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The Capelli identity and unitary representations

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Introduction

Let $\Omega = G/K$ be an irreducible Hermitian symmetric space of tube type and of rank n. Its Shilov boundary is of the form G/P where $P = LN$ is a maximal parabolic subgroup of G. It is known that $\mathfrak{n} = \mathrm{Lie}(N)$ is an abelian Lie algebra with a natural *Jordan* algebra structure. There is a distinguished polynomial (the Jordan norm) φ on $\mathfrak{n}$. The polynomial φ has degree n and transforms by a positive character ν^{-2} of L.

Let $\bar{P}$ be the opposite parabolic and consider for each $t \in \mathbf{R}$ the induced representation $I(t) = \mathrm{Ind}_{\bar{P}}^{G}(\nu^t \otimes 1)$ (normalized induction). Using the Gelfand–Naimark decomposition $G \approx NL\bar{N}$ and the exponential map, we may realize $I(t)$ as a subspace $E(t)$ of $C^\infty(\mathfrak{n})$.

The Killing form and Cartan involution yield an isomorphism from $\mathfrak{n}$ to $\mathfrak{n}^*$. Thus we get a differential operator $D = \partial(\varphi)$ corresponding to φ in the usual manner. It is known that for each $m \in \mathbf{Z}$, D^m intertwines $I(m)$ with its Hermitian dual $I(-m)$. So if $\langle , \rangle$ is the Hermitian pairing between $I(m)$ and $I(-m)$, then $(f_1, f_2)_m = \langle D^m f_1, f_2 \rangle$ is an invariant Hermitian form on $I(m)$.

Our main result is an explicit formula for the signature of this form on each K-type!

To state this result we introduce some notation. Let $\mathfrak{h}$ and $\mathfrak{h}^s$ be compact and maximally split Cartan subalgebras, respectively, of $\mathfrak{g} = \mathrm{Lie}(G)$. The K-types in $I(t)$ occur with multiplicity one, and have highest $\mathfrak{h}$-weights of the form $b_1\gamma_1 + \cdots + b_n\gamma_n$, where γ_i are the Harish-Chandra strongly orthogonal roots and $b_1 \geqslant \cdots \geqslant b_n$ are integers. Also, if $\mathfrak{a}^s$ is the split part of $\mathfrak{h}^s$, then the restricted root system $\Sigma(\mathfrak{a}^s, \mathfrak{g})$ is of type C_n, and all the short root spaces have a common dimension d. Our main result is

THEOREM 1. *For $\beta = b_1\gamma_1 + \cdots + b_n\gamma_n$, let $I(m)_\beta$ be the corresponding K-isotypic subspace of $I(m)$. Then for any $f \in I(m)_\beta$, we have*

$$(f,f)_m = q_m(\beta) \int_K |f(k)|^2 \, \mathrm{d}k$$

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where

$$q_m(\beta) = (-1)^{mn} \prod_{i=1}^{n} \prod_{j=1}^{m} \left(b_i + \frac{d}{4}(n - 2i + 1) + \frac{1}{2}(m - 2j + 1) \right).$$

Let us write $R(m)$ for the kernel of D^m and $Q(m)$ for the quotient $I(m)/R(m)$. We consider next the question of the unitarizability of $Q(m)$. Various parity considerations arise—there is an important difference between the cases of even and odd d.

THEOREM 2. *Suppose that either* (a) $d \equiv 0 (\mathrm{mod}\, 4)$ *and m is odd; or that* (b) $d \equiv 2 (\mathrm{mod}\, 4)$ *and* $m \equiv n (\mathrm{mod}\, 2)$. *Then $Q(m)$ is a direct sum of $n + 1$ unitary representations* $Q_0, \ldots, Q_n$.

This is proved in 3.1, and in 3.2 we extend this result to degenerate series representations of the universal covering group $\tilde{G}$ of G.

It may be shown (although we do not do so in this paper) that the Q_i are irreducible, Q_0 and Q_n are holomorphic and anti-holomorphic discrete series representations and that the other Q_i are cohomologically induced representations (indeed they are $A(\mathfrak{q}, \lambda)$'s for suitable θ-stable parabolics $\mathfrak{q}$).

For odd d the situation is completely different. The representation $Q(m)$ is either irreducible, or it breaks up into approximately $n/2$ pieces of which at most 2 are unitary. These are holomorphic and anti-holomorphic discrete series for G. The other "interesting" unitary representations occur for half-integral values of m, whose analysis requires an extension of the present techniques. This we leave to a future paper.

In the appendix we study the degenerate series of $\tilde{G}$. Using results of [G] we give a complete determination of the points of reducibility of this series. This part is independent of the rest of the paper and we include it only for the sake of completeness.

Here are the main ideas and techniques of this paper.

The first ingredient is the generalized Capelli identity of [KosS]. This is a formula for the radial part of L-invariant differential operator $\varphi^m D^m$ on $L/L \cap K$.

The second ingredient is the Cayley transform, as discussed in [KorW], for example. This is an element c of $\mathrm{Ad}(\mathfrak{g}_c)$ which interchanges $\mathfrak{k}_c$ with $\mathfrak{l}_c$, and $\mathfrak{h}_c^s$ with $\mathfrak{h}_c$.

Now, conjugation by c transforms $\varphi^m D^m$ to a $\mathfrak{k}_c$-invariant differential operator Δ_m on $I(m)$. We show that Δ_m acts by the scalar $q_m(\beta)$ on $I(m)_\beta$. The explicit formula for $q_m(\beta)$ then follows from the generalized Capelli identity!

We now describe some of the relevant literature. Since the work of [W] and [RV], two papers have obtained explicit formulas for the inner product on holomorphic discrete series with one-dimensional lowest K-types. Of these, [FK] which considers holomorphic functions on Ω, is closest to our approach. Since holomorphic functions are determined by their values on the Shilov boundary, the boundary value map imbeds these spaces inside the $I(m)$. It is possible to rederive some of the results of [FK] from our main result.

The paper [G] considers Hermitian representations with one-dimensional K-types (semi-spherical) which are annihilated by a certain ideal in $\mathscr{U}(\mathfrak{g})$, deriving an explicit formula for the inner product. It is easy to see that, while any such representation must be a constituent of $I(t)$, most unitary constituents of $I(t)$ are not semi-spherical. If m is large enough, then among the unitary representations in Theorem 2 only the discrete series Q_0 and Q_n have one dimensional K-types.

The representations $I(0)$ have been studied for $SU(n, n)$ and $Sp(n, \mathbf{R})$ in [KaV] and the general $I(t)$ is considered for $SU(2, 2)$ in [JV] and [S].

It is a pleasure to thank Bert Kostant for invaluable advice and encouragement. While various ideas in this paper were worked out in Fall 1987, the whole picture became much clearer to me after the results of [KosS] were obtained. The final version of this paper has also benefited from helpful discussions with David Vogan and Gregg Zuckerman.

Finally, it would be a shame to permit the 'Cayley coincidence' to go unremarked. Indeed a provisional title for this paper was "The Cayley Transform of the Cayley Operator and Unitary Representations", which was abandoned as being too cumbersome.

1. Preliminaries

The material of this section is well-known. All the basic facts may be found in [KorW], [KosS] and [B]. In what follows, all Lie algebras will be real unless complexified with a subscript "c".

1.1. The Cayley transform

Let $(\mathfrak{g}, \mathfrak{k})$ be an irreducible Hermitian symmetric pair of tube type. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and let $\mathfrak{a}^s$ be a Cartan subspace of $\mathfrak{p}$. Choose $\mathfrak{t}^s \subseteq \mathfrak{k}$ so that $\mathfrak{h}^s = \mathfrak{a}^s \oplus \mathfrak{t}^s$ is a maximally split Cartan subalgebra of $\mathfrak{g}$.

It is known ([M]) that the restricted root system is of type C_n where $n = \dim(\mathfrak{a}^s)$. Thus we may choose a basis $\varepsilon_1, \ldots, \varepsilon_n$ for $(\mathfrak{a}^s)^*$ such that

$$\sum(\mathfrak{g}, \mathfrak{a}^s) = \{\pm\varepsilon_i \pm \varepsilon_j\} \cup \{\pm 2\varepsilon_j\}.$$

The root spaces for $\pm\varepsilon_i \pm \varepsilon_j$ have a common dimension which we will denote by d. The root spaces for $\pm 2\varepsilon_j$ are one-dimensional. Thus $\pm 2\varepsilon_j$ may be regarded as a root of $\mathfrak{h}^s$ in $\mathfrak{g}$, vanishing on $\mathfrak{t}^s$.

To each $2\varepsilon_j$ we attach in a standard manner (see [KosR]) an S-triple $\{h_j, e_j, f_j\}$, contained in $\mathfrak{g}$. These S-triples commute, and if

$$h = \sum_{j=1}^{n} h_j,\ e = \sum_{j=1}^{n} e_j,\ f = \sum_{j=1}^{n} f_j$$

then $\{h, e, f\}$ is also an S-triple.

The eigenvalues of $\mathrm{ad}(h)$ on $\mathfrak{g}$ are $-2, 0$ and 2. Let us write $\bar{\mathfrak{n}}$, $\mathfrak{l}$ and $\mathfrak{n}$ for the corresponding eigenspaces. Then

$$\Sigma(\bar{\mathfrak{n}}, \mathfrak{a}^s) = \{-\varepsilon_i - \varepsilon_j\} \cup \{-2\varepsilon_j\},\ \Sigma(\mathfrak{l}, \mathfrak{a}^s) = \{\pm(\varepsilon_i - \varepsilon_j)\},\ \Sigma(\bar{\mathfrak{n}}, \mathfrak{a}^s) = \{\varepsilon_i + \varepsilon_j\} \cup \{2\varepsilon_j\}.$$

Clearly $\mathfrak{q} = \mathfrak{l} + \mathfrak{n}$ is a maximal parabolic subalgebra whose one-dimensional center is $\mathfrak{a} = \mathbf{R}h$. Let $\mathfrak{m} \oplus \mathfrak{a}$ be the Langlands decomposition of $\mathfrak{l}$.

We fix positive restricted root systems of $\mathfrak{g}$ and $\mathfrak{l}$, lexicographically with respect to $h_1, \ldots h_n$. Thus

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}^s) = \{e_i \pm \varepsilon_j | i \leqslant j\} \cup \{2\varepsilon_j\},\ \Sigma^+(\mathfrak{l}, \mathfrak{a}^s) = \{\varepsilon_i - \varepsilon_j | i \leqslant j\}.$$

The Cayley transform is the element (of order 4) in $\mathrm{Ad}(\mathfrak{g}_c)$ defined by $c = \exp \pi i/4(e + f)$. It is known that $c(\mathfrak{l}_c) = \mathfrak{k}_c$ and $c(\mathfrak{k}_c) = \mathfrak{l}_c$, and that c^2 is the Cartan involution for the symmetric pairs $(\mathfrak{k}, \mathfrak{l} \cap \mathfrak{k})$ and $(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k})$.

Let us write $\mathfrak{b} = \mathfrak{k} \cap c(\mathfrak{h}_c^s)$. Then $\mathfrak{b}$ is a compact Cartan subalgebra for $\mathfrak{g}$ (and $\mathfrak{k}$), and we have the decomposition $\mathfrak{b} = \mathfrak{t} + \mathfrak{t}^s$, where $\mathfrak{t} = ic(\mathfrak{a}^s)$. The Harish-Chandra strongly orthogonal roots in $\Sigma(\mathfrak{b}_c, \mathfrak{g}_c)$ are $\gamma_1, \ldots, \gamma_n$ where $\gamma_i = c \circ (2\varepsilon_i)$.

1.2. The Capelli identity

Let $\mathscr{P}$ be the polynomial algebra on $\mathfrak{n}$. Then $\mathfrak{l}$ acts naturally on $\mathscr{P}$ by

$$\pi(u)f(x) = \frac{d}{dt} f(x + t[u, x])|_{t=0} \quad \text{for } u \in \mathfrak{l},\ f \in \mathscr{P},\ x \in \mathfrak{n}.$$

It is known (see [KosS] for details and references) that the algebra of lowest weight vectors of $(\pi, \mathscr{P})$ is a polynomial algebra in n generators $\varphi_1, \ldots, \varphi_n$ with $\deg(\varphi_j) = j$ and $\mathrm{weight}(\varphi_j) = -2\nu_j$, where $\nu_j = \varepsilon_1 + \cdots + \varepsilon_j$.

Furthermore, $\mathfrak{n}$ has the structure of a formally real Jordan algebra with e (see 1.1) as the identity. We normalize the φ_j by requiring that $\varphi_j(e) = 1$ and write ν for ν_n and φ for φ_n. Then φ is the Jordan norm polynomial for $\mathfrak{n}$.

Let θ be the Cartan involution on $\mathfrak{g}$, then $(x, y) = B(x, \theta y)/B(e, f)$ is a positive definite inner product on $\mathfrak{n}$ (where B is the Killing form). Thus we have an isomorphism ∂ from $\mathscr{P}$ to the algebra of constant coefficient differential operators on $\mathfrak{n}$. We write D for $\partial(\varphi)$.

It is easy to see that $\varphi^m D^m$ commutes with $\pi(\mathfrak{l})$ on $\mathscr{P}$. It follows from Schur's Lemma that it acts by a scalar on each irreducible constituent of $\mathscr{P}$.

PROPOSITION ([KosS]). *If V is an irreducible subrepresentation of $(\pi, \mathscr{P})$ with highest weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$, then for all f in V,*

$$\varphi^m D^m f = \prod_{i=1}^{n}\prod_{j=1}^{m}\left(-\frac{\lambda_i}{2} + \frac{d}{2}(i-1) - j + 1\right) f.$$

Moreover, there is an $\operatorname{ad}(\mathfrak{l} \cap \mathfrak{k})$*-invariant element X_m in the enveloping algebra* $\mathscr{U}(\mathfrak{l})$ *such that*

$$\pi(X_m) = \varphi^m D^m.$$

Proof. This is a reformulation of the main result of [KosS]. The function denoted by f^λ in (3.16) of that paper is a highest weight vector of weight $-\Sigma_i(\lambda_i - d/2(n - 2i + 1))\varepsilon_i$, and (4.4) in that paper shows that

$$\varphi^m D^m f^\lambda = \prod_{i=1}^{n}\prod_{j=1}^{m}\left(\frac{\lambda_i}{2} + \frac{d}{4}(n-1) - j + 1\right) f^\lambda.$$

Thus if f is a highest weight vector of weight λ, then

$$\varphi^m D^m f = \prod_{i=1}^{n}\prod_{j=1}^{m}\left(-\frac{1}{2}\left(\lambda_i + \frac{d}{2}(n - 2i + 1)\right) + \frac{d}{4}(n-1) - j + 1\right) f,$$

which, after simplification, proves the first part of the Proposition.

Also, as shown in Section 4 of [KosS] (although with slightly different notation), $\varphi^m D^m$ is an invariant differential operator on $L/L \cap K$. Now the existence of X_m follows from the fact that the natural map from $\mathscr{U}(\mathfrak{l})^{L \cap K}$ to $\mathbf{D}(L/L \cap K)$ is surjective. □

1.3. The induced representation

Let $\Omega = G/K$ be the Hermitian symmetric space corresponding to the pair $(\mathfrak{g}, \mathfrak{k})$ and let P be the normalizer of $\mathfrak{q}$ in G. Then G/P is the Shilov boundary of Ω in the bounded (Harish–Chandra) realization. Let $\bar{P}$ be the opposite parabolic subgroup and let $\bar{P} = L\bar{N} = MA\bar{N}$ be its Levi and Langlands decompositions.

Let δ be half the sum of restricted roots in $\mathfrak{n}$. Then an easy calculation shows that $\delta = r\nu$ where $\nu = \varepsilon_1 + \cdots + \varepsilon_n$ as in 1.2 and $r = (nd - d + 2)/2$. For each $t \in \mathbf{R}$, we define the character χ_t of L by $\chi_t(m \exp(H)) = \exp(t\nu(H))$ for all $m \in M$ and $H \in \mathfrak{a}$. We define

$$I(t) = \{f \in C^\infty(G) | f(l\bar{n}g) = \chi_{t-r}(l) f(g) \quad \text{all } l \in L,\ \bar{n} \in \bar{N} g \in G\},$$

and we write π_t for the representation of G on $I(t)$ by right translations.

Since $G = \bar{P}K$, restriction to K is a K-isomorphism between $I(t)$ and $C^\infty((L \cap K)\backslash K)$. Now, as remarked in 1.1, $(K, L \cap K)$ is a symmetric pair. It follows that each K-type in $I(t)$ has multiplicity one. Furthermore, the K-types which do occur are exactly those that have an $L \cap K$-fixed vector. By the Cartan–Helgason theorem, this happens exactly when the highest weight β of the K-type satisfies $\beta|\mathfrak{t} = 0$ and

$$\beta|\mathfrak{t}^s = \sum b_j \gamma_j \quad \text{where } b_j \in \mathbf{Z},\ b_1 \geqslant \cdots \geqslant b_n$$

where the γ_j are as in 1.1.

We will write $I(t)_\beta$ for the K-type of $I(t)$ of highest weight β.

Since $\bar{P}N$ is open and dense in G, restriction to N gives an injection from $I(t)$ to $C^\infty(N)$. Using the exponential map to identify $\mathfrak{n}$ and N, we get an embedding $f \mapsto \tilde{f}$ from $I(t)$ into $C^\infty(\mathfrak{n})$. The image $E(t)$ will be referred to as the non-compact picture.

In this picture, $\mathfrak{g}$ acts by polynomial coefficient vector fields and so π_t gives a homomorphism from the enveloping algebra $\mathscr{U}(\mathfrak{g})$ to the Weyl algebra $\mathscr{W}$ of polynomial coefficient differential operators on $\mathfrak{n}$.

The action of $\mathfrak{q}$ is particularly simple, and we have

$$\pi_t(x) = \partial(x) \quad \text{for all } x \in \mathfrak{n}$$

$$\pi_t(u) = (t - r)\nu(u) + \pi(u) \quad \text{for all } u \in \mathfrak{l}$$

where $\pi(u)$ is as in 1.2 and ν is regarded as a character of $\mathfrak{l}$ vanishing on $\mathfrak{m}$.

If $D = \partial(\varphi)$ is as in 1.2, then D: $E(t) \to C^\infty(\mathfrak{n})$. The crucial property of D^m is the following:

PROPOSITION ([B]). *For each integer $m \geqslant 0$, D^m intertwines π_m and π_{-m}.* □

For a different proof see Proposition V.6.1 in [JV].

For $f_1 \in I(t)$ and $f_2 \in I(-t)$ we define

$$\langle f_1, f_2 \rangle = \int_{\bar{P}\backslash G} f_1(g) \bar{f}_2(g)\, d\dot{g}$$

where $d\dot{g}$ is the right-invariant measure on $\bar{P}\backslash G$.

It is easily checked that the above expression is well-defined, and that it gives a non-degenerate, Hermitian, G-invariant pairing between $I(t)$ and $I(-t)$.

The pairing has a very simple expression in the non-compact picture. It becomes

$$\langle f_1, f_2 \rangle = \int_{\mathfrak{n}} f_1 \bar{f}_2 \, dx$$

for all $f_1 \in E(t)$ and $f_2 \in E(-t)$.

2. The signature of the hermitian form

In this section we prove Theorem 1.

2.1. Finite dimensional subrepresentations

Let $\nu = \varepsilon_1 + \cdots + \varepsilon_n$ be as in 1.2. Regard ν as a character of $\mathfrak{h}^s$ by extending it trivially on $\mathfrak{t}^s$. Then by the Cartan–Helgason theorem, for each nonnegative integer m, there is a finite-dimensional spherical representation of G with highest weight $2m\nu$. An easy calculation of central characters shows that these finite dimensional representations occur as constituents of certain of the $E(t)$'s. The next lemma makes this precise.

Let $r = (nd - d + 2)/2$ be as in 1.3, and let $m \geqslant 0$ be an integer.

LEMMA 1. *$E(r + 2m)$ has a finite-dimensional subrepresentation F_{2m} with highest weight $2m\nu$. Furthermore, if $f \in F_{2m}$, then f is a polynomial on $\mathfrak{n}$.*

Proof. As remarked in 1.3, π_t gives a homomorphism of $\mathcal{U}(\mathfrak{g})$ into $\mathcal{W}$. Thus π_t may be extended to an action on all of $C^\infty(\mathfrak{n})$.

For $t = r + 2m$, the constant function is $\pi_t(\mathfrak{n})$-invariant and transforms by the character $2m\nu$ of $\mathfrak{l}$. Thus it generates a finite-dimensional subrepresentation F_{2m}, say, of $C^\infty(\mathfrak{n})$. This exponentiates to a spherical representation of the group G. It follows that F_{2m} consists of K-finite vectors and so is contained in $E(t)$.

Finally, if f is in F_{2m} then $f = \pi_t(u)\cdot 1$ for some u in $\mathcal{U}(\mathfrak{g})$. Since $\pi_t(u) \in \mathcal{W}$, f must be a polynomial. □

Let φ be the Jordan norm as in 1.2. Define the polynomial ψ (of degree $2n$) by

$$\psi(x) = \varphi(e + x^2) \quad \text{for all } x \in \mathfrak{n}$$

where e is as in 1.1 and x^2 is the square in the Jordan algebra.

LEMMA 2. *The functions* 1 *and* φ^{2m} *are the highest and lowest weight vectors in* F_{2m}. *The functions* φ^m *and* ψ^m *span the one-dimensional spaces of L-fixed and K-fixed vectors respectively.*

Proof. The formula for $\pi_{r+2m}(\mathfrak{l})$ shows that the only polynomials transforming by a character of $\pi_{r+2m}(\mathfrak{l})$ are the powers φ^k for $k = 0, 1, \ldots$. The first part of the Lemma follows after we note that $1, \varphi^m, \varphi^{2m}$ transform by the appropriate weights of $\mathfrak{a}$.

It remains to show that ψ^m is K-fixed. Complexifying the action of $\mathfrak{g}$ on F_{2m} we get a holomorphic representation of $\mathfrak{g}_c$. Let c be the Cayley transform as in 1.2. Then c acts on F_{2m} and it suffices to show that c takes φ^m to ψ^m.

Let $e_1 \cdots e_n$ be as in 1.1. Then the set $\{\mathrm{ad}(L \cap K)(\alpha_1 e_1 + \cdots + \alpha_n e_n) | \alpha_j \in \mathbf{R}\}$ is open in $\mathfrak{n}$. (Indeed as remarked in Proposition 1 of [KosS], this is an open cone.) On the other hand, since φ^m and ψ^m are both $L \cap K$-invariant, it suffices to show that

$$c(\varphi^m)(\alpha_1 e_1 + \cdots + \alpha_n e_n) = (1 + \alpha_1^2)^m \cdots (1 + \alpha_n^2)^m.$$

However, this follows from the definition of c and an easy $\mathfrak{sl}_2$-calculation. □

Each of the representations π_t is spherical. Let f_t be the K-fixed vector in $I(t)$; then we have $f_t(ma\bar{n}k) = (\alpha^\vee)^{t-r}$. The next Lemma gives an explicit formula for $\tilde{f}_t$.

LEMMA 3. *The spherical vector in* $E(t)$ *is* $\psi^{t-r/2}$.

Proof. The formula for f_t shows that $f_{t+r}(g) = (f_{2+r}(g))^{t/2}$. On the other hand, Lemma 2 shows that the spherical vector in $E(2 + r)$ is ψ. Thus the spherical vector in $E(t + r)$ is $\psi^{t/2}$ and the result follows. □

LEMMA 4. *For each w and t in* $\mathbf{R}$, *the map* $f \mapsto \psi^t f$ *is a K-isomorphism between* $E(w)$ *and* $E(w + 2t)$.

Proof. The definition of $I(t)$ in 1.3 shows that if $f \in I(w)$ and $f' \in I(2t + r)$, then $f'f$ belongs to $I(w + 2t)$. In particular, $f_{2t+r}f$ belongs to $I(w + 2t)$ for all f in $I(w)$. Going over to the compact picture, we see that the map $f \mapsto f_{2t+r}f$ is a K-isomorphism – indeed it is just the identity map in the compact picture. On the other hand, by Lemma 3, this map becomes $f \mapsto \psi^t f$ in the noncompact picture. □

As remarked in 1.3, for each m, π_m gives a homomorphism of $\mathscr{U}(\mathfrak{g})$ into the Weyl algebra $\mathscr{W}$. Let us write $\mathscr{W}_m$ for the image. Then $\mathscr{W}_m$ contains all constant coefficient differential operators and is stable under the adjoint action of $\pi_m(G)$. The next Lemma is due to Kostant.

LEMMA 5. *For each* $p \in F_{2m}$, *the operator* pD^m *belongs to* $\mathscr{W}_m$. *Moreover, the* *map* $p \mapsto pD^m$ *intertwines* F_{2m} *with a subrepresentation of the adjoint action of G on* $\mathcal{W}_m$.

Proof. (Kostant) Fix $f \in E(m)$, $g \in G$ and let $f' = D^m \pi_m(g)^{-1} f$. Then by Lemma 2.3, $f' \in E(-m)$ and $\pi_{-m}(g) f' = D^m f$.

Now, if $p \in F_{2m} \subseteq E(r + 2m)$, then as remarked in the proof of Lemma 4, $pf' \in E(m)$. Further, it is easy to check that $\pi_m(g)(pf') = (\pi_{r+2m}(g)p)(\pi_{-m}(g)f')$. Rewriting this in terms of f, we get

$$(\pi_{r+2m}(g)p)(D^m f) = \pi_m(g)(pD^m \pi_m(g)^{-1} f). \tag{1}$$

Now set $p = 1$ in (1). This shows that

$$(\pi_{r+2m}(g) \cdot 1) D^m = \pi_m(g) D^m \pi_m(g)^{-1}. \tag{2}$$

By the remarks preceding this Lemma, the right side of (2) is in $\mathcal{W}_m$. On the other hand, since F_{2m} is irreducible, the set $\{\pi_{r+2m}(g) \cdot 1 \mid g \in G\}$ spans F_{2m}. This proves the first part of the Lemma. The second part is now just the identity (1). □

Let us write Δ_m for the differential operator $\psi^m D^m$. In Lemma 2 we showed that in the representation of $\mathfrak{g}_c$ on F_{2m}, the Cayley transform c maps φ^m to ψ^m. Combining this with Lemma 5 gives us the following key result.

PROPOSITION. $\Delta_m = \pi_m(c)\varphi^m D^m \pi_m(c)^{-1}$. □

2.2. *Proof of Theorem 1*

For each integer $m \geqslant 0$, we define a Hermitian form $(,)_m$ on $E(m)$ by

$$(f, f')_m = \langle D^m f, f' \rangle, \tag{1}$$

where f and f' belong to $E(m)$ and $\langle , \rangle$ is as in 1.2.

Proposition 1.3 shows that $(,)_m$ is $\pi_m(\mathfrak{g})$-invariant. Consequently, for each β as in 1.3 there is a constant $q_m(\beta)$ such that for all $f \in I(m)_\beta$,

$$(\tilde{f}, \tilde{f})_m = q_m(\beta) \int_K |f(k)|^2 \, dk. \tag{2}$$

The right side of (2) can be expressed in the non-compact picture as follows:

LEMMA 1. *Let* ψ *be as in* 2.1, *then for all* $f \in I(m)$,

$$\int_K |f(k)|^2 \, dk = \int_{\mathfrak{n}} |\tilde{f}|^2 \psi^m \, dx.$$

Proof. This is easy to check for $m = 0$. The general case follows by Lemma 2.1.4. □

COROLLARY. *For each* $f \in E(m)_\beta$, $\psi^m D^m f = q_m(\beta) f$.

Proof. This follows by combining (1), (2) and the Lemma. □

Now consider the restriction of $(\pi_m, C^\infty(\mathfrak{n}))$ to $\mathfrak{l}$. The formula in 1.3 shows that if $f \in \mathscr{P}$ has $\pi_m(\mathfrak{a}^s)$-weight $\Sigma \lambda_i \varepsilon_i$, then it has $\pi(\mathfrak{a}^s)$-weight $\Sigma(\lambda_i + m - r)\varepsilon_i$. Combining this with Proposition 1.2 gives

LEMMA 2. *Let V be a finite dimensional, irreducible $\pi_m(\mathfrak{l})$-subrepresentation of $C^\infty(\mathfrak{n})$ with highest weight $\Sigma_i \lambda_i \varepsilon_i$, then for all $f \in V$*

$$\varphi^m D^m f = (-1)^{mn} \prod_{i=1}^{n} \sum_{j=1}^{m} \left(\frac{\lambda_i}{2} + \frac{d}{4}(n - 2i + 1) + \frac{1}{2}(m - 2j + 1)\right) f.$$

Further, there is an $ad(\mathfrak{l} \cap \mathfrak{k})$-invariant element Y_m in $\mathscr{U}(\mathfrak{l})$ such that $\pi_m(Y_m) = \varphi^m D^m$. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let $Z_m = \mathrm{ad}(c) Y_m$, then $Z_m \in \mathscr{U}(\mathfrak{k}_c)^{\mathfrak{l} \cap \mathfrak{k}}$ and Proposition 2.1 shows that $\pi_m(Z_m) = \Delta_m$.

Now if f is a highest weight vector for $\pi_m(\mathfrak{l})$ with weight $2\Sigma_i b_i \varepsilon_i$ then 1.3 implies that $\pi_m(c) f$ is a highest weight vector for $\pi_m(\mathfrak{k}_c)$ with weight $\beta = \Sigma_i b_i \gamma_i$. Thus by Lemma 2,

$$\begin{aligned} q_m(\beta)\pi_m(c) f &= \Delta_m \pi_m(c) f = \pi_m(Z_m)\pi_m(c) f = \pi_m(c)\pi_m(Y_m) f \\ &= \pi_m(c)\left((-1)^{mn} \prod_{i=1}^{n} \prod_{j=1}^{m} \left(b_i + \frac{d}{4}(n - 2i + 1) + \frac{1}{2}(m - 2j + 1)\right) f\right). \end{aligned}$$ □

3. Certain unitary representations

In this section we prove Theorem 2 and extend our results to the universal covering group.

3.1. Proof of Theorem 2

Let $E(m)$ and $(,)_m$ be as in 2.2, and write $V(m)$ for the Harish-Chandra module of K-finite vectors in $E(m)$. If $R(m)$ is the radical of $(,)_m$, then $R(m)$ is a submodule of $V(m)$; and the quotient $Q(m) = V(m)/R(m)$ is a Harish-Chandra module with a non-degenerate Hermitian form.

Theorem 1 completely determines the K-structure and the form on $Q(m)$. We will show that the form is definite on certain submodules of $Q(m)$. For this we need an elementary Lemma. Fix a positive integer m and let $\mathcal{Q}$ and $\mathcal{R}$ be the sets of highest weights of K-types of in $Q(m)$ and $R(m)$ respectively.

LEMMA. *Suppose S is a subset of $\mathcal{Q}$ with the property that for each $\beta \in S$, $\beta \pm \gamma_j \in S \cup \mathcal{R}$ for each j. Then $\Sigma\{Q(m)_\beta \mid \beta \in S\}$ is a $\mathfrak{g}$-submodule of $Q(m)$.*

Proof. From 1.1 we see that the $\mathfrak{t}$-weights of the adjoint representation are $\pm\gamma_i$ and $\frac{1}{2}(\pm\gamma_i \pm \gamma_j)$. So if V_β is a K-type of $V(m)$, then $\pi_m(\mathfrak{g})V_\beta \subseteq \Sigma\{V_\alpha \mid \alpha = \beta, \beta \pm \gamma_j\}$.

Now let $W = \Sigma_{\beta \in S} V(m)_\beta$. Then the assumption of the Lemma shows that $\pi_m(\mathfrak{g})W \subseteq W + R(m)$, which clearly implies the result. □

Proof of Theorem 2. First suppose that (a) $d = 4k$ and $m = 2p - 1$ with $k, p \in \mathrm{N}$. Then $q_m(\beta)$ becomes

$$(-1)^n \prod_{i=1}^{n} \prod_{j=1}^{2p-1} (b_i + k(n - 2i + 1) + p - j).$$

Let us write $\mathcal{Q}$ for the K-types of $Q(m)$ and a_i for $k(n - 2i + 1)$. The formula shows that $\mathcal{Q}$ consists of $\beta = \Sigma_i b_i \gamma_i$ such that (a) $b_1 \geqslant \cdots \geqslant b_n \in \mathbf{Z}$ and (b) for each i, either $b_i + a_i \leqslant -p$ or $b_i + a_i \geqslant p$.

Let $S_l = \{\beta \in \mathcal{Q} \mid b_1 \geqslant \cdots \geqslant b_l \geqslant p - a_l$ and $-p - a_{l+1} \geqslant b_{l+1} \geqslant \cdots \geqslant b_n\}$. Then $\mathcal{Q}$ is the disjoint union of $S_0, \ldots, S_n$ and each S_l satisfies the hypothesis of the Lemma above. Thus if we write Q_l for $\Sigma\{Q(m)_\beta \mid \beta \in S_l\}$, then each Q_l is a submodule, and $Q(m) = Q_0 + \cdots + Q_n$. It is easily checked that for β in a fixed S_l, all $q_m(\beta)$ have the same sign. Consequently Q_l is unitarizable.

The argument for case (b), when $d \equiv 2 (\mathrm{mod}\ 4)$ and $m \equiv n (\mathrm{mod}\ 2)$, is similar. □

It can be shown that the representations Q_l are irreducible, but the proof is a bit involved. Once this is proved, Proposition 6.1 of [VZ] can be applied to show that Q_l is an $A(\mathfrak{q}_l, \lambda)$, where $\mathfrak{q}_l$ is a θ-stable parabolic whose Levi component is a real form of $\mathfrak{k}$. We postpone this to a future paper.

The representations Q_0 and Q_n are the holomorphic (and anti-holomorphic) discrete series considered by Wallach in [W]. They have 1-dimensional lowest K-types. Renormalizing our form to be 1 on the lowest K-type gives an explicit formula for the form considered by [W]! This yields Wallach's results on the analytic continuation of these series. Since these results are already known (see [FK]), we omit the details.

Finally, let us point out that Theorem 2 applies to the groups $SO^*(4n)$ (with $d = 4$), $O(4p + 1, 2)$ (with $d = 4p$), $E_{7(-25)}$ (with $d = 8$), $U(n, n)$ (with $d = 2$) and $O(4p + 3, 2)$ (with $d = 4p + 2$).

3.2. Degenerate series for the universal covering group

Let $\tilde{G}$ be the universal covering group of G and let $\tilde{L}$ be the inverse image of L in G. Then it may be shown that $\tilde{L} \approx L \times \mathbf{Z}$. The characters of $\mathbf{Z}$ may be identified with the set $[0, 1)$ through the correspondence $\alpha \mapsto \xi_\alpha$ where $\xi_\alpha(\eta) = \exp(2\pi i \alpha \eta)$.

We define $\pi_{\alpha,t}$ to be the representation of $\tilde{G}$ by right translations on

$$\begin{aligned} I(\alpha, t) = \{ f \in C^\infty(\tilde{G}) \mid & f(\eta l \bar{n} \tilde{g}) \\ & = \xi_\alpha(\eta) \chi_{t-r}(l) f(\tilde{g}) \quad \text{all } \eta \in \mathbf{Z},\ l \in L,\ \bar{n} \in \bar{N},\ \tilde{g} \in \tilde{G} \}. \end{aligned}$$

Observe that $I(0, t)$ is just $I(t)$.

An easy calculation shows that the $\tilde{K}$-types occur with multiplicity one and have highest weights β such that $\beta|\mathfrak{t} = 0$ and

$$\beta|\mathfrak{t}^s = \sum b_j \gamma_j \quad \text{where } b_1 \geqslant \cdots \geqslant b_n \in \mathbf{Z} + \alpha.$$

As before, restricting to N and using the exp map, we get an injection from $I(\alpha, t)$ to $C^\infty(\mathfrak{n})$. Let us write $E(\alpha, t)$ for the image. The space $E(\alpha, t)$ depends on both α and t, however the action of $\pi_{\alpha,t}(\mathfrak{g})$ depends only on t.

We observe also that the Hermitian dual of $E(\alpha, t)$ is $E(\alpha, -t)$. As before, D^m intertwines $E(\alpha, m)$ with its Hermitian dual, so that we get a Hermitian form on $E(\alpha, m)$ for each m. The signature of this form is given by Theorem 1 (the only difference is that the b_j are in $\mathbf{Z} + \alpha$).

Let us write $V(\alpha, m)$ for the Harish-Chandra module of $E(\alpha, m)$ and $Q(\alpha, m)$ for the quotient of $V(\alpha, m)$ by the radical of the form.

Arguing as in the proof of Theorem 2 we find that if (a) $d = 4k$ and m is even, or if (b) $d = 4k + 2$ and $m \not\equiv n \pmod 2$, then $Q(\frac{1}{2}, m)$ is a direct sum of $n + 1$ unitary representations whose $\mathfrak{k}$-types may be explicitly computed as before.

Appendix

In this appendix we study the reducibility of $I(\alpha, t)$ using a result of Guillemonat. For the group $Sp(n, \mathbf{R})$, Kudla and Rallis [KuR] describe how this may be done. However their result depends crucially on Proposition 1.2 of [KuR] which does not obviously hold for other tube domains.

Guillemonat in [G], considers a problem very similar to the one discussed in our paper. He studies representations of $\mathfrak{g}$ which have a one-dimensional $\mathfrak{k}$-type annihilated by a certain ideal $\mathbf{J}$ in the enveloping algebra. These representations are constructed as follows:

Let $\mathfrak{a}$ be as in 1.1 and let $\mathfrak{z}$ be the center of $\mathfrak{k}$. Fix $\lambda \in \mathfrak{a}^*$ and $\chi \in \mathfrak{z}^*$ and let $\varphi_{\chi,\lambda}$ be the $i\chi$-semi-spherical function on $\tilde{G}$ with infinitesimal character corresponding to λ. The representation $V'_{\chi,\lambda}$ of [G] is obtained by the right translations of $\varphi_{\chi,\lambda}$. This representation has a natural Hermitian form (normalized to be 1 on $\varphi_{\chi,\lambda}$). The quotient of $V'_{\chi,\lambda}$ by this form is an irreducible representation denoted $V_{\chi,\lambda}$ in Section 5 of [G].

The connection with our paper is that $V_{\chi,\lambda}$ is a constituent of $I(\alpha, t)$ for

$$\alpha = i\chi(H)/2n \pmod{\mathbf{Z}} \quad \text{and } t = \lambda(E)/n$$

where H and E are in Section 2.2 and Section 3.1 of [G]. (This follows by elementary infinitesimal character and $\mathfrak{k}$-type considerations.) Furthermore, for these values of the parameters, $V'_{\chi,\lambda}$ and $I(\alpha, t)$ have the same $\mathfrak{k}$-types. Therefore, $I(\alpha, t)$ is irreducible if and only if the corresponding $V_{\chi,\lambda}$ is irreducible. We claim that this happens if and only if the form on $V'_{\chi,\lambda}$ is non-degenerate. This depends on the following simple observation:

If the form is degenerate then the representation is clearly reducible. Suppose the form is non degenerate. Since $\varphi_{\chi,\lambda}$ is a cyclic vector, it suffices to show that the representation is completely reducible. This is a consequence of the next Lemma.

LEMMA. *Suppose V is a Harish–Chandra module with a non-degenerate invariant Hermitian form. If each K-type in V has multiplicity one, then V is completely reducible.*

Proof. If W be a submodule of V, then $W^{\perp} \equiv \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}$ is also a submodule. On the other hand, since each K-type in V has multiplicity one, the form is definite on each K-type. Since different K-types are orthogonal, it follows that $W^{\perp}$ contains exactly the K-types not in W. Consequently $V = W \oplus W^{\perp}$. □

In Section 6 of [G] it is shown that the form on $V'_{\chi,\lambda}$ is non-degenerate if and only if the expressions in formulas (1′) and (2′) of [G Section 6.9] are non-zero. Rewriting these formulas in our notation the requirement becomes

$$\prod_{i=1}^{n}\prod_{j=1}^{m_i}\left[\left(\pm\alpha-\frac{1}{2}+\frac{d}{4}(n+1-2i)+j\right)^2-\frac{t^2}{4}\right]\neq 0 \quad \text{for all } 0\leqslant m_n\leqslant\cdots\leqslant m_1\in\mathbf{Z}.$$

This is equivalent to the condition that for each $i = 1, \ldots, n$,

$$\left(\pm\alpha\pm\frac{t}{2}\right)\not\equiv\left(\frac{1}{2}+\frac{d}{4}(n+1-2i)\right)\pmod{\mathbf{Z}}.$$

Thus for $d \equiv 0 \pmod 4$, $I(0, t)$ is reducible for odd integral values of t, and

$I(\frac{1}{2}, t)$ is reducible for even integral values of t. While for $d \equiv 2 \pmod 4$, $I(0, t)$ is reducible when $t - n$ is an even integer and $I(\frac{1}{2}, t)$ is reducible when $t - n$ is an odd integer.

For odd values of d, the representations $I(0, t)$ and $I(\frac{1}{2}, t)$ are reducible for integral values of t and also for half-integral values. There are unitary representations at these other points which are not amenable to the present analysis. We intend to discuss these and other unitary constituents of $I(t)$ in a future paper.

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