COMPOSITIO MATHEMATICA

RAYMOND ROSS

K_2 of elliptic curves with sufficient torsion over Q

Compositio Mathematica, tome 81, n° 2 (1992), p. 211-221
<http://www.numdam.org/item?id=CM_1992__81_2_211_0>

Compositio Mathematica **81**: 211–221, 1992.

K_2 of elliptic curves with sufficient torsion over Q

RAYMOND ROSS

University of Southern California, Los Angeles, CA 90089-1113, U.S.A.

Received 13 February 1991; accepted 22 April 1991

1. Introduction

The conjectures of Beilinson and Bloch ([1]–[3]) relate the conjectural behavior at $s=0$ of the Hasse–Weil L-function $L(s, E)$ of an elliptic curve E defined over $\mathbf{Q}$ to K_2E via a regulator which generalizes that of Dirichlet. Part of what the conjectures assert is that K_2E is a finitely generated abelian group of rank $1+|\mathrm{Spl}(E)|$, where $\mathrm{Spl}(E)$ denotes the set of primes where E has split multiplicative reduction [3].

In case E has complex multiplication, there is partial evidence in support of the part of the conjecture concerning the rank of K_2E: in this case the conjectural rank of K_2E is 1, and Bloch has constructed a rank 1 subgroup ([2], also see [8]). But in case E does not have CM, there were only finitely many examples for which one knew that K_2E had positive rank. In this paper, we show that for all but finitely many elliptic curves $E/\mathbf{Q}$ possessing a rational torsion point of order at least 3, K_2E has positive rank. Our method is as follows. In the case of an elliptic curve E defined over $\mathbf{C}$, one may view the regulator as a homomorphism $K_2E \to \mathbf{C}$. Parametrize elliptic curves in the usual manner by points in the complex upper half-plane $\mathscr{H}$; denote by E_λ the elliptic curve corresponding to $\lambda \in \mathscr{H}$. For each λ, we construct an element $\alpha_\lambda \in K_2E_\lambda$ using torsion points on E_λ, and show that the map $\lambda \mapsto \mathrm{reg}_{E_\lambda}(\alpha_\lambda)$ is real analytic on $\mathscr{H}$ and behaves well near the cusps. (Here, we are denoting by reg_{E_λ} the regulator homomorphism on K_2E_λ.) This allows us to conclude our result with $\mathbf{Q}$ replaced by $\mathbf{R}$; using the twisting theory of elliptic curves allows us to descend to $\mathbf{Q}$.

This work is part of the author's 1990 Rutgers Ph.D. thesis written under the direction of David Rohrlich, to whom the author expresses thanks for the support and encouragement received during that project.

2. Analytic behavior of the regulator

Let E be an elliptic curve defined over $\mathbf{C}$. In this section, we only care about the $\mathbf{C}$-isomorphism class of E, and thus identify $E(\mathbf{C})$ with a complex torus $\mathbf{C}/\Lambda$, where Λ is a lattice in $\mathbf{C}$. Let ω be a nonzero holomorphic 1-form on $E(\mathbf{C})$. In [1], Beilinson defines a regulator

$$\mathrm{reg}_E\colon K_2\mathbf{C}(E) \to \mathbf{C}$$

by

$$\mathrm{reg}_E(\{f, g\}) = \frac{1}{2\pi i}\int_{E(\mathbf{C})} \log|f|\,\overline{\mathrm{d}\log g} \wedge \omega.$$

Note that this depends on the choice of ω. To eliminate this dependence, we normalize the regulator as follows. The period lattice of ω is homothetic to $\Lambda = \mathbf{Z} + \mathbf{Z}\lambda$ for some $\lambda \in \mathscr{H}$. Let Γ_E denote the element of $H_1(E(\mathbf{C}), \mathbf{Z})$ determined by the segment of the real axis connecting 0 to 1. Put

$$\Omega_E = \int_{\Gamma_E} \omega.$$

Then define $\rho_E\colon K_2\mathbf{C}(E) \to \mathbf{C}$ by $\rho_E(\{f, g\}) = \Omega_E^{-1}\,\mathrm{reg}_E(\{f, g\})$. We want to express $\rho_E(\{f, g\})$ in terms of the homothety class of Λ, $\mathrm{div}(f)$, and $\mathrm{div}(g)$.

Let $r, s \in \mathbf{R}$, $\lambda = x + iy \in \mathscr{H}$, and define $\mathscr{E}(r, s; \lambda)$ by:

$$\mathscr{E}(r, s; \lambda) = \sum_{(m,n)}{}' (m\lambda + n)|m\lambda + n|^{-4}\,\mathrm{e}^{2\pi i(mr+ns)}.$$

Here, the prime indicates that the sum is over all pairs of integers $(m, n) \neq (0, 0)$. Note that $\mathscr{E}$ depends on r and s only mod $\mathbf{Z}$. $\mathscr{E}$ has the following modular behavior: If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, then

$$\mathscr{E}(r, s; \gamma\lambda) = \frac{|c\lambda + \mathrm{d}|^4}{c\lambda + \mathrm{d}}\,\mathscr{E}(\mathrm{d}r - bs, as - cr; \lambda). \tag{1}$$

Beilinson, in [1], gives a formula for $\rho_E(\{f, g\})$, which we state in the following lemma.

LEMMA 2.1. *Let E be an elliptic curve defined over* $\mathbf{C}$, *and let $\lambda \in \mathscr{H}$ be such that the period lattice of E is homothetic to $\Lambda = \mathbf{Z} + \mathbf{Z}\lambda$. For $z \in \mathbf{C}/\Lambda$, write $z = u(z)\lambda + v(z) \bmod \Lambda$ with $u(z)$ and $v(z)$ in* $[0, 1)$. *Let $f, g \in \mathbf{C}(E)^*$, and identify f and g with functions on $\mathbf{C}/\Lambda$. Then*

$$\rho_E(\{f, g\}) = \frac{(\mathrm{Im}\,\lambda)^2}{\pi^2} \sum_{z, w \in \mathbf{C}/\Lambda} (\mathrm{ord}_z f)(\mathrm{ord}_w g)\mathscr{E}(v(z - w), -u(z - w); \lambda).$$

Proof. [4], Lemma (3.2). □

We now examine the analytic properties of this expression for ρ_E. We begin with the following lemma, which gives a Fourier expansion for $\mathscr{E}(r, s; \lambda)$. Let $\lambda = x + iy$.

LEMMA 2.2. *Suppose that $s \in \mathbf{Q}$ and $N \in \mathbf{N}$ satisfy $Ns \in \mathbf{Z}$. Then*

$$\operatorname{Re} \mathscr{E}(r, s; \lambda) = y^{-2} \sum_{k=0}^{\infty} a_k \mathrm{e}^{-2\pi k y} + y^{-1} \sum_{k=0}^{\infty} b_k \mathrm{e}^{-2\pi k y/N}$$

and

$$\operatorname{Im} \mathscr{E}(r, s; \lambda) = 4\pi^3 B(s) + y^{-1} \sum_{k=0}^{\infty} c_k \mathrm{e}^{-2\pi k y/N}$$

where a_k, b_k, $c_k \in \mathbf{R}$ depend only on r, s, N, and x, and $B(s) = \frac{1}{3}s^3 - \frac{1}{2}s^2 + \frac{1}{6}s$ for $s \in [0, 1]$, and for general s, $B(s) = B(s - [s])$, where $[s]$ denotes the greatest integer less than or equal to s.

Proof. For $z \in \mathbf{C}$, $\operatorname{Re} z > \frac{3}{4}$, define $\mathscr{E}(r, s; \lambda, z)$ by

$$\mathscr{E}(r, s; \lambda, z) = \sum_{(m,n)}' (m\lambda + n)|m\lambda + n|^{-4z} \mathrm{e}^{2\pi i(mr + ns)}.$$

For z in this half-plane, the sum converges absolutely and uniformly on compact sets. Assume now that $\operatorname{Re} z > 1$. Letting

$$S(\lambda, z) = \sum_{m \neq 0} \sum_{n} \frac{1}{m} |m\lambda + n|^{2-4z} \mathrm{e}^{2\pi i(mr + ns)},$$

we may write

$$\begin{aligned} \mathscr{E}(r, s; \lambda, z) &= \sum_{n \neq 0} \frac{n}{|n|^{4z}} \mathrm{e}^{2\pi i n s} - \frac{1}{2z-1} \frac{\partial}{\partial \bar{\lambda}} S(\lambda, z) \\ &= 4\pi i B(s) - \frac{1}{2z-1} \frac{\partial}{\partial \bar{\lambda}} S(\lambda, z). \end{aligned}$$

We have

$$\begin{aligned} S(\lambda, z) &= \frac{-\pi^{2z-1}}{\Gamma(2z)} \sum_{m \neq 0} \sum_{n} \frac{1}{m} \mathrm{e}^{2\pi i(mr + ns)} \int_0^{\infty} \mathrm{e}^{-\pi t |m\lambda + n|^2} t^{2z-1} \frac{\mathrm{d}t}{t} \\ &= \frac{-\pi^{2z-1}}{\Gamma(2z)} \sum_{m \neq 0} \frac{1}{m} \mathrm{e}^{2\pi i m(r - sx)} \int_0^{\infty} \left(\sum_{n} \mathrm{e}^{-\pi t(n + mx - is/t)^2} \right) \mathrm{e}^{-\pi(ty^2 m^2 + s^2/t)} t^{2z-1} \frac{\mathrm{d}t}{t}. \end{aligned}$$

By Poisson summation,

$$\sum_n e^{-\pi t(n+mx-is/t)^2} = \frac{1}{\sqrt{t}}\sum_n e^{-\pi n^2/t}\, e^{2\pi in(mx-is/t)}.$$

Substituting the right-hand side into the expression for S and simplifying, we obtain

$$\begin{aligned} S(\lambda, z) &= \frac{-\pi^{2z-1}}{\Gamma(2z)} \sum_{m\neq 0} \frac{1}{m} e^{2\pi im(r-sx)} \sum_n e^{2\pi imnx} \int_0^\infty e^{-\pi(tm^2y^2+(n-s)^2/t)} t^{2z-3/2}\frac{dt}{t} \\ &= \frac{-\pi^{2z-1}}{\Gamma(2z)} \sum_{m\neq 0} \frac{1}{m} e^{2\pi im(r-sx)} \sum_n e^{2\pi imnx} K_{2z-3/2}(\sqrt{\pi}\,|m|y, \sqrt{\pi}\,|n-s|), \end{aligned}$$

where, following [5],

$$K_v(a,b) = \int_0^\infty e^{-(a^2t+b^2/t)} t^v \frac{dt}{t}.$$

By analytic continuation, the expression above for $S(\lambda, z)$ holds for all z. In particular, it holds for $z=1$.

We have (see [5], pp. 270–271)

$$K_{1/2}(\sqrt{\pi}\,|m|y, \sqrt{\pi}\,|n-s|) = |my|^{-1} e^{-2\pi|m|\,|n-s|y}.$$

Hence, we have the following expression for $S(\lambda, 1)$:

$$S(\lambda, 1) = -\pi y^{-1} \sum_{m\neq 0}\sum_n \frac{1}{m|m|} e^{2\pi i(m(r-sx)+nmx+iy|n-s|\,|m|)}, \tag{2}$$

and therefore obtain the following expression for $\mathscr{E}$:

$$\mathscr{E}(r, s; \lambda) = 4\pi^3 iB(s) + \frac{\pi i}{2} y^{-2} S(\lambda, 1) + \pi y^{-1}\frac{\partial S(\lambda, 1)}{\partial\bar{\lambda}}.$$

Noting that $S(\lambda, 1)$ is totally imaginary, we find that

$$\operatorname{Re}\mathscr{E}(r, s; \lambda) = \frac{\pi i}{2} y^{-2} S(\lambda, 1) + \pi y^{-1}\operatorname{Re}\frac{\partial S(\lambda, 1)}{\partial\bar{\lambda}}$$

and

$$\operatorname{Im}\mathscr{E}(r, s; \lambda) = 4\pi^3 B(s) + \pi y^{-1}\operatorname{Im}\frac{\partial S(\lambda, 1)}{\partial\bar{\lambda}}.$$

In view of (2), the lemma now follows. □

We now turn our attention to functions of the form

$$\phi(\lambda) = \sum_{j=1}^{r} m_j \mathscr{E}(r_j, s_j; \lambda)$$

where $m_j \in \mathbf{Z}$, and $r_j, s_j \in [0, 1)$ with $s_j \in \mathbf{Q}$. For such a ϕ, we will choose a natural number N such that for all j, $Ns_j \in \mathbf{Z}$. It is clear that ϕ is a complex-valued real analytic function on $\mathscr{H}$. We now proceed to examine the behavior of ϕ near the cusps.

We will need the following simple lemma.

LEMMA 2.3. *For $y > 0$ consider the function*

$$\Phi(y) = y^{-1} \sum_{k=0}^{\infty} a_k \mathrm{e}^{-2\pi ky/N} + \sum_{k=0}^{\infty} b_k \mathrm{e}^{-2\pi ky/N},$$

where $a_k, b_k \in \mathbf{R}$ and $N \in \mathbf{N}$. Suppose that Φ is not identically zero. Then for all y sufficiently large, $\Phi(y) \neq 0$.

Proof. Let $w = \mathrm{e}^{-2\pi y/N}$. It suffices to show that for all $w > 0$ sufficiently small, the function

$$f(w) = -\frac{2\pi}{N}(\log w)^{-1} \sum_{k=0}^{\infty} a_k w^k + \sum_{k=0}^{\infty} b_k w^k$$

has no zeros. This is straightforward. □

We now return to ϕ. For $x \in \mathbf{R}$, we let L_x denote the vertical ray in $\mathscr{H}$ defined by $L_x = \{x + iy: y > 0\}$.

LEMMA 2.4. *Let $x \in \mathbf{Q}$. Suppose that* $\operatorname{Re}\phi$ *(resp.* $\operatorname{Im}\phi$*) is not identically zero on L_x. Then* $\operatorname{Re}\phi$ *(resp.* $\operatorname{Im}\phi$*) has at most finitely many zeros on L_x.*

Proof. We prove this only for $\operatorname{Re}\phi$, the proof for $\operatorname{Im}\phi$ being similar.

By Lemma 2.2 we have

$$\operatorname{Re}\phi(x+iy) = y^{-1}\left(y^{-1} \sum_{k=0}^{\infty} A_k \mathrm{e}^{-2\pi ky/N} + \sum_{k=0}^{\infty} B_k \mathrm{e}^{-2\pi ky/N}\right)$$

which, by Lemma 2.3, has no zeros for y sufficiently large.

If $x \neq 0$, write $x = A/C$ with $A, C \in \mathbf{Z}$, $C > 0$, and $(A, C) = 1$. Let B and D be integers such that $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_2(\mathbf{Z})$. Put $x' = -D/C$. If $x = 0$, let $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $x' = 0$. Give each L_x the orientation induced by the usual ordering on y. Note that γ gives an orientation-reversing map of $L_{x'}$ onto L_x. Thus, by Equation (1), we are led to examine

$$\Phi(y) = C^3 y^3 \operatorname{Im} \tilde{\phi}(x' + iy)$$

for large values of y, where

$$\tilde{\phi}(\lambda) = \sum_{j=1}^{r} m_j \mathscr{E}(Dr_j - Bs_j, As_j - Cr_j; \lambda).$$

Im $\tilde{\phi}$ is not identically zero on $L_{x'}$ because Re ϕ is not identically zero on L_x. Then, by Lemma 2.3, we conclude that Im $\tilde{\phi}$ has no zeros on $L_{x'}$ for y large enough.

Therefore the zeros of Re ϕ on L_x are contained in a compact subset of L_x. Since Re ϕ is real analytic, it follows that it has only finitely many zeros on L_x. □

3. The main theorem

We now construct elements in the K_2 groups of elliptic curves defined over **Q** with a rational torsion point of order at least three, and study the relevant regulator expression.

We begin by standardizing our choice of period lattice for E. Let O denote the identity element for the group law on E.

LEMMA 3.1. *Let E be an elliptic curve defined over* **R**. *Fix an orientation on $E(\mathbf{R})^\circ$, the connected component of the identity in $E(\mathbf{R})$. Then there exists a unique pair (Λ, θ) where $\Lambda \subset \mathbf{C}$ is a lattice and $\theta: \mathbf{C}/\Lambda \to E(\mathbf{C})$ is a complex analytic isomorphism such that*:

(a) *θ is defined over* **R**.

(b) *$\Lambda \cap \mathbf{R} = \mathbf{Z}$ and $\theta|_{\mathbf{R}/\mathbf{Z}}$ maps $\mathbf{R}/\mathbf{Z}$ isomorphically onto $E(\mathbf{R})^\circ$ in an orientation-preserving manner, where $\mathbf{R}/\mathbf{Z}$ is given the orientation induced by the usual order on* **R**. *Hence $\Gamma_E = E(\mathbf{R})^\circ$ with the specified orientation.*

(c) *$\Lambda = \mathbf{Z} + \mathbf{Z}\lambda$ with* Re $\lambda = 0$ *or* $1/2$ *and* Im $\lambda > 0$. *Furthermore,* Re $\lambda = 0$ *(resp.* $1/2$) *if* $[E(\mathbf{R}): E(\mathbf{R})^\circ] = 2$ *(resp.* 1).

Proof. Let ω be a non-zero holomorphic 1-form on $E(\mathbf{C})$ defined over **R**. Let Λ be the period lattice of ω. Then Λ is invariant under complex conjugation, whence $\Lambda \cap \mathbf{R} \neq \varnothing$. By suitably renormalizing ω, we may assume that $\Lambda \cap \mathbf{R} = \mathbf{Z}$. Let ψ denote the Abel–Jacobi map:

$$\psi: E(\mathbf{C}) \to \mathbf{C}/\Lambda \qquad \psi: P \mapsto \int_O^P \omega \bmod \Lambda.$$

Then ψ is defined over $\mathbf{R}$. Let $\theta=\psi^{-1}$. By replacing θ with $-\theta$ if necessary, we may assume that $\theta|_{\mathbf{R}/\mathbf{Z}}$ preserves orientations. This shows (a) and (b).

Now let $\Lambda=\mathbf{Z}\lambda_1+\mathbf{Z}\lambda_2$. Then there exist integers a and b such that $1=a\lambda_1+b\lambda_2$. Because $\Lambda\cap\mathbf{R}=\mathbf{Z}$, a and b must be relatively prime. Choose integers c and d such that $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})$, and let $\lambda=c\lambda_1+\mathrm{d}\lambda_2$. Then $\Lambda=\mathbf{Z}+\mathbf{Z}\lambda$. By replacing λ with $-\lambda$ if necessary, we may assume that $\lambda\in\mathscr{H}$. Since $\bar{\lambda}\in\Lambda$, we find that $\operatorname{Re}\lambda\in\frac{1}{2}\mathbf{Z}$. Adding a suitable integer to λ allows us to assume that $\operatorname{Re}\lambda=0$ or $1/2$.

Suppose that $\operatorname{Re}\lambda=0$, and put $\lambda=iy$, $y>0$. Let $X=\{x+\frac{1}{2}iy: 0\leqslant x<1\}$. Then $\bar{X}\equiv X \bmod \Lambda$, where the bar denotes complex conjugation, and $\bar{X}\not\equiv\{x: 0\leqslant x<1\} \bmod \Lambda$. So $E(\mathbf{R})$ has two components.

Suppose that $\operatorname{Re}\lambda=\frac{1}{2}$. Note then that $\Lambda=\mathbf{Z}\lambda+\mathbf{Z}\bar{\lambda}$, and that the fundamental parallelogram $\mathscr{P}$ defined by λ and $\bar{\lambda}$ is invariant under complex conjugation. So if $z\in\mathscr{P}$ satisfies $z\equiv\bar{z} \bmod \Lambda$, then $z=\bar{z}$, whence $z\in\mathbf{R}$. So in this case $E(\mathbf{R})$ has only one component.

To verify the uniqueness of (Λ,θ), assume that we have another pair (Λ',θ') satisfying (a), (b), and (c) above. Then $\phi=\theta'^{-1}\circ\theta: \mathbf{C}/\Lambda\to\mathbf{C}/\Lambda'$ is a complex analytic isomorphism defined over $\mathbf{R}$. Therefore, $\Lambda=c\Lambda'$ for some $c\in\mathbf{C}^*$, (a) implies that $c\in\mathbf{R}$ and then (b) implies that $c=1$. □

Now let E be defined over $\mathbf{Q}$, and let $N\in\{3,4,5,6,7,8,9,10,12\}$. We assume that E has a rational torsion point of exact order N. For each of these values of N, there are infinitely many such $E/\mathbf{Q}$, because the modular curve $X_1(N)$ has genus zero in these cases. A well-known theorem of Mazur implies that these values of N, together with 1 and 2, are the only ones possible.

Let $P\in E(\mathbf{Q})$ be a point of exact order N, and write $P=\theta(u\lambda+a/N)$ where θ and λ are as in Lemma 3.1, and a is unique modulo N. Since $2P\in E(\mathbf{R})^{\circ}$, we may assume that $u=0$ or $\frac{1}{2}$. If $\operatorname{Re}\lambda=\frac{1}{2}$, so that $E(\mathbf{R})$ has only one component, we necessarily have $u=0$.

LEMMA 3.2. *For each N, let $P\in E(\mathbf{Q})$ be a point of exact order N. Then there exist functions f and g in $\mathbf{Q}(E)$ such that* $\operatorname{div}(f)=N(P)-N(O)$, $\operatorname{div}(g)=N(-P)-N(O)$, *and* $\{f,g\}\in\ker\tau$, *where τ is the global tame symbol on* $K_2\mathbf{Q}(E)$ [7].

Proof. Since P is of order N and defined over $\mathbf{Q}$, there exist functions f and g defined over $\mathbf{Q}$ having the indicated divisors. By multiplying these functions by suitable rational numbers, we may assume that $f(-P)=g(P)=1$. Weil Reciprocity implies that the symbol $\{f,g\}\in\ker\tau$. □

Let f and g be as in Lemma 3.2. An easy calculation gives:

$$\rho_E(\{f,g\})=\frac{N^2(\operatorname{Im}\lambda)^2}{\pi^2}\left(\mathscr{E}\left(\frac{2a}{N},0;\lambda\right)-2\mathscr{E}\left(\frac{a}{N},u;\lambda\right)\right),$$

where $E=E_\lambda$ and λ is given by Lemma 3.1. Let $\phi(u,a,N;\lambda)=\mathscr{E}(2a/N,0;\lambda)-2\mathscr{E}(a/N,u;\lambda)$. Note that $\phi(u,a,N;\lambda)\in\mathbf{R}$ for $\operatorname{Re}\lambda=0$ or $\frac{1}{2}$.

LEMMA 3.3. *Let u, a, and N be as above. Then $\phi(u,a,N;\lambda)$ has only finitely many zeros on L_0 and $L_{1/2}$.*

Proof. Let $\sigma=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ and $\gamma=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$. Note that $\sigma(L_0)=L_0$ and $\gamma(L_{-1/2})=L_{1/2}$.Note also that if $E/\mathbf{R}$ has period lattice $\mathbf{Z}+\mathbf{Z}\lambda$ with $\operatorname{Re}\lambda=\frac{1}{2}$, then $E(\mathbf{R})=E(\mathbf{R})^\circ$; hence in this case $u=0$.

By Lemma 2.4, it suffices to show that $\operatorname{Re}\phi(u,a,N;\sigma\lambda)$ is not identically zero on L_0 and that $\operatorname{Re}\phi(0,a,N;\gamma\lambda)$ is not identically zero on $L_{-1/2}$ for each of the values of u, a, and N which can occur. Computing using equation (1) and discarding an automorphy factor which never vanishes, it suffices to show that

$$\operatorname{Im}\left(\mathscr{E}\left(0,\frac{2a}{N};\lambda\right)-2\mathscr{E}\left(-u,\frac{a}{N};\lambda\right)\right)$$

is not identically zero on L_0, and that

$$\operatorname{Im}\left(\mathscr{E}\left(\frac{2a}{N},\frac{-4a}{N};\lambda\right)-2\mathscr{E}\left(\frac{a}{N},\frac{-2a}{N};\lambda\right)\right)$$

is not identically zero on $L_{-1/2}$. We do this by examining the Fourier coefficients of these expressions, using Lemma 2.2.

Note that the leading term of the first expression is $4\pi^3\left(B\left(\frac{2a}{N}\right)-2B\left(\frac{a}{N}\right)\right)$. Since $B(2t)-2B(t)=2t^3-t^2$ for t between 0 and 1, we see that this term is nonzero for all admissible values of a and N.

As for the second expression, note that its leading term is $4\pi^3\left(B\left(-\frac{4a}{N}\right)-2B\left(-\frac{2a}{N}\right)\right)$, which is nonzero for all admissible values of a and N except $N=4$ and $a=\pm 1$.

To take care of this case, we return to

$$\phi(0,\pm 1,4;\lambda)=\pm(\mathscr{E}(\tfrac{1}{2},0;\lambda)-2\mathscr{E}(\tfrac{1}{4},0;\lambda)),$$

where we have used the fact that $\mathscr{E}(-r,-s;\lambda)=-\mathscr{E}(r,s;\lambda)$. This fact also implies in particular that $\mathscr{E}(\frac{1}{2},0;\lambda)=0$. Returning to the proof of Lemma 2.2, we find that

$$\mathscr{E}(\tfrac{1}{4},0;\tfrac{1}{2}+iy)=-\pi\frac{\partial}{\partial\bar{\lambda}}y^{-1}\sum_{m\neq 0}\sum_n\frac{1}{m|m|}e^{2\pi i(m/4+mnx+iy|mn|)}\Big|_{\lambda=1/2+iy}.$$

Break this into two sums, one for which $n=0$ and one for which $n \neq 0$. Denote this latter sum by $S(x, y)$. We thus obtain

$$\operatorname{Re} \mathscr{E}(\tfrac{1}{4}, 0; \tfrac{1}{2} + iy) = -\pi y^{-2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} + S(\tfrac{1}{2}, y).$$

The term $S(\frac{1}{2}, y)$ decays like $y^{-1} e^{-2\pi y}$ as $y \to \infty$. Hence,

$$\lim_{y \to \infty} y^2 \mathscr{E}(\tfrac{1}{4}, 0; \tfrac{1}{2} + iy) = -\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \neq 0. \qquad \square$$

PROPOSITION 3.1. *Let K be a perfect field of characteristic $\neq 2, 3$. Let $j \in K$, $j \neq 0$, and let $N \geqslant 3$ be an integer. Then there are only finitely many K-isomorphism classes of elliptic curves E/K such that $j(E) = j$ and $E(K)$ has a point of exact order N.*

Proof. Suppose that $j \neq 1728$. Let E/K have invariant j. Choose a Weierstrass equation for E:

$$E\colon y^2 = x^3 + Ax + B$$

with $A, B \in K$. The set of K-isomorphism classes of elliptic curves E'/K such that $j(E') = j$ is in one-to-one correspondence with K^*/K^{*2}; this correspondence is given explicitly by

$$D \bmod K^{*2} \leftrightarrow E_D\colon y^2 = x^3 + D^2Ax + D^3B$$

and an isomorphism $\phi_D\colon E \to E_D$, defined over $\bar{K}$, is given by

$$\phi_D\colon (x, y) \mapsto (Dx, D^{3/2}y),$$

where $D^{3/2}$ is some fixed square root of D^3 [10].

Let $(x, y) \in E(\bar{K})$ be of exact order N; since $N \geqslant 3$, we know that $y \neq 0$. We claim that there is at most one $D \bmod K^{*2}$ such that $\phi_D(x, y) \in E_D(K)$. For suppose that D' were also such that $\phi_{D'}(x, y) \in E_{D'}(K)$. Then both $\sqrt{D}y$ and $\sqrt{D'}y$ belong to K. Since $y \neq 0$, we conclude that $D \equiv D' \bmod K^{*2}$. Hence we obtain the proposition in case $j \neq 1728$.

If $j = 1728$, consider the following elliptic curve

$$E\colon y^2 = x^3 + x.$$

The set of K-isomorphism classes of elliptic curves E'/K with $j(E') = 1728$ is in one-to-one correspondence with K^*/K^{*4}; this correspondence is given explicitly by

$$D \bmod K^{*4} \leftrightarrow E_D\colon y^2 = x^3 + Dx,$$

and an isomorphism $\psi_D\colon E \to E_D$, defined over $\bar{K}$, is given by

$$\psi_D\colon (x, y) \mapsto (\delta^2 x, \delta^3 y)$$

where δ is any fourth-root of D [10].

Let $(x, y) \in E(\bar{K})$ be of exact order N; since $N \geqslant 3$, we know that $xy \neq 0$. Again there is at most one $D \bmod K^{*4}$ such that $\psi_D(x, y) \in E_D(K)$. For if $D' \bmod K^{*4}$ were also such that $\psi_{D'}(x, y) \in E_{D'}(K)$, then, letting δ' be a fourth-root of D', we have $\delta'^2 x$ and $\delta'^3 y$ belonging to K. Since $xy \neq 0$, we have $(\delta/\delta')^2 \in K^*$ and $(\delta/\delta')^3 \in K^*$. So $\delta/\delta' \in K^*$, that is, $D \equiv D' \bmod K^{*4}$. □

REMARKS. (1) In the case $K = \mathbf{Q}$, this is a weak version of the main result of [6].

(2) As stated, the proposition is false for curves of j invariant 0. As a counterexample, consider the family E_d of curves defined over $\mathbf{Q}$ by

$$E_d\colon y^2 = x^3 + d^2$$

where $d \in \mathbf{Q}^{*2}$. Then the 3-torsion in $E_d(\mathbf{Q})$ consists of $(0, d)$, $(0, -d)$, and ∞.

We may now state our main result:

THEOREM 3.1. *Let N be an integer greater than or equal to 3. Then for all but finitely many $\mathbf{Q}$-isomorphism classes of elliptic curves $E/\mathbf{Q}$ such that $E(\mathbf{Q})$ possesses a torsion point of order N, there exists $\alpha \in K_2 E$ such that $\rho_E(\alpha) \neq 0$.*

Proof. If $j(E) = 0$, then the statement follows from Bloch's theorem [2]. Hence, we may assume that $j(E) \neq 0$. For each such curve, choose a point P of exact order N defined over $\mathbf{Q}$ and construct $\{f, g\}$ as in Lemma 3.2. Since $\{f, g\}$ is in the kernel of the tame symbol, it follows from the localization sequence in K-theory that $\{f, g\}$ represents an element $\alpha \in K_2 E$. Let λ be the point in $\mathscr{H}$ corresponding to E, as determined in Lemma 3.1. Then $\rho_E(\alpha) = \phi(u, a, N; \lambda)$ for some admissible choice of u, a, N.

By Lemma 3.3, there are at most finitely many values λ_0 for λ such that the corresponding value $\rho_E(\alpha)$ is zero. By Proposition 3.1, to each of these values λ_0 there are associated only finitely many elliptic curves of the type we are considering. The theorem follows. □

Using the functoriality of the regulator, we immediately obtain the following:

THEOREM 3.2. *For all but finitely many elliptic curves $E/\mathbf{Q}$ which are isogenous* *over* $\mathbf{Q}$ *to an elliptic curve defined over* $\mathbf{Q}$ *containing a rational torsion point of order at least three,* K_2E *contains an element of infinite order.*

We remark that this generalization is non-vacuous, since any elliptic curve defined over $\mathbf{Q}$ is isogenous over $\mathbf{Q}$ to an elliptic curve $E'/\mathbf{Q}$ such that $|E'(\mathbf{Q})_{\mathrm{tors}}| = 1$ or 2 ([9]).

References

[1] A.A. Beilinson: Higher regulators and values of L-functions of curves, *Functional Analysis and its Applications* 14 (1980), 116–118.
[2] S. Bloch: *Lectures on Algebraic Cycles*, Duke Mathematical Series, Duke University Press, 1980.
[3] S. Bloch and D. Grayson: K_2 of Elliptic Curves and Values of L-Functions: Computer Calculations, in *Contemporary Mathematics* 55, American Mathematical Society, 1983, 79–88.
[4] C. Deninger and K. Wingberg: On the Beilinson Conjectures for Elliptic Curves with Complex Multiplication, in *Beilinson's Conjectures on Special Values of L-Functions*, Academic Press, 1988.
[5] S. Lang: *Elliptic Functions*, Addison-Wesley, 1973.
[6] L. Olsen: Torsion points on elliptic curves with given j-invariant, *Manuscripta Math.* (1975), 145–150.
[7] D. Ramakrishnan: Regulators, Algebraic Cycles, and Values of L-Functions, in *Contemporary Mathematics* 83, American Mathematical Society, 1989, 183–310.
[8] D. Rohrlich: Elliptic Curves and Values of L-Functions, in *CMS Conf. Proc.* 7, 1987, 371–387.
[9] R. Ross: Minimal torsion in isogeny classes of elliptic curves, in preparation.
[10] J. Silverman: *The Arithmetic of Elliptic Curves*, Springer-Verlag, 1986.