COMPOSITIO MATHEMATICA

DEVRA GARFINKLE

On the classification of primitive ideals for complex classical Lie algebras, II

Compositio Mathematica, tome 81, nº 3 (1992), p. 307-336
<http://www.numdam.org/item?id=CM_1992__81_3_307_0>

Compositio Mathematica **81**: 307–336, 1992.

On the classification of primitive ideals for complex classical Lie algebras, II

DEVRA GARFINKLE*
Department of Mathematics, Rutgers University, Newark, NJ, U.S.A. 07102

Received 9 October 1990; accepted 1 May 1991

Introduction

This paper is the continuation of 'On the Classification of Primitive Ideals for Complex Classical Lie Algebras, I' [4]. References to items, the first of whose three digits is the numeral 1, are references to items contained in that paper. Unexplained notation refers implicitly to part I, as well.

The first aim of this series of papers, as explained in the introduction to part I, is to classify the primitive ideals in the enveloping algebra of a complex semisimple Lie algebra of classical type by determining the fibres of the Duflo map ([3]), that is, determining explicitly when two irreducible highest weight modules have the same annihilator. Joseph [7] and [8], first accomplished this for $\mathfrak{g}$ of type A_{n-1}, by showing that the Robinson–Schensted algorithm applied to the symmetric group on n letters calculated a complete invariant of the annihilator of an irreducible highest weight module. (Since this algorithm has an explicit inverse this solves the problem in a very nice way.) In part I, the existence of an analogous algorithm, called A, for the Weyl groups of types B_n, C_n, and D_n, was demonstrated. It produces a pair of domino tableaux. Completely new, however, was the phenomenon of cycles which was investigated in that paper, and the completely dissimilar algorithm, S (involving the notion of cycles, peculiar to the domino situation) which, given a domino tableau, produces one in a special shape (corresponding to Lusztig's notion of special irreducible Weyl group representation [10]: in the A_n case, every irreducible representation is special). This last product will be shown, in part III of this series (for types B_n and C_n; the treatment of type D_n will appear in part IV) to be the complete invariant sought. (The inverses to both algorithms were studied in part I.)

In this part we study the relationship between the map α (the main ingredient of A) and the various $T_{\alpha\beta}$'s (for α and β appropriate pairs of adjacent simple roots), since we intend, in part III, to prove Vogan's conjecture (in the cases B_n

*Partially supported by NSF grants MCS-8211320, DMS 8503781

and C_n, i.e. the cases where it is true) that the generalized τ-invariant is a complete invariant of a primitive ideal [11].

This study falls into two main halves. The first half, contained in Section 1 of this paper, is, modulo what one may call the obvious complications induced by the introduction of dominos where squares used to suffice, basically analogous to Joseph's analysis of the A_n situation, and obtains when α and β have the same length. In Section 2 we then deal with the essentially new sort of complications obtaining when α and β have differing lengths. (The case of D_n would be a third half and its complications are postponed until part IV.) We must then study how the step α of the algorithm A disturbs what we call the cycle structure of the domino tableaux (produced by the previous steps of A). After this key result is proved we can then finish our analysis of the relation between A and the $T_{\alpha\beta}$'s, in Section 3.

In this way, the significance of the cycles has broadened considerably from part I, where they played no role in the discussion of A.

In [5] this author showed how to calculate the annihilators of irreducible Harish-Chandra modules for type A_n using a hybrid cross between the Robinson–Schensted algorithm and a completely different procedure. By making somewhat similar (but more complicated) hybridizations between A and other items (and then of course applying S) one can do the same for other families of classical groups. In order, however, to even define these hybrids we will need the results of this paper on the cycle structure of domino tableaux. (To show further that these hybrids compute the annihilators we will need the results on the generalized τ-invariant from part III, in cases B_n and C_n, and on the generalized generalized τ-invariant, for Lie (G) of type D_n, from part IV.)

It is best to take the results of this paper, to the extent that they refer to simple roots, etc., as applying to types B_n and C_n. (In particular W is the Weyl group of type B_n or C_n.) Most of these results do apply equally well to type D_n (although not those which refer to α_1); this will be explained in part IV. Also, it is known that the classification of primitive ideals depends only on the Weyl group. So the discussion of tableaux for type B_n is unnecessary for the immediate purposes of this series of papers. On the other hand, most of it will be needed for the study of Harish-Chandra modules.

In [4] we gave two definitions of the map A. We will use the second of these, Definition 1.2.7, in all proofs in this paper.

Section 1

The various definitions and theorems of this section are stated for $T \in \mathscr{T}(M)$, etc.; they hold for $T \in \mathscr{T}^0(M)$, etc., as well.

2.1.1. NOTATION. (1) If $T \in \mathscr{T}(M)$ and $i \in M$, write $T - i$ for $T \backslash D(i, T)$.

(2) If $P = \{S_{i,j}, S_{i+1,j}\}$ (respectively $P = \{S_{ij}, S_{i,j+1}\}$) we set $\rho^1(P) = i$, $\rho^2(P) = i+1$, $\kappa^1(P) = \kappa^2(P) = j$ (respectively $\rho^1(P) = \rho^2(P) = i$, $\kappa^1(P) = j$, and $\kappa^2(P) = j+1$). Such a P is called horizontal (respectively vertical). If $T \in \mathscr{T}(M)$ and $k \in M$ we set $\rho^1(k, T) = \rho^1(P(k, T))$; similarly $\rho^2(k, T)$, $\kappa^1(k, T)$, and $\kappa^2(k, T)$.

We define various maps, all called transpose.

2.1.2. NOTATION. If $F \subseteq \mathscr{F}$ let ${}^tF = \{S_{ji} \mid S_{ij} \in F\}$. For $T \in \mathscr{T}(M)$ let

$$ {}^tT = \{(k, S_{ji}) \mid (k, S_{ij}) \in T\}. $$

(Then $\rho_i({}^tT) = \kappa_i(T)$.) We define similarly ${}^t\colon \mathscr{T}(M_1, M_2) \to \mathscr{T}(M_1, M_2)$. We define ${}^t\colon \mathscr{C}(M) \to \mathscr{C}(M)$ by ${}^t(T, v, \varepsilon) = ({}^tT, v, -\varepsilon)$ and ${}^t\colon \mathscr{D}(M) \to \mathscr{D}(M)$ by ${}^t(T, P) = ({}^tT, {}^tP)$.

PROPOSITION. *$\alpha({}^tX) = {}^t(\alpha(X))$ for $X \in \mathscr{C}(M)$.*

Proof. The proof uses induction on $|M|$. It is then trivial from the definitions. $\square$

2.1.3. DEFINITION. Let $w \in W$. We define $\tau^L(w)$ and $\tau^R(w)$ as subsets of Π, the simple roots, by:

(1) $\alpha \in \tau^R(w)$ if and only if $w\alpha \notin \Delta^+$ which is in turn equivalent to $l(ws_\alpha) < l(w)$,
(2) $\alpha \in \tau^L(w)$ if and only if $\alpha \in \tau^R(w^{-1})$ which is in turn equivalent to $l(s_\alpha w) < l(w)$.

2.1.4. DEFINITION. (1) If α and β are adjacent simple roots we define

$$ D^L_{\alpha\beta}(W) = \{w \in W \mid \beta \in \tau^L(w),\ \alpha \notin \tau^L(w)\}; $$

similarly we define $D^R_{\alpha\beta}(W)$.

(2) When α and β are adjacent simple roots of the same length, we define a map

$$ T^L_{\alpha\beta}\colon D^L_{\alpha\beta}(W) \to D^L_{\beta\alpha}(W) $$

by $T^L_{\alpha\beta}(w) = z$ where

$$ \{z\} = \{s_\alpha w,\ s_\beta w\} \cap D^L_{\beta\alpha}(W). $$

Then $T^L_{\alpha\beta}$ is a well-defined bijection with inverse $T^L_{\beta\alpha}$. (This is obvious if the Lie algebra is of type A_2. The general case is reduced to this case by Exercise 3, §1, [2].)

(3) When α and β are adjacent simple roots with different lengths, we define a

map $T^L_{\alpha\beta}$ with domain $D^L_{\alpha\beta}(W)$ by: for $w \in D^L_{\alpha\beta}(W)$,

$$T^L_{\alpha\beta}(w) = \{s_\alpha w, s_\beta w\} \cap D^L_{\beta\alpha}(W).$$

Then $T^L_{\alpha\beta}(w)$ is a one or two element set. Furthermore we have:

(i) if $T^L_{\alpha\beta}(w) = \{w'\}$ then $T^L_{\beta\alpha}(w') = \{w, w''\}$ with $w'' \neq w$ and $T^L_{\alpha\beta}(w'') = \{w'\}$.
(ii) If $T^L_{\alpha\beta}(w) = \{w_1, w_2\}$ with $w_1 \neq w_2$ then

$$T^L_{\beta\alpha}(w_1) = T^L_{\beta\alpha}(w_2) = \{w\}.$$

(Again, the general case is reduced to that of B_2.)
(4) We define similarly $T^R_{\alpha\beta}$, with analogous properties.

2.1.5. DEFINITION

(1) Suppose $\gamma \in \mathscr{S}(M_1, M_2)$ and $\{i-1, i\} \subseteq M_1$. Let k, l, ε_1, ε_2 be such that $\{(i-1, k, \varepsilon_1), (i, l, \varepsilon_2)\} \subseteq \gamma$.
   (a) We say that $\alpha_i \in \tau^L(\gamma)$ if either both $k > l$ and $\varepsilon_1 = 1$ or both $k < l$ and $\varepsilon_2 = -1$.
   (b) We say $\alpha_i \notin \tau^L(\gamma)$ if either both $k > l$ and $\varepsilon_1 = -1$ or both $k < l$ and $\varepsilon_2 = 1$.
   (c) If $1 \in M_1$ and $(1, k, \varepsilon) \in \gamma$ we say $\alpha_1 \in \tau^L(\gamma)$ if $\varepsilon = -1$; otherwise we say $\alpha_1 \notin \tau^L(\gamma)$.
(2) We define similarly $\alpha_i \in \tau^R(\gamma)$ and $\alpha_i \notin \tau^R(\gamma)$ by interchanging the roles of M_1 and M_2 in the above definitions. (This could be phrased as $\alpha_i \in \tau^R(\gamma)$ if and only if $\alpha_i \in \tau^L(\gamma^{-1})$ (see Definition 1.1.5).)

REMARK. If $M_1 = \{1, \ldots, n\}$ then it is natural to consider (and we do) the above as defining $\tau^L(\gamma)$ as a subset of Π, the simple roots. Otherwise the statements $\alpha_i \in \tau^L(\gamma)$, $\alpha_i \notin \tau^L(\gamma)$ are considered as formal statements to be made only under condition of the stated hypothesis on M. Thus, for example, if $1 \notin M$, then the statement $\alpha_1 \notin \tau^L(\gamma)$ is considered, not true, but meaningless, and as such, false. (Similarly for $\tau^R(\gamma)$.)

2.1.6. NOTATION. Let $\gamma \in \mathscr{S}(M_1, M_2)$.

(1) If $\{i, j\} \subseteq M_1$ we define $\mathrm{In}^L(i, j; \gamma) \in \mathscr{S}(M_1, M_2)$ as follows: suppose $\{(i, k_i, \varepsilon_i), (j, k_j, \varepsilon_j)\} \subseteq \gamma$ then

$$\mathrm{In}^L(i, j;\, \gamma) = (\gamma \backslash \{(i, k_i, \varepsilon_i), (j, k_j, \varepsilon_j)\}) \cup \{(j, k_i, \varepsilon_i), (i, k_j, \varepsilon_j)\}.$$

(2) If $i \in M_1$ we define $\mathrm{SC}^L(i; \gamma) \in \mathscr{S}(M_1, M_2)$ as follows: suppose $(i, k_i, \varepsilon_i) \in \gamma$ then

$$\mathrm{SC}^L(i;\, \gamma) = (\gamma \backslash \{(i, k_i, \varepsilon_i)\}) \cup \{(i, k_i, -\varepsilon_i)\}.$$

(3) We define similarly In^R and SC^R (in relation to M_2).

(4) Let $M_1 = \{i_1, \ldots, i_m\}$ with $|M_1| = m$. We define

$$\mathrm{SC}(\gamma) = \mathrm{SC}^L(i_m; \ldots; \mathrm{SC}^L(i_2; \mathrm{SC}^L(i_1; \gamma)) \ldots).$$

REMARK. (1) For $w \in W$ and $2 \leqslant i \leqslant n$ we have $\delta(s_i w) = \mathrm{In}^R(i-1, i; \delta(w))$ and $\delta(ws_i) = \mathrm{In}^L(i-1, i; \delta(w))$. We have $\delta(s_1 w) = \mathrm{SC}^R(1; \delta(w))$ and $\delta(ws_1) = \mathrm{SC}^L(1; \delta(w))$. We have also $\delta(w_0 w) = \delta(ww_0) = \mathrm{SC}(\delta(w))$. (Recall Definition 1.1.3 for δ.)

(2) We have $A(\mathrm{SC}(\gamma)) = {}^t A(\gamma)$ (by Proposition 2.1.2 and induction).

2.1.7. DEFINITION. (1) If $\{\alpha, \beta\} = \{\alpha_i, \alpha_{i+1}\}$ and either $i = 1$ and $\{1, 2\} \subseteq M_1$ or $i \geqslant 2$ and $\{i-1, i, i+1\} \subseteq M_1$ we define

$$D^L_{\alpha\beta}(\mathscr{S}(M_1, M_2)) = \{\gamma \in \mathscr{S}(M_1, M_2) \mid \beta \in \tau^L(\gamma), \alpha \notin \tau^L(\gamma)\}.$$

(2) If $\{\alpha, \beta\} = \{\alpha_i, \alpha_{i+1}\}$ with $i \leqslant 2$ and $\{i-1, i, i+1\} \subseteq M_1$ we define

$$T^L_{\alpha\beta}\colon D^L_{\alpha\beta}(\mathscr{S}(M_1, M_2)) \to D^L_{\beta\alpha}(\mathscr{S}(M_1, M_2))$$

by $T^L_{\alpha\beta}(\gamma) = \gamma'$ where

$$\{\gamma'\} = \{\mathrm{In}^L(i-1, i; \gamma), \mathrm{In}^L(i, i+1; \gamma)\} \cap D^L_{\beta\alpha}(\mathscr{S}(M_1, M_2)).$$

It is easy to verify (see the list in the proof of Theorem 2.1.19) that $T^L_{\alpha\beta}$ is a well-defined bijection with inverse $T^L_{\beta\alpha}$.

(3) If $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$ and $\{1, 2\} \subseteq M_1$ we define a map $T^L_{\alpha\beta}$ with domain $D^L_{\alpha\beta}(\mathscr{S}(M_1, M_2))$ by

$$T^L_{\alpha\beta}(\gamma) = \{\mathrm{In}^L(1, 2; \gamma), \mathrm{SC}^L(1; \gamma)\} \cap D^L_{\beta\alpha}(\mathscr{S}(M_1, M_2)).$$

Again, it is easy to verify (see the list in the proof of Theorem 2.3.7) that $T^L_{\alpha\beta}(\gamma)$ is a one or two element set, and furthermore that $T^L_{\alpha\beta}$ verifies the statements (i) and (ii) of part (3) of Definition 2.1.4 above.

(4) We define similarly $D^R_{\alpha\beta}(\mathscr{S}(M_1, M_2))$ and $T^R_{\alpha\beta}$, with analogous properties.

REMARK. (1) Clearly we have, for $w \in W$, $\tau^L(\delta(w)) = \tau^R(w)$ and $\tau^R(\delta(w)) = \tau^L(w)$. For $w \in D^L_{\alpha\beta}(W)$ we have $\delta(T^L_{\alpha\beta}(w)) = T^R_{\alpha\beta}(\delta(w))$; similarly with L and R interchanged. (This follows directly from Remark 2.1.6(1).)

(2) If $\gamma \in D^L_{\alpha\beta}(\mathscr{S}(M_1, M_2))$ then $\mathrm{SC}(\gamma) \in D^L_{\beta\alpha}(\mathscr{S}(M_1, M_2))$ and $T^L_{\beta\alpha}(\mathrm{SC}(\gamma)) = \mathrm{SC}(T^L_{\alpha\beta}(\gamma))$; similarly with R in place of L.

2.1.8. NOTATION. Let $T \in \mathscr{T}(M)$. If $i, j \in M$ we define

$$\mathrm{In}(i, j; T) = (T \backslash (D(i, T) \cup D(j, T))) \cup \{(S, j) | S \in P(i, T)\} \cup \{(S, i) | S \in P(j, T)\}.$$

If $i \in M$ and $j \notin M$ we define

$$\mathrm{Re}(i, j; T) = (T \setminus D(i, T)) \cup \{(S, j) \mid S \in P(i, T)\}.$$

These need not be domino tableaux, as they need not satisfy condition (4) of Definition 1.1.8. When defined, $\mathrm{Re}(i, i+1; T)$ and $\mathrm{Re}(i, i-1; T)$ will be domino tableaux.

2.1.9. DEFINITION. (1) Let $T \in \mathscr{T}(M)$ and suppose $i \geqslant 2$ and $\{i-1, i\} \subseteq M$.

(a) We say $\alpha_i \in \tau(T)$ if $\rho^2(i-1, T) < \rho^1(i, T)$, or, equivalently, if $\kappa^1(i-1, T) \geqslant \kappa^1(i, T)$ and $\kappa^2(i-1, T) \geqslant \kappa^2(i, T)$.

(b) We say $\alpha_i \notin \tau(T)$ if $\kappa^2(i-1, T) < \kappa^1(i, T)$, or, equivalently, if $\rho^1(i-1, T) \geqslant \rho^1(i, T)$ and $\rho^2(i-1, T) \geqslant \rho^2(i, T)$.

(The equivalences follow from condition (4) of Definition 1.1.8.)

(2) Suppose $1 \in M$. Then we say $\alpha_1 \in \tau(T)$ if $\kappa^2(1, T) = \kappa^1(1, T)$ $(=1)$ (i.e. if $D(1, T)$ is vertical), otherwise we say $\alpha_1 \notin \tau(T)$.

REMARK. As in Definition 2.1.5, these are formal statements, except when $M = \{1, \ldots, n\}$, in which case we consider these definitions as defining $\tau(T)$ as a subset of Π.

2.1.10. DEFINITION. (1) Let

$$F_1(a; j, k) = \{(a, S_{j,k}), (a, S_{j+1,k}), (a+1, S_{j,k+1}), (a+1, S_{j,k+2}), (a+2, S_{j+1,k+1}), (a+2, S_{j+1,k+2})\}.$$

Let

$$\tilde{F}_1(a; j, k) = \{(a, S_{j,k}), (a, S_{j,k+1}), (a+1, S_{j+1,k}), (a+1, S_{j+1,k+1}), (a+2, S_{j,k+2}), (a+2, S_{j+1,k+2})\}.$$

Define $F_2(a; j, k) = {}^t(\tilde{F}_1(a; k, j))$ and $\tilde{F}_2(a; j, k) = {}^t(F_1(a; k, j))$.

(2) If $\{\alpha, \beta\} = \{\alpha_i, \alpha_{i+1}\}$ and either $i = 1$ and $\{1, 2\} \subseteq M$ or $i \geqslant 2$ and $\{i-1, i, i+1\} \subseteq M$ we define

$$D_{\alpha\beta}(\mathscr{T}(M)) = \{T \in \mathscr{T}(M) \mid \beta \in \tau(T), \alpha \notin \tau(T)\}.$$

(3) If $\{\alpha, \beta\} = \{\alpha_i, \alpha_{i+1}\}$ with $i \geqslant 2$ and if $\{i-1, i, i+1\} \subseteq M$ we define

$$T_{\alpha\beta}: D_{\alpha\beta}(\mathscr{T}(M)) \to D_{\beta\alpha}(\mathscr{T}(M))$$

as follows: if $F_r(i-1; j, k) \subseteq T$ (respectively $\tilde{F}_r(i-1; j, k) \subseteq T$) then

$$T_{\alpha\beta}(T) = (T \setminus F_r(i-1; j, k)) \cup \tilde{F}_r(i-1; j, k)$$

(respectively $T_{\alpha\beta}(T)=(T\backslash\tilde{F}_r(i-1;j,k))\cup F_r(i-1;j,k)$). (In this case we say that $T_{\alpha\beta}(T)$ is obtained from T by an 'F-type' interchange.) If not, then $T_{\alpha\beta}(T)=Z$ where

$$\{Z\} = \{\mathrm{In}(i-1,\ i;\ T),\ \mathrm{In}(i,\ i+1;\ T)\} \cap D_{\beta\alpha}(\mathscr{T}(M)).$$

Clearly, then, $T_{\alpha\beta}$ and $T_{\beta\alpha}$ are inverses. (The fact that, under the stated hypotheses, this intersection consists of precisely one element can be shown to follow from condition (4) of Definition 1.1.8.)

2.1.11. DEFINITION. If $X=(T_1, T_2)\in\mathscr{T}(M_1, M_2)$ we say $\alpha_i\in\tau^L(X)$ if and only if $\alpha_i\in\tau(T_1)$; similarly for $\alpha_i\notin\tau^L(X)$, $\alpha_i\in\tau^R(X)$, and $\alpha_i\notin\tau^R(X)$. We then have the usual definition of $D^L_{\alpha\beta}(\mathscr{T}(M_1, M_2))$, and define, for $(T_1, T_2)\in D^L_{\alpha\beta}(\mathscr{T}(M_1, M_2))$ and $\{\alpha, \beta\}=\{\alpha_i, \alpha_{i+1}\}$ with $i\geqslant 2$, $T^L_{\alpha\beta}((T_1, T_2))=(T_{\alpha\beta}(T_1), T_2)$; similarly with R in place of L.

2.1.12. REMARK. (1) If $T\in\mathscr{T}(M)$, and if $\{i-1, i\}\subseteq M$ (respectively $1\in M$) then we have $\alpha_i\in\tau(T)$ if and only if $\alpha_i\notin\tau({}^tT)$ (respectively, $\alpha_1\in\tau(T)$ if and only if $\alpha_1\notin\tau({}^tT)$).

(2) If $T\in D_{\alpha\beta}(\mathscr{T}(M))$ then (for $\{\alpha, \beta\}=\{\alpha_i, \alpha_{i+1}\}$ with $i\geqslant 2$) we have ${}^t(T_{\alpha\beta}(T))=(T_{\beta\alpha}({}^tT))$.

2.1.13. PROPOSITION. *Suppose* $\{i, i+1\}\subseteq M, (T', v, \varepsilon)\in\mathscr{C}(M)$ *and let* $(T, P)=\alpha((T', v))$.

(a) *If* $v=i+1$ *and* $\varepsilon=1$ *then* $\alpha_{i+1}\notin\tau(T)$.
(b) *If* $v=i+1$ *and* $\varepsilon=-1$ *then* $\alpha_{i+1}\in\tau(T)$.
(c) *If* $v=i$ *and* $\varepsilon=1$ *then* $\alpha_{i+1}\in\tau(T)$.
(d) *If* $v=i$ *and* $\varepsilon=-1$ *then* $\alpha_{i+1}\notin\tau(T)$.
(e) *If* $v\notin\{i, i+1\}$ *and* $\alpha_{i+1}\in\tau(T')$ *then* $\alpha_{i+1}\in\tau(T)$.
(f) *If* $v\notin\{i, i+1\}$ *and* $\alpha_{i+1}\notin\tau(T')$ *then* $\alpha_{i+1}\notin\tau(T)$.

Proof. Let $e=\sup M$. Suppose first $e\neq i+1$. Then let $T_1'=T'-e$ and $(T_1, P_1)=\alpha((T_1', v, \varepsilon))$. By induction on $|M|$ (the induction starts at $|M|=2$, in which case necessarily $i+1=\sup M$) the implications are true with T_1' in place of T'. (Clearly (T_1', v, ε) satisfies the hypotheses of any of the above implications when (T', v, ε) does.) But $P(k, T)=P(k, T_1)$ for $k\in M$, $k\neq e$, in particular for $k\in\{i, i+1\}$, so we are done.

Henceforth assume $i+1=\sup M$. By Proposition 2.1.2 and Remark 2.1.12(1) it suffices to prove statements (b), (c), and (e). Now (b) is clear since $\rho^1(i+1, T)>\rho^2(k, T)$ for all $k\in M\backslash\{i+1\}$. Also, (c) is clear unless $\rho^1(i+1, T')=1$ (since if not then $\kappa^1(i, T)=\rho_1(T')+1>\kappa^2(k, T)$ for any $k\in M$), so assume $\rho^1(i+1, T')=1$. But in that case we see directly that $\rho^1(i+1, T)=\rho^2(i+1, T)=2$; on the other hand we know $\rho^2(i, T)=\rho^2(i, T)=1$, so done.

To prove (e) let $T_1'=(T'-(i+1))-i$, and set $(T_1, P_1)=\alpha((T_1', v, \varepsilon))$. We use the definition of the map α to compute $P(i, T)$ and $P(i+1, T)$ in the various situations which arise. First, we see that (e) is clear if any of the following three conditions holds:

(i) $P_1 \cap (P(i, T') \cup P(i+1, T')) = \varnothing$,
(ii) $P_1 \cap P(i+1, T') = \varnothing$ and $\rho^2(i, T) < \rho^1(i+1, T)-1$,
(iii) $P_1 \cap P(i, T') = \varnothing$ and either $P(i+1, T')$ is horizontal or $P_1 \neq P(i+1, T')$.

Suppose now that $P_1 \cap P(i, T') = \varnothing$ and $P_1 = P(i+1, T')$ is vertical. Then $\kappa^1(i, T') = \kappa^2(i, T')$ and $\kappa^1(i+1, T) = \kappa^2(i+1, T) = \kappa^2(i, T')+1$ so we are done unless $\kappa^1(i+1, T') = \kappa^1(i, T)$. But this latter contradicts our hypotheses.

We are reduced to the case $P_1 \cap P(i, T') \neq \varnothing$ and $\rho^2(i, T') = \rho^1(i+1, T')-1$. Assume first $P(i, T')$ is vertical. Then the result holds since then $\rho^2(i, T) \leqslant \rho^2(i, T')$, and, by our hypotheses (including that $\rho^2(i, T') < \rho^1(i+1, T')$ which implies $P_1 \cap P(i+1, T') = \varnothing$) we also see that $(P_1 \cup P(i, T)) \cap P(i+1, T') = \varnothing$.

So, finally, assume $P_1 \cap P(i, T') \neq \varnothing$ and $\rho^2(i, T') = \rho^1(i+1, T')-1$ and $P(i, T')$ is horizontal. There are two cases: firstly, that $P_1 \cap P(i, T') = \{S_{k,l}\}$ for some k, l. In this case, then we have that $P(i, T') = \{S_{k,l}, S_{k,l+1}\}$, and

$$P(i+1, T') = \{S_{k+1,l}, S_{k+1,l+1}\} \quad \text{or} \quad \{S_{k+1,l}, S_{k+2,l}\}.$$

Then we compute: $P(i, T) = \{S_{k,l+1}, S_{k+1,l+1}\}$ and $\rho^1(i+1, T) = \rho^2(i+1, T) = k+2$ so we are done with this case.

Secondly, we have the case that $P_1 = P(i, T')$. In this case, then we compute

$$\rho^1(i, T) = \rho^2(i, T) = \rho^1(i, T') + 1$$

and

$$\rho^1(i+1, T) = \rho^2(i+1, T) = \rho^1(i+1, T') + 1 = \rho^1(i, T') + 2. \qquad \square$$

2.1.14. PROPOSITION. *Suppose* $\{i-1, i, i+1\} \subseteq M$.

(a) *Suppose* $(T_1', i+1, 1) \in \mathscr{C}(M)$ *and* $\alpha_i \in \tau(T_1')$. *Let* $T_2' = \mathrm{Re}(i, i+1; T_1')$; *let* $(T_1, P_1) = \alpha((T_1', i+1, 1))$, *and let* $(T_2, P_2) = \alpha((T_2', i, 1))$. *Then* $T_2 = T_{\alpha_{i+1},\alpha_i}(T_1)$ (*in particular* $P_1 = P_2$).
(b) *Suppose* $(T_1', i+1, -1) \in \mathscr{C}(M)$ *and* $\alpha_i \notin \tau(T_1')$. *Let* $T_2' = \mathrm{Re}(i, i+1; T_1')$, *let* $(T_1, P_1) = \alpha((T_1', i+1, -1))$ *and let* $(T_2, P_2) = \alpha((T_1', i, -1))$. *Then* $T_2 = T_{\alpha_i,\alpha_{i+1}}(T_1)$ (*in particular* $P_1 = P_2$).
(c) *Suppose* $(T_1', i-1, 1) \in \mathscr{C}(M)$ *and* $\alpha_{i+1} \notin \tau(T_1')$. *Let* $T_2' = \mathrm{Re}(i, i-1; T_1')$, *let* $(T_1, P_1) = \alpha((T_1', i-1, 1))$, *and let* $(T_2, P_2) = \alpha((T_2', i, 1))$. *Then* $T_2 = T_{\alpha_{i+1},\alpha_i}(T_1)$ (*in particular* $P_1 = P_2$).
(d) *Suppose* $(T_1', i-1, -1) \in \mathscr{C}(M)$ *and* $\alpha_{i+1} \in \tau(T_1')$. *Let* $T_2' = \mathrm{Re}(i, i-1; T_1')$, *let* $(T_1, P_1) = \alpha((T_1', i-1, -1))$ *and let* $(T_2, P_2) = \alpha((T_2', i, -1))$. *Then* $T_2 = T_{\alpha_i,\alpha_{i+1}}(T_1)$ (*in particular* $P_1 = P_2$).

Proof. By Proposition 2.1.2 and Remark 2.1.12, it suffices to prove (a) and (c). By induction, we may assume that $i+1=\sup M$. By Proposition 2.1.13 it suffices to prove that T_2 is related to T_1 by one of the interchanges listed in the definition of $T_{\alpha\beta}$. We prove first (a). Now it is clear that $T_2=\mathrm{In}(i, i+1, T_1)$ unless $\rho^1(i, T_1')+1$, which latter would contradict $\alpha_i \in \tau(T_1')$, so done.

We next prove (c). If $\rho^1(i, T_1') > 1$ then it is clear that $T_2=\mathrm{In}(i-1, i; T_1)$, so assume $\rho^1(i, T_1')=1$. This assumption yields three possibilities; setting $r=\rho_1(T_1')$ they are:

(1) $P(i, T_1')=\{S_{1,r-3}, S_{1,r-2}\}$ and $P(i+1, T_1')=\{S_{1,r-1}, S_{1,r}\}$,
(2) $P(i, T_1')=\{S_{1,r-2}, S_{2,r-2}\}$ and $P(i+1, T_1')=\{S_{1,r-1}, S_{1,r}\}$,
(3) $P(i, T_1')=\{S_{1,r-1}, S_{2,r-1}\}$ and $P(i+1, T_1')=\{S_{1,r}, S_{2,r}\}$.

In each case we compute explicitly that the proposition is true. More precisely, we find that in case 1 we have $T_2=\mathrm{In}(i, i+1; T_1)$; in case 2, setting $T=(T_1'-(i+1))-i$, that $T_1=T \cup \tilde{F}_1(i-1; 1, r-2)$ and $T_2=T \cup F_1(i-1; 1, r-2)$; and in case 3, that, with T as in case 2, $T_1=T \cup \tilde{F}_1(i-1; 1, r-1)$ and $T_2=T \cup F_1(i-1; 1, r-1)$. □

2.1.15. PROPOSITION. *Suppose $(T_1', v, \varepsilon) \in \mathscr{C}(M)$ with $v \notin \{i-1, i, i+1\} \subseteq M$. Suppose further that $T_1' \in D_{\alpha_i, \alpha_{i+1}}(\mathscr{T}(M\backslash\{v\}))$ and let $T_2'=T_{\alpha_i, \alpha_{i+1}}(T)$. Let $(T_j, P_j)=\alpha((T_j', v, \varepsilon))$ for $j=1, 2$. Then $T_1 \in D_{\alpha_i, \alpha_{i+1}}(\mathscr{T}(M))$ and $T_2=T_{\alpha_i, \alpha_{i+1}}(T_1)$, and in particular $P_1=P_2$.*

Proof. As before, we may assume (by induction on $|M|$) that $i+1=\sup M$. Again, by Proposition 2.1.13 it suffices to check that T_2 is related to T_1 by one of the relevant interchanges. Let

$$\bar{T}'=((T_1'-(i+1))-i)-(i-1)=((T_2'-(i+1))-i)-(i-1).$$

Set $(\bar{T}, \bar{P})=\alpha((\bar{T}', v, \varepsilon))$.

The basic idea of this proof is as follows: in most cases $T_2'=\mathrm{In}(a, b; T_1')$ for $\{a, b\} \subseteq \{i-1, i, i+1\}$; we may take $b=a+1$. (The case where T_2' is obtained from T_1' by an F-type interchange is handled separately.) Now in many situations it will be true that, again, $T_2=\mathrm{In}(a, a+1; T_1)$; we refer here to the situations where $P(a, T_1) \cap P(a+1, T_1')=\varnothing$ and $P(a, T_2) \cap P(a+1, T_2')=\varnothing$. So in the proof we will first show under what hypotheses these two conditions hold, and then in the remaining cases we will calculate explicitly the positions of $i-1$, i, and $i+1$ in T_1 and T_2, and thus prove the proposition. There is also a symmetry involving transpose which we will exploit to reduce the number of cases we have to consider. This symmetry arises in that ${}^tT_2'$ satisfies the hypotheses on T_1', and that it suffices to prove the proposition with ${}^tT_2'$ in place of T_1'.

Now the proposition is obvious unless $\bar{P}$ intersects at least one of $P(i-1, T_1')$, $P(i, T_1')$, or $P(i+1, T_1')$, so assume that it does. There are three major cases.

(1) $T_2' = \mathrm{In}(i, i+1; T_1)$.

We will show that $T_2 = \mathrm{In}(i, i+1; T_1)$. By symmetry we may assume $P(i-1, T_1')$ (which equals $P(i-1, T_2')$) is horizontal, i.e. $\rho^1(i-1, T_1') = \rho^2(i-1, T_2')$. By the above it suffices to show that

$$P(i, T_1) \cap P(i+1, T_1') = \varnothing \tag{2.1.16}$$

and

$$P(i, T_2) \cap P(i+1, T_2') = \varnothing. \tag{2.1.17}$$

For (2.1.16), obviously we need only consider situations in which it is possible that $P(i, T_1) \neq P(i, T_1')$. Note first that since $\rho^1(i, T_1') \leqslant \rho^1(i-1, T_1')$, it follows from the definition of α that if $\bar{P} \cap P(i-1, T_1') \neq \varnothing$ then $(\bar{P} \cup P(i-1, T_1)) \cap P(i, T_1') = \varnothing$, and thus that $P(i, T_1) = P(i, T_1')$ and thus (2.1.16). So we are left with the possibility that $\bar{P} \cap P(i-1, T_1') = \varnothing$ and $\bar{P} \cap P(i, T_1') \neq \varnothing$. But in this case we must have $\rho^1(i, T_1') < \rho^1(i-1, T_1')$ (or else $\bar{P}$ would meet $P(i-1, T_1')$) and, since $\rho^1(i+1, T_1') > \rho^1(i-1, T_1')$, this implies (2.1.16).

For (2.1.17), note that we have $\kappa^2(i, T_2') < \kappa^1(i+1, T_2')$, so it suffices to exclude the possibility that $\kappa^1(i, T_2') = \kappa^2(i, T_2') = \kappa^1(i+1, T_2') - 1$. So assume this. Now, since $T_2' = \mathrm{In}(i, i+1; T_1')$ and since $\alpha_i \notin \tau(T_1')$ we have $\kappa^2(i-1, T_2') < \kappa^1(i+1, T_2')$. Thus $\kappa^2(i-1, T_2') \leqslant \kappa^1(i, T_2')$. Since we have assumed that $P(i-1, T_1') = P(i-1, T_2')$ is horizontal we have $\kappa^1(i-1, T_2') < \kappa^1(i, T_2')$. But this, together with the fact that, since $\alpha_i \in \tau(T_2')$, $\rho^2(i-1, T_2') < \rho^1(i, T_2')$, contradicts the satisfaction of condition (4) of Definition 1.1.8 by T_2'.

(2) $F_r(i-1; j, k) \subseteq T_1'$ for some j, k and $r \in \{1, 2\}$.

Then (since we assume that $\bar{P} \cap F_r(i-1; j, k) \neq \varnothing$) we have either $\bar{P} = \{S_{j,k}, S_{j,k+1}\}$ or $\bar{P} = \{S_{j,k}, S_{j+1,k}\}$. In each of the four situations that we have listed (i.e. either possibility for $\bar{P}$ and either $r=1$ or $r=2$) we compute directly that $T_2 = \mathrm{In}(i, i+1; T_1)$. (In fact up to transpose there are only two situations.)

(3) $T_2' = \mathrm{In}(i-1, i; T_1')$.

Since $\kappa^1(i, T_1') > \kappa^2(i-1, T_1')$, it is clear from the definition of α that $P(i-1, T_1) \cap P(i, T_1') = \varnothing$ unless $\kappa^1(i-1, T_1') = \kappa^2(i-1, T_1')$ (i.e. $P(i-1, T_1')$ is vertical) and $\bar{P} = P(i-1, T_1')$ and $\kappa^1(i, T_1') = \kappa^1(i-1, T_1') + 1$. Similarly, it is clear that $P(i-1, T_2) \cap P(i, T_2') = \varnothing$ unless $\rho^1(i-1, T_2') = \rho^2(i-1, T_2')$ and $\bar{P} = P(i-1, T_2')$ and $\rho^1(i, T_2') = \rho^1(i-1, T_2') + 1$. These situations are related by transpose (as discussed above) so it suffices to consider the situation in which the last-mentioned three conditions hold. Again there are two cases:

(a) $\rho^1(i, T_2') = \rho^2(i, T_2')$,

(b) $\rho^1(i, T_2') < \rho^2(i, T_2')$.

In case (a) our hypotheses imply that $D(i+1, T_2')$ is a horizontal domino in the same row as $D(i, T_2')$ and we then compute that $T_2 = \mathrm{In}(i, i+1; T_1)$; in case (b) our hypotheses imply that $\rho^1(i+1, T_2') = \rho^1(i, T_2')$; we then compute that $T_1 = \bar{T} \cup F_1(i-1; j, k)$ and $T_2 = \bar{T} \cup \tilde{F}_1(i-1; j, k)$ where $P(i, T_2') = \{S_{j,k}, S_{j+1,k}\}$. □

2.1.18. PROPOSITION. *Let* $\gamma \in \mathscr{S}(M_1, M_2)$.

(a) *Suppose* $1 \in M_1$. *Then* $\alpha_1 \in \tau^L(\gamma)$ *if and only if* $\alpha_1 \in \tau^L(A(\gamma)) = \tau(L(\gamma))$.
(b) *Suppose* $\{i-1, i\} \subseteq M_1$. *Then* $\alpha_i \in \tau^L(\gamma)$ *if and only if* $\alpha_i \in \tau^L(A(\gamma))$.
(c) *Suppose* $1 \in M_2$. *Then* $\alpha_1 \in \tau^R(\gamma)$ *if and only if* $\alpha_1 \in \tau^R(A(\gamma))$.
(d) *Suppose* $\{i-1, i\} \subseteq M_2$. *Then* $\alpha_i \in \tau^R(\gamma)$ *if and only if* $\alpha_i \in \tau^R(A(\gamma))$.

Proof. We note first that it suffices to prove (a) and (b). For, given (a) and (b), we have (assuming M_2 contains the appropriate numbers) $\alpha \in \tau^R(\gamma)$ if and only if $\alpha \in \tau^L(\gamma^{-1})$ if and only if $\alpha \in \tau^L(A(\gamma^{-1}))$ if and only if $\alpha \in \tau(L(\gamma^{-1}))$ if and only if $\alpha \in \tau(R(\gamma))$ which in turn is equivalent to $\alpha \in \tau^R(A(\gamma))$; where the next to last implication uses Proposition 1.2.3.

Note next that (a) follows directly from the definition of A (here we refer to Definition 1.2.7) and of the map α.

It remains to prove (b). We prove first that $\alpha_i \in \tau^L(\gamma)$ implies $\alpha_i \in \tau^L(A(\gamma))$. Assume first $i = \sup M_1$. Now by definition, $\alpha_i \in \tau^L(\gamma)$ implies that one of the following holds:

(i) $\{(i-1, b, 1), (i, a, \varepsilon)\} \subseteq \gamma$ with $a < b$,
(ii) $\{(i-1, a, \varepsilon), (i, b, -1)\} \subseteq \gamma$ with $a < b$.

If case (i) holds, the fact that $\alpha_i \in \tau(L(\gamma))$ follows directly from the definition of A (again we refer to Definition 1.2.7) and Proposition 2.1.13 part (c). If case (ii) holds then similarly, using Proposition 2.1.13(b), we see that $\alpha_i \in \tau(L(\gamma))$. If instead $i < \sup M_1$, then we use Proposition 2.1.13(e) and induction on $|M_1|$.

The implication $\alpha_i \notin \tau^L(\gamma)$ implies $\alpha_i \notin \tau^L(A(\gamma))$ is proved similarly, using Proposition 2.1.13(a), (d), (f). This completes the proof of (b), and thus of the proposition. □

We would like to prove that A and the $T_{\alpha\beta}$'s commute. When $\{\alpha, \beta\} = \{1, 2\}$ this is difficult (we have not even defined $T_{\alpha\beta}$ on tableaux in this case yet) and we must postpone it until Section 3. Meanwhile we can prove:

2.1.19. THEOREM. (a) *Suppose* $\{i-1, i, i+1\} \subseteq M_1$ *and* $\{\alpha, \beta\} = \{\alpha_i, \alpha_{i+1}\}$. *Suppose* $\gamma \in D^L_{\alpha\beta}(\mathscr{S}(M_1, M_2))$. *Then* $A(T^L_{\alpha\beta}(\gamma)) = T^L_{\alpha\beta}(A(\gamma))$.

(b) *Suppose* $\{i-1, i, i+1\} \subseteq M_2$, $\{\alpha, \beta\} = \{\alpha_i, \alpha_{i+1}\}$, *and* $\gamma \in D^R_{\alpha\beta}(\mathscr{S}(M_1, M_2))$. *Then* $A(T^R_{\alpha\beta}(\gamma)) = T^R_{\alpha\beta}(A(\gamma))$.

Proof. (a) Since $T^L_{\alpha\beta}$ and $T^L_{\beta\alpha}$ are inverses, we may assume $\alpha = \alpha_{i+1}$ and $\beta = \alpha_i$. Define for $j \in M_1$, k_j and ε_j by $(j, k_j, \varepsilon_j) \in \gamma$. Set $\bar{\gamma} = \gamma \backslash \{(j, k_j, \varepsilon_j) \mid j \in \{i-1, i, i+1\}\}$.

Let $d < e < f$ be such that $\{d, e, f\} = \{k_{i-1}, k_i, k_{i+1}\}$. Set $\gamma' = T^L_{\alpha\beta}(\gamma)$. Then one of the following eight cases holds:

(1) $\gamma \supseteq \{(i-1, e, 1), (i, d, \varepsilon), (i+1, f, 1)\}$ and

$$\gamma' = \bar{\gamma} \cup \{(i-1, e, 1), (i, f, 1), (i+1, d, \varepsilon)\}.$$

(2) $\gamma \supseteq \{(i-1, d, \varepsilon), (i, e, -1), (i+1, f, 1)\}$ and

$$\gamma' = \bar{\gamma} \cup \{(i-1, d, \varepsilon), (i, f, 1), (i+1, e, -1)\}.$$

(3) $\gamma \supseteq \{(i-1, f, 1), (i, d, \varepsilon), (i+1, e, 1)\}$ and

$$\gamma' = \bar{\gamma} \cup \{(i-1, d, \varepsilon), (i, f, 1), (i+1, e, 1)\}.$$

(4) $\gamma \supseteq \{(i-1, f, 1), (i, e, -1), (i+1, d, \varepsilon)\}$ and

$$\gamma' = \bar{\gamma} \cup \{(i-1, e, -1), (i, f, 1), (i+1, d, \varepsilon)\}.$$

Cases (5)–(8) are the situations obtained from cases (1)–(4) by replacing γ with $SC(\gamma')$. By Remarks 2.1.6(2), 2.1.7(2), and 2.1.12(2) we need only consider cases (1)–(4).

Assume first $i+1 = \sup M_1$. Then the proposition is a consequence of Propositions 2.1.13 and 2.1.14: more precisely, case (1) uses Proposition 2.1.13(c),(e) (applied to α_i) and Proposition 2.1.14(a). Case (2) uses Proposition 2.1.13(b),(e) (again, applied to α_i) and Proposition 2.1.14(a). Case (3) uses Proposition 2.1.13(a), (f) and Proposition 2.1.14(c). Case (4) uses Proposition 2.1.13(d), (f) and Proposition 2.1.14(c).

Assume next $i+1 \neq \sup M_1$. Then the proposition is easily seen to follow from Proposition 2.1.15 (using induction on $|M_1|$).

(b) This now follows easily from part (a) and Proposition 1.2.3: clearly $T^R_{\alpha\beta}(\gamma) = (T^L_{\alpha\beta}(\gamma^{-1}))^{-1}$. So

$$\begin{aligned} A(T^R_{\alpha\beta}(\gamma)) &= A((T^L_{\alpha\beta}(\gamma^{-1}))^{-1}) = (R(T^L_{\alpha\beta}(\gamma^{-1})), L(T^L_{\alpha\beta}(\gamma^{-1}))) \\ &= (R(\gamma^{-1}), T_{\alpha\beta}(L(\gamma^{-1}))) = T^R_{\alpha\beta}(R(\gamma^{-1}), L(\gamma^{-1})) \\ &= T^R_{\alpha\beta}(L(\gamma), R(\gamma)) = T^R_{\alpha\beta}(A(\gamma)). \qquad \square \end{aligned}$$

We now record two propositions which are analogous to Propositions 2.1.13, 2.1.14, and 2.1.15. They, as well as Propositions 2.1.13, 2.1.14, and 2.1.15, will be used in subsequent papers on annihilators of Harish-Chandra modules. (One could have proved these initially instead of Propositions 2.1.13, 2.1.14, and 2.1.15 and then derived Propositions 2.1.13, 2.1.14, and 2.1.15 from these.)

2.1.20. PROPOSITION. *Let* $T \in \mathscr{T}(M)$, $\{l_1, l_2\} \cap M = \varnothing$, $l_1 < l_2$.

(a) *Let* $(T_1, P_1) = \alpha((T, l_1, \varepsilon))$ *for some* $\varepsilon \in \{1, -1\}$, *and let* $(T_{12}, P_{12}) = \alpha((T_1, l_2, 1))$. *Then* $\kappa^2(P_1) < \kappa^1(P_{12})$.
(b) *Let* $(T'_{12}, P'_{12}) = \alpha((T_1, l_2, -1))$. *Then* $\rho^2(P_1) < \rho^1(P'_{12})$.
(c) *Let* $(T_2, P_2) = \alpha((T, l_2, 1))$ *and let* $(T_{21}, P_{21}) = \alpha((T_2, l_1, \varepsilon))$ *for some* $\varepsilon \in \{1, -1\}$. *Then* $\rho^2(P_2) < \rho^1(P_{21})$.
(d) *Let* $(T'_{21}, P'_{21}) = \alpha((T_2, l_1, \varepsilon))$ *for some* $\varepsilon \in \{1, -1\}$. *Then* $\kappa^2(P_2) < \kappa^1(P'_{21})$.

Proof. The proofs are similar; we prove (a). Let $\gamma = A^{-1}((T, T)) \in \mathscr{T}(M)$. Pick $i > \sup M$, and let $\gamma' = \gamma \cup \{(l_1, i, \varepsilon), (l_2, i+1, 1)\}$. Then $T_{12} = L(\gamma')$, $P_1 = P(i, R(\gamma'))$, and $P_{12} = P(i+1, R(\gamma'))$. Since $\alpha_{i+1} \notin \tau^R(\gamma')$, our proposition is a consequence of Proposition 2.1.18. □

For the next proposition, let $\alpha((T, (v, \varepsilon))) = \alpha((T, v, \varepsilon))$.

2.1.21. PROPOSITION. *Let* $T \in \mathscr{T}(M)$, *and let* $l_1 < l_2 < l_3$ *be such that* $\{l_1, l_2, l_3\} \cap M = \varnothing$. *Suppose one of the following:*

(i) $x_1 = (l_1, \varepsilon)$, $x_2 = (l_2, 1)$, $x_3 = (l_3, 1)$,
(ii) $x_1 = (l_2, -1)$, $x_2 = (l_1, \varepsilon)$, $x_3 = (l_3, 1)$,
(iii) $x_1 = (l_3, -1)$, $x_2 = (l_1, \varepsilon)$, $x_3 = (l_2, 1)$,
(iv) $x_1 = (l_3, -1)$, $x_2 = (l_2, -1)$, $x_3 = (l_1, \varepsilon)$.

Set $(T_i, P_i) = \alpha((T_i, x_i))$ *for* $i \in \{1, 2, 3\}$, $(T_{ij}, P_{ij}) = \alpha((T_i, x_j))$ *for* $j \in \{1, 2, 3\} \backslash \{i\}$, *and* $(T_{ijk}, P_{ijk}) = \alpha((T_{ij}, x_k))$ *for* $k \in \{1, 2, 3\} \backslash \{i, j\}$. *Then we have the following:*

(1) $T_{213} = T_{231}$ *(and both* $P_{21} = P_{231}$ *and* $P_{23} = P_{213}$*).*
(2) $T_{312} = T_{132}$ *and further,*
  (a) *if* $P_1 \cap P_3 = \varnothing$ *then* $P_{13} = P_3$ *and* $P_{31} = P_1$.
  (b) *If* $P_1 \cap P_3 = \{S_{jk}\}$ *then either*
    (i) $P_1 = \{S_{j,k}, S_{j+1,k}\}$, $P_{13} = \{S_{j,k+1}, S_{j,k+2}\}$, *and* $P_{132} = \{S_{j+1,k+1}, S_{j+2,k+2}\}$

  or

    (ii) $P_1 = \{S_{j,k}, S_{j+1,k}\}$, $P_{13} = \{S_{j,k+1}, S_{j+1,k+1}\}$ *and* $P_{132} = \{S_{j+2,k}, S_{j+2,k+1}\}$.
  (c) *If* $P_1 = P_3$ *then* $P_{13} = P_{312}$ *and* $P_{31} = P_{132}$.

Proof. The eight statements parallel the eight cases of the proof of Theorem 2.1.19. As an example we will prove (2) when x_1, x_2, x_3 satisfy condition (ii); the other seven statements are proved in an analogous fashion. Set $\gamma = A^{-1}((T, T))$ and pick $i-1 > \sup M$. Let

$$\gamma^1 = \gamma \cup \{(l_3, i-1, 1), (l_2, i, -1), (l_1, i+1, \varepsilon)\}$$

and

$$\gamma^2 = \gamma \cup \{(l_2, i-1, -1), (l_3, i, 1), (l_1, i+1, \varepsilon)\}.$$

Then $T_{312} = L(\gamma^1)$, $P_3 = P(i-1, R(\gamma^1))$, $P_{31} = P(i, R(\gamma^1))$, and $P_{312} = P(i+1, R(\gamma^1))$, and $T_{132} = L(\gamma^2)$, $P_1 = P(i-1, R(\gamma^2))$, $P_{13} = P(i, R(\gamma^2))$, and $P_{132} = P(i+1, R(\gamma^2))$. On the other hand, $\gamma^2 = T^R_{\alpha_{i+1}, \alpha_i}(\gamma^1)$, so the proposition follows from Theorem 2.1.19. □

Section 2

In [4] we introduced two basic objects: the map α and cycles. We now have to describe the relations between them. There are several results which will be needed either for Theorem 2.3.8 of Section 3 of this paper or for subsequent papers. We prove the first one in detail as a model for the others.

Throughout this section, we will be discussing tableaux with grid. The results we prove are valid only for elements of $\mathscr{T}_B(M)$, $\mathscr{T}_C(M)$, or $\mathscr{T}_D(M)$. We will indicate this in the hypotheses of theorems by writing e.g. $\mathbf{T} \in \mathscr{T}_K(M)$; the theorem is then true with K either B, C, or D.

2.2.1. NOTATION. (1) Let $T \in \mathscr{T}(M)$ or $\mathscr{T}^0(M)$, let ϕ be a grid, and let $\mathbf{T} = (T, \phi)$. Objects defined in relation to T will be extended in the obvious way to $\mathbf{T}$, e.g. $\text{Shape}(\mathbf{T}) = \text{Shape}(T)$, $\mathbf{T} - e = (T - e, \phi)$, ${}^t\mathbf{T} = ({}^tT, \phi)$, and $P(e, \mathbf{T}) = P(e, T)$.

We extend the definition of α to tableaux with grid: for $K = B$, C, or D write

$$\mathscr{C}_K(M) = \{(\mathbf{T}, v, \varepsilon) : \mathbf{T} \in \mathscr{T}_K(M \backslash \{v\}) \quad \text{and} \quad \varepsilon \in \{1, -1\}\}.$$

Similarly we have $\mathscr{D}_K(M)$, and we have as usual the bijection α from $\mathscr{C}_K(M)$ to $\mathscr{D}_K(M)$, with inverse β.

(2) If $|M| = 0$ and $\mathbf{T}$ is the unique element of $\mathscr{T}_B(M)$, respectively $\mathscr{T}_C(M)$, we introduce the convention that $\mathbf{T}$ has one open cycle, c, which is of course empty, and we say that $S_f(c) = S_b(c) = S_{1,1}$. We define $\mathbf{E}(\mathbf{T}, c)$ as in Definition 1.5.26(2), respectively, 1.5.26(3), so that $\mathbf{E}(\mathbf{T}, c) \in \mathscr{T}_C(M)$, respectively, $\mathbf{E}(\mathbf{T}, c) \in \mathscr{T}_B(M)$.

(3) We will modify the notation of Section 1.5.31 to replace the comma by a semicolon, that is, if U is a set of cycles in $\mathbf{T}$, $U = \{c_1, \ldots, c_n\}$, with $|U| = n$, then $\mathbf{E}(\mathbf{T}; U) = \mathbf{E}(\mathbf{T}, c_1, \ldots, c_n)$, and extend this definition to the case when $U = \varnothing$, that is, we put $\mathbf{E}(\mathbf{T}; \varnothing) = \mathbf{T}$.

Suppose $(\mathbf{T}', v, \varepsilon) \in \mathscr{C}_K(M)$ and $(\mathbf{T}, P) = \alpha((\mathbf{T}', v, \varepsilon))$. We need first to describe the relations between the cycles of $\mathbf{T}'$ and the cycles of $\mathbf{T}$. Now (as may be guessed) α does not preserve the cycles of $\mathbf{T}'$. But it does behave well in relation to what might be called the cycle structure of $\mathbf{T}'$. We will not define formally a notion of cycle structure, but loosely speaking, by cycle structure of a tableau $\mathbf{T} \in \mathcal{T}_K(M)$ we mean the list of which corners (Definition 1.5.5) of $\mathbf{T}$ are connected to which holes of $\mathbf{T}$ by cycles in $\mathbf{T}$. (A corner C and hole H of $\mathbf{T}$ are connected by a cycle c if either $C = S_b(c)$ and $H = S_f(c)$ or $C = S_f(c)$ and $H = S_b(c)$.)

2.2.2. DEFINITION. Let $\mathbf{T}_1 \in \mathcal{T}_K(M_1)$, $\mathbf{T}_2 \in \mathcal{T}_K(M_2)$ and suppose $U_1 \subseteq \mathrm{OC}(\mathbf{T}_1)$ and $U_2 \subseteq \mathrm{OC}(\mathbf{T}_2)$. A bijection $\mu: U_1 \to U_2$ is called a cycle structure preserving bijection (and abbreviated c.s.p.b.) if for all $c \in U_1$ we have $S_b(\mu(c)) = S_b(c)$ and $S_f(\mu(c)) = S_f(c)$.

(Of course, given U_1 and U_2, any such map is unique.)

2.2.3. THEOREM. *Let* $(\mathbf{T}', v, \varepsilon) \in \mathscr{C}_K(M)$ (*for* $K = B, C$, *or* D) *and let* $(\mathbf{T}, P) = \alpha((\mathbf{T}', v, \varepsilon))$. *Suppose* $P = \{S_{ij}, S_{i,j+1}\}$.

(1) *Suppose* $S_{i,j+1}$ *is* ϕ_K*-variable (so that* $S_{i,j+1}$ *is a corner or hole of* $\mathbf{T}$).

  (a) *Suppose* $S_{i,j-1}$ *is a corner or hole of* $\mathbf{T}'$, *i.e.* $j > 1$ *and* $S_{i+1,j-1} \notin \mathrm{Shape}(\mathbf{T}')$. *Then there exists* $c' \in \mathrm{OC}(\mathbf{T}')$ *and* $c \in \mathrm{OC}(\mathbf{T})$ *such that* $S_b(c') = S_{i,j-1}$, $S_b(c) = S_{i,j+1}$, *and* $S_f(c) = S_f(c')$, *and furthermore such that there is a c.s.p.b.* $\mu: \mathrm{OC}(\mathbf{T}') \setminus \{c'\} \to \mathrm{OC}(\mathbf{T}) \setminus \{c\}$.

  (b) *Suppose* $S_{i,j-1}$ *is not a corner or hole of* $\mathbf{T}'$, *i.e.* $j = 1$ *or* $S_{i+1,j-1} \in \mathrm{Shape}(\mathbf{T}')$. (*Then* $S_{i+1,j}$ *is an empty hole or corner of* $\mathbf{T}$ *which is not a corner or hole of* $\mathbf{T}'$.) *There are two possibilities:*

    (i) *There is a cycle* $c \in \mathrm{OC}(\mathbf{T})$ *such that* $S_b(c) = S_{i,j+1}$ *and* $S_f(c) = S_{i+1,j}$ *and such that there is a c.s.p.b.* $\mu: \mathrm{OC}(\mathbf{T}') \to \mathrm{OC}(\mathbf{T}) \setminus \{c\}$.

    (ii) *There is a cycle* $c' \in \mathrm{OC}(\mathbf{T}')$ *and cycles* $c_1, c_2 \in \mathrm{OC}(\mathbf{T})$ *such that* $S_f(c_1) = S_f(c')$, $S_b(c_1) = S_{i,j+1}$, $S_f(c_2) = S_{i+1,j}$, $S_b(c_2) = S_b(c')$, *and such that there is a c.s.p.b.* $\mu: \mathrm{OC}(\mathbf{T}') \setminus \{c'\} \to \mathrm{OC}(\mathbf{T}) \setminus \{c_1, c_2\}$.

(2) *Suppose* $S_{i,j+1}$ *is* ϕ_K*-fixed (so* S_{ij} *is an empty corner or hole of* $\mathbf{T}'$).

  (a) *Suppose* $S_{i,j+2}$ *is a corner or hole of* $\mathbf{T}$ (*i.e.* $i = 1$ *or* $S_{i-1,j+2} \in \mathrm{Shape}(T)$). *Then there exists a* $c' \in \mathrm{OC}(\mathbf{T}')$ *and* $c \in \mathrm{OC}(\mathbf{T})$ *such that* $S_f(c') = S_{ij}$, $S_f(c) = S_{i,j+2}$, *and* $S_b(c) = S_b(c')$, *and such that there is a c.s.p.b.* $\mu: \mathrm{OC}(\mathbf{T}') \setminus \{c\} \to \mathrm{OC}(\mathbf{T}) \setminus \{c\}$.

  (b) *Suppose* $S_{i,j+2}$ *is not a corner or hole of* $\mathbf{T}$ (*i.e.* $i > 1$ *and* $S_{i-1,j+2} \notin \mathrm{Shape}(T)$). (*Then* $S_{i-1,j+1}$ *is a corner or hole of* $\mathbf{T}'$ *which is not a corner or hole of* $\mathbf{T}$.) *Then there is a* $u \in M$, *with* $u > v$, *and such that* $\{u\} = c' \in \mathrm{OC}(\mathbf{T}')$, *with* $S_f(c') = S_{ij}$ *and* $S_b(c') = S_{i-1,j+1}$, *and such that there is a c.s.p.b.* $\mu: \mathrm{OC}(\mathbf{T}') \setminus \{c'\} \to \mathrm{OC}(\mathbf{T})$.

If instead $P = \{S_{j,i}, S_{j+1,i}\}$ *we have the obvious transposed statements of the above.*

To prove this theorem we will make use of the following observation:

2.2.4. PROPOSITION. *Let* $\mathbf{T} \in \mathcal{T}_K(M)$. *Let* $e = \sup M$ *and let* $\bar{\mathbf{T}} = \mathbf{T} - e$. *Suppose* $P(e, \mathbf{T}) = \{S_{ij}, S_{i,j+1}\}$.

(1) *Suppose* $S_{i,j+1}$ *is* ϕ_K*-variable.*

(a) *Suppose* $S_{i,j-1}$ *is a corner or hole in* $\bar{\mathbf{T}}$. *Let* $\bar{c} \in \mathrm{OC}(\bar{\mathbf{T}})$ *be such that* $S_b(\bar{c}) = S_{i,j-1}$. *Then* $c = \bar{c} \cup \{e\} \in \mathrm{OC}(\mathbf{T})$, $S_f(\bar{c}) = S_f(c)$, $S_b(c) = S_{i,j+1}$ *and* $\mathrm{OC}(\bar{\mathbf{T}}) \backslash \{\bar{c}\} = \mathrm{OC}(\mathbf{T}) \backslash \{c\}$ *and this equality is a c.s.p.b.*

(b) *Suppose* $S_{i,j-1}$ *is not a corner or hole in* $\bar{\mathbf{T}}$. *Then* $c = \{e\} \in \mathrm{OC}(\mathbf{T})$, $S_b(c) = S_{i,j+1}$, $S_f(c) = S_{i+1,j}$, *and* $\mathrm{id} : \mathrm{OC}(\bar{\mathbf{T}}) \to \mathrm{OC}(\mathbf{T}) \backslash \{c\}$ *is a c.s.p.b.*

(2) *Suppose* S_{ij+1} *is* ϕ_K*-fixed.*

(a) *Suppose* $S_{i,j+2}$ *is a corner or hole in* $\mathbf{T}$. *Let* $\bar{c} \in \mathrm{OC}(\bar{\mathbf{T}})$ *be such that* $S_f(\bar{c}) = S_{ij}$. *Then* $c = \bar{c} \cup \{e\} \in \mathrm{OC}(\mathbf{T})$, $S_f(c) = S_{i,j+2}$, $S_b(c) = S_b(\bar{c})$ *and* $\mathrm{id} : \mathrm{OC}(\bar{\mathbf{T}}) \backslash \{\bar{c}\} \to \mathrm{OC}(\mathbf{T}) \backslash \{c\}$ *is a c.s.p.b.*

(b) *Suppose* $S_{i,j+2}$ *is not a corner or hole in* $\mathbf{T}$. *There are two possibilities.*

(i) *There exists* $\bar{c} \in \mathrm{OC}(\bar{\mathbf{T}})$ *with* $S_b(\bar{c}) = S_{i-1,j+1}$ *and* $S_f(\bar{c}) = S_{ij}$. *Then* $\bar{c} \cup \{e\}$ *is a closed cycle in* $\mathbf{T}$ *and* $\mathrm{OC}(\mathbf{T}) = \mathrm{OC}(\bar{\mathbf{T}}) \backslash \{\bar{c}\}$ *(with the identity map a c.s.p.b.).*

(ii) *There exists* $\bar{c}_1, \bar{c}_2 \in \mathrm{OC}(\bar{\mathbf{T}})$ *with* $\bar{c}_1 \neq \bar{c}_2$ *such that* $S_b(\bar{c}_1) = S_{i-1,j+1}$ *and* $S_f(\bar{c}_2) = S_{ij}$. *Let* $c = \bar{c}_1 \cup \bar{c}_2 \cup \{e\}$. *Then* $c \in \mathrm{OC}(\mathbf{T})$, $S_b(c) = S_b(\bar{c}_2)$, $S_f(c) = S_f(\bar{c}_1)$, *and* $\mathrm{OC}(\mathbf{T}) \backslash \{c\} = \mathrm{OC}(\bar{\mathbf{T}}) \backslash \{\bar{c}_1, \bar{c}_2\}$ *(and the identity is a c.s.p.b.).*

Again, we have the corresponding transposed statements.

Proof. It suffices to note that $P'(e, \mathbf{T})$ is determined by Shape ($\mathbf{T}$) (and how it is determined by Shape($\mathbf{T}$)), and that if $u \in M$ and $u \neq e$ then $P'(u, \mathbf{T}) = P'(u, \bar{\mathbf{T}})$. The rest follows easily. □

Proof of Theorem 2.2.3. Let $e = \sup M$. Assume first $v = e$. Then the theorem is reduced to Proposition 2.2.4 (note that since $i = 1$, P cannot satisfy the hypothesis of 2(b), and that if P satisfies the hypothesis of 1(b), then it satisfies the conclusion of 1(b)(i) with $c = \{e\}$).

Henceforth assume $v \neq e$. Let $\bar{\mathbf{T}}' = \mathbf{T}' - e$ and set $(\bar{\mathbf{T}}, \bar{P}) = \alpha((\bar{\mathbf{T}}', v, \varepsilon))$ (so $\bar{\mathbf{T}} = \mathbf{T} - e$ by definition of α). We will assume by induction on $|M|$ that the theorem is true for $(\bar{\mathbf{T}}', v, \varepsilon)$ (since if $|M| = 1$ then $v = e$ and we have already proved the theorem in this case). Write $P'_e = P(e, \mathbf{T}')$ and $P_e = P(e, \mathbf{T})$. If P and $\mathbf{T}'$ (respectively, $\bar{P}$ and $\bar{\mathbf{T}}'$) satisfy the hypotheses of 1(a) of the theorem we will say that P (respectively, $\bar{P}$) is in situation 1(a), and similarly for the other parts of the theorem, and also similarly for P_e and P'_e in relation to the various parts of Proposition 2.2.4.

We will prove the theorem by treating in turn the various cases which can arise, where each case is specified by the situation of $\bar{P}$ and of P'_e and by the location of P'_e relative to $\bar{P}$. (The situation of P of course will be determined by this.) Many cases are basically trivial; we will treat the more interesting cases in detail. By symmetry (i.e. transpose) we may assume $\bar{P}$ is horizontal, so henceforth assume $\bar{P} = \{S_{ij}, S_{i,j+1}\}$.

The point of this proof is that from Proposition 2.2.4 we have relations between the cycle structures of $\bar{\mathbf{T}}'$ and $\mathbf{T}'$, and between the cycle structures of $\bar{\mathbf{T}}$

and **T**, and by induction we relate the cycle structure of $\bar{\mathbf{T}}'$ to that of $\bar{\mathbf{T}}$. Putting these three sets of information together gives the desired relation between the cycle structures of **T**′ and of **T**. To do this we have to list the various cases and then carry out this program in each case.

Case A. $\bar{P}$ is in situation 1(a) and $P'_e = \bar{P}$ (so P'_e is also in situation (1a) and Shape($\mathbf{T}'$) = Shape($\bar{\mathbf{T}}$)). Let $\bar{c}' \in \mathrm{OC}(\bar{\mathbf{T}}')$ be such that $S_b(\bar{c}') = S_{i,j-1}$. Let $\bar{c} \in \mathrm{OC}(\bar{\mathbf{T}})$ and $c' \in \mathrm{OC}(\mathbf{T}')$ be such that $S_f(\bar{c}) = S_f(\bar{c}') = S_f(c')$. Then by induction and Proposition 2.2.4 we have $S_b(\bar{c}) = S_{i,j+1} = S_b(c')$. Also, we have c.s.p.b.'s

$$\mathrm{OC}(\mathbf{T}')\backslash\{c'\} \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}}')\backslash\{\bar{c}'\} \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}})\backslash\{\bar{c}\}.$$

Since $S_b(c') = S_b(\bar{c})$ and $S_f(c') = S_f(\bar{c})$ the above composition of c.s.p.b.'s extends to a c.s.p.b. $\mathrm{OC}(\mathbf{T}') \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}})$. Now $P = P_e = \{S_{i+1,r+1}, S_{i+1,r+2}\}$ where $r = \rho_{i+1}(\mathbf{T}')$, and so P lies in one of the situations 1(a), 1(b), or 2(a) (since by hypothesis $\rho_{i+1}(\mathbf{T}') < \rho_i(\mathbf{T}') - 2$). The relation which we want between OC(**T**) and OC(**T**′) is, by Proposition 2.2.4(1a), (1b), or (2a), respectively, holding as a relation between OC(**T**) and OC($\bar{\mathbf{T}}$). Since, as noted above, we have a c.s.p.b. between OC(**T**′) and OC($\bar{\mathbf{T}}$), we are done.

Case B. Here, $\bar{P}$ is in situation 1(a) and $P'_e = \{S_{i+1,j-1}, S_{i+2,j-1}\}$ (so P'_e is in situation 1(a)). Then $P = \bar{P}$, $P_e = P'_e$, and both P and P_e are in situation 1(b). Let $\bar{c}' \in \mathrm{OC}(\bar{\mathbf{T}}')$ be such that $S_b(\bar{c}') = S_{i,j-1}$. Let $\bar{c} \in \mathrm{OC}(\bar{\mathbf{T}})$ and $c' \in \mathrm{OC}(\mathbf{T}')$ be such that $S_f(\bar{c}) = S_f(\bar{c}') = S_f(c')$ (so $c' = \bar{c}' \cup \{e\}$). Then by induction and Proposition 2.2.4, $S_b(\bar{c}) = S_{i,j+1}$ and $S_b(c') = S_{i+2,j-1}$. Let $c_1 = \bar{c}$ and $c_2 = \{e\}$, then by Proposition 2.2.4, $c_1, c_2 \in \mathrm{OC}(\mathbf{T})$, $S_b(c_1) = S_b(\bar{c})$, $S_f(c_1) = S_f(\bar{c})$, $S_b(c_2) = S_{i+2,j-1}$ and $S_f(c_2) = S_{i+1,j}$. Also by induction and Proposition 2.2.4, we have c.s.p.b.'s

$$\mathrm{OC}(\mathbf{T}')\backslash\{c'\} \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}}')\backslash\{\bar{c}'\} \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}})\backslash\{\bar{c}\} \leftrightarrow \mathrm{OC}(\mathbf{T})\backslash\{c_1, c_2\}.$$

Since $S_b(c') = S_{i+2,j-1} = S_b(c_2)$, $S_f(c') = S_f(\bar{c}') = S_f(\bar{c}) = S_f(c_1)$, $S_f(c_2) = S_{i+1,j}$, and $S_b(c_1) = S_b(\bar{c}) = S_{i,j+1}$ we see that P verifies the conclusion of situation 1(b)(ii).

Case C. Here, $\bar{P}$ is in situation 1(a), $P'_e = \{S_{i+1,j-2}, S_{i+1,j-1}\}$, and P'_e is in situation (2b)(i). (That is there is a $\bar{c}' \in \mathrm{OC}(\bar{\mathbf{T}}')$ such that $S_b(\bar{c}') = S_{i,j-1}$ and $S_f(\bar{c}') = S_{i+1,j-2}$.) Then $P = \bar{P}$, $P_e = P'_e$, P is in situation 1(b), and P_e is in situation 2(a). Also, $\bar{c}' \cup \{e\}$ is a closed cycle in **T**′. Let $\bar{c} \in \mathrm{OC}(\bar{\mathbf{T}})$ be such that $S_f(\bar{c}) = S_f(\bar{c}')$. Then by induction $S_b(\bar{c}) = S_{i,j+1}$. Let $c \in \mathrm{OC}(\mathbf{T})$ be such that $S_b(c) = S_b(\bar{c})$. Then by Proposition 2.2.4, $S_f(c) = S_{i+1,j}$. By induction and Proposition 2.2.4 we have c.s.p.b.'s

$$\mathrm{OC}(\mathbf{T}') \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}}')\backslash\{\bar{c}'\} \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}})\backslash\{\bar{c}\} \leftrightarrow \mathrm{OC}(\mathbf{T})\backslash\{c\}.$$

Since $S_f(c) = S_{i+1,j}$ and $S_b(c) = S_b(\bar{c}) = S_{i,j+1}$, we see that P verifies the conclusion of situation 1(b)(i).

Case D. Here $\bar{P}$ is in situation 1(a), $P'_e=\{S_{i+1,j-2}, S_{i+1,j-1}\}$, and P'_e is in situation 2(b)(ii), i.e., there are cycles $\bar{c}'_1$, $\bar{c}'_2\in\mathrm{OC}(\bar{\mathbf{T}}')$ with $\bar{c}'_1\neq\bar{c}'_2$ such that $S_b(\bar{c}'_1)=S_{i,j-1}$ and $S_f(\bar{c}'_2)=S_{i+1,j-2}$. Then $P=\bar{P}$, $P_e=P'_e$, P is in situation 1(b), and P_e is in situation 2(a). Let $c'=\bar{c}'_1\cup\bar{c}'_2\cup\{e\}$, then $c'\in\mathrm{OC}(\bar{\mathbf{T}}')$, $S_b(c')=S_b(\bar{c}'_2)$, and $S_f(c')=S_f(\bar{c}'_1)$. Let $\bar{c}_1$, $\bar{c}_2\in\mathrm{OC}(\bar{\mathbf{T}})$ be such that $S_f(\bar{c}_1)=S_f(\bar{c}'_1)$ and $S_b(\bar{c}_2)=S_b(\bar{c}'_2)$. Then by induction $S_b(\bar{c}_1)=S_{i,j+1}$ and $S_f(\bar{c}_2)=S_f(\bar{c}'_2)$. Let c_1, $c_2\in\mathrm{OC}(\mathbf{T})$ be such that $S_b(c_2)=S_b(\bar{c}_2)$ and $S_f(c_1)=S_f(\bar{c}_1)$. Then by Proposition 2.2.4, $S_b(c_1)=S_b(\bar{c}_1)$ and $S_f(c_2)=S_{i+1,j}$. By induction and Proposition 2.2.4, we have c.s.p.b.'s

$$\mathrm{OC}(\mathbf{T}')\backslash\{c'\}\leftrightarrow\mathrm{OC}(\bar{\mathbf{T}}')\backslash\{\bar{c}'_1,\bar{c}'_2\}\leftrightarrow\mathrm{OC}(\bar{\mathbf{T}})\backslash\{\bar{c}_1,\bar{c}_2\}\leftrightarrow\mathrm{OC}(\mathbf{T})\backslash\{c_1,c_2\}.$$

Since $S_b(c')=S_b(\bar{c}'_2)=S_b(\bar{c}_2)=S_b(c_2)$, $S_f(c')=S_f(\bar{c}'_1)=S_f(\bar{c}_1)=S_f(c_1)$, $S_f(c_2)=S_{i+1,j}$ and $S_b(c_1)=S_b(\bar{c}_1)=S_{i,j+1}$, we see that P verifies the conclusion of situation 1(b)(ii).

Case E. Here, $\bar{P}$ is in situation 1(a) and P_e does not satisfy the hypotheses of any of the previous cases. Then $P=\bar{P}$, $P_e=P'_e$, and P is also in situation 1(a). One now checks (tediously but trivially) using induction and Proposition 2.2.4 that P satisfies the conclusions of situation 1(a).

Case F. Here $\bar{P}$ is in situation 1(b)(i) and $P'_e=P$ (so P'_e is in situation 1(b)). Let $\bar{c}\in\mathrm{OC}(\bar{\mathbf{T}})$ be such that $S_b(\bar{c})=S_{i,j+1}$ and $S_f(\bar{c})=S_{i+1,j}$. We have $P=P_e=\{S_{i+1,j}, S_{i+1,j+1}\}$ so P is in situation 2(b) and we see that P_e is in situation 2(b)(i) (with $\bar{c}$ as in the statement of that situation). Then, setting $c'=\{e\}\in\mathrm{OC}(\mathbf{T}')$, we have c.s.p.b.'s

$$\mathrm{OC}(\mathbf{T}')\backslash\{c'\}\leftrightarrow\mathrm{OC}(\bar{\mathbf{T}}')\leftrightarrow\mathrm{OC}(\bar{\mathbf{T}})\backslash\{\bar{c}\}\leftrightarrow\mathrm{OC}(\mathbf{T}),$$

which yields the conclusion of situation 2(b) with $u=e$.

Case G. Here $\bar{P}$ is in situation 1(b)(ii) and $P'_e=\bar{P}$ (so P'_e is in situation 1(b)). Then $P=P_e=\{S_{i+1,j}, S_{i+1,j+1}\}$. Let $\bar{c}'\in\mathrm{OC}(\bar{\mathbf{T}}')$ and $\bar{c}_1$, $\bar{c}_2\in\mathrm{OC}(\bar{\mathbf{T}})$ be such that $S_f(\bar{c}_1)=S_f(\bar{c}')$, $S_b(\bar{c}_1)=S_{i,j+1}$, $S_f(\bar{c}_2)=S_{i+1,j}$ and $S_b(\bar{c}_2)=S_b(\bar{c}')$. Now P is in situation (2b) and P_e is in situation 2(b)(ii), with $\bar{c}_1$ and $\bar{c}_2$ as in the statement of that situation. Let $c'=\{e\}$ and $\tilde{c}'=\bar{c}'$, then, by Proposition 2.2.4, c', $\tilde{c}'\in\mathrm{OC}(\mathbf{T}')$, $S_b(\tilde{c}')=S_b(\bar{c}')$, $S_f(\tilde{c}')=S_f(\bar{c}')$, $S_b(c')=S_{i,j+1}$ and $S_f(c')=S_{i+1,j}$. Let $c=\bar{c}_1\cup\bar{c}_2\cup\{e\}$, then $c\in\mathrm{OC}(\mathbf{T})$, $S_b(c)=S_b(\bar{c}_2)$, and $S_f(c)=S_f(\bar{c}_1)$. We have c.s.p.b.'s

$$\mathrm{OC}(\mathbf{T}')\backslash\{c',\tilde{c}'\}\leftrightarrow\mathrm{OC}(\bar{\mathbf{T}}')\backslash\{\bar{c}'\}\leftrightarrow\mathrm{OC}(\bar{\mathbf{T}})\backslash\{\bar{c}_1,\bar{c}_2\}\leftrightarrow\mathrm{OC}(\mathbf{T})\backslash\{c\}.$$

Since $S_b(c)=S_b(\bar{c}_2)=S_b(\bar{c}')=S_b(\tilde{c}')$ and $S_f(c)=S_f(\bar{c}_1)=S_f(\bar{c}')=S_f(\tilde{c}')$ we can extend the above-given composite of c.s.p.b.'s to a c.s.p.b. $\mathrm{OC}(\mathbf{T}')\backslash\{c'\}\leftrightarrow\mathrm{OC}(\mathbf{T})$. We have thus the conclusion of situation 2(b) with $u=e$.

Case H. Here, $\bar{P}$ is in situation 1(b)(i) and $P'_e=\{S_{ij}, S_{i+1,j}\}$. Then

$P=\{S_{i,j+1}, S_{i+1,j+1}\}$ is in situation 2(b) and $P_e=\{S_{i+1,j}, S_{i+1,j+1}\}$ is in situation 2(b)(i). This case is entirely similar to Case F.

Case I. Here $\bar{P}$ is in situation 1(b)(ii) and $P'_e=\{S_{ij}, S_{i+1,j}\}$. This case is entirely similar to case G.

Case J. Here, $\bar{P}$ is in situation 1(b)(i) or (ii) and P'_e does not satisfy the hypotheses of any of the previous four cases (that is, $\bar{P}\cap P'_e=\varnothing$). Then $P=\bar{P}$ is in the same situation as $\bar{P}$ (with one exception), and one verifies the appropriate conclusion using induction and Proposition 2.2.4. We discuss the exception as an example. Suppose $\bar{P}$ is in situation 1(b)(ii). Let $\bar{c}_1, \bar{c}_2\in \mathrm{OC}(\bar{\mathbf{T}})$ be such that $S_b(\bar{c}_1)=S_{i,j+1}$ and $S_f(\bar{c}_2)=S_{i+1,j}$. Let $\bar{c}'\in\mathrm{OC}(\bar{\mathbf{T}}')$ be such that $S_f(\bar{c}')=S_f(\bar{c}_1)$ and $S_b(\bar{c}')=S_b(\bar{c}_2)$. Suppose further that for some k and l we have $S_f(\bar{c}_1)=S_{k,l}$ (respectively $S_{l,k}$), $S_b(\bar{c}_2)=S_{k-1,l+1}$ (respectively $S_{l+1,k-1}$), and $P'_e=\{S_{k,l}, S_{k,l+1}\}$ (respectively $\{S_{l,k}, S_{l+1,k}\}$). Then $S_f(\bar{c}')=S_{k,l}$ (respectively $S_{l,k}$) and $S_b(\bar{c}')=S_{k-1,l+1}$ (respectively $S_{l+1,k-1}$), so P'_e is in situation 2(b)(i) with $\bar{c}'$ the distinguished cycle of that situation (so $c'=\bar{c}'\cup\{e\}$ is a closed cycle in $\mathbf{T}'$). Then we see that P_e is in situation 2(b)(ii) with $\bar{c}_1$ and $\bar{c}_2$ the distinguished cycles of that situation and $c=\bar{c}_1\cup\bar{c}_2\cup\{e\}$ the corresponding open cycle in $\mathbf{T}$, that is $S_b(c)=S_b(\bar{c}_1)$ and $S_f(c)=S_f(\bar{c}_2)$. Also, we have c.s.p.b.'s

$$\mathrm{OC}(\mathbf{T}')\leftrightarrow\mathrm{OC}(\bar{\mathbf{T}}')\backslash\{\bar{c}'\}\leftrightarrow\mathrm{OC}(\bar{\mathbf{T}})\backslash\{\bar{c}_1,\bar{c}_2\}\leftrightarrow\mathrm{OC}(\mathbf{T})\backslash\{c\}.$$

Since $S_b(c)=S_b(\bar{c}_1)=S_{i,j+1}$ and $S_f(c)=S_f(\bar{c}_2)=S_{i+1,j}$, we see that P verifies the conclusion of situation 1(b)(i).

Case K. Here $\bar{P}$ is in situation 2(a) and $P'_e=\bar{P}$ (so P'_e is also in situation 2(a)). This is entirely similar to case A (again, our hypotheses rule out the possibility that P is in situation 2(b)).

Case L. Here $\bar{P}$ is in situation 2(a), $P'_e=\{S_{ij}, S_{i+1,j}\}$, and P'_e is in situation 2(a) (i.e. $j=1$ or $S_{i+2,j-1}\in\mathrm{Shape}(\mathbf{T}')$). This case follows the pattern of case B (let $\bar{c}'\in\mathrm{OC}(\bar{\mathbf{T}}')$ be such that $S_f(\bar{c}')=S_{ij},\ldots$) and P satisfies situation 1(b)(ii).

Cases M and N follow the patterns of Cases C and D, respectively.

Case M. Here $\bar{P}$ is in situation 2(a), $P'_e=\{S_{ij}, S_{i+1,j}\}$ and P'_e is in situation 2(b)(i) (so $j>1$ and $S_{i+2,j-1}\notin\mathrm{Shape}(\mathbf{T}')$). Then P satisfies situation 1(b)(i).

Case N. Hypotheses as in Case M, except that P'_e is in situation 2(b)(ii). Then P satisfies situation 1(b)(ii).

Case O. Here $\bar{P}$ is as in situation 2(a) and P'_e does not satisfy the hypotheses of any of the previous four cases. This is as Case E.

Case P. Here $\bar{P}$ is in situation 2(b) and $P'_e=\bar{P}$. Then since by induction there is a $u\in M\backslash\{v,e\}$ such that $\{u\}=\bar{c}'\in\mathrm{OC}(\bar{\mathbf{T}}')$ with $S_f(\bar{c}')=S_{ij}$ and $S_b(\bar{c}')=S_{i-1,j+1}$, we see that P'_e is in situation 2(b)(i) with $\bar{c}'$ the distinguished cycle of this case. We have c.s.p.b.'s

$$\mathrm{OC}(\mathbf{T}')\leftrightarrow\mathrm{OC}(\bar{\mathbf{T}}')\backslash\{\bar{c}'\}\leftrightarrow\mathrm{OC}(\bar{\mathbf{T}}),$$

and thus a c.s.p.b. $\mathrm{OC}(\mathbf{T}') \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}})$. The rest is as in cases A and K.

Case Q. Here $\bar{P}$ is in situation 2(b), $P'_e = \{S_{ij}, S_{i+1,j}\}$, and P'_e is in situation 2(a). Then $P = \{S_{i,j+1}, S_{i+1,j+1}\}$ is in situation 1(a) and $P_e = \{S_{i+1,j}, S_{i+1,j+1}\}$ is in situation 1(b). By induction we have $\{u\} = \bar{c}' \in \mathrm{OC}(\bar{\mathbf{T}}')$ with $S_b(\bar{c}') = S_{i-1,j+1}$ and $S_f(\bar{c}') = S_{ij}$. By Proposition 2.2.4 we have $\bar{c}' \cup \{e\} = c' \in \mathrm{OC}(\mathbf{T}')$ with $S_b(c') = S_b(\bar{c}')$ and $S_f(c') = S_{i+2,j}$, and also $\{e\} = c \in \mathrm{OC}(\mathbf{T})$ with $S_b(c) = S_{i+1,j+1}$ and $S_f(c) = S_{i+2,j}$. Then we have c.s.p.b.'s

$$\mathrm{OC}(\mathbf{T}')\backslash\{c'\} \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}}')\backslash\{\bar{c}'\} \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}}) \leftrightarrow \mathrm{OC}(\mathbf{T})\backslash\{c\}.$$

Since $S_f(c') = S_{i+2,j} = S_f(c)$, $S_b(c') = S_b(\bar{c}') = S_{i-1,j+1}$, and $S_b(c) = S_{i+1,j+1}$, we see that P verifies the conclusion of situation 1(a).

Case R. Here $\bar{P}$ is in situation 2(b), $P'_e = \{S_{ij}, S_{i+1,j}\}$, and P'_e is in situation 2(b). Then $P = \{S_{i,j+1}, S_{i+1,j+1}\}$ and $P_e = \{S_{i+1,j}, S_{i+1,j+1}\}$ are in situation 1(a). By induction we have $\{u\} = \bar{c}'_2 \in \mathrm{OC}(\bar{\mathbf{T}}')$ such that $S_b(\bar{c}'_2) = S_{i-1,j+1}$ and $S_f(\bar{c}'_2) = S_{ij}$. Let $\bar{c}'_1 \in \mathrm{OC}(\bar{\mathbf{T}}')$ be such that $S_b(\bar{c}'_1) = S_{i+1,j-1}$. We see then that P'_e is in situation 2(b)(ii) (with $\bar{c}'_1$ and $\bar{c}'_2$ the distinguished cycles of that situation). Let $\bar{c}'_1 \cup \bar{c}'_2 \cup \{e\} = c' \in \mathrm{OC}(\mathbf{T}')$, so $S_b(c') = S_b(\bar{c}'_2)$ and $S_f(c') = S_f(\bar{c}'_1)$. Let $\bar{c} \in \mathrm{OC}(\bar{\mathbf{T}})$ be such that $S_b(\bar{c}) = S_b(\bar{c}'_1)$; then $S_f(\bar{c}) = S_f(\bar{c}'_1)$. Let $\bar{c} \cup \{e\} = c \in \mathrm{OC}(\mathbf{T})$, so $S_f(c) = S_f(\bar{c})$ and $S_b(c) = S_{i+1,j+1}$. We have c.s.p.b.'s

$$\mathrm{OC}(\mathbf{T}')\backslash\{c'\} \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}}')\backslash\{\bar{c}'_1, \bar{c}'_2\} \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}})\backslash\{\bar{c}\} \leftrightarrow \mathrm{OC}(\mathbf{T})\backslash\{c\}.$$

Since $S_f(c') = S_f(\bar{c}'_1) = S_f(\bar{c}) = S_f(c)$, $S_b(c') = S_b(\bar{c}'_2) = S_{i-1,j+1}$, and $S_b(c) = S_{i+1,j+1}$, we see that P verifies the conclusion of situation 1(a).

Case S. Here $\bar{P}$ is in situation 2(b) and $P'_e = \{S_{i-2,j+2}, S_{i-1,j+2}\}$ (so P'_e is in situation 2(b)). Then $P = \bar{P}$, $P_e = P'_e$, and P and P_e are in situation 2(a). Let $\{u\} = \bar{c}'_1 \in \mathrm{OC}(\bar{\mathbf{T}}')$ be such that $S_b(\bar{c}'_1) = S_{i-1,j+1}$ and $S_f(\bar{c}'_1) = S_{ij}$, and let $\bar{c}'_2 \in \mathrm{OC}(\bar{\mathbf{T}}')$ be such that $S_f(\bar{c}'_2) = S_{i-2,j+2}$. Then we see that P'_e is in situation 2(b)(ii) (with $\bar{c}'_1$ and $\bar{c}'_2$ the distinguished cycles of that situation). Let $\bar{c}'_1 \cup \bar{c}'_2 \cup \{e\} = c' \in \mathrm{OC}(\mathbf{T}')$, so $S_b(c') = S_b(\bar{c}'_2)$ and $S_f(c') = S_f(\bar{c}'_1)$. Let $\bar{c} \in \mathrm{OC}(\bar{\mathbf{T}})$ be such that $S_b(\bar{c}) = S_b(\bar{c}'_2)$; then $S_f(\bar{c}) = S_f(\bar{c}'_2)$. Finally, let $\bar{c} \cup \{e\} = c \in \mathrm{OC}(\mathbf{T})$, so $S_b(c) = S_b(\bar{c})$ and $S_f(c) = S_{i,j+2}$. We have the following c.s.p.b.'s:

$$\mathrm{OC}(\mathbf{T}')\backslash\{c'\} \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}}')\backslash\{\bar{c}'_1, \bar{c}'_2\} \leftrightarrow \mathrm{OC}(\bar{\mathbf{T}})\backslash\{\bar{c}\} \leftrightarrow \mathrm{OC}(\mathbf{T})\backslash\{c\}.$$

Since $S_b(c) = S_b(\bar{c}) = S_b(\bar{c}'_2) = S_b(c')$, $S_f(c') = S_f(\bar{c}'_1) = S_{ij}$, and $S_f(c) = S_{i,j+2}$, we see that P verifies the conclusion of situation 2(a).

Case T. Here $\bar{P}$ is in situation 2(b) and $P'_e = \{S_{i-1,j+2}, S_{i-1,j+3}\}$ (so P'_e is in situation 1(a)). Then $P = \bar{P}$, $P_e = P'_e$, P is in situation 2(a) and P_e is in situation 1(b). This case follows the pattern of case Q.

Case U. Here $\bar{P}$ is in situation 2(b) and P'_e does not satisfy the hypotheses of any of the previous five cases. Then $P=\bar{P}$ and P is also in situation 2(b). As in cases E, J, and O, the conclusion follows by induction.

This completes the proof of Theorem 2.2.3. □

2.2.5. DEFINITION. Let $\mathbf{T}'$ and $\mathbf{T}$ be as in Theorem 2.2.3. A cycle $c \in \mathrm{OC}(\mathbf{T})$ is said to correspond to a cycle $c' \in \mathrm{OC}(\mathbf{T}')$ if either $S_b(c')=S_b(c)$ or $S_f(c')=S_f(c)$ (or both). Equivalently, by Theorem 2.2.3 (and in that theorem's notation), either $c=\mu(c')$ for μ the applicable c.s.p.b., or P is in one of the situations 1(a), 1(b)(ii), or 2(a) of the theorem and c and c' are amongst the distinguished cycles of P's situation. In particular note that an open cycle in $\mathbf{T}$ corresponds to at most one cycle in $\mathbf{T}'$.

2.2.6. PROPOSITION. *With the notation as in Theorem* 2.2.3, *suppose* $m \in M$ *and* $m < v$. *Suppose* $c(m, \mathbf{T}') \in \mathrm{OC}(\mathbf{T}')$. *Then* $c(m, \mathbf{T}) \in \mathrm{OC}(\mathbf{T})$ *and* $c(m, \mathbf{T})$ *corresponds to* $c(m, \mathbf{T}')$.

Proof. This can be proved by induction as in the proof of Theorem 2.2.3 by examining the cases which arise in that proof. (When $|M|=1$ the proposition is vacuously true.) As examples we treat cases G and R (with notation as in those cases). Let $c'_m=c(m, \mathbf{T}')$ and let $c_m=c(m, \mathbf{T})$.

Case G. Suppose first $c'_m=\tilde{c}'$; then since $\bar{c}'=\tilde{c}'$, we have $m \in \bar{c}'$. By induction then either $m \in \bar{c}_1$ or $m \in \bar{c}_2$. Since $c=\bar{c}_1 \cup \bar{c}_2 \cup \{e\}$ we have $m \in c$, i.e. $c=c_m$. By definition, c and $\tilde{c}'$ correspond, as desired. Suppose next $c'_m \neq \tilde{c}'$. Let $\tilde{c}$ be the cycle in $\mathbf{T}$ corresponding to c'_m. Then $c'_m \in \mathrm{OC}(\bar{\mathbf{T}}')$, $\tilde{c} \in \mathrm{OC}(\bar{\mathbf{T}})$, and c'_m and $\tilde{c}$ correspond as cycles of $\bar{\mathbf{T}}'$ and $\bar{\mathbf{T}}$. Thus by induction $m \in \tilde{c}$, so we are done.

Case R. If $m \notin c'$ then the argument is as in the second half of the previous case, so assume $c'=c'_m$. Now $c'=\bar{c}'_1 \cup \bar{c}'_2 \cup \{e\}$, and $\bar{c}'_2=\{u\}$ with $u > v$. Since $m < v$ by hypothesis, we have $m \in \bar{c}'_1$. By induction (since $\bar{c}$ corresponds to $\bar{c}'_1$) we have $m \in \bar{c}$. Since $c=\bar{c} \cup \{e\}$ we have $m \in c$, as desired. □

2.2.7. DEFINITION. Let $\mathbf{T}'$, etc., be as in Theorem 2.2.3. Suppose $m \in M$, $m < v$, and suppose $c'=c(m, \mathbf{T}')$ is closed. We define $c=c(m, \mathbf{T})$ to be the cycle in $\mathbf{T}$ corresponding to c'.

2.2.8. PROPOSITION. (1) *The above is well-defined, i.e. does not depend on the choice of* $m < v$ *in* c'.

(2) *Either*

(a) *c is closed*

or

(b) *P is in the situation* 1(b)(i) *of Theorem* 2.2.3 *and c is the open cycle in* $\mathbf{T}$ *with no corresponding open cycle in* $\mathbf{T}'$ (*i.e.* $S_b(c) \in P$).

Proof. The proof uses induction on $|M|$. (The proposition is vacuously true when $|M|=1$.) Let $e=\sup M$. The proposition is obvious if $e=v$ (since then $c(m, \mathbf{T})=c(m, \mathbf{T}')$ is closed) so assume $e \neq v$. Let $\bar{\mathbf{T}}$, etc. be as in the proof of

Theorem 2.2.3, and let $c(m)=c(m,\mathbf{T})$, $c'(m)=c(m,\mathbf{T}')$, $\bar{c}(m)=c(m,\bar{\mathbf{T}})$ and $\bar{c}'(m)=c(m,\bar{\mathbf{T}}')$. Note that $\bar{c}'(m)$ is closed if and only if $e\notin c'(m)$ which in turn is equivalent to $c'(m)=\bar{c}'(m)$. Assume first that both $\bar{c}'(m)$ and $\bar{c}(m)$ are closed. Then (since $\bar{c}(m)$ closed implies that $c(m)=\bar{c}(m)$) the proposition is true by induction.

Assume next $e\in c'(m)$. We may assume $P(e,\mathbf{T}')$ is horizontal, so let $P(e,\mathbf{T}')=\{S_{k,l}, S_{k,l+1}\}$. Since $c'(m)$ is closed we have $P'(e,\mathbf{T}')=\{S_{k-1,l+1}, S_{k,l+1}\}$. Now $\bar{c}'(m)$ is open in $\bar{\mathbf{T}}'$, $S_b(\bar{c}'(m))=S_{k-1,l+1}$, and $S_f(\bar{c}'(m))=S_{k,l}$. By the previous proposition, $\bar{c}(m)$ corresponds to $\bar{c}'(m)$ (in the sense of Definition 2.2.5). By Theorem 2.2.3, and using Proposition 2.2.6, there are four possibilities:

(1) $S_b(\bar{c}(m))=S_b(\bar{c}'(m))$ and $S_f(\bar{c}(m))=S_f(\bar{c}'(m))$. Then $c(m)=\bar{c}(m)\cup\{e\}$ is closed.

(2) $\bar{P}=\{S_{k-1,l+2}, S_{k-1,l+3}\}$, $S_f(\bar{c}(m))=S_f(\bar{c}'(m))$, and $S_b(\bar{c}(m))=S_{k-1,l+3}$ (so $P=\bar{P}$ and $P(e,\mathbf{T})=P(e,\mathbf{T}')$) (cf. case C of Theorem 2.2.3). Then P is in situation 1(b)(i) and $c(m)=\bar{c}(m)\cup\{e\}$ is the open cycle in $\mathbf{T}$ which has no corresponding open cycle in $\mathbf{T}'$.

(3) $\bar{P}=\{S_{k,l}, S_{k+1,l}\}$, $S_b(\bar{c}(m))=S_b(\bar{c}'(m))$, and $S_f(\bar{c}(m))=S_{k+3,l}$ (cf. case M of Theorem 2.2.3). Then P is in situation 1(b)(i) and $c(m)=\bar{c}(m)\cup\{e\}$ is the open cycle in $\mathbf{T}$ which has no corresponding open cycle in $\mathbf{T}'$.

(4) $\bar{P}$ is in situation 1(b)(ii), $\bar{P}=\{S_{r,t}, S_{r,t+1}\}$ (respectively $\{S_{r,t}, S_{r+1,t}\}$) and there are cycles $\bar{c}_1$, $\bar{c}_2\in \mathrm{OC}(\bar{\mathbf{T}})$ such that $S_f(\bar{c}_1)=S_f(\bar{c}'(m))$, $S_b(\bar{c}_1)= S_{r,t+1}$ (respectively, $S_{r+1,t}$), $S_f(\bar{c}_2)=S_{r+1,t}$ (respectively $S_{r,t+1}$), and $S_b(\bar{c}_2)=S_b(\bar{c}'(m))$. By Proposition 2.2.6 either $m\in\bar{c}_1$ or $m\in\bar{c}_2$. Then $c(m)=\bar{c}_1\cup\bar{c}_2\cup\{e\}$ is the open cycle in $\mathbf{T}$ with no corresponding open cycle in $\mathbf{T}'$ ($P=\bar{P}$ but now P is in situation 1(b)(i)). (This is the exceptional subcase of case J of Theorem 2.2.3.)

This completes the proof of the proposition in the case where $\bar{c}'(m)$ is open. (Note that part 1 of the proposition is a consequence of Proposition 2.2.6 and is implicit in what we have said above.)

Suppose finally $\bar{c}'(m)$ is closed (i.e. $e\notin\bar{c}'(m)$) and $\bar{c}(m)$ is open. Then by induction $\bar{P}$ is in situation 1(b)(i) and $S_b(\bar{c}(m))\in\bar{P}$. Thus we are in one of the cases F, H, or J of the proof of Theorem 2.2.3. If we are in case J then $P=\bar{P}$, $c(m)=\bar{c}(m)$ and $S_b(c(m))=S_b(\bar{c}(m))$. In cases F and H we have $c(m)=\bar{c}(m)\cup\{e\}$ is closed. This gives part 2 of the proposition and part 1 by induction. □

2.2.9. THEOREM. *Let* $(\mathbf{T}'_1, v, \varepsilon)\in\mathscr{C}_K(M)$ ($K=B, C$, or D). *Let* U' *be a set of cycles in* $\mathbf{T}'_1$ *such that for every* $c\in U'$ *either* c *is open or there exists an* $m\in c$ *such that* $m<v$. *Set* $\mathbf{T}'_2=\mathbf{E}(\mathbf{T}'_1; U')$. *Let* $(\mathbf{T}_i, P_i)=\alpha((\mathbf{T}'_i, v, \varepsilon))$ *for* $i=1,2$. *Let* U *be the set of all cycles in* $\mathbf{T}_1$ *which correspond to cycles in* U'. *Then* $\mathbf{T}_2=\mathbf{E}(\mathbf{T}_1; U)$.

Proof. By induction on $|U'|$ it suffices to prove the theorem when $|U'|=1$ (since a cycle in $\mathbf{T}_1$ corresponds to at most one cycle in $\mathbf{T}'_1$, so that, writing $U'=U'_1\cup U'_2$ with $|U'_1|+|U'_2|=|U'|$, the sets U_1 and U_2 of corresponding cycles in $\mathbf{T}_1$ are disjoint). So assume $|U'|=1$; write $U'=\{c'_0\}$. Let $e=\sup M$ and assume first $v = e$. By symmetry (via transpose) we may assume here $\varepsilon = 1$. Then $P(e, \mathbf{T}_1) = \{S_{1,r}, S_{1,r+1}\}$ where $r = \rho_1(\mathbf{T}'_1) + 1$. Assume first $c'_0 \in \mathrm{OC}(\mathbf{T}'_1)$ and $S_b(c'_0) = S_{1,r-1}$. Then $\rho_1(\mathbf{T}'_2) = r - 2$ so $P(e, \mathbf{T}_2) = \{S_{1,r-1}, S_{1,r}\}$. On the other hand it is clear that, setting $c_0 = c'_0 \cup \{e\}$, we have $c_0 \in \mathrm{OC}(\mathbf{T}_1)$ is the cycle corresponding to c'_0, and that $\mathbf{T}_2 = \mathbf{E}(\mathbf{T}_1, c_0)$. This also handles the case when $S_f(c'_0) = S_{1,r}$ (by interchanging $\mathbf{T}'_1$ and $\mathbf{T}'_2$ in the previous case). In all other cases $\rho_1(\mathbf{T}'_1) = \rho_1(\mathbf{T}'_2)$, so $P(e, \mathbf{T}_1) = P(e, \mathbf{T}_2)$, and the theorem is clear.

Henceforth assume $v \neq e$. The rest of the proof uses induction on $|M|$ (when $|M| = 1$ we have, necessarily, $v = e$, and thus have already proved the theorem). Let $\bar{\mathbf{T}}'_j = \mathbf{T}'_j - e$ for $j = 1, 2$, and set $(\bar{\mathbf{T}}_j, \bar{P}_j) = \alpha((\bar{\mathbf{T}}'_j, v, \varepsilon))$. Henceforth the proof follows the cases of the proof of Theorem 2.2.3. Note that if $\bar{P}_1$ is in case A and $c'_0 = c'$ then $\bar{P}_2$ is in case K, and similarly case B corresponds to case L, case C to case M, case D to case N, case E to case O, case Q to case T, and case R to case S.

We treat cases A and B; the rest are handled along the same lines. In general we use induction to show that every domino except the one containing e is in the correct position in $\mathbf{T}_2$ and then inspection to show that the domino containing e is in the correct position in $\mathbf{T}_2$. 'Correct position' means that if a number k is in one of the cycles corresponding to c'_0 then, in $\mathbf{T}_2$, it occupies the position $P'(k, \mathbf{T}_1)$; otherwise it occupies the position $P(k, \mathbf{T}_1)$.

Case A. Here $\bar{P}_1 = P(e, \mathbf{T}_1) = \{S_{ij}, S_{i,j+1}\}$ and $\bar{P}_1$ is in situation 1(a). (We use the notation of case A, but with $\mathbf{T}'_1$ in place of $\mathbf{T}'$, etc.) Suppose first $c'_0 = c' = c(e, \mathbf{T}'_1)$. Then $\bar{c}' = c' \backslash \{e\} \in \mathrm{OC}(\bar{\mathbf{T}}'_1)$ and $\bar{c} \in \mathrm{OC}(\bar{\mathbf{T}}_1)$ corresponds to $\bar{c}'$. By induction $\bar{\mathbf{T}}_2 = \mathbf{E}(\bar{\mathbf{T}}_1, \bar{c})$.

Now $P(e, \mathbf{T}_1) = \{S_{i+1,r}, S_{i+1,r+1}\}$ where $r = \rho_{i+1}(\mathbf{T}'_1) + 1$. We distinguish two cases:

(1) $S_f(c') \neq S_{i+1,r}$. Then $\rho_{i+1}(\mathbf{T}'_2) = \rho_{i+1}(\mathbf{T}'_1)$ and $P(e, \mathbf{T}_2) = P(e, \mathbf{T}_1)$; setting $c = \bar{c}$ we have $c \in \mathrm{OC}(\mathbf{T}_1)$ is the cycle corresponding to c', and hence, finally, $\mathbf{T}_2 = \mathbf{E}(\mathbf{T}_1, c)$.

(2) $S_f(c') = S_{i+1,r}$. Then $\rho_{i+1}(\mathbf{T}'_2) = r$ and $P(e, \mathbf{T}_2) = \{S_{i+1,r+1}, S_{i+1,r+2}\}$. Set $c = \bar{c} \cup \{e\}$, then $c \in \mathrm{OC}(\mathbf{T}_1)$ is the cycle corresponding to c', and again we see that $\mathbf{T}_2 = \mathbf{E}(\mathbf{T}_1, c)$.

Suppose instead $c'_0 \neq c(e, \mathbf{T}'_1)$. Let $\bar{c}'_0 \in \mathrm{OC}(\bar{\mathbf{T}}'_1)$ with $\bar{c}'_0 = c'_0$; let $\bar{c}_0$ be the corresponding cycle in $\bar{\mathbf{T}}$; then either both $\bar{c}'_0$ and c'_0 are closed or $S_b(\bar{c}_0) = S_b(c'_0)$ and $S_f(\bar{c}_0) = S_f(c'_0)$. As before by induction, $\bar{\mathbf{T}}_2 = \mathbf{E}(\bar{\mathbf{T}}_1, \bar{c}_0)$. Then the two cases are again as above.

Case B. Suppose $\bar{P}_1$ is as in case B of the proof of Theorem 2.2.3. If $c'_0 \neq c'$ (notation as in that case, but with $\mathbf{T}'_1$ in place of $\mathbf{T}'$, etc.) then the theorem is clearly true by induction so assume $c'_0 = c'$. Then $P(e, \mathbf{T}'_2) = \{S_{i,j-1}, S_{i+1,j-1}\}$ and, by induction applied to $\bar{\mathbf{T}}'_1$ and $\bar{U}' = \{\bar{c}'\}$, we have $\bar{\mathbf{T}}_2 = \mathbf{E}(\bar{\mathbf{T}}_1, \bar{c})$. Thus $\bar{P}_2 = \{S_{i,j-1}, S_{ij}\}$ so $P(e, \mathbf{T}_2) = \{S_{i+1,j-1}, S_{i+1,j}\}$. On the other hand, $U = \{c_1, c_2\}$, so by induction (since $\bar{c}_1 = \bar{c}$) and the above computation of $P(e, \mathbf{T}_2)$ (since $c_2 = \{e\}$) we have $\mathbf{T}_2 = \mathbf{E}(\mathbf{T}_1'; U)$. $\square$

Section 3

We now proceed to define $T_{\alpha\beta}$ on tableaux for $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$. To do this we must define extended cycles.

2.3.1. DEFINITION. (1) Let $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_K(M_1, M_2)$ (for $K = B, C,$ or D). We define an equivalence relation $\sim_{ec}$ on M_1 (depending on $\mathbf{T}_1$ and $\mathbf{T}_2$) as the relation generated by the following two types of relations:

(i) $k \sim_{ec} k'$ if $k' \in c(k, \mathbf{T}_1)$,

(ii) $k \sim_{ec} k'$ if there exists a cycle $c_2 \in \mathrm{OC}(\mathbf{T}_2)$ such that $S_b(c_2) \in P(k, \mathbf{T}_1)$ and $S_f(c_2) \in P'(k', \mathbf{T}_1)$.

The equivalence class containing $k \in M_1$ is written $ec(k, \mathbf{T}_1; \mathbf{T}_2)$ and is called the extended cycle of k in $\mathbf{T}_1$ relative to $\mathbf{T}_2$. (Clearly it depends only on $\mathbf{T}_1$ and the cycle structure of $\mathbf{T}_2$.) Similarly we define an equivalence relation on M_2, and, for $k \in M_2$, write $ec(k, \mathbf{T}_2; \mathbf{T}_1)$ for the equivalence class containing k. Obviously an extended cycle is a union of cycles of $\mathbf{T}_1$, and, in fact, consists either of one closed cycle of $\mathbf{T}_1$ or of one or more open cycles of $\mathbf{T}_1$. If $c = ec(k, \mathbf{T}_1; \mathbf{T}_2)$ consists of one or more open cycles of $\mathbf{T}_1$ we can write $c = c_1^1 \cup \cdots \cup c_r^1$ so that, if $c_i^2 \in \mathrm{OC}(\mathbf{T}_2)$ is such that $S_b(c_i^2) = S_b(c_i^1)$ then $S_f(c_r^2) = S_f(c_1^1)$ and for $1 \leqslant i \leqslant r-1$, $S_f(c_i^2) = S_f(c_{i+1}^1)$. Then clearly $c_1^2 \cup \cdots \cup c_r^2$ is an extended cycle in $\mathbf{T}_2$ relative to $\mathbf{T}_1$ which we define to be the extended cycle corresponding to $ec(k, \mathbf{T}_1; \mathbf{T}_2)$.

(2) If c is an extended cycle in $\mathbf{T}_1$ relative to $\mathbf{T}_2$ we define $\mathbf{E}((\mathbf{T}_1, \mathbf{T}_2), c, L)$ to be equal to

$$(\mathbf{E}(\mathbf{T}_1, c), \mathbf{T}_2)$$

if c consists of one closed cycle in $\mathbf{T}_1$, but to be equal to

$$(\mathbf{E}(\mathbf{T}_1, c_1^1, \ldots, c_r^1), \mathbf{E}(\mathbf{T}_2, c_1^2, \ldots, c_r^2))$$

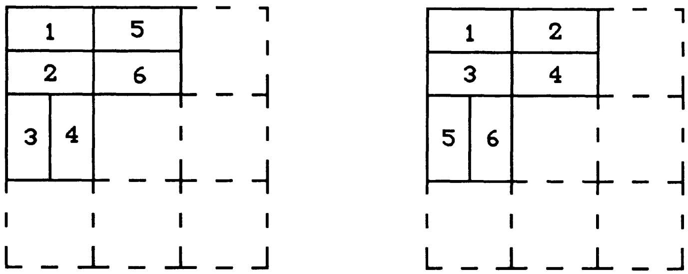
if c is a union of open cycles and the c_i^j are as above.

Similarly we define $\mathbf{E}((\mathbf{T}_1, \mathbf{T}_2); U, L)$ if U is a union of extended cycles in $\mathbf{T}_1$ relative to $\mathbf{T}_2$. We define similarly $\mathbf{E}((\mathbf{T}_1, \mathbf{T}_2), c, R)$ when c is an extended cycle in $\mathbf{T}_2$ relative to $\mathbf{T}_1$, and $\mathbf{E}((\mathbf{T}_1, \mathbf{T}_2); U, R)$ when U is a union of extended cycles in $\mathbf{T}_2$ relative to $\mathbf{T}_1$.

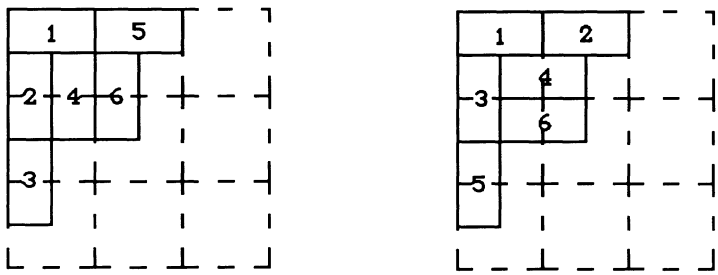
REMARK. The point of the definition of extended cycle is that, if $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_K(M_1, M_2)$ and $c = ec(k, \mathbf{T}_1; \mathbf{T}_2)$ for some $k \in M_1$, and, setting

$(\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2) = \mathbf{E}((\mathbf{T}_1, \mathbf{T}_2), c, L)$, we have $\text{Shape}(\bar{\mathbf{T}}_1) = \text{Shape}(\bar{\mathbf{T}}_2)$. More precisely, if $(\mathbf{T}_1, \mathbf{T}_2) \in \mathcal{T}_C(M_1, M_2)$ and if $u = \inf M_1$ then $\mathbf{E}((\mathbf{T}_1, \mathbf{T}_2), c, L) \in \mathcal{T}_C(M_1, M_2)$ if and only if $u \notin c$. If $u \in c$ then $\mathbf{E}((\mathbf{T}_1, \mathbf{T}_2), c, L) \in \mathcal{T}_B(M_1, M_2)$. Similarly if C and B are interchanged. If $(\mathbf{T}_1, \mathbf{T}_2) \in \mathcal{T}_D(M_1, M_2)$ then $\mathbf{E}((\mathbf{T}_1, \mathbf{T}_2), c, L) \in \mathcal{T}_D(M_1, M_2)$. Similarly for R.

EXAMPLE. Suppose $(\mathbf{T}_1, \mathbf{T}_2)$ is:



Then $c(2, \mathbf{T}_1) = \{2, 3, 4\}$ but $ec(2, \mathbf{T}_1; \mathbf{T}_2) = \{2, 3, 4, 6\}$ (and the corresponding extended cycle in $\mathbf{T}_2$, relative to $\mathbf{T}_1$, is $\{3, 4, 5, 6\}$). Thus $\mathbf{E}((\mathbf{T}_1, \mathbf{T}_2), ec(2, \mathbf{T}_1; \mathbf{T}_2), L) = (\tilde{\mathbf{T}}_1, \tilde{\mathbf{T}}_2)$ where $(\tilde{\mathbf{T}}_1, \tilde{\mathbf{T}}_2)$ is:



2.3.2. PROPOSITION. *Let* $(\mathbf{T}'_1, \tilde{\mathbf{T}}'_1) \in \mathcal{T}_K(M, \tilde{M})$. *Suppose* $v \in \mathbb{N}^* \backslash M$, $e \in \mathbb{N}^* \backslash \tilde{M}$, $\varepsilon \in \{1, -1\}$, *and* $e > \sup \tilde{M}$. *Let* $(\mathbf{T}_1, P_1) = \alpha((\mathbf{T}'_1, v, \varepsilon))$ *and let* $\tilde{\mathbf{T}}_1 = \text{Adj}(\tilde{\mathbf{T}}'_1, P_1, e)$ (*i.e., in the notation of* 1.1.13, $\tilde{\mathbf{T}}_1 = \tilde{\mathbf{T}}'_1 \cup \{(e, S) \mid S \in P_1\}$).

(a) *Let* $c' = ec(m, \mathbf{T}'_1; \tilde{\mathbf{T}}'_1)$ *for some* $m < v$. *If* c' *consists of the cycles* $c'_1, \ldots, c'_k$ *and* $c_1, \ldots, c_r$ *are the cycles in* $\mathbf{T}_1$ *corresponding to* $c'_1, \ldots, c'_k$ (*without repetition*) *then* $\bigcup_{1 \leqslant i \leqslant r} c_1 = ec(m, \mathbf{T}_1; \tilde{\mathbf{T}}_1)$ (*which we define to be the extended cycle corresponding to* c').

(b) *Let* U' *be a set of extended cycles in* $\mathbf{T}'_1$ *relative to* $\tilde{\mathbf{T}}'_1$ *each of whose elements satisfies the hypothesis of* (a) *and let* $(\mathbf{T}'_2, \tilde{\mathbf{T}}'_2) = \mathbf{E}((\mathbf{T}'_1, \tilde{\mathbf{T}}'_1); U, L)$. *Let* $(\mathbf{T}_2, P_2) = \alpha((\mathbf{T}'_2, v, \varepsilon))$ *and let* $\tilde{\mathbf{T}}_2 = \text{Adj}(\tilde{\mathbf{T}}'_2, P_2, e)$. *Let* U *be the set of extended cycles in* $\mathbf{T}_1$ *relative to* $\tilde{\mathbf{T}}_1$ *corresponding to the cycles in* U'. *Then* $(\mathbf{T}_2, \tilde{\mathbf{T}}_2) = \mathbf{E}((\mathbf{T}_1, \tilde{\mathbf{T}}_1); U, L)$.

Proof. To prove (a) we note first that Proposition 2.2.8 proves the case when $k=1$ and c'_1 is closed. So assume c'_1 is open. By Proposition 2.2.6 it suffices to prove that $\bigcup_{1\leqslant i\leqslant r} c_i$ is an extended cycle in $\mathbf{T}_1$ relative to $\tilde{\mathbf{T}}_1$. Then (a) of the proposition is a consequence of Theorem 2.2.3 and Proposition 2.2.4. The only difficult case is when P_1 is in situation 2(b) of Theorem 2.2.3 and the cycle $\{u\}$ of that situation is one of the c'_i. By hypothesis then $k\geqslant 2$ and $r=k-1$. Then $P(e,\tilde{\mathbf{T}}_1)$ is in situation 2(b)(ii) of Proposition 2.2.4 and the two distinguished cycles of that situation are contained in the extended cycle in $\tilde{\mathbf{T}}'_1$ relative to $\mathbf{T}'_1$ which corresponds to c'. The desired conclusion is now clear.

To prove (b) let $(\mathbf{T}_3,\tilde{\mathbf{T}}_3)=\mathbf{E}((\mathbf{T}_1,\tilde{\mathbf{T}}_1); U, L)$. Then Theorem 2.2.9 says that $\mathbf{T}_3=\mathbf{T}_2$. It remains to show that $\tilde{\mathbf{T}}_3=\tilde{\mathbf{T}}_2$. We may assume that U' consists of one extended cycle, c'. Suppose first c' is a closed cycle in $\mathbf{T}'_1$. Then $\tilde{\mathbf{T}}'_2=\tilde{\mathbf{T}}'_1$. If the cycle c in $\mathbf{T}_1$ corresponding to c' is also closed then $\mathrm{Shape}(\mathbf{T}_2)=\mathrm{Shape}(\mathbf{T}_1)$ so $\tilde{\mathbf{T}}_2=\tilde{\mathbf{T}}_1$. On the other hand, since $U=\{c\}$ and c is closed, we have $\tilde{\mathbf{T}}_3=\tilde{\mathbf{T}}_1$, so we are done in this case. If c is open, and, say, $P_1=\{S_{ij}, S_{i,j+1}\}$, then by Proposition 2.2.8 and Theorem 2.2.9 we have $P_2=\{S_{ij}, S_{i+1,j}\}$, that is, $\{e\}$ is the extended cycle in $\tilde{\mathbf{T}}_1$ relative to $\mathbf{T}_1$ corresponding to $U=\{c\}$ and $\tilde{\mathbf{T}}_2=\mathbf{E}(\mathbf{T}_2,\{e\})$, as was to be shown.

The other cases are similar. For example, consider the case treated in the proof of (a). Again, assume $P_1=\{S_{ij}, S_{i,j+1}\}$. Let $\tilde{U}'$ be the extended cycle in $\tilde{\mathbf{T}}'_1$ relative to $\mathbf{T}'_1$ which corresponds to U' and let $\tilde{U}$ be the extended cycle in $\tilde{\mathbf{T}}_1$ relative to $\mathbf{T}_1$ which corresponds to U. Then there are cycles $\tilde{c}'_1$ and $\tilde{c}'_2$ contained in $\tilde{U}'$ with $\tilde{c}'_1\neq\tilde{c}'_2$, $S_b(\tilde{c}'_1)=S_{i-1,j+1}$, and $S_f(\tilde{c}'_2)=S_{ij}$. Then $\tilde{c}'_1\cup\tilde{c}'_2\cup\{e\}$ is an open cycle in $\tilde{\mathbf{T}}_1$ and $\tilde{U}$ consists of it and the remaining cycles of $\tilde{U}'$. As a consequence of Theorem 2.2.9 we have $P_2=\{S_{i-1,j+1}, S_{i,j+1}\}$, and thus $\tilde{\mathbf{T}}_2=\tilde{\mathbf{T}}_3$, as desired. □

2.3.3. PROPOSITION. *Let* $(\mathbf{T}'_1,\tilde{\mathbf{T}}'_1)\in\mathscr{T}_K(M,\tilde{M})$. *Suppose* $v\in\mathbb{N}^*\backslash\tilde{M}$, $e\in\mathbb{N}^*\backslash M$, $\varepsilon\in\{1,-1\}$, *and* $e>\sup M$. *Let* $(\tilde{\mathbf{T}}_1,P_1)=\alpha((\tilde{\mathbf{T}}'_1,v,\varepsilon))$ *and let* $\mathbf{T}_1=\mathrm{Adj}(\mathbf{T}'_1,P_1,e)$.

- (a) $\{e\}$ *is an extended cycle in* $\mathbf{T}_1$ *relative to* $\tilde{\mathbf{T}}_1$ *if and only if* P_1 *is in situation* 1(b)(i) *of Theorem* 2.2.3.
- (b) *Extended cycles in* $\mathbf{T}'_1$ *relative to* $\tilde{\mathbf{T}}'_1$ *are of the form* $c\cap(M\backslash\{e\})$ *where* c *is an extended cycle in* $\mathbf{T}_1$ *relative to* $\tilde{\mathbf{T}}_1$.
- (c) *Let* U *be a set of extended cycles in* $\mathbf{T}_1$ *relative to* $\tilde{\mathbf{T}}_1$ *such that no element of* U *is equal to* $\{e\}$. *Let* $(\mathbf{T}_2,\tilde{\mathbf{T}}_2)=\mathbf{E}((\mathbf{T}_1,\tilde{\mathbf{T}}_1); U, L)$. *Let* $\mathbf{T}'_2=\mathbf{T}_2-e$ *and let* $(\tilde{\mathbf{T}}'_2,v_2,\varepsilon)=\beta((\tilde{\mathbf{T}}_2,P(e,\mathbf{T}_2)))$. *Let* $U=\{c\cap(M\backslash\{e\})\mid c\in U\}$. *Then* $v_2=v$, $\varepsilon_2=\varepsilon$, *and* $(\mathbf{T}'_2,\tilde{\mathbf{T}}'_2)=\mathbf{E}((\mathbf{T}'_1,\tilde{\mathbf{T}}'_1); U', L)$.

Proof. (a) is obvious. (b) is a direct consequence of Theorem 2.2.3 and Proposition 2.2.4.

To prove (c), we may assume U consists of one extended cycle; say $U=\{c\}$. Let $c'=c\cap(M\backslash\{e\})$, so $U'=\{c'\}$ and by (b), c' is an extended cycle in $\mathbf{T}'_1$ relative to $\tilde{\mathbf{T}}'_1$. Set $(\mathbf{T}'_3, \tilde{\mathbf{T}}'_3) = \mathbf{E}((\mathbf{T}'_1, \tilde{\mathbf{T}}'_1), c', L)$, let $(\tilde{\mathbf{T}}_3, P_3) = \alpha((\tilde{\mathbf{T}}'_3, v, \varepsilon))$ and let $\mathbf{T}_3 = \mathrm{Adj}(\mathbf{T}'_3, P_3, e)$. Since α and β are inverses, it suffices to prove that $(\mathbf{T}_3, \tilde{\mathbf{T}}_3) = (\mathbf{T}_2, \tilde{\mathbf{T}}_2)$. Since $c' = c \cap (M \backslash \{e\})$ we have $\mathbf{T}'_3 = \mathbf{T}_2 - e$. Since $\mathrm{Shape}(\mathbf{T}_j) = \mathrm{Shape}(\tilde{\mathbf{T}}_j)$ for $j = 2, 3$ and since $\mathbf{T}_3 = \mathbf{T}'_3 \cup D(e, \mathbf{T}_3)$, it now suffices to prove that $\tilde{\mathbf{T}}_3 = \tilde{\mathbf{T}}_2$. If c consists of one or more open cycles in $\mathbf{T}_1$ then let $\tilde{c}$ be the corresponding extended cycle in $\tilde{\mathbf{T}}_1$, otherwise set $c = \varnothing$, and define similarly $\tilde{c}'$ in relation to c'. Note that $\tilde{c}$ is the union of the cycles in $\tilde{\mathbf{T}}_1$ which correspond (in the sense of Definition 2.2.5) to the cycles in $\tilde{\mathbf{T}}'_1$ contained in $\tilde{c}'$; this follows from Definitions 2.2.5 and 2.3.1, and our hypothesis that $c \neq \{e\}$. Then the fact that $\tilde{\mathbf{T}}_3 = \tilde{\mathbf{T}}_2$ is just Theorem 2.2.9. □

REMARK. We have also the obvious analogues of Propositions 2.3.2 and 2.3.3, in which left and right are interchanged.

We now restrict our attention to types B and C, and define $T_{\alpha\beta}$ on tableaux when $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$. Unlike the situation when α and β have the same length, here we have to define $T_{\alpha\beta}$ on pairs of tableaux.

2.3.4. DEFINITION. (1) We treat first type C. Let

$$F_2 = \{(1, S_{1,1}), (1, S_{1,2}), (2, S_{2,1}), (2, S_{2,2})\},$$
$$\tilde{F}_2 = \{(1, S_{1,1}), (1, S_{1,2}), (2, S_{2,1}), (2, S_{3,1})\},$$

let $F_1 = {}^tF_2$, and let $\tilde{F}_1 = {}^t\tilde{F}_2$. Suppose $\{1,2\} \subseteq M_1$ and suppose $(\mathbf{T}_1, \mathbf{T}_2) \in D^L_{\alpha\beta}(\mathscr{T}_C(M_1, M_2))$ with $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$. Suppose first $\beta = \alpha_2$. Then clearly either $F_2 \subseteq \mathbf{T}_1$ or $\tilde{F}_2 \subseteq \mathbf{T}_1$. If $\tilde{F}_2 \subseteq \mathbf{T}_1$, then let $(\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2) = \mathbf{E}((\mathbf{T}_1, \mathbf{T}_2), c)$ where $c = ec(2, \mathbf{T}_1; \mathbf{T}_2)$ (so $F_2 \subseteq \bar{\mathbf{T}}_1$), let $\tilde{\mathbf{T}}_1 = (\bar{\mathbf{T}}_1 \backslash F_2) \cup F_1$, and define $T^L_{\alpha_1\alpha_2}((\mathbf{T}_1, \mathbf{T}_2)) = \{(\tilde{\mathbf{T}}_1, \bar{\mathbf{T}}_2)\}$. If instead $F_2 \subseteq \mathbf{T}_1$ let $\mathbf{T}'_1 = (\mathbf{T}_1 \backslash F_2) \cup F_1$. If $1 \in ec(2, \mathbf{T}'_1; \mathbf{T}_2)$, define $T^L_{\alpha_1,\alpha_2}((\mathbf{T}_1, \mathbf{T}_2)) = \{(\mathbf{T}'_1, \mathbf{T}_2)\}$. If $1 \notin ec(2, \mathbf{T}'_1; \mathbf{T}_2)$ then let

$$(\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2) = \mathbf{E}((\mathbf{T}'_1, \mathbf{T}_2), ec(2, \mathbf{T}'_1; \mathbf{T}_2), L),$$

and define

$$T^L_{\alpha_1,\alpha_2}((\mathbf{T}_1, \mathbf{T}_2)) = \{(\mathbf{T}'_1, \mathbf{T}_2), (\bar{\mathbf{T}}_1, \bar{\mathbf{T}}_2)\}.$$

Suppose next $\beta = \alpha_1$. We define $T^L_{\alpha_2,\alpha_1}((\mathbf{T}_1, \mathbf{T}_2))$ as in the definition of $T^L_{\alpha_1,\alpha_2}((\mathbf{T}_1, \mathbf{T}_2))$ but interchanging F_1 (respectively $\tilde{F}_1$) and F_2 (respectively $\tilde{F}_2$). We define similarly $T^R_{\alpha\beta}$ for $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$ and $\{1, 2\} \subseteq M_2$.

(2) T_{α_1,α_2} and T_{α_2,α_1} are defined similarly for type B, putting

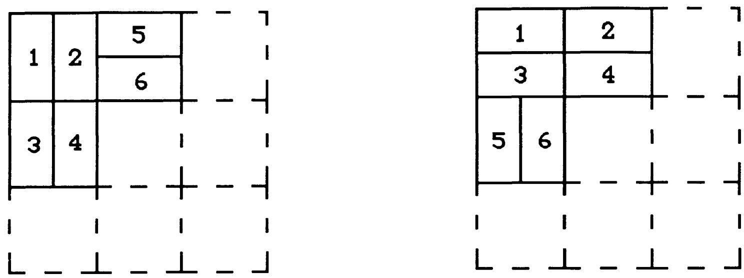
$$F_2 = \{(1, S_{1,2}), (1, S_{1,3}), (2, S_{2,1}), (2, S_{3,1})\}$$

and

$$\tilde{F}_2 = \{(1, S_{1,2}), (1, S_{1,3}), (2, S_{2,1}), (2, S_{2,2})\}.$$

The rest of the definition is the same.

EXAMPLE. Let $(\mathbf{T}'_1, \mathbf{T}'_2) =$



Then $1 \in ec(2, \mathbf{T}'_1; \mathbf{T}'_2)$, so

$$T^L_{\alpha_2,\alpha_1}((\mathbf{T}'_1, \mathbf{T}'_2)) = \{(\mathbf{T}_1, \mathbf{T}_2), (\tilde{\mathbf{T}}_1, \tilde{\mathbf{T}}_2)\},$$

where $(\mathbf{T}_1, \mathbf{T}_2)$ and $(\tilde{\mathbf{T}}_1, \tilde{\mathbf{T}}_2)$ are as in Example 2.3.1.

REMARK. Let $(\mathbf{T}_1, \mathbf{T}_2) \in \mathscr{T}_K(M_1, M_2)$ where $K = B$ or C and suppose $F_1 \subseteq \mathbf{T}_1$. Let $\tilde{\mathbf{T}}_1 = (\mathbf{T}_1 \setminus F_1) \cup F_2$. Then $1 \in ec(2, \mathbf{T}_1; \mathbf{T}_2)$ if and only if $1 \notin ec(2, \tilde{\mathbf{T}}_1; \mathbf{T}_2)$.

The proof of this remark is as follows. One can check this directly if $|M_1| = 2$. Then the general statement follows by induction on $|M_1|$ and Proposition 2.3.3(b); i.e. let $e = \sup M_1$, let $\mathbf{T}'_1 = \mathbf{T}_1 - e$, let $(\mathbf{T}'_2, v, \varepsilon) = \beta((\mathbf{T}_2, P(e, \mathbf{T}_1)))$, and let $\tilde{\mathbf{T}}'_1 = (\mathbf{T}'_1 \setminus F_1) \cup F_2$. Then $1 \in ec(2, \mathbf{T}_1; \mathbf{T}_2)$ if and only if $1 \in ec(2, \mathbf{T}'_1; \mathbf{T}'_2)$ which is equivalent to $1 \notin ec(2, \tilde{\mathbf{T}}'_1; \mathbf{T}'_2)$ which in turn is equivalent to $1 \notin ec(2, \tilde{\mathbf{T}}_1; \mathbf{T}_2)$.

2.3.5. PROPOSITION. *Let $x \in D^L_{\alpha\beta}(\mathscr{T}_K(M_1, M_2))$ for $K = B$ or C and $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$. If $T^L_{\alpha\beta}(x) = \{y, z\}$ with $y \neq z$ then $T^L_{\beta\alpha}(y) = T^L_{\beta\alpha}(z) = \{x\}$. If $T^L_{\alpha\beta}(x) = \{y\}$ then $T_{\beta\alpha}(y) = \{x, z\}$ with $z \neq x$ and $T^L_{\alpha\beta}(z) = \{y\}$. Similarly for $T^R_{\alpha\beta}$'s.*

Proof. This follows from Remark 2.3.4. □

2.3.6. PROPOSITION. *If $(\mathbf{T}_1, \mathbf{T}_2) \in D^L_{\alpha\beta}(\mathscr{T}_K(M_1, M_2))$ for $K = B$ or C with $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$ then ${}^t(\mathbf{T}_1, \mathbf{T}_2) \in D^L_{\beta\alpha}(\mathscr{T}_K(M_1, M_2))$ and ${}^tT^L_{\alpha\beta}((\mathbf{T}_1, \mathbf{T}_2)) = T^L_{\beta\alpha}({}^t(\mathbf{T}_1, \mathbf{T}_2))$. (Similarly with R in place of L.)*

Proof. The statement about $D^L_{\alpha\beta}$ is contained in Remark 2.1.12(1). For the second statement, note first that for any grid, ϕ, S_{ij} is ϕ-fixed if and only if S_{ji} is ϕ-fixed. (In fact the ϕ-fixedness of S_{ij} depends only on the parity of $i + j$.) It follows from this and Definition 1.5.8 that $P'(k, {}^t\mathbf{T}) = {}^tP'(k, \mathbf{T})$ for any $\mathbf{T} = (T, \phi)$. As a consequence cycles in $\mathbf{T}$ are cycles in ${}^t\mathbf{T}$ and vice versa, and furthermore for any such cycle c we have $\mathbf{E}({}^t\mathbf{T}, c) = {}^t\mathbf{E}(\mathbf{T}, c)$. The rest follows easily.

We now prove the analogue of Theorem 2.1.19 for $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$. In place of Proposition 2.1.14 we will use the following lemma. We state it for type C; it has an obvious analogue for type B, with an analogous proof.

2.3.7. LEMMA. *Let* $\mathbf{T}'_1 \in \mathscr{T}_C(M)$, *and suppose* $1 \in M$, $2 \notin M$, *and* $P(1, \mathbf{T}'_1) = \{S_{1,1}, S_{1,2}\}$. *Let* $\mathbf{T}'_2 = \mathrm{Re}(1, 2; \mathbf{T}'_1)$. *Let* $(\mathbf{T}_1, P_1) = \alpha((\mathbf{T}'_1, 2, -1))$ *and* $(\mathbf{T}_2, P_2) = \alpha((\mathbf{T}'_2, 1, 1))$.

(1) *If* $S_{2,1} \notin \mathrm{Shape}(\mathbf{T}'_1)$ *then* $\mathbf{T}_2 = \mathbf{E}(\mathbf{T}_1, c)$ *where* $c = \{2\}$.
(2) *If* $S_{2,1} \in \mathrm{Shape}(\mathbf{T}'_1)$ *then* $\mathbf{T}_2 = \mathbf{E}(\mathbf{T}_1, c)$ *where* $c = c(2, \mathbf{T}_1)$ *is closed (and in fact* $|c| = 2$).

We have also the obvious transposed version of the lemma.

Proof. (1) is obvious. For (2), let $e = \sup M$ and assume first $S_{2,1} \in P(e, \mathbf{T}'_1)$, so either $P(e, \mathbf{T}'_1) = \{S_{2,1}, S_{3,1}\}$ or $P(e, \mathbf{T}'_2) = \{S_{2,1}, S_{2,2}\}$. In either case the proposition is obvious by inspection.

If $S_{2,1} \notin P(e, \mathbf{T}'_1)$ let $\bar{\mathbf{T}}'_i = \mathbf{T}'_i - e$ (for $i = 1, 2$), then by induction the proposition is true for $(\bar{\mathbf{T}}_1, \bar{P}_1) = \alpha((\bar{\mathbf{T}}'_1, 2, -1))$. (To start the induction, note that if $|M| = 2$ then $S_{1,2} \in P(e, \mathbf{T}'_1)$.) But then $\mathrm{Shape}(\bar{\mathbf{T}}_1) = \mathrm{Shape}(\bar{\mathbf{T}}_2)$ (where $(\bar{\mathbf{T}}_2, \bar{P}_2) = \alpha((\bar{\mathbf{T}}'_2, 1, 1))$) so $P(e, \mathbf{T}_1) = P(e, \mathbf{T}_2)$. Since by induction $c(2, \bar{\mathbf{T}}_1)$ is closed we have $c(2, \mathbf{T}_1) = c(2, \bar{\mathbf{T}}_1)$. Hence $c(2, \mathbf{T}_1)$ is closed and $\mathbf{T}_2 = \mathbf{E}(\mathbf{T}_1, c(2, \mathbf{T}_1))$. □

2.3.8. THEOREM. (a) *Suppose* $\{1, 2\} \subseteq M_1$, $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$. *Suppose* $\gamma \in D^L_{\alpha\beta}(\mathscr{S}(M_1, M_2))$. *Then* $A(T^L_{\alpha\beta}(\gamma)) = T^L_{\alpha\beta}(A(\gamma))$.

(b) *Suppose* $\{1, 2\} \subseteq M_2$, $\{\alpha, \beta\} = \{\alpha_1, \alpha_2\}$, *and* $\gamma \in D^R_{\alpha\beta}(\mathscr{S}(M_1, M_2))$. *Then* $A(T^R_{\alpha\beta}(\gamma)) = T^R_{\alpha\beta}(A(\gamma))$.

Proof. As in the proof of Theorem 2.1.19, it suffices to prove (a). Let $k < l$ be such that $\{(1, k_1, \varepsilon_1), (2, k_2, \varepsilon_2)\} \subseteq \gamma$ and $\{k, l\} = \{k_1, k_2\}$. If $\alpha = \alpha_1$ then either

(a) $\{(1, k, 1), (2, l, -1)\} \subseteq \gamma$,
(b) $\{(2, k, 1), (1, l, 1)\} \subseteq \gamma$, or
(c) $\{(2, k, -1), (1, l, 1)\} \subseteq \gamma$.

We see then, using Proposition 2.3.5, the corresponding statement for $T^L_{\alpha\beta}$'s on $\mathscr{S}(M_1, M_2)$, Remark 2.1.6(2), and Proposition 2.3.6, that it suffices to prove the following: suppose $\gamma \in \mathscr{S}(M_1, M_2)$ and suppose $\{(1, k, 1), (2, l, -1)\} \subseteq \gamma$ with $k < l$. Let $\bar{\gamma} = \gamma \backslash \{(1, k, 1), (2, l, -1)\}$, let $\gamma' = \bar{\gamma} \cup \{(2, k, 1), (1, l, -1)\}$, and let $\tilde{\gamma} = \bar{\gamma} \cup \{(2, k, 1), (1, l, 1)\}$ (so $T^L_{\alpha\beta}(\gamma) = T^L_{\alpha\beta}(\tilde{\gamma}) = \{\gamma'\}$). Set $(\mathbf{T}_1, \mathbf{T}_2) = A(\gamma)$, $(\mathbf{T}'_1, \mathbf{T}'_2) = A(\gamma')$ and $(\tilde{\mathbf{T}}_1, \tilde{\mathbf{T}}_2) = A(\tilde{\gamma})$. Then

$$\tilde{F}_2 \subseteq \mathbf{T}_1, \quad F_1 \subseteq \mathbf{T}'_1 \quad \text{and} \quad F_2 \subseteq \tilde{\mathbf{T}}_1, \tag{2.3.9}$$

$$\tilde{\mathbf{T}}_1 = (\mathbf{T}'_1 \backslash F_1) \cup F_2 \quad \text{and} \quad \tilde{\mathbf{T}}_2 = \mathbf{T}'_2, \tag{2.3.10}$$

and

$$(\tilde{\mathbf{T}}_1, \tilde{\mathbf{T}}_2) = \mathbf{E}((\mathbf{T}_1, \mathbf{T}_2), ec(2, \mathbf{T}'_1, \mathbf{T}_2), L). \tag{2.3.11}$$

Now (2.3.9) and (2.3.10) follow easily from the definition of the map A (i.e. Definition 1.2.7). For (2.3.11), it is clear that if $l = \sup M_2$ this follows from Lemma 2.3.7; if $l \neq \sup M_2$ then (using induction on $|M_2|$) this is a consequence of Proposition 2.3.2(b). □

References

1. Barbasch, D. and Vogan, D.: Primitive ideals and orbital integrals in complex classical groups. *Math. Ann.* 259 (1982), 153–199.
2. Bourbaki, N.: Groupes et algebres de Lie, Chapitres 4, 5, et 6, Éléments de Mathématiques XXXIV. Paris: Hermann 1968.
3. Duflo, M.: Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple. *Ann. of Math.* 105 (1977), 107–120.
4. Garfinkle, D.: On the classification of primitive ideals for complex classical Lie algebras, I. *Comp. Math.* 75 (1990), 135–169.
5. Garfinkle, D.: The annihilators of irreducible Harish-Chandra modules for SU(p, q) and other type A_{n-1} groups. *Am. J. Math.*, to appear.
6. Jantzen, J.: Moduln mit einem höchsten Gewicht, *Lecture Notes in Mathematics*, vol 750. Springer, Berlin (1979).
7. Joseph, A.: A characteristic variety for the primitive spectrum of a semisimple Lie algebra, preprint. Short version in: Carmona, J. and Vergne, M. (eds), *Non-commutative Harmonic Analysis*: Lecture Notes in Mathematics vol. 587, pp. 102–118. Springer: Berlin (1977).
8. Joseph, A.: Towards the Jantzen conjecture, II. *Comp. Math.* 40 (1980), 69–78.
9. Kazhdan, D. and Lusztig, G.: Representations of Coxeter groups and Hecke algebras. *Inv. Math.* 53 (1979), 165–184.
10. Lusztig, G.: A class of irreducible representations of a Weyl group. *Proc. Kon. Nederl. Akad.*, Series A 82 (1979), 323–335.
11. Vogan, D.: A generalized τ-invariant for the primitive spectrum of a semisimple Lie algebra. *Math. Ann.* 242 (1979), 209–224.