $T\colon S_p \to \mathbb{R} \cup \infty$ is upper semicontinuous.

The following theorem is proved in [P1]. A weaker version, which requires an upper curvature bound, can be found in [Be] or [PD]. A proof of this weaker version is also indicated at the end of the proof of Proposition 3.7.

THEOREM 3.2. *Let X be a complete, locally compact inner metric space. If $B(p, r)$ is a strictly convex, geodesically complete region of curvature bounded below, then T_p is isometric to $\mathbb{R}^n$ for some n, and $\exp_p|_{B(0,r)}$ is a homeomorphism.*

Without geodesic completeness, the situation is somewhat less simple. In particular, finite dimensionality is not guaranteed by local compactness, nor is the direction space always compact. Suppose X is a complete, locally compact inner metric space of locally bounded curvature. Let $\bar{S}_p$ be the metric completion of S_p; then elements of the metric completion $\bar{T}_p$ of T_p can clearly be written in the form $t\bar{\gamma}$, where $\bar{\gamma} \in \bar{S}_p$, $t \in \mathbb{R}^+$, and $0\bar{\gamma} = 0$. For any $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3 \in \bar{S}_p$, $\bar{\gamma}_2$ is said to be *between* $\bar{\gamma}_1$ and $\bar{\gamma}_3$ if $\alpha(\bar{\gamma}_1, \bar{\gamma}_3) = \alpha(\bar{\gamma}_1, \bar{\gamma}_2) + \alpha(\bar{\gamma}_2, \bar{\gamma}_3)$. For any distinct $\bar{\gamma}_1, \bar{\gamma}_2 \in \bar{\bar{S}}_p$, the *span* $\mathrm{sp}\{\bar{\gamma}_1, \bar{\gamma}_2\} \subseteq \bar{T}_p$ of $\bar{\gamma}_1, \bar{\gamma}_2$ is the set of all $t\bar{\gamma}$ such that one of $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}$ is between the other two. In general, given distinct $\bar{\gamma}_1, \ldots, \bar{\gamma}_k \in \bar{S}_p$, $k > 1$, the span of $\bar{\gamma}_1, \ldots, \bar{\gamma}_k$ is the smallest subset $\mathrm{sp}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\} \subseteq \bar{T}_p$ containing $\bar{\gamma}_1, \ldots, \bar{\gamma}_k$ such that if $\bar{\alpha}, \bar{\gamma} \in \mathrm{sp}\{\bar{c}_1, \ldots, \bar{\gamma}_k\}$, then $\mathrm{sp}\{\bar{\alpha}, \bar{\gamma}\} \subset \mathrm{sp}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\}$. The elements $\bar{\gamma}_1, \bar{\gamma}_2, \ldots$ are said to be *independent* if $\bar{\gamma}_{j+1}$ does not lie in $\mathrm{sp}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_j\}$ for any j. The notions of angle (not as a metric!) and betweeness can be generalized to the space $\bar{T}_p$ in the obvious way; e.g., for $t_1, \ldots, t_k > 0$, $\mathrm{sp}\{t_1\bar{\gamma}_1, \ldots, t_k\bar{\gamma}_k\} = \mathrm{sp}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\}$.

NOTATION. For 3.3–3.7 and 3.9, let $B = B(p, r)$ be a strictly convex region of curvature $\geqslant k$ and $\leqslant K$ in a locally compact, complete inner metric space X.

LEMMA 3.3. *Let $\{\gamma_i\}$ and $\{\eta_i\}$ be Cauchy sequences in S_p. Given any positive $s_i \to 0$ and $t_i \to 0$ such that $s_i \leqslant T(\gamma_i)$ and $t_i \leqslant T(\eta_i)$, if $d_i = \mathrm{d}(\gamma_i(s_i), \eta_i(t_i))$, then*

$$\lim_{i\to\infty} \alpha(\gamma_i, \eta_i) = \lim_{i\to\infty} \cos^{-1}[(s_i^2 + t_i^2 - d_i^2)/2 \cdot s_i \cdot t_i].$$

Proof. By A1 we have, for large i, $\alpha_k(p; \gamma_i(s_i), \eta_i(t_i)) \leqslant \alpha(\gamma_i, \eta_i) \leqslant \alpha_K(p; \gamma_i(s_i), \eta_i(t_i))$, and combining this with the Cosine Laws (cf. Section 36, [R]) we have

$$\begin{aligned}\liminf_{i\to\infty} \alpha_k(p; \gamma_i(s_i), \eta_i(t_i)) &= \liminf_{i\to\infty} \alpha_K(p; \gamma_i(s_i), \eta_i(t_i)) \\ &\geqslant \lim_{i\to\infty} \alpha(\gamma_i, \eta_i) \\ &\geqslant \limsup_{i\to\infty} \alpha_k(p; \gamma_i(s_i), \eta_i(t_i)) \\ &= \limsup_{i\to\infty} \alpha_K(p; \gamma_i(s_i), \eta_i(t_i)).\end{aligned}$$

□

LEMMA 3.4. *For every $\bar{\eta}_1, \bar{\eta}_2 \in \bar{S}_p$ and $a \in [0, \alpha(\bar{\eta}_1, \bar{\eta}_2)]$, there exists some $\bar{\zeta} \in \bar{S}_p$ between $\bar{\eta}_1, \bar{\eta}_2$ such that $\alpha(\bar{\zeta}, \bar{\eta}_1) = a$. Furthermore, if $\alpha(\bar{\eta}_1, \bar{\eta}_2) < \pi$, $\bar{\zeta}$ is unique.*

Proof. Since S_p is dense in $\bar{S}_p$, for the existence part of the lemma it suffices to show the following: Given minimal curves γ_{pb} and γ_{pc} with $a = \alpha(\gamma_{pb}, \gamma_{pc})$, then for each $\varepsilon > 0$ there exists a minimal curve γ starting at p such that

$$|\alpha(\gamma_{pb}, \gamma) - a/2| \leqslant \epsilon, \quad \text{and } \alpha(\gamma_{pc}, \gamma) - a/2| \leqslant \varepsilon.$$

Suppose first that $a < \pi$. For all $i = 1, 2, 3, \ldots$ let b_i denote $\gamma_{pb}(2^{-1})$, c_i denote $\gamma_{pc}(2^{-1})$, α_i: $[0, 1] \to B$ be minimal from b_i to c_i, and d_i be the midpoint of α_i. Since $a < \pi$, $\mathrm{d}(b_i, c_i) < \mathrm{d}(p, b_i) + \mathrm{d}(p, c_i)$ for large i, which implies $p \neq d_i$ and the minimal curve γ_i from p to d_i is nonconstant.

Let P_1, P_2, P_3, P_4 be points in the plane such that $\mathrm{d}(P_1, P_2) = \mathrm{d}(P_1, P_3) = 1$, $\alpha(\overline{P_1P_2}, \overline{P_1P_3}) = a$, P_4 lies on $\overline{P_2P_3}$, and $\alpha(\overline{P_1P_2}, \overline{P_1P_4}) = a/2$. A2 implies $\liminf_{i\to\infty} 2^i \cdot L(\gamma_i) \geqslant \mathrm{d}(P_1, P_4)$, and the definition of angle implies $\lim_{i\to\infty} 2^i \cdot L(\alpha_i) = \mathrm{d}(P_2, P_3)$. By A1 and elementary Euclidean geometry,

$$\begin{aligned}\limsup_{i\to\infty} \alpha(\alpha_{pa}, \gamma_i) &\leqslant \limsup_{i\to\infty} \alpha_K(p; b_i, d_i) \\ &= \limsup_{i\to\infty} \alpha_0(p; b_i, d_i) \\ &\leqslant a/2.\end{aligned}$$

Similarly, $\limsup_{i\to\infty} \alpha(\alpha_{pc}, \gamma_i) \leqslant a/2$, and existence for $a < \pi$ follows from $\alpha(\alpha_{pa}, \gamma_i) + \alpha(\alpha_{pc}, \gamma_i) \geqslant a$.

If $a = \pi$, choose a minimal curve γ' in a third direction. If $\alpha(\gamma', \alpha_{pa}) = \pi/2$, then we are finished. Otherwise, we can use the above special case repeatedly to obtain the desired minimal curves.

To prove uniqueness, suppose $a < \pi$, and assume that, contrary to uniqueness, there exists a $\delta > 0$ and, for $k = 1, 2$, sequences $\{\gamma_{ik}\}$ of minimal curves starting at p such that

$$|\alpha(\gamma_{pb}, \gamma_{ik}) - a/2| \leqslant 2^{-i}, |\alpha(\gamma_{pc}, \gamma_{ik}) - a/2| \leqslant 2^{-i},$$

with $\alpha(\gamma_{i1}, \gamma_{i2}) > \delta$ for all i. Let $t_i \leqslant \min\{T(\gamma_{ik})\}$ with $t_i \to 0$ and $s_i = t_i/\cos(a/2)$. Let $\eta_i, \zeta_{1i}, \zeta_{2i}$ be the minimal curves in $B(p, r)$ from $\gamma_{pb}(s_i)$ to $\gamma_{pc}(s_i)$, $\gamma_{1i}(t_i)$, and $\gamma_{2i}(t_i)$, respectively. The assumptions on the γ_{ki} and A2 (applied with both curvature bounds) imply that

$$\lim_{i\to\infty} [\mathrm{d}(\gamma_{pb}(s_i), \gamma_{ki}(t_i)) + d(\gamma_{ki}(t_i), \gamma_{pc}(s_i))]/t_i = \lim_{i\to\infty} \mathrm{d}(\gamma_{pb}(s_i), \gamma_{pc}(s_i))/t_i.$$

Choosing a representative of $(\gamma_{pb}(s_i); \gamma_{ki}(t_i), \gamma_{pc}(s_i))$ in S_K and applying A1 proves that $\lim_{i\to\infty} \alpha(\eta_i, \zeta_{ki}) = 0$. From the triangle inequality it follows that $\lim_{i\to\infty} \alpha(\zeta_{1i}, \zeta_{2i}) = 0$. Let Z_{1i}, Z_{2i} be unit minimal curves in S_k, with common endpoint y and other endpoints z_{1i} and z_{2i}, respectively, such that $l(Z_{1i}) = l(\zeta_{1i})$, $l(Z_{2i}) = l(\zeta_{2i})$, and $\alpha(Z_{1i}, Z_{2i}) = \alpha(\zeta_{1i}, \zeta_{2i})$. Then

$$\begin{aligned} 0 &= \lim_{i\to\infty} \mathrm{d}(z_{1i}, z_{2i})/l(Z_{1i}) \\ &\geqslant \lim_{i\to\infty} \mathrm{d}(\gamma_{1i}(t_i), \gamma_{2i}(t_i))/l(\zeta_{1i}) \\ &\geqslant \cot(a/2) \cdot \lim_{i\to\infty} \mathrm{d}(\gamma_{1i}(t_i), \gamma_{2i}(t_i))/t_i. \end{aligned}$$

From $\lim_{i\to\infty} \mathrm{d}(\gamma_{1i}(t_i), \gamma_{2i}(t_i))/t_i = 0$ and $A1$ we obtain $\lim_{i\to\infty} \alpha(\gamma_{1i}, \gamma_{2i}) = 0$, a contradiction. □

The proof of the following lemma is essentially the same as the proof of the uniqueness part of Lemma 3.4 (the case $\alpha(\bar{\eta}_1, \bar{\eta}_3) = \pi$ below follows from the absence of branch points).

LEMMA 3.5. *If $\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3, \bar{\eta}_4 \in \bar{S}_p$ and $\bar{\eta}_2$ is between $\bar{\eta}_1$ and $\bar{\eta}_3$, and between $\bar{\eta}_1$ and $\bar{\eta}_4$, with $\alpha(\bar{\eta}_1, \bar{\eta}_3) = \alpha(\bar{\eta}_1, \bar{\eta}_4)$, then $\bar{\eta}_3 = \bar{\eta}_4$.*

LEMMA 3.6. *Let $\bar{\gamma}_1, \ldots, \bar{\gamma}_4 \in \bar{S}_p$ be distinct and, setting $\alpha_{ij} = \alpha(\bar{\gamma}_i, \bar{\gamma}_j)$, suppose $\alpha_{12} + \alpha_{23} = \alpha_{13} < \pi$. Then there exist unit vectors $v_i \in \mathbb{R}^3$ such that $\alpha(v_i, v_j) = \alpha_{ij}$, and a choice of v_4 any two of v_1, v_2, v_3 determines the remaining v_i.*

Proof. Letting $a = \alpha_{13}$, we need only consider the case $\alpha_{12} = \alpha_{23} = a/2$. There exist $X_i \in \mathbb{R}^3 \backslash 0$ such that X_1, X_2, and X_3 are colinear, with

$\alpha(\overline{0X_1}, \overline{0X_2}) = a/2, \alpha(\overline{0X_2}, \overline{0X_3}) = a/2, \alpha(\overline{0X_1}, \overline{0X_4}) = \alpha_{14}$, and $\alpha(\overline{0X_3}, \overline{0X_4}) = \alpha_{34}$. Choose $\gamma_{ij} \in S_p$ such that $\gamma_{ij} \to \gamma_i$, $i = 1, \ldots, 4$, and positive $t_j \to 0$ such that $s_{ij} = \|t_j \cdot X_i\| \leqslant T(\gamma_{ij})$. Let $\beta^j_{ik}: [0, 1] \to B$ be minimal from $x_{ij} = \gamma_{ij}(s_{ij})$ to $x_{kj} = \gamma_{kj}(s_{kj})$, and let α'_{2j} be the unique minimal curve from p to $x'_{2j} = \beta^j_{13}(1/2)$. To prove the first part of the lemma, it suffices, by Lemma 3.3, to show that $\lim_{j\to\infty} \alpha(\alpha'_{2j}, \alpha_{4j}) = \alpha(\overline{0X_2}, \overline{0X_4})$; i.e., $\lim_{j\to\infty} \mathrm{d}(x'_{2j}, x_{4j})/t_j = X_2X_4$. Lemma 3.3 and Lemma 1.3 imply that

$$\lim_{j\to\infty} \mathrm{d}(x_{1j}, x_{3j})/t_j = \mathrm{d}(X_1, X_3),$$

$$\lim_{j\to\infty} \mathrm{d}(x_{1j}, x_{4j})/t_j = \mathrm{d}(X_1, X_4),$$

$$\lim_{j\to\infty} \mathrm{d}(x_{3j}, x_{4j})/t_j = \mathrm{d}(X_3, X_4),$$

and it follows from curvature $\geqslant k$ (A1) and the definition of angle (applied in S^k) that $\liminf_{j\to\infty} \alpha(\beta^j_{13}, \beta^j_{14}) \geqslant \alpha(\overline{X_1X_3}, \overline{X_1X_4})$. Curvature $\leqslant K$ similarly applied shows $\liminf_{j\to\infty} \mathrm{d}(x'_{2j}, x_{4j})/t_j \geqslant \mathrm{d}(X_2, X_4)$. On the other hand, $\limsup_{j\to\infty} \mathrm{d}(x'_{2j}, x_{4j})/t_j \leqslant \mathrm{d}(X_2, X_4)$ can be proved in the same way, using the opposite the curvature bounds.

The last part of the lemma is elementary linear algebra. □

PROPOSITION 3.7. *If $\bar\gamma_1, \ldots, \bar\gamma_m \in \bar S_p$ are independent, then* $\mathrm{sp}\{\bar\gamma_1, \ldots, \bar\gamma_m\}$ *is isometric to the closure of an open convex radial cone in* $\mathbb{R}^m$.

Proof. Using Lemmas 3.4 and 3.5, one can now easily map $\mathrm{sp}\{\bar\gamma_1, \bar\gamma_2\}$ isometrically onto the closure C^2 of a convex open cone in $\mathbb{R}^2$, taking each $\{t\cdot\bar\gamma: t \in \mathbb{R}^+\}$ isometrically onto a radial ray. Note that if $\bar\gamma_1$ and $\bar\gamma_2$ lie in S_p and have extensions past p as geodesics, then the image of this map is all of $\mathbb{R}^2$.

Suppose such a map φ has been inductively constructed from $\mathrm{sp}\{\bar\gamma_1, \ldots, \bar\gamma_k\}$ onto the closure C^k of a convex open cone in $\mathbb{R}^k$. In $\mathbb{R}^{k+1}$ there exists some unit vector v_{k+1} such that $\alpha(\bar\gamma_1, \bar\gamma_{k+1}) = \alpha(v_i, v_{K+1})$ for all $1 \leqslant i \leqslant k$. Note that if $\bar\gamma \in \mathrm{sp}\{\bar\alpha, \bar\gamma_{k+1}\}$ for some $\bar\alpha \in \mathrm{sp}\{\bar\gamma_1, \ldots, \bar\gamma_k\} \cap \bar S_p$, then $\bar\alpha$ is the unique such element of $\mathrm{sp}\{\bar\gamma_1, \ldots, \bar\gamma_k\} \cap \bar S_p$. For if $\bar\gamma \in \mathrm{sp}\{\bar\alpha', \bar\gamma_{k+1}\}$, with $\bar\alpha \neq \bar\alpha', \bar\gamma_{k+1} \in \mathrm{sp}\{\bar\alpha, \bar\alpha'\}$, and hence $\bar\gamma_{k+1} \in \mathrm{sp}\{\bar\gamma_1, \ldots, \bar\gamma_k\}$, a contradiction.

One can now extend φ to the union C of the spans $\mathrm{sp}\{\bar\alpha, \bar\gamma_{k+1}\}$ for all $\bar\alpha \in \mathrm{sp}\{\bar\gamma_1, \ldots, \bar\gamma_k\}$ to an injective map onto the closure of a convex open radial cone in $\mathbb{R}^{k+1}$, which is an isometry on each $\mathrm{sp}\{\bar\alpha, \bar\gamma_{k+1}\}$. This extension is actually an isometry: Given $\bar\beta \in \mathrm{sp}\{\bar\alpha, \bar\gamma_{k+1}\}$, with $\bar\alpha \in \mathrm{sp}\{\bar\gamma_1, \ldots, \bar\gamma_k\}$, and $\bar\zeta \in \mathrm{sp}\{\bar\gamma_1, \ldots, \bar\gamma_k\} \cap \bar S_p$, apply Lemma 3.6 to $\bar\zeta, \bar\beta, \bar\gamma_{k+1}, \bar\alpha$, with $v_1 = \varphi(\bar\zeta)$, $v_3 = \varphi(\bar\gamma_{k+1})$, and $v_4 = \varphi(\bar\alpha)$. The unique unit vector $v_2 \in \mathbb{R}^{k+1}$ determined by these choices is the only unit vector in $\mathbb{R}^{k+1}$ to satisfy $\alpha(v_1, v_2) = \alpha(\bar\zeta, \bar\beta)$ and $\alpha(v_2, v_3) = \alpha(\bar\beta, \bar\gamma_{k+1})$, and so must coincide with $\varphi(\bar\beta)$; i.e., $\alpha(\bar\beta, \bar\alpha) = \alpha(v_2, v_4) = \alpha(\varphi(\bar\beta), \varphi(\bar\alpha))$. Now suppose $\bar\beta$ and $\bar\zeta$ are arbitrary elements of C. Then

$\bar{\zeta} \in \mathrm{sp}\{\bar{\gamma}_{k+1}, \bar{\alpha}\}$, with $\bar{\alpha} \in \mathrm{sp}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\}$. The above special case shows that $\alpha(\bar{\beta}, \bar{\alpha}) = \alpha(\varphi(\bar{\beta}), \varphi(\bar{\alpha}))$, and repeating the argument shows $\alpha(\bar{\beta}, \bar{\zeta}) = \alpha(\varphi(\bar{\beta}), \varphi(\bar{\zeta}))$.

In a similar fashion, the map φ can now be extended to an isometry on the union C' of the all $\mathrm{sp}\{\bar{\beta}, \bar{\zeta}\}$ with $\bar{\beta}, \bar{\zeta} \in C$. One need only show that if $\bar{\eta} \in \mathrm{sp}\{\bar{\beta}, \bar{\zeta}\} \cap \mathrm{sp}\{\bar{\beta}', \bar{\zeta}'\}$, then the extensions defined using the two different spans coincide; but this follows from Lemma 3.6 as in the above argument. If $\bar{\gamma} \in C'$ is strictly between any two elements of C', then the fact that C is the closure of a convex open cone implies that $\varphi(\bar{\gamma})$ is contained an open Euclidean subset contained in $\varphi(C')$. In addition, for any element $\bar{\mu} \in C'$ there are arbitrarily close elements strictly between $\bar{\mu}$ and some other element; in otherwords, the interior points of $\varphi(C')$ are dense in $\varphi(C')$. Since $\varphi(C')$ is a convex cone by construction, C' satisfies the requirements of the inductive step.

Finally, $C' = \mathrm{sp}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_{k+1}\}$. This follows from the fact that, since $\varphi(C)$ has non-empty interior, every element of $\mathbb{R}^{k+1}$ lies in the span of some two elements of $\varphi(C)$. Suppose $\bar{\eta} \in \mathrm{sp}\{\bar{\beta}, \bar{\zeta}\}$, with $\bar{\beta}, \bar{\zeta} \in C' \cap \bar{S}_p$. As before φ can be extended as an isometry to $C' \cup \mathrm{sp}\{\bar{\beta}, \bar{\zeta}\}$. But then $\varphi(\bar{\eta})$ lies in the span of some $\varphi(\bar{\alpha})$, $\varphi(\bar{\gamma}) \in \varphi(C)$. Since φ is an isometry, $\bar{\eta}$ lies in $\mathrm{sp}\{\bar{\alpha}, \bar{\gamma}\}$ and so $\bar{\eta} \in C'$. This completes the inductive step.

Note that if, in addition, $C^k = \mathbb{R}^k$, $\gamma_{k+1} \in T_p$, and γ_{k+1} has a continuation past p, then $C^{k+1} = \mathbb{R}^{k+1}$. Theorem 3.2, with the additional assumption of an upper curvature bound on $B(p, r)$, can now be proved: If U is geodesically complete, each geodesic starting at p is defined for at least length r, and so local compactness implies S_p is compact, and $T_p = \bar{T}_p$. Finally, compactness of S_p implies that T_p is spanned by at most finitely many elements, so $T_p = \mathbb{R}^n$ for some n. □

The space $\mathrm{sp}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_m\}$ will now be identified with its image in Euclidean space. For example, given any $\bar{\alpha}, \bar{\beta} \in \bar{S}_p$ and $s \in [0, 1]$, $s \cdot \bar{\alpha} + (1 - s) \cdot \bar{\beta}$ will denote the unique element of $\bar{S}_p$ between $\bar{\alpha}$ and $\bar{\beta}$ such that $\alpha(\bar{\alpha}, (s \cdot \bar{\alpha} + (1 - s) \cdot \bar{\beta})) = s \cdot \alpha(\bar{\alpha}, \bar{\beta})$.

It is not obvious at this stage that, if S_p contains m independent elements, T_p (and let alone $\exp_p^{-1}(B(p, r))$!) contains an open subset of $\mathbb{R}^m$ (e.g., $\mathrm{sp}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_m\} \backslash T_p$ could be dense in $\mathrm{sp}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_m\}$).

DEFINITION 3.8. Let $x \in X$. Then a subset $A \subset X$ is said to be *transverse* to x if each minimal geodesic starting at x intersects A in at most one point.

Note that if, in the above definition, x and A both lie in a strictly convex set C, one need only consider minimal curves that lie in C.

LEMMA 3.9. *Suppose $\bar{\gamma}_1, \ldots, \bar{\gamma}_m \in \bar{S}_p$ are independent. Then for every $\varepsilon > o$ there exists a subset C_ε of S_p homeomorphic to an open subset of S^{m-1} that is ε-close to $\mathrm{sp}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_m\} \cap S_p$, such that the map T has a positive lower bound on C_ε.*

Proof. Recall that subsets X and Y of a metric space are ε-close (in the Hausdorff sense) if X is in an ε-neighborhood of Y, and vice versa.

The case $m = 1$ follows from the fact that S_p is dense in $\bar{S}_p$. For $m = 2$, choose $\alpha_1, \alpha_2 \in S_p$ such that for all $s \in [0, 1]$,

$$\alpha(s \cdot \alpha_1 + (1 - s) \cdot \alpha_2, s \cdot \gamma_1 + (1 - s) \cdot \gamma_2) < \varepsilon/2.$$

Let β_t: $[0, 1] \to B$ be the minimal curve from $\alpha_1(t)$ to $\alpha_2(t)$, $t \leqslant \min\{T(\alpha_1), T(\alpha_2)\}$, and let $\zeta_{s,t}$ be unit minimal from p to $\beta_t(s)$. From the proof of Lemma 3.4, for each $s \in [0, 1]$ there exists a maximal $\delta(s) \in [0, r]$ such that for all $t \leqslant \delta(s)$,

$$\alpha(\zeta_{s,t}, s \cdot \alpha_1 + (1 - s) \cdot \alpha_2) \leqslant \varepsilon/2.$$

For any fixed t, as $s \to s'$, $\zeta_{s,t} \to \zeta_{s,t}$, so δ: $[0, 1] \to (0, r]$ is continuous, with a positive maximum. Hence for any fixed positive $T < \delta(s)$, $\{\zeta_{s,T}: s \in (0, 1)\}$ satisfies the requirements for C_ϵ.

Now suppose $m > 2$. For any k, set

$$C_k = \{t_1\bar{\gamma}_1 + \cdots + t_k\bar{\gamma}_k: t_1 + \cdots + t_k = 1\},$$

and suppose, as we have already shown for $k = 2$, there exist homeomorphisms φ_i: $C_{m-1} \to B$ whose images are transverse to p, with the following property. For any $a \in C_{m-1}$, let α_a^i be the unit minimal curve from p to $\varphi_i(a)$ in B and B_a be the unit vector on the radial line from 0 to a. Then $\alpha(\alpha_a^i, \beta_a)$ is uniformly small for all $a \in C_{m-1}$ and sufficiently large i. In particular, for large i, the interior of $\{\gamma \in S_p: \gamma(s) \in \varphi_1(C_{m-1}), s > 0\}$ satisfies the requirements for C_ϵ.

Let $\beta_i \in S_p$ be such that $\alpha(\beta_i, \bar{\gamma}_m) \leqslant 1/i$ and $T_i = \min\{T(\beta_i), 1/i\}$. For any $a, b \in C_{m-1}$, there exists an $I > 0$ such that for all $i \geqslant I$, $\beta_i(T_i)$ is transverse to $\{\varphi_i(a), \varphi_i(b)\}$. If otherwise, there would exist $i_k \to \infty$ and minimal curves from $\beta_{i_k}(T_{i_k})$ through both $\varphi_{i_k}(a)$ and $\varphi_{i_k}(b)$. But then by Lemma 3.3, $\bar{\gamma}_m$ would lie in $\mathrm{sp}\{\lim \varphi_i(a), \lim \varphi_i(b)\}$, and hence in $\mathrm{sp}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_{m-1}\}$, a contradiction. Furthermore, the choice of minimal such I is an upper semicontinuous function of $C_{m-1} \times C_{m-1}$ into the positive integers, since if $a_j \to a$ and $b_j \to b$, then for any i, the limit of minimal curves from $\beta_i(T_i)$ through both $\varphi_i(a_j)$ and $\varphi_i(b_j)$ is a minimal curve from $\beta_i(T_i)$ through both $\varphi_i(a)$ and $\varphi_i(b)$.

One can therefore choose $I > 0$ such that for all $i > I$, $\varphi_i(C_{m-1})$ is transverse to $\beta_i(T_i)$, and φ_i can be extended to a homeomorphism on C_m by letting $\varphi_i(t_1\bar{\gamma}_1 + \cdots + t_m\bar{\gamma}_m) = \gamma(t_m)$, where γ is the geodesic from $\varphi_i(t_1\bar{\gamma}_1 + \cdots + t_{m-1}\bar{\gamma}_{m-1})$ to $\beta_i(T_i)$. By Lemma 3.3, for any $a \in C_m$ and $\varepsilon > 0$, there is a $K > 0$ such that for all $i > K$, $\alpha(\alpha_a^i, \beta_a) < \varepsilon$. As in the above argument, the choice of a minimal such K for each $a \in C_m$ is upper semicontinuous; in other words, $\alpha(\alpha_a^1, \beta_a)$ is uniformly small for large i. Finally, $\varphi_i(C_m)$ is transverse to p

for large enough i, by an argument similar to the proof in the above paragraph, and this completes the proof of the inductive step. $\square$

PROOF OF THEOREM A. $(a) \Rightarrow (b)$. Suppose $\dim X = n < \infty$. For any $p \in X$, let $B(p, r)$ be a strictly convex region of curvature $\leqslant k$ and $\geqslant K$, and $\bar{\gamma}_1, \ldots, \bar{\gamma}_m$ be independent elements of $\bar{S}_p$. Choose ε small enough that the set C_ε of Lemma 3.9 is non-empty. Then $\exp_p$ is (a homeomorphism) defined on $\{t \cdot \gamma : \gamma \in C_\varepsilon, t < \varepsilon\}$, which is an open radial cone in $\mathbb{R}^m$; i.e., $B(p, r)$ contains a subset of dimension m, and so $m \leqslant n$.

$(b) \Rightarrow (c)$. Suppose $\bar{S}_p$ is spanned by independent elements $\bar{\gamma}_1, \ldots, \bar{\gamma}_n$ and let $B(p, r)$ be a region of curvature $\geqslant k$ and $\leqslant K$. By Proposition 3.7 $\bar{S}_p$ is isometric to the closure of an open subset of S^{n-1} and $\bar{T}_p$ is the closure of an open cone in $\mathbb{R}^n$.

For all i, let C_i be from Lemma 3.9 for $\bar{\gamma}_1, \ldots, \bar{\gamma}_n$ and $\varepsilon = 2^{-1}$, and let $C = \exp_p\{t \cdot \gamma : \gamma \in C_1$ for some $i, 0 < t < \min\{T(\gamma), r\}\}$. Since each C_i is open in $\bar{S}_p$ and $\exp_p$ is surjective, each set $\exp_p\{t \cdot \gamma : \gamma \in C_i 0 < t < \min\{T(\gamma), r\}\}$, and hence C, is open in X. In fact, C is an open dense subset of $B(p, r)$ homeomorphic to an open subset of $\mathbb{R}^n$. For any $z \in C$ and small $\rho > 0$, $B(z, \rho)$ is homeomorphic to an open subset of $\mathbb{R}^n$ and $\bar{B}(z, \rho)$ is a compact, n-dimensional inner metric space of curvature $\geqslant k$ and $\leqslant K$. By the step $(a) \Rightarrow (b)$ proved above, $\bar{S}_z$ is the span of some independent $\bar{\alpha}_1, \ldots, \bar{\alpha}_m, m \leqslant n$. Now $\exp_z^{-1}(B(z, \rho))$ is open in T_z and homeomorphic to an open subset of $\mathbb{R}^n$, so in fact $m = n$, and $\bar{T}_z = T_z = \mathbb{R}^n$. In other words, $S_z = S^{n-1}$ and z is not a geodesic terminal. All geodesic terminals in $B(p, r)$ therefore lie in $B(p, r) \backslash C$, a nowhere dense set.

Let Y be the subset of all points $y \in X$ such that for some $\rho > 0$, the geodesic terminals in $B(y, \rho)$ are nowhere dense. Y is obviously open, non-empty by the above argument, and also closed: Let $w \in \bar{Y}$, and suppose $B(w, \rho)$ is a strictly convex region of curvature $\leqslant K$ and $\geqslant k$. There exists in $B(w, \rho/2)$ a point $y \in Y$ and hence a point z which is contained in a geodesically complete open ball; i.e., by Theorem 3.2, $\bar{S}_z = S_z$ is spanned by finitely many independent elements. But then by the preceding paragraph the geodesic terminals in the ball $B(z, \rho/2)$, which is a strictly convex region of curvature $\leqslant K$ and $\geqslant k$, are nowhere dense. It follows that $w \in Y$ and $Y = X$. Finally, since the set $\mathscr{T}$ of geodesic terminals is nowhere dense in a region of every point, $\mathscr{T}$ is nowhere dense in all of X.

$(c) \Rightarrow (d)$. We will show first that X is a manifold with boundary; by invariance of domain we need only show that every $x \in X$ has a neighborhood homeomorphic to a neighborhood of a Euclidean upper half-space (whose dimension might *a priori* depend on x). Choose a strictly convex region $B(x, 3r)$ of curvature $\leqslant K$ and $\geqslant k$ containing x, and a point $p \in B(x, r) \backslash \mathscr{T}$. Since $\mathscr{T}$ is nowhere dense, there exists a geodesically complete ball $B(p, \rho) \subset B(x, 3r)$. By Theorem 3.2, $\bar{T}_p = T_p$ is isometric to $\mathbb{R}^n$ for some n, and $\exp_p|_{B(0,\rho)}$ is a homeomorphism. For all $q \in B = B(p, r)$, let γ_q denote the unique minimal curve from p to q.

We will show that if x is a terminal of a geodesic in B starting at p, then x is a boundary point, and if x is not such a terminal, it is an interior point. As argued previously, the map $T: S_p \to \mathbb{R}^+$ is upper semicontinuous; at any $v \in S_p$ such that $\exp_p(tv) \in B$ for all $t \in [0, T(v)]$, T is lower semicontinuous (and hence continuous), as the following claim shows.

CLAIM 3.10. *If for some $v \in S_p$ and $c > 0$, $\exp_p(tv) \in B$ for all $t \in [0, c]$, then* $\liminf_{w \to v} T(w) \geqslant c$.

Proof. Let $y = \exp(cv)$. Note that $\bar{T}_y = \mathrm{sp}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\}$ for some $\bar{\gamma}_1, \ldots, \bar{\gamma}_n$. In fact, Lemma 3.9 implies that if $\bar{S}_y$ contains m independent elements, B contains a closed subset C of dimension m. Since $\exp_p^{-1}(C) \cap B(0, r)$ is a closed m-dimensional subset of the (closed) domain of definition of $\exp_p$ in $B(0, r)$, $m \leqslant n$. Since $p \in B(y, r) \subset B(x, 3r)$ we can apply the same argument to obtain the reverse inequality for an independent set spanning $\bar{T}_y$.

Let $U = \exp_y^{-1}(B(p, \rho))$; then by Invariance of Domain U is open in $\bar{T}_y \subset \mathbb{R}^n$. Let $S(c + \varepsilon)$ denote the intersection of T_y with the $(n - 1)$-sphere in $\mathbb{R}^n$ centered at 0 in of radius c + ε. For some small ε, $S(c + \varepsilon) \cap U$ contains an $(n - 1)$-disk $\mathscr{D}$ such that $(c + \varepsilon) \cdot v$ lies in the interior of $\mathscr{D}$. Let S denote the $(n - 1)$-sphere in T_y which is the union of $\mathscr{D}$ with all radial lines from $\partial\mathscr{D}$ to 0.

The set $Z = \exp_p^{-1}(\exp_y(S))$ is a topological $(n - 1)$-sphere containing cv such that for all $t \in [0, c)$, tv lies in the n-dimensional ball bounded by Z. In particular, if $\alpha(v, w)$ is small, then the radial line through w must intersect Z near cv. In other words, $\exp_p(tw)$ is defined for t not much smaller than c. This completes the proof of the claim.

Suppose now that x is a terminal of the minimal curve γ_x. Let $D = \{w \in T_p: w = T(v) \cdot v, v \in S_p$, and $\alpha(\exp(tw), \gamma_x)) < \varepsilon N$, where ε is chosen, using the continuity of T, small enough that $\exp(tw) \in V$, for all $w \in D$ and $t \in [0, 1]$. The continuity of T shows furthermore that D is a topological $(n - 1)$-ball, and that $D' = \{tw: w \in D$ and $t \in (0, \|w\|]\}$ is homeomorphic to a boundary ball in n-dimensional half-space. Finally, $\exp(D')$ is an open subset containing x; for if x' is sufficiently close to x, then $\alpha(\gamma_x, \gamma_{x'}) < \varepsilon$ and $d(x', p) > r/3$. By definition, the terminal of γ_p, lies in $\exp(D)$, and so $x' \in \exp(D')$.

If x is not a terminal of γ_x, then Claim 3.10 shows that nearby points are also not terminals, and so the exponential map provides a neighborhood of x homeomorphic to an open subset of Euclidean space.

X has now been shown to be a manifold with boundary, and $\partial X = \mathscr{T}'$, where $\mathscr{T}'$ is the set of terminals x with the following property: in some strictly convex region V of curvature bounded above and below containing x, there is a minimal curve γ_{px}, with $p \in V \backslash \bar{\mathscr{T}}$. Since ∂X is closed, the proof of the theorem will be complete if it is shown that each point $z \in \mathscr{T}$ is $z = \lim z_i$, with $z_i \in \mathscr{T}'$. Choose a region V containing z with curvature bounded above and below, and pick $q \in V$ so that z is a terminal of a minimal curve γ starting at q. Now choose points

$q_i \to q$ such that $q_i \in V \backslash \bar{\mathscr{T}}$, and let γ_i denote the unique geodesic starting from q_i, with maximal domain of definition. Then as in the proof of the upper semi-continuity of the map T, as $i \to \infty$, the geodesics must have terminals z_i, and $z_i \to z$.

$(d) \Rightarrow (a)$ is, of course, a classical result.

The last statement of the theorem follows from the proofs of $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$. □

The only part of Corollary B which cannot be proved as in the classical Hopf–Rinow Theorem is (a) ⇒ (b), which is immediate from Theorem A.

Proof of Corollary C. If the geodesic terminals $\mathscr{T}$ in a space X are not dense, then their complement contains an open ball $B(p,r)$; but such a ball is by definition geodesically complete, and hence by Theorem 3.2 S_p contains at most finitely many independent elements. Theorem A now implies X is finite dimensional. □

Any finite dimensional space can be embedded in Euclidean space, and so completeness and finite dimensionality together imply local compactness. Theorem A therefore implies that any finite dimensional space X with a complete metric of locally bounded curvature is a topological manifold with boundary. More generally, since the induced inner metric on a convex subset C of X is the same the original metric of C (as a subset of X), the following corollary holds (in the Riemannian case this was proved by Cheeger and Gromoll, cf. [CE], Chapter 8).

COROLLARY 3.11. *If X is a finite dimensional, complete inner metric space of locally bounded curvature, then every closed, convex subset of X is a manifold with boundary.*

In general the boundary of a space of bounded curvature need not be smooth in the "normal" coordinates of the type constructed in Theorem A: A square in the plane with the induced inner metric is flat, but in no choice of normal coordinates is the boundary smooth.

LEMMA 3.12. *Let X be a finite dimensional complete inner metric space of locally bounded curvature. Then ∂X is transverse to every interior point.*

Proof. We need only prove the following: For interior point $p \in X$ and $\gamma \in S_p$, if $x \in \partial X$ is the first boundary point along γ from p (since ∂X is closed there is such a point), then γ terminates at x. Since x is the first boundary point on γ, we can choose an interior point q on γ and a strictly convex region $B(q,r)$ of curvature bounded above and below containing x. If γ were defined beyond x, then Claim 3.10 would imply that $\exp_p^{-1}$ is a homeomorphism on an open set containing x; that is, x is contained in a Euclidean neighborhood, and so is not a boundary point. This contradiction completes the proof. □

DEFINITION 3.13. Suppose $B = B(p, r)$ is a strictly convex region of curvature bounded above and below, and $A, A' \subset B$ are transverse to p. Then A and A' are said to be *r-equivalent* in B if there is a (possibly not continuous) bijection $\varphi: A \to A'$ such that a and $\varphi(a)$ lie on the same radial geodesic from p. The radial distance $\delta_r(A, A')$ is defined to be the supremum of the distances $\mathrm{d}(a, \varphi(a))$.

LEMMA 3.14. *Let $U = B(p, r)$ and $V = B(q, s)$ be strictly convex regions of curvature bounded above and below. Suppose A is a compact subset of $U \cap V$ such that A is transverse to both p and q. Then there exists an $\varepsilon > 0$ such that if $A' \subset U \cap V$ is r-equivalent to A in U and $\delta_r(A, A') < \varepsilon$, then A' is transverse to q.*

Proof. The set $C = \{\gamma \in S_p: \gamma(t) \in A$ for some $t\}$ is compact. If $\gamma(s), \alpha(t) \in A$, there exists some $\delta > 0$ such that for all $\zeta \in (-\delta, \delta)$, $\{\gamma(s + \zeta), \alpha(t + \zeta)\}$ is transverse to q. For if otherwise, one could find $t_i \to 0$ with geodesics β_i to q starting at q passing through both $\gamma(s + t_i)$ and $\alpha(t + t_i)$. But $\lim \beta_i$ would be a minimal curve in V starting at q and passing through both $\gamma(s)$ and $\alpha(t)$, a contradiction. A similar argument shows that the function which assigns to each element of $C \times C$ the infimum of all such δ is lower semicontinuous, and therefore has a positive minimum on $C \times C$; this minimum is the desired ε. □

Proof of Theorem D. Suppose X is a smooth manifold with boundary. Endow the interior of X with a Riemannian metric which is a product metric near the boundary. Extend the metric (distance) to all of X by continuity. Then X is isometrically embedded as a convex subset of the Riemannian manifold $\tilde{X}$ obtained by adding a small open collar (with the product metric) to X along the boundary. In particular, all angle comparisons in X can be carried out in $\tilde{X}$, which has locally bounded sectional curvature, and hence locally bounded curvature in the present sense.

Conversely, suppose X is finite dimensional with a complete inner metric of locally bounded curvature. For simplicity, assume X is compact, and let $\{B_i(x_i, r_i)\}$, $1 \leqslant i \leqslant k$, be a cover of X by balls with the following properties: (1) $\bar{B}_i$ is contained in a strictly convex region of curvature bounded above and below, (2) $x_i \in \operatorname{int} X$ for all i, (3) there exist coordinates (B_i, ψ_i) having C^1 overlap on the complement of the boundary of X (cf. [Be]). The terminal map $T_1: S_{x_1} \to (0, \infty]$ was shown to be continuous on $T_1^{-1}((0, r_1])$ in the proof of Theorem A. $U_1 = T_1^{-1}((0, r_1))$ is an open subset of the unit sphere S_{x_1} homeomorphic to $W_1 = B_1 \cap \partial X$ via the map $\varphi_1(v) = \exp_{x_1} \circ T_1(v) \cdot v$. One can choose a smooth map $\tau_1: U_1 \to (0, r)$ having a continuous extension equal to T_1 on $S_{x_1} \backslash U_1$ such that, on $U_1, \tau_1 < T_1$ and τ_1 approximates T_1 near enough that the following holds: Let $D_1 = \{\gamma(\tau_1(\gamma)): \gamma \in U_1\}$, that is, D_1 is obtained by 'pushing' W_1 inward along radial geodesics starting at x_1 by the amount $T_1 - \tau_1$. D_1 is r-equivalent to W_1, and so by Lemma 3.14, if τ_1 is chosen close enough to $T_1, D_1 \cap B_i$ is still transverse to x_i for all i such that $D_1 \cap B_i \neq \varnothing$. The set

$$X_1 = X \backslash \{\gamma(t): t > \tau_1(\gamma), \gamma \in U_1\}$$

is homeomorphic to X and has smooth boundary in B_1. Let B_1' be an open subset of B_1 such that $\bar{B}_1' \subset B_1$ and $\{B_1', B_2, \ldots, B_k\}$ covers X. Let T_2 be the 'terminal' map for X_1, i.e., for each $\gamma \in S_{x_2}$, $T_2(\gamma) = t$ provided $\gamma(t) \in \partial X_1$, and $T_2(\gamma) = \infty$ if no such t exists. Since $\partial X_1 \cap B_2$ is transverse to x_2, T_2 is well-defined and continuous. $U_2 = T_2^{-1}((0, r_2))$ is an open subset of the unit sphere S_{x_2} homeomorphic to $W_2 = B_2 \cap \partial X_1$ via the map $\varphi_2(v) = \exp_{x_2} \circ T_2(v) \cdot v$. The overlap between B_1 and B_2 is C^1 on their interiors, and $\partial X_1 \cap B_1 \cap B_2 \subset \operatorname{int} B_1 \cap \operatorname{int} B_2$; this implies that T_2 is smooth on $Y = \varphi_2^{-1}(\partial X_1 \cap B_1 \cap B_2)$. Setting $Y' = \varphi_2^{-1}(\partial X_1 \cap B_1' \cap B_2)$, one can now choose a smooth approximation τ_2 of T_2 on U_2 which agrees with T_2 on Y', and so that the new manifold with boundary, X_2, constructed as above, has boundary whose intersection with any B_i is transverse to x_i, and has smooth coordinates (i.e. the restrictions of ψ_i) in $B_1' \cup B_2$. This inductive procedure can be continued for a finite number of steps to obtain a manifold with boundary X_k contained in, and homeomorphic to, X, such that the restrictions of $\{\psi_i\}$ are C^1 coordinates for X_k.

In the noncompact case, one can use the above procedure to 'smooth out' $B(p, 2)$ for some point p. On can then cover $B(p, 3)$ by $B(p, 1.5)$ and a finite number of open sets which do not intersect $B(p, 1)$. A C^1 structure can now be constructed on $B(p, 3)$ which agrees with a the previous smooth structure on $B(p, 1)$. This process can now be continued for a countable number of steps to put a C^1 structure on all of X. □

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References

[B] Busemann, H. *The Geometry of Geodesics*, Academic Press, New York, 1955.

[Be] Berestovskii, V. N. Introduction of a Riemann structure into certain metric spaces, *Siberian Math. J.*, **16** (1975), 499–507.

[CE] Cheeger, J., and Ebin, D. G. *Comparison Theorems in Riemannian Geometry*, North Holland, Amsterdam, 1975.

[F] Fukaya, K. A boundary of the set of the Riemannian manifolds with bounded curvatures and diameters, *J. Diff. Geo.* **28** (1988), 1–21.

[FY] Fukaya, K. and Yamaguchi, T. Almost non-positively curved manifolds, *J. Diff. Geo.* **33** (1991), 67–90.

[G] Gromov, M. Groupes of polynomial growth and expanding maps, *Publ. Math. I.H.E.S.* **53** (1981), 53–78.

[GLP] Gromov, M., Lafontaine, J., and Pansu, P. *Structure Métrique pour les Variétés Riemanniennes*, Cedic/Fernant Nathan, Paris, 1981.

[GP1] Grove, K. and Petersen, P. Bounding homotopy types by geometry, *Ann. of Math.* **128** (1988) 195–206.

[GP2] Grove, K. and Petersen, P. Manifolds near the boundary of existence, *J. Diff. Geo.* **33** (1991), 379–394.

[GPW] Grove, K., Petersen, P., and Wu, J. Y. Geometric finiteness theorems via controlled topology, *Invent. math.* **99** (1990) 205–213.

[GW] Greene, R. E. and Wu, H. Lipschitz convergence of Riemannian manifolds, *Pacific Math. J.* **131** (1988), 119–141.

[K] Karcher, H. *Riemannian Comparison Constructions*, preprint.

[N] Nikolaev, I. G. Smoothness of the metric in spaces with bilaterally bounded curvature in the sense of A. P. Aleksandrov, *Siberian Math. J.* **24** (1983) 247–263.

[P] Peters, S. Convergence of Riemannian manifolds, *Compositio Math.* **62** (1987), 3–16.

[P1] Plaut, C. Almost Riemannian Spaces, *J. Diff. Geo.*, to appear.

[P2] Plaut, C. Metric curvature, convergence, and topological finiteness, preprint.

[PD] Plaut, C. *Riemannian Geometry of Non-Riemannian Spaces*, dissertation, University of Maryland, 1989.

[R] Rinow, W. *Die Innere Geometrie der Metrischen Raume*, Springer-Verlag, Berlin, 1961.