COMPOSITIO MATHEMATICA

BENEDICT H. GROSS
STEPHEN S. KUDLA

Heights and the central critical values of triple product *L*-functions

Compositio Mathematica, tome 81, nº 2 (1992), p. 143-209
<http://www.numdam.org/item?id=CM_1992__81_2_143_0>

Compositio Mathematica **81**: 143–209, 1992.

Heights and the central critical values of triple product L-functions

BENEDICT H. GROSS[1] and STEPHEN S. KUDLA[2,*]

[1]*Harvard University, Cambridge, MA 02138, U.S.A.;* [2]*University of Maryland, College Park, MD 20742, U.S.A.*

Received 25 September 1990; accepted 20 March 1991

Introduction

Let $N \geqslant 1$ be a square-free integer, and let f, g and h be three holomorphic cusp forms of weight 2 for the group $\Gamma_0(N)$. We assume that f, g and h are all normalized eigenforms for the Hecke algebra, and are all newforms of level N. The function $F(z_1, z_2, z_3) = f(z_1)g(z_2)h(z_3)$ is then a newform of weight (2, 2, 2) for $\Gamma_0(N)^3$.

The triple product L-function $L(f \otimes g \otimes h, s) = L(F, s)$ is defined by a convergent Euler product (1.6) in the half-plane $\mathrm{Re}(s) > 5/2$. Using an integral representation of this function discovered by Garrett [9], we show that it has a holomorphic continuation to the entire s-plane and satisfies a simple functional equation (Theorem 1.1) when s is replaced by $4 - s$. The sign in this functional equation is given by the formula $\varepsilon = -\Pi_{p|N}\,\varepsilon_p$, with $\varepsilon_p = -a_p(f)a_p(g)a_p(h) = \pm 1$ given by the product of the p-th Fourier coefficients. The proof follows the argument given by Garrett for the case $N = 1$, but the genus 3 Eisenstein series $E(Z, s-2)$ which appears in the integral representation of $L(F, s)$ depends on the local constants ε_p, for all primes $p \mid N$. Because of these considerations, which are essentially local in nature, we use an adèlic version of Garrett's argument, following Piatetski-Shapiro and Rallis [31].

The Eisenstein series which occurs in the integral considered in [31] is constructed from a section $\Phi(s)$ of a certain family of induced representations $I(s)$. If this section is factorizable, then the global integral, (7.7) below, can be unfolded and written as a product of local 'zeta' integrals (2.3), which involve the local Whittaker functions determined by f, g and h, and the local components of $\Phi(s)$. At primes p not dividing the level, there is a natural choice of $\Phi_p^{\natural}(s) \in I_p(s)$ and, for this choice, the local zeta integral gives precisely the local factor $L_p(F, s+2)$ times a normalizing factor, cf. (2.5). This 'spherical' vector $\Phi_p^{\natural}(s)$ is also an eigenvector for the local intertwining operator $M_p(s)\colon I_p(s) \to I_p(-s)$. Moreover, its value $\Phi_p^{\natural}(0)$ lies in an irreducible subspace of the induced

*Partially supported by NSF Grant DMS-9003109.

representation $I_p(0)$ which arises via the Weil representation associated to a split quadratic form in 4 variables, (i) of Proposition 8.2. Our main local problem is to find an analogous section $\Phi_p^\natural(s) \in I_p(s)$ for primes $p \mid N$. As mentioned above, this $\Phi_p^\natural(s)$ will depend on ε_p, and it will lie in the space of Iwahori fixed vectors in $I_p(s)$. The calculation of the local zeta integrals for such vectors is given in Section 4. Our choice of $\Phi_p^\natural(s)$, given by (5.10), is an eigenvector for the action of an 'Atkin-Lehner' operator on the space of Iwahori fixed vectors, see (3.9), and has the property that its value at $s = 0$ arises from the Weil representation associated to the quaternion algebra over $\mathbb{Q}_p$ which is ramified if $\varepsilon_p = -1$ and split if $\varepsilon_p = +1$, see (ii) of Proposition 8.2. This particular choice is *not* an eigenvector for the local intertwining operator, but is shifted by a vector which gives zero in the local zeta integral, see (5.5) and (5.6). These facts allow us to obtain a good functional equation for the complete L-function, Proposition 1.1. In fact, our calculations of local zeta integrals etc. at primes $p \mid N$, together with some additional work at the archimedean place (cf. Section 6) will allow us to give a precise functional equation and to prove holomorphy for any triple of newforms of arbitrary weights ($\geqslant 2$) for $\Gamma_0(N)$ and the same square free level N.

We then turn to a study of the central critical value $L(F, 2)$, using the Weil-Siegel formula [24, 33] for $E(Z, 0)$. We obtain an expression for $L(F, 2) = \Omega(F) \cdot A(F)$, where $\Omega(F)$ is the period

$$\Omega(F) = \frac{2^{8-t}\pi^5}{N} \int_{\Gamma_0(N)^3 \backslash \mathfrak{H}^3} |F(z_1, z_2, z_3)|^2 \, dx_1 \, dy_1 \, dx_2 \, dy_2 \, dx_3 \, dy_3$$

with $t = \#\{p : p \mid N\}$ and $A(F)$ is a real algebraic number in the subfield of $\mathbb{C}$ generated by the coefficients of the Dirichlet series of $L(F, s)$. We then give an interpretation of $A(F)$ as a "height pairing", which allows us to show that $A(F) \geqslant 0$, and to give a simple algebraic criterion for its vanishing.

More precisely, assume that the sign in the functional equation of $L(F, s)$ is $+1$, and let B be the definite quaternion algebra over $\mathbb{Q}$ which is ramified at the odd set of primes where $\varepsilon_p = -1$. Let R be an Eichler order of reduced discriminant N in B, and let X be the curve over $\mathbb{Q}$ associated to R, which was introduced in [13]. Then X is the disjoint union of n rational curves, where n is the class number of R. Let Δ be the codimension 2 cycle consisting of X embedded diagonally in the 3-fold X^3. We show that $A(F)$ is the "height pairing" of the F-isotypic component Δ_F in the cycle group.

Unwinding the definition of this pairing gives us the following formula. The global correspondence of Jacquet-Langlands and Shimizu between automorphic forms on GL_2 and on $B^\times$ (which, in this case, can be proved by Eichler's methods [7]) shows that the eigenforms f, g and h determine real valued eigenfunctions $\lambda_i(f)$, $\lambda_i(g)$ and $\lambda_i(h)$ on the set of left ideal classes of R. The theorem of multiplicity one for these groups shows that each eigenfunction is well defined up to scaling. If $2w_i$ is the order of the group of units in the right order R_i of the class I_i, we have

$$A(F) = \frac{(\sum_{i=1}^{n} w_i^2 \lambda_i(f)\lambda_i(g)\lambda_i(h))^2}{\sum_{i=1}^{n} w_i\lambda_i(f)^2 \sum_{i=1}^{n} w_i\lambda_i(g)^2 \sum_{i=1}^{n} w_i\lambda_i(h)^2}.$$

We note that when $N = p$ is a prime, R is a maximal order in B and the left ideal classes of R correspond to the isomorphism classes of supersingular elliptic curves in characteristic p. For eigenforms f on $\Gamma_0(p)$ with rational Fourier coefficients, the eigenfunctions $\lambda_i(f)$ on the set of supersingular curves have been extensively tabulated by Mestre and Oesterlé [28]. For square-free $N < 300$ and f with rational Fourier coefficients, Birch [2] has calculated the integers $\lambda_i(f)$ using the theory of ternary quadratic forms. We thank these authors for generously sharing their data with us, some of which appears tabulated in Section 12. We also wish to thank Buhler and Zagier for their computational assistance on the problem of determining the values $L(F, 2)$; this was a welcome check on the many constants in the final formula.

We end the paper with a conjecture on the first derivative $L'(F, 2)$ when the sign in the functional equation for $L(F, s)$ is -1 (and, hence, $L(F, 2) = 0$). Let X be the Shimura curve associated to an Eichler order of reduced discriminant N in the indefinite quaternion algebra B ramified at the even set of finite primes where $\varepsilon_p = -1$. Let Δ be the codimension 2 cycle of X embedded diagonally in the threefold X^3. Loosely speaking, we conjecture that

$$L'(F, 2) = \Omega(F) \cdot \langle \Delta_F, \Delta_F \rangle^{\mathrm{BB}}$$

where Δ_F is the F-eigencomponent of Δ in the Chow group and $\langle\, ,\rangle^{\mathrm{BB}}$ is the Bloch-Beilinson height pairing. Actually, one must first modify Δ to obtain a class which is homologically trivial, and the eigencomponent Δ_F only is known to exist in the quotient of the Chow group by the radical of the height pairing.

Special values of the triple product L-function have been considered by a number of people, beginning with the fundamental work of Garrett [9]. Other work includes that of Blasius and Orloff [3, 30], Garrett and Harris [10] and Harris and Kudla [17]. This last paper describes the central critical value in a more general case and proves a conjecture of Jacquet concerning the vanishing of such values, but for the special case of weight (2, 2, 2) and square free level it gives less precise information than is obtained in the present paper. Of fundamental importance in [17] and, implicitly, in the present paper is the work of D. Prasad [32], which gives the uniqueness of certain invariant trilinear forms and characterizes their existence in terms of epsilon factors.

The existence of a good function functional equation for the triple product L-function was first shown by Shahidi [36]. His more recent work also establishes the finiteness of the number of poles and holomorphy at the center of the critical strip [37], Cor. 6.9, p. 582. Finally, T. Ikeda considered the functional equation [19] and gave a precise description of the poles [20] of triple product L-functions.

1. The triple product L-function

Let $N \geqslant 1$ be square free, and let f, g, and h be three (not necessarily distinct) cuspidal newforms of weight 2 on $\Gamma_0(N)$. Assume that the Fourier expansions of f, g, and h are given by

$$\begin{cases} f = \Sigma_{n \geqslant 1} a_n q^n \\ g = \Sigma_{n \geqslant 1} b_n q^n \\ h = \Sigma_{n \geqslant 1} c_n q^n \end{cases} \tag{1.1}$$

with $a_1 = b_1 = c_1 = 1$. For a prime l not dividing N, we write:

$$\begin{cases} 1 - a_l x + l x^2 = (1 - \alpha_l x)(1 - \alpha_l' x) \\ 1 - b_l x + l x^2 = (1 - \beta_l x)(1 - \beta_l' x) \\ 1 - c_l x + l x^2 = (1 - \gamma_l x)(1 - \gamma_l' x). \end{cases} \tag{1.2}$$

Then $|\alpha_l| = |\beta_l| = |\gamma_l| = l^{1/2}$ [8, 40, Chapt. 7]. For a prime p dividing N, the coefficients a_p, b_p, and c_p are equal to ± 1. We put

$$\varepsilon_p = -a_p b_p c_p. \tag{1.3}$$

Define the modular form $F = f \otimes g \otimes h$ of weight (2, 2, 2) for $\Gamma_0(N)^3$ by:

$$F(z_1, z_2, z_3) = f(z_1) g(z_2) h(z_3) \tag{1.4}$$

for $(z_1, z_2, z_3) \in \mathfrak{H}^3$. For $p \mid N$, we have an involution $u_p = w_p \times w_p \times w_p$ on the space of forms of weight (2, 2, 2) where w_p is the Atkin-Lehner involution on the space of forms on $\Gamma_0(N)$, and

$$F \mid u_p = \varepsilon_p \cdot F \qquad \text{for all } p \mid N. \tag{1.5}$$

Indeed, $f \mid w_p = -a_p \cdot f$, and similarly for g and h.

We define the triple product product L-function $L(f \otimes g \otimes h, s) = L(F, s)$ by an Euler product, convergent in the half plane $\mathrm{Re}(s) > \frac{5}{2}$:

$$L(F, s) = \prod_{l \nmid N} L_l(F, s) \cdot \prod_{p \mid N} L_p(F, s). \tag{1.6}$$

Here

$$L_l(F, s) = (1 - \alpha_l\beta_l\gamma_l l^{-s})^{-1}(1 - \alpha_l\beta_l\gamma_l' l^{-s})^{-1} \cdots (1 - \alpha_l'\beta_l'\gamma_l' l^{-s})^{-1} \tag{1.7}$$

has degree 8 in l^{-s}. The bad Euler factors

$$L_p(F, s) = (1 - a_p b_p c_p p^{-s})^{-1}(1 - a_p b_p c_p p^{1-s})^{-2} \tag{1.8}$$

have degree 3 in p^{-s}. The absolute convergence in $\mathrm{Re}(s) > \frac{5}{2}$ follows from a comparison of $L(F, s)$ with $\zeta(s - \frac{3}{2})^8$.

Our choice of local L-factors for F follows a general recipe of Serre [35], applied to the 8-dimensional l-adic representation $\sigma_f \otimes \sigma_g \otimes \sigma_h$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Here σ_f, σ_g, and σ_h are the 2-dimensional Galois representations associated to the newforms f, g, and h. Similarly, we define the archimedean L-factor. Set

$$\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s).$$

Then

$$L_\infty(F, s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s - 1)^3 = (2\pi)^{3-4s}\Gamma(s)\Gamma(s - 1)^3, \tag{1.9}$$

as the Hodge numbers of the motive attached to F are $h^{3,0} = h^{0,3} = 1$ and $h^{2,1} = h^{1,2} = 3$.

Let

$$\Lambda(F, s) = L_\infty(F, s)L(F, s) \tag{1.10}$$

in $\mathrm{Re}(s) > \frac{5}{2}$, and define

$$\varepsilon(F, s) = \varepsilon \cdot N^{10-5s} \tag{1.11}$$

where $\varepsilon = -\Pi_{p|N}\,\varepsilon_p = \pm 1$.

PROPOSITION 1.1. *The function $\Lambda(F, s)$ has an analytic continuation to the entire s-plane and satisfies the functional equation:*

$$\Lambda(F, s) = \varepsilon(F, s)\Lambda(F, 4 - s).$$

This will be proved in Section 7 using the local results of Sections 2–6. If we put

$$\Lambda^*(F, s) = N^{\frac{5}{2}s}\Lambda(F, s), \tag{1.12}$$

then the functional equation of Proposition 1.1 takes the simpler form

$$\Lambda^*(F, s) = \varepsilon \cdot \Lambda^*(F, 4 - s). \tag{1.13}$$

In particular, we see that

$$\varepsilon = -\prod_{p|N} (-a_p b_p c_p) = (-1)^{\mathrm{ord}_{s=2} L(F,s)} \tag{1.14}$$

as all other factors in $\Lambda^*(F, s)$ are non-zero at $s = 2$.

2. Local factors

In this section we turn to local considerations and describe the local zeta integrals which arise in the Garrett and Piatetski-Shapiro, Rallis integral representation of the triple product L-function.

We begin with the p-adic case. Let $G = GSp_3(\mathbb{Q}_p)$ be the similitude group of the six dimensional symplectic vector space $\mathbb{Q}_p^6$ (row vectors) with standard symplectic form given by

$$J = \begin{pmatrix} & 1_3 \\ -1_3 & \end{pmatrix},$$

and let $v\colon G \to \mathbb{Q}_p^\times$ be the scale map. Let $P = MN$ be the maximal parabolic subgroup of G with

$$M = \left\{ m(a, v) = \begin{pmatrix} a & \\ & v\,{}^t a^{-1} \end{pmatrix} \middle| a \in \mathrm{GL}_3(\mathbb{Q}_p),\ v \in \mathbb{Q}_p^\times \right\}, \tag{2.0}$$

and

$$N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \middle| b = {}^t b \in M_3(\mathbb{Q}_p) \right\}.$$

Let $K = \mathrm{GSp}_3(\mathbb{Z}_p)$, and let $Z_G \simeq \mathbb{Q}_p^\times$ be the center of G.

For $s \in \mathbb{C}$, we consider the induced representation $I(s) = \mathrm{Ind}_P^G\, {}^{2s}$ consisting of smooth functions (i.e., locally constant) $\Phi(s)$ on G such that

$$\Phi(nm(a, v)g, s) = |\det a|^{2s+2} |v|^{-3s-3} \Phi(g, s). \tag{2.1}$$

Note that such functions automatically have compact support modulo P, and that Z_G acts trivially in this representation.

Let ψ be the standard additive character of $\mathbb{Q}_p$ with conductor $\mathbb{Z}_p$, and, for an irreducible admissible and infinite dimensional representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$, let $\mathscr{W}(\pi)$ be the Whittaker model of π with respect to ψ [21].

Let

$$H = \{h = (h_1, h_2, h_3) \in \mathrm{GL}_2(\mathbb{Q}_p)^3 \mid \det h_1 = \det h_2 = \det h_3\},$$

and recall that H is embedded as a subgroup of G [9, 31]. Let

$$U = \left\{ u = \left(\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_3 \\ 0 & 1 \end{pmatrix} \right) \middle| \, x_i \in \mathbb{Q}_p \right\},$$

and let

$$U_0 = \{u \in U \mid x_1 + x_2 + x_3 = 0\}.$$

For a triple of irreducible admissible infinite dimensional representations of π_1, π_2 and π_3 of $\mathrm{GL}_2(\mathbb{Q}_p)$, and for Whittaker functions $W_i \in \mathscr{W}(\pi_i)$, $i = 1, 2, 3$, define a function W on H by

$$W(h) = W_1(h_1)W_2(h_2)W_3(h_3). \tag{2.2}$$

Then, for $\Phi(s) \in I(s)$, the local zeta integral associated to W and $\Phi(s)$ is

$$Z(s, W, \Phi(s)) = \int_{Z_G U_0 \backslash H} \Phi(\delta h, s) W(h) \, \mathrm{d}h, \tag{2.3}$$

where

$$\delta = \begin{bmatrix} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \in \mathrm{Sp}_3(\mathbb{Z}_p), \tag{2.4}$$

and $\mathrm{d}h$ is an H-invariant measure on $Z_G U_0 \backslash H$ which will be specified below. Note that δ is a representative for the unique 'non-negligible' orbit of H in the flag variety $P\backslash G$. Recall [31] that $P\backslash G$ may be identified with the space of maximal isotropic subspaces in the symplectic vector space $\mathbb{Q}_p^6$. The group H

has a finite number of orbits in this space, and the orbit of δ is the only one for which the stabilizer of a point does not contain the unipotent radical of a proper parabolic subgroup of H. The isotropic subspace corresponding to δ is the span of the bottom three rows of this matrix. This particular choice of δ is due to Garrett and Harris [10].

If π_1, π_2 and π_3 are unramified principal series representations of $GL_2(\mathbb{Q}_p)$, let $W_i^0 \in \mathscr{W}(\pi_i)$ be the unique $GL_2(\mathbb{Z}_p)$ fixed vector with $W_i^0(e) = 1$, and let W^0 be the corresponding function on H via (2.2). Also let $\Phi^0(s) \in I(s)$ be the unique K fixed vector with $\Phi^0(k, s) = 1$ for $k \in K$. Note that, because of the Iwasawa decomposition $G = PK$, any function in $I(s)$ is determined by its restriction to K. Then [9 and 31, p. 57]

$$Z(s, W^0, \Phi^0(s)) = \frac{1}{b_p(s)} L_p\left(\pi, s + \frac{1}{2}\right), \tag{2.5}$$

where $\pi = \pi_1 \otimes \pi_2 \otimes \pi_3$, and $L_p(\pi, s)$ is the local Langlands L-factor associated to π and the degree 8 representation of the L-group of $GL_2(\mathbb{Q})^3$, and

$$b_p(s) = \zeta_p(2s + 2)\zeta_p(4s + 2). \tag{2.5.1}$$

Note that if π_1, π_2, and π_3 are the representations of $GL_2(\mathbb{Q}_p)$ determined by the p-th Hecke eigenvalues of our cusp forms f, g, and h of Section 1 when $p \nmid N$, then

$$L_p(\pi, s) = L_p(F, s + \tfrac{3}{2}) \tag{2.6}$$

for the local factor defined in Section 1. Note that the shift here is due to the convention that the Langlands L-function will have a functional equation relating s to $1 - s$, while the functional equation of the Eisenstein series and hence of the global analogue of $Z(s, W, \Phi(s))$ involves s and $-s$.

In order to obtain precise information about the central value of the triple product L-function, we will need to have an analogue of (2.5) when $p \mid N$. Note that a local factor $L_p(F, s)$ has been defined for such p by (1.8) of Section 1. We then *define* $L_p(\pi, s)$ by (2.6), i.e.,

$$L_p(\pi, s + \tfrac{1}{2}) = L_p(F, s + 2) = (1 + \varepsilon_p p^{-s-2})^{-1}(1 + \varepsilon_p p^{-s-1})^{-2}. \tag{2.7}$$

Our main result in the next two sections will relate $L_p(\pi, s + \frac{1}{2})$ to a zeta integral as in (2.5), but with W^0 and Φ^0 replaced by functions invariant under the Iwahori subgroup.

Since N was assumed to be square free, the triple of local representations π_1, π_2, and π_3 determined by the newforms f, g, and h for $p \mid N$ are twists of the special representation Sp on the locally constant functions on $\mathbb{P}^1 \simeq B\backslash GL_2(\mathbb{Q}_p)$

modulo the constant functions (B the standard Borel subgroup) [5, 6], by unramified quadratic characters ω_i satisfying

$$\omega_1(p) = a_p, \qquad \omega_2(p) = b_p, \quad \text{and} \quad \omega_3(p) = c_p.$$

We need a formula for the unique Whittaker vector in these representations which is fixed by the Iwahori subgroup

$$\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \,\middle|\, c \equiv 0 \bmod p \right\}.$$

The following result can be extracted from [5]:

LEMMA 2.1. *Let $\sigma = \mathrm{Sp} \otimes \eta$ be an unramified special representation with trivial central character, and let η be the associated unramified quadratic character. Let $W^0 \in \mathcal{W}(\sigma)$ be the unique $\Gamma_0(p)$ fixed vector with $W^0(e) = 1$. Then*

$$W^0\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) = \begin{cases} \eta(a)\,|a| & \textit{if } \mathrm{ord}_p a \geqslant 0 \\ 0 & \textit{otherwise} \end{cases}$$

and

$$W^0\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) = \begin{cases} -p^{-1}\eta(a)|a| & \textit{if } \mathrm{ord}_p a \geqslant -1 \\ 0 & \textit{otherwise.} \end{cases}$$

Note that since

$$\mathrm{GL}_2(\mathbb{Z}_p) = \Gamma_0(p) \cup \bigcup_{x \bmod p} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\Gamma_0(p), \tag{2.8}$$

the values given in Lemma 2.1 give a complete description of W^0.

For our triple of local components π_1, π_2, and π_3, with quadratic characters ω_1, ω_2, and ω_3, we set

$$\omega = \omega_1\omega_2\omega_3, \tag{2.9}$$

and

$$\varepsilon = \varepsilon_p = -a_p b_p c_p = -\omega(p). \tag{2.10}$$

For the Iwahori fixed Whittaker vectors $W_i^0 \in \mathcal{W}(\pi_i)$ furnished by Lemma 2.1, we again let W^0 be the function on H given by (2.2). As in Section 1, let

$$u_p = (w_p, w_p, w_p) \in H. \tag{2.11}$$

Then

LEMMA 2.2.

$$W^0 \mid u_p = -\omega(p) W^0 = -a_p b_p c_p W^0.$$

Proof. This follows from the fact that $W_i^0 \mid w_p = -\omega_i(p) \cdot W_i^0$. □

3. Iwahori fixed vectors

Next we must choose a function in $I(s)$ for use with W^0 of Section 2, in the local zeta integral. Let

$$K_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K = \mathrm{GSp}_3(\mathbb{Z}_p) \,\middle|\, c \equiv 0 \bmod pM_3(\mathbb{Z}_p) \right\},$$

and note that the reduction mod p of $K_0(p)$ is the maximal parabolic subgroup $P(\mathbb{F}_p)$ in $\mathrm{GSp}_3(\mathbb{F}_p)$. The usual double coset decomposition of $\mathrm{GSp}_3(\mathbb{F}_p)$ with respect to this parabolic yields a decomposition

$$K = \coprod_{i=0}^{3} K_0(p) w_i K_0(p), \tag{3.1}$$

where

$$w_i = \begin{bmatrix} 1_{3-i} & & & \\ & 0 & & 1_i \\ & & 1_{3-i} & \\ & -1_i & & 0 \end{bmatrix} \in K. \tag{3.2}$$

Note that the double coset of w_i is precisely the set of all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ such that the rank of $c \pmod p$ is i.

For $0 \leqslant i \leqslant 3$ let $\Phi^i(s) \in I(s)$ be the function whose restriction to K is the characteristic function of $K_0(p) w_i K_0(p)$. Note that $\Phi^0(s)$ is the characteristic function of $K_0(p)$; in the next section we will compute its zeta integral against

W^0. Also note that the normalized K fixed vector in $I(s)$ is

$$\Phi_K(s) = \sum_{i=0}^{3} \Phi^i(s). \tag{3.3}$$

Moreover, for any fixed s, the four functions $\Phi^i(s)$ form a basis for the space $I(s)^{K_0(p)}$ of $K_0(p)$ fixed vectors. This basis will not turn out to be the most convenient one.

Since the image in G of the element u_p defined above normalizes $K_0(p)$, we may consider the action of this element (which acts as an involution) in the space $I(s)^{K_0(p)}$.

LEMMA 3.1.

$$\Phi^i(s)|u_p = p^{(2i-3)(s+1)}\Phi^{3-i}(s).$$

Proof. For convenience, let

$$\eta = \begin{pmatrix} 0 & 1_3 \\ -p\cdot 1_3 & \end{pmatrix} \in \mathrm{GSp}_3(\mathbb{Q}_p).$$

We want to calculate the value $\Phi^i(w_j\eta, s)$ as a function of i and j. Note that

$$w_j\eta = \begin{pmatrix} 1_3 & \\ & p\cdot 1_3 \end{pmatrix} \begin{bmatrix} 1_{3-j} & & & \\ & p\cdot 1_j & & \\ & & 1_{3-j} & \\ & & & p^{-1}\cdot 1_j \end{bmatrix} \begin{bmatrix} 0 & & 1_{3-j} & \\ & -1_j & & 0 \\ -1_{3-j} & & 0 & \\ & 0 & & 1_j \end{bmatrix},$$

where the last factor on the right hand side lies in K_p and has c of rank $3-j \pmod p$. Thus this last factor lies in the double coset $K_0(p)w_{3-j}K_0(p)$ in (3.1) and we get

$$\Phi^i(w_j\eta, s) = |p|^{-3s-3}|p^j|^{2s+2}\delta_{i,3-j} = p^{(2i-3)(s+1)}\delta_{i,3-j},$$

as claimed. □

Setting

$$\tilde{\Phi}^i(s) = p^{-i(s+1)}\Phi^i(s), \tag{3.4}$$

we have

$$\tilde{\Phi}^i(s) \mid u_p = \tilde{\Phi}^{3-i}(s). \tag{3.5}$$

Next observe that conjugation by u_p carries the fixed maximal compact subgroup $K = \mathrm{GSp}_3(\mathbb{Z}_p)$ into another maximal compact subgroup $K' = u_p K u_p^{-1}$, and that these are the only maximal compact subgroups containing $K_0(p)$. The normalized fixed vectors then have the expressions

$$\Phi_K(s) = \sum_{i=0}^{3} p^{i(s+1)} \tilde{\Phi}^i(s), \tag{3.6}$$

and

$$\Phi_{K'}(s) = \sum_{i=0}^{3} p^{(3-i)(s+1)} \tilde{\Phi}^i(s) = \Phi_K(s) \mid u_p. \tag{3.7}$$

Note that these two vectors actually coincide at $s = -1$, and this is due to the fact that the trivial representation of G occurs as a submodule of $I(-1)$ $(\simeq C^\infty(P\backslash G))$.

Define two more functions in $I(s)^{K_0(p)}$ by

$$\Phi_R^{\pm}(s) = \sum_{i=0}^{3} (\pm 1)^i \tilde{\Phi}^i(s) = \sum_{i=0}^{3} (\pm 1)^i p^{-i(s+1)} \Phi^i(s). \tag{3.8}$$

These functions, whose significance will be explained in Section 4 below, satisfy

$$\Phi_R^{\pm}(s) \mid u_p = \pm \Phi_R^{\pm}(s). \tag{3.9}$$

Moreover, we remark that for $s \neq -1$ fixed, the four functions $\Phi_K(s)$, $\Phi_{K'}(s)$, and $\Phi_R^{\pm}(s)$ form a basis for $I(s)^{K_0(p)}$.

4. In which we compute a p-adic zeta integral

We now consider the zeta integral $Z(s, W^0, \Phi(s))$ with W^0 as above and with $\Phi(s) \in I(s)^{K_0(p)}$. Recall that $\varepsilon = \varepsilon_p = -a_p b_p c_p$ is fixed as above. It is easily checked that

$$Z(s, W^0, \Phi(s) \mid u_p) = Z(s, W^0 \mid u_p, \Phi(s)) = \varepsilon Z(s, W^0, \Phi(s)), \tag{4.1}$$

and, hence, that $Z(s, W^0, \Phi(s))$ vanishes identically if $\Phi(s)|u_p = -\varepsilon\Phi(s)$. Moreover, by a direct calculation, we have

LEMMA 4.1.

$$Z(s, W^0, \Phi_K(s)) = 0.$$

Proof. Now

$$Z(s, W^0, \Phi_K) = \int_{Z_G U_0 \backslash H} \Phi_K(\delta h, s) W^0(h)\, \mathrm{d}h.$$

Since Φ_K is right K-invariant, we may substitute for W^0 its K_H-invariant projection:

$$(W^0)_{K_H}(h) = \int_{K_H} W^0(hk)\, \mathrm{d}k,$$

where $\mathrm{vol}(K_H, \mathrm{d}k) = 1$. But this projection is zero. In fact, via the Iwasawa decomposition of H and the triviality of the central characters, it suffices to verify this for

$$h = \left(\begin{pmatrix} aa_1^2 & \\ & 1 \end{pmatrix}, \begin{pmatrix} aa_2^2 & \\ & 1 \end{pmatrix}, \begin{pmatrix} aa_3^2 & \\ & 1 \end{pmatrix} \right),$$

in which case we obtain, via (2.8), a sum of values $W^0(hk)$ as k runs over a set of representatives for the $(p+1)^3$ cosets of $K_H/(\Gamma_0(p)^3 \cap H)$. These values are given by Lemma 2.1, and it is easy to check that their sum is zero. $\square$

Thus the zeta integral $Z(s, W^0, \Phi(s))$ for any $\Phi(s) \in I(s)^{K_0(p)}$ is proportional to $Z(s, W^0, \Phi_R^\varepsilon(s))$.

Using (4.1) we find that

$$Z(s, W^0, \Phi_R^\varepsilon(s)) = 2[Z(s, W^0, \tilde{\Phi}^0(s)) + \varepsilon Z(s, W^0, \tilde{\Phi}^1(s))].$$

By the preceeding Lemma we find that

$$Z(s, W^0, \tilde{\Phi}^1(s)) = -p^{s+1} \frac{(1 + \varepsilon p^{-3s-3})}{(1 + \varepsilon p^{-s-1})} Z(s, W^0, \tilde{\Phi}^0(s)), \tag{4.2}$$

and thus

$$\begin{aligned} Z(s, W^0, \Phi_R^\varepsilon(s)) &= 2\left[1 - \varepsilon p^{s+1} \frac{(1 + \varepsilon p^{-3s-3})}{(1 + \varepsilon p^{-s-1})}\right] Z(s, W^0, \tilde{\Phi}^0(s)) \\ &= -2\varepsilon p^{s+1}(1 - \varepsilon p^{-s-1})^2 Z(s, W^0, \tilde{\Phi}^0(s)). \end{aligned} \tag{4.3}$$

Next we (brutally) compute the zeta integral $Z(s, W^0, \tilde{\Phi}^0(s))$, where we recall that $\tilde{\Phi}^0(s) = \Phi^0(s)$ is supported on $PK_0(p)$.

PROPOSITION 4.2.

$$Z(s, W^0, \tilde{\Phi}^0(s)) = -(p+1)^{-3}p^{-2s-2}\frac{1}{(1+\varepsilon_p p^{-s-2})(1+\varepsilon_p p^{-s-1})^2}.$$

Proof. First write

$$H = H^0 \cup \begin{pmatrix} p & \\ & 1 \end{pmatrix} H^0, \tag{4.4}$$

where

$$H^0 = \{h \in H \mid \mathrm{ord}_p(\det h_i) \equiv 0 \pmod 2\}. \tag{4.5}$$

Here we continue to abuse notation and view the matrix $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$ as embedded diagonally in H. Setting

$$K_H = \mathrm{GL}_2(\mathbb{Z}_p)^3 \cap H,$$

we have

$$H^0 = Z_G \cdot U \cdot T \cdot K_H \tag{4.6}$$

with U and Z_G as above, and with

$$T = \left\{ t(a) = \left(\begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & a_2^{-1} \end{pmatrix}, \begin{pmatrix} a_3 & \\ & a_3^{-1} \end{pmatrix} \right) \middle| a_1, a_2, a_3 \in \mathbb{Q}_p^\times \right\}. \tag{4.7}$$

Note that $T \cap Z_G$ has order 2.

We now fix our choice of measures. First, note that any $h \in H^0$ can be written as $h = z \cdot u \cdot t(a) \cdot k$ for the Iwasawa decomposition (4.6), and that the map

$$Z_G \times U \times T \times K_H \to H^0$$
$$(z, u, t, k) \mapsto zutk$$

is proper and surjective. On the product space we take the measure

$$|a|^{-2}\, d^\times z\, du\, d^\times a\, dk,$$

where $|a| = |a_1 a_2 a_3|$ and $\mathrm{d}^\times a = \mathrm{d}^\times a_1 \,\mathrm{d}^\times a_2 \,\mathrm{d}^\times a_3$ with $\mathrm{vol}(\mathbb{Z}_p^\times, \mathrm{d}a_i) = 1$. We also take $\mathrm{d}u = \mathrm{d}u_1 \,\mathrm{d}u_2 \,\mathrm{d}u_3$ with $\mathrm{vol}(\mathbb{Z}_p, \mathrm{d}u_i) = 1$, and require $\mathrm{vol}(\mathbb{Z}_p^\times, \mathrm{d}^\times z) = 1$ and $\mathrm{vol}(K_H, \mathrm{d}k) = 1$. The pushforward of this measure to H^0 defines a Haar measure $\mathrm{d}h$ on this group, and since H^0 is an open subgroup of H, determines a unique Haar measure on H, which we also denote by dh. Note that, if we let

$$u(x) = \left(\begin{pmatrix} 1 & x/3 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & x/3 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & x/3 \\ & 1 \end{pmatrix}\right), \tag{4.8}$$

then $\{u(x) \mid x \in \mathbb{Q}_p\}$ is a set of coset representatives for $U_0\backslash U$. If we write a coset representative $\dot{h} = u(x)t(a)k \in H^0$, then the measure $\mathrm{d}\dot{h}$ on the open set $Z_G U_0\backslash H^0$ is given by $|a|^{-2}\,\mathrm{d}x\,\mathrm{d}^\times a\,\mathrm{d}k$, while on the open set

$$Z_G U_0\backslash\begin{pmatrix} p & \\ & 1 \end{pmatrix} H^0 \simeq Z_G U_0\backslash H^0,$$

where we again use representatives $\dot{h}$ on the right hand side, the measure is

$$p^2|a|^{-2}\,\mathrm{d}x\,\mathrm{d}^\times a\,\mathrm{d}k,$$

as is easily checked.

Thus

$$\begin{aligned} Z(s, W^0, \Phi^0(s)) = & \int_{Z_G U_0\backslash H^0} \Phi^0(\delta h, s) W^0(h)\,\mathrm{d}\dot{h} \\ & + p^2 \int_{Z_G U_0\backslash H^0} \Phi^0\left(\delta\begin{pmatrix} p & \\ & 1\end{pmatrix} h, s\right) W^0\left(\begin{pmatrix} p & \\ & 1\end{pmatrix} h\right) \mathrm{d}\dot{h}, \end{aligned} \tag{4.9}$$

and we write $Z_1(s)$ and $Z_2(s)$ for the two terms on the right hand side.

First, noting that the inverse image in $Z_G \times U \times T \times K_H$ of a fundamental domain for the action of $Z_G U_0$ in H^0 is a fundamental domain for the action of this same group in $Z_G \times U \times T \times K_H$, we obtain

$$Z_1(s) = \int_{\mathbb{Q}_p \times (\mathbb{Q}_p^\times)^3 \times K_H} \Phi^0(\delta u(x)t(a)k, s) W^0(u(x)t(a)k)|a|^{-2}\,\mathrm{d}x\,\mathrm{d}^\times a\,\mathrm{d}k \tag{4.10}$$

with the normalization of measures fixed above. We decompose K_H using (2.8) componentwise, and we must evaluate the two functions in the integrand in (4.10) on each coset.

Note that the support of $\Phi^0(g, s)$ is the set $P\cdot K_0(p)$, and write

$$g = \delta u(x)t(a)k = nm(\alpha, \nu)\mathbf{k},$$

with $\mathbf{k} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in K_0(p)$, $\alpha \in \mathrm{GL}_3(\mathbb{Q}_p)$ and $\nu \in \mathbb{Q}_p^\times$. Note that $c \equiv 0 \pmod p$ and $d \in \mathrm{GL}_3(\mathbb{Z}_p)$, and that it is possible, by adjusting α if necessary, to assume that $d = 1$. Since $\nu(\delta) = 1$ and $g \in \delta H^0$, we may set $\nu = 1$ and absorb any further scale into the $K_0(p)$ component. Now, using the second expression for g,

$$(0, 1_3)\cdot g = {}^t\alpha^{-1}(c, 1_3), \tag{4.11}$$

while, using the first expression,

$$(0, 1_3)\cdot g = \left[\begin{array}{ccc|ccc} a_1 & a_2 & a_3 & \dfrac{x}{3a_1} & \dfrac{x}{3a_2} & \dfrac{x}{3a_3} \\ 0 & 0 & 0 & -a_1^{-1} & a_2^{-1} & 0 \\ 0 & 0 & 0 & -a_1^{-1} & 0 & a_3^{-1} \end{array}\right]\cdot k = (C, D),$$

with $C, D \in M_3(\mathbb{Q}_p)$. Since the rows of $(c, 1_3)$ span a free summand of $\mathbb{Z}_p^6$, we must have

$$|\det\alpha|^{-1} = |a_1a_2a_3|^{-1}\max_{i,j}(|x|, |a_ia_j|).$$

Here, the right hand side is the maximum of the absolute values of the 3×3 minors of the matrix $(C, D)k^{-1}$.

Observe that by right $K_0(p)$ invariance, it suffices to calculate our functions when each component of k is one of the coset representatives 1 or $\begin{pmatrix} 1 & x_i' \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ in (2.8). Since D is invertible, we see that k can involve at most one $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

Suppose that $k = 1$, so that,

$${}^t\alpha^{-1} = \begin{bmatrix} \dfrac{x}{3a_1} & \dfrac{x}{3a_2} & \dfrac{x}{3a_3} \\ -a_1^{-1} & a_2^{-1} & 0 \\ -a_1^{-1} & 0 & a_3^{-1} \end{bmatrix}, \tag{4.12}$$

and

$$ {}^t\alpha^{-1}c = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.13} $$

Thus

$$ |\det\alpha|^{-1} = |a_1a_2a_3|^{-1}|x| = |a_1a_2a_3|^{-1}\max_{i,j}(|x|, |a_ia_j|), \tag{4.14} $$

and the condition

$$ c = x^{-1}(a_ia_j) \equiv 0 \pmod p \tag{4.15} $$

(so, $c_{i,j} = x^{-1}a_ia_j$) is equivalent to $|a_ia_j| < |x|$ for all i, j. This amounts to

$$ |a_i^2| < |x|, \qquad i = 1, 2, 3. \tag{4.16} $$

Note that this last condition implies (4.14).

LEMMA 4.3. *If* $k \in \Gamma_0(p)^3 \cap H$, *then*

$$ \Phi^0(\delta u(x)t(a)k, s) = \begin{cases} |a|^{2s+2}|x|^{-2s-2} & \textit{if } |a_i^2| < |x|, i = 1, 2, 3 \\ 0 & \textit{otherwise,} \end{cases} $$

with $|a| = |a_1a_2a_3|$, *and*

$$ W^0(u(x)t(a)k) = \begin{cases} \psi(x)|a|^2 & \textit{if } |a_i| \leqslant 1, i = 1, 2, 3 \\ 0 & \textit{otherwise.} \end{cases} $$

Proof. From the calculation above, it only remains to note that

$$ W_i\left(\begin{pmatrix} a_i & \\ & a_i^{-1} \end{pmatrix}\right) = W_i\left(\begin{pmatrix} a_i^2 & \\ & 1 \end{pmatrix}\right) = \begin{cases} |a_i|^2 & \text{if } |a_i| \leqslant 1 \\ 0 & \text{otherwise,} \end{cases} \tag{4.17} $$

via our assumption of trivial central character and Lemma 2.1. □

Thus we obtain the contribution

$$ \operatorname{vol}(\Gamma_0(p)^3 \cap H) \int_{\substack{|a_i^2| < |x| \\ |a_i| \leqslant 1}} |a|^{2s+2}|x|^{-2s-2}\psi(x)\,\mathrm{d}a\,\mathrm{d}x \tag{4.18} $$

from the trivial coset. This integral is relatively easy to evaluate.

LEMMA 4.4. *The expression* (4.18) *is equal to*

$$(p+1)^{-3}(1-p^{-2s-2})^{-3}(1-p^{-2s-4})^{-1}$$
$$\times[-p^{-2s-2}(1-p^{-2s-4})+(1-p^{-1})p^{-4s-5}(1+p^{-2s-1})].$$

Proof. First the piece of the integral in which $|x|>1$ gives

$$\int_{|a_i|\geq 1}|a|^{2s+2}\,d^\times a\cdot\int_{|x|>1}\psi(x)|x|^{-2s-2}\,dx=(1-p^{-2s-2})^{-3}\cdot(-p^{-2s-2}). \tag{4.19}$$

The remaining part is

$$\int_{|a_i^2|<|x|\leq 1}|a|^{2s+2}|x|^{-2s-2}\,dx\,d^\times a$$
$$=(1-p^{-1})p^{-4s-5}(1+p^{-2s-1})(1-p^{-2s-2})^{-3}(1-p^{-2s-4})^{-1}. \tag{4.20}$$

□

Next suppose that

$$k=\left(\begin{pmatrix}1 & x_1'\\ & 1\end{pmatrix}\begin{pmatrix} & 1\\ -1 & \end{pmatrix},1,1\right), \tag{4.21}$$

so that

$$(C,D)=\left[\begin{array}{ccc|ccc} -\dfrac{x}{3a_1}-a_1x_1' & a_2 & a_3 & a_1 & \dfrac{x}{3a_2} & \dfrac{x}{3a_3}\\ a_1^{-1} & 0 & 0 & 0 & a_2^{-1} & 0\\ a_1^{-1} & 0 & 0 & 0 & 0 & a_3^{-1}\end{array}\right]={}^t\alpha^{-1}(c,1). \tag{4.22}$$

Here $(c,1)$ is the bottom 3 by 6 block of an element of $K_0(p)$. Thus

$$|\det\alpha|^{-1}=|a_1a_2a_3|^{-1}\cdot|a_1|^2=|a_1a_2a_3|^{-1}\max_{i,j}(|x+a_1x_1'|,|a_ia_j|), \tag{4.23}$$

and the condition

$$c={}^t\alpha\begin{bmatrix} -\dfrac{x}{3a_1}-a_1x_1' & a_2 & a_3\\ a_1^{-1} & 0 & 0\\ a_1^{-1} & 0 & 0\end{bmatrix}\equiv 0\ (\mathrm{mod}\ p) \tag{4.24}$$

amounts to

$$|a_1|^2 > |x + a_1^2 x_1'|, \quad \text{and} \quad |a_1| > |a_2|, |a_3|. \tag{4.25}$$

Note that this condition again implies (4.23), and we obtain

LEMMA 4.5. *If k is in the $\Gamma_0(p)^3 \cap H$ coset with representative given by* (4.21), *then*

$$\Phi^0(\delta u(x)t(a)k, s) = \begin{cases} |a|^{2s+2}|a_1|^{-4s-4} & \text{if (4.25) holds} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$W^0(u(x)t(a)k) = \begin{cases} -p^{-1}\psi(x + a_1^2 x_1')|a|^2 & \text{if } |a_1^2| \leqslant p, \text{ and } |a_2|, |a_3| \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the condition $|a_1^2| \leqslant p$ is equivalent to $|a_1| \leqslant 1$.

The contribution of this coset to (4.7) is then

$$-p^{-1}(p+1)^{-3} \int_{\substack{|a_2|,|a_3|<|a_1|\leqslant 1 \\ |x+a_1^2x_1'|<|a_1^2|}} |a|^{2s+2}|a_1|^{-4s-4}\psi(x + a_1^2 x_1')\, \mathrm{d}x\, \mathrm{d}^\times a. \tag{4.26}$$

The factor $\psi(x+a_1^2x_1')$ is 1 in this range, and we obtain

$$\begin{aligned} &\int_{\substack{|a_2|,|a_3|<|a_1|\leqslant 1 \\ |x+a_1^2x_1'|<|a_1^2|}} |a|^{2s+2}|a_1|^{-4s-4}\, \mathrm{d}x\, \mathrm{d}^\times a \\ &\quad = \int_{\substack{|a_1|\leqslant 1 \\ |a_2|,|a_3|<1 \\ |x|<1}} |a_1|^{2s+4}|a_2a_3|^{2s+2}\, \mathrm{d}x\, \mathrm{d}^\times a \\ &\quad = p^{-4s-5}(1 - p^{-2s-2})^{-2}(1 - p^{-2s-4})^{-1}. \end{aligned} \tag{4.27}$$

We remark that result is independent of x_1'. Moreover, it is easy to check that the other cosets with precisely one component involving $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ yield the same answer. The total contribution of these $3p$ cosets is thus

$$-3(p+1)^{-3}p^{-4s-5}(1 - p^{-2s-2})^{-3}(1 - p^{-2s-4})^{-1}. \tag{4.28}$$

Finally, we observe that if more than one component of k involves $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, then the argument of Φ^0 fall outside of $P \cdot K_0(p)$, and the coset in question yields zero. Combining these results, we obtain

$$Z_1(s) = -p^{-2s-2}(p+1)^{-3}(1-p^{-2s-2})^{-3}(1-p^{-2s-4})^{-1} \times [1 + 2p^{-2s-3} - p^{-4s-4} - 2p^{-4s-5}]. \tag{4.29}$$

Next we consider $Z_2(s)$. First, it is not difficult to check that

$$\Phi^0\left(\delta\begin{pmatrix} p & \\ & 1\end{pmatrix} h, s\right) = p^{-s-1}\Phi^0(\delta h, s). \tag{4.30}$$

Thus we may use the expressions given in the Lemmas above for $\Phi^0(s)$ on the relevant cosets. The corresponding expressions for W^0 are given in the following:

LEMMA 4.6. *If $k \in \Gamma_0(p)^3$, then*

$$W^0\left(\begin{pmatrix} p & \\ & 1\end{pmatrix} u(x)t(a)k\right) = \begin{cases} \psi(px)\omega(p)|p|^3|a|^2 & \text{if } |pa_1^2| \leqslant 1,\ |pa_2^2| \leqslant 1,\ |pa_3^2| \leqslant 1 \\ 0 & \text{otherwise.}\end{cases}$$

Here $|pa_i^2| \leqslant 1$ is equivalent to $|a_i| \leqslant 1$.

If k is in the $\Gamma_0(p)^3$ coset with representative given by (2.7) *above, then*

$$W^0\left(\begin{pmatrix} p & \\ & 1\end{pmatrix} u(x)t(a)k\right) = \begin{cases} -p^{-1}\psi(p(x + a_1^2x_1'))\omega(p)|p|^3|a|^2 & \text{if } |pa_1^2| \leqslant p,\ |pa_2^2| \leqslant 1,\ |pa_3^2| \leqslant 1 \\ 0 & \text{otherwise.}\end{cases}$$

Note that in this last expression, the condition for non-vanishing may be written as

$$|a_1| \leqslant p \quad \text{and} \quad |a_2|, |a_3| \leqslant 1.$$

The contribution of the coset of $k = 1$ is then

$$p^{-s-2}(p+1)^{-3}\omega(p)\int_{\substack{|a_i| < |x| \\ |a_i| \leqslant 1}} |a|^{2s+2}|x|^{-2s-2}\psi(px)\,\mathrm{d}x\,\mathrm{d}^\times a, \tag{4.31}$$

and this is equal to

$$\omega(p)p^{-3s-3}(1+p)^{-3}(1-p^{-2s-2})^{-3}(1-p^{-2s-4})^{-1}$$
$$\times[1-p^{-1}-p^{-2s-2}+p^{-4s-5}]. \tag{4.32}$$

Next, each of the terms in which precisely one component of k involves $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ contributes:

$$-(p+1)^{-3}p^{-s-3}\omega(p)\int_{\substack{|a_2|,|a_3|<|a_1|\leqslant p \\ |x+a_1^2x_1'|<|a_1^2|}} |a|^{2s+2}|a_1|^{-4s-4}\psi(p(x+a_1^2x_1'))\,\mathrm{d}x\,\mathrm{d}^\times a. \tag{4.33}$$

Again the factor $\psi(p(x+a_1^2x_1'))$ is 1 in the given range, and the integral becomes

$$\int_{\substack{|x|<1 \\ |a_2|,|a_3|<1,|a_1|\leqslant p}} |a_1|^{2s+4}|a_2a_3|^{2s+2}\,\mathrm{d}x\,\mathrm{d}^\times a = p^{-2s-1}(1-p^{-2s-2})^{-2}(1-p^{-2s-4})^{-1}. \tag{4.34}$$

Thus the total contribution of such cosets is

$$3(1+p)^{-3}\varepsilon p^{-3s-3}(1-p^{-2s-2})^{-2}(1-p^{-2s-4})^{-1}. \tag{4.35}$$

This yields

$$Z_2(s)=(1+p)^{-3}\varepsilon p^{-3s-3}(1-p^{-2s-2})^{-3}(1-p^{-2s-4})^{-1}$$
$$\times[2+p^{-1}-2p^{-2s-2}-p^{-4s-5}]. \tag{4.36}$$

Combining these terms we obtain

$$\begin{aligned}
&Z(s, W^0, \Phi^0(s)) \\
&\quad = Z_1(s)+Z_2(s) \\
&\quad = -(p+1)^{-3}p^{-2s-2}(1-p^{-2s-2})^{-3}(1-p^{-2s-4})^{-1} \\
&\qquad [1+2p^{-2s-3}-p^{-4s-4}-2p^{-4s-5}-2\varepsilon p^{-s-1}-\varepsilon p^{-s-2} \\
&\qquad\quad +2ep^{-3s-3}+\varepsilon p^{-5s-6}] \\
&\quad = -(p+1)^{-3}p^{-2s-2}\frac{(1-\varepsilon p^{-s-2})(1-\varepsilon p^{-s-1})^2(1-p^{-2s-2})}{(1-p^{-2s-2})^3(1-p^{-2s-4})} \\
&\quad = -(p+1)^{-3}p^{-2s-2}\frac{1}{(1+\varepsilon p^{-s-2})(1+\varepsilon p^{-s-1})^2}.
\end{aligned} \tag{4.37}$$

This is the claimed value! □

COROLLARY 4.7.

$$Z(s, W^0, \Phi_R^\varepsilon(s)) = \xi_p(s)\frac{1}{b_p(s)} L_p(F, s+2),$$

where $L_p(F, s+2)$ is given by (2.7), $b_p(s) = \zeta_p(2s+2)\zeta_p(4s+2)$, and

$$\xi_p(s) = 2\varepsilon_p(p+1)^{-3}p^{-s-1}\cdot(1-\varepsilon_p p^{-s-1})^2 b_p(s).$$

5. Local intertwining operators

Several additional facts will be needed when we consider the functional equation. Recall that there is an intertwining operator

$$M_p(s): I_p(s) \to I_p(-s)$$

defined, for large real part of s, by the integral

$$M_p(s)\Phi(g) = \int_N \Phi(w_3 ng, s)\,\mathrm{d}n. \tag{5.1}$$

This integral has a meromorphic analytic continuation to the whole s plane and the normalized operator

$$M_p^*(s) = \frac{1}{a_p(s)} M_p(s),$$

with

$$a_p(s) = \zeta_p(2s-1)\zeta_p(4s-1) \tag{5.2}$$

is entire and non-vanishing for all s [31]. Since $M_p(s)$ is G intertwining, it carries $I_p(s)^{K_0(p)}$ to $I_p(-s)^{K_0(p)}$ and respects eigenspaces of u_p and K and K' fixed vectors. For example,

$$M_p(s)\Phi_K(s) = \frac{a_p(s)}{b_p(s)}\Phi_K(-s) \tag{5.3}$$

with

$$b_p(s) = \zeta_p(2s+2)\zeta_p(4s+2) \tag{5.4}$$

as before. Similarly, we must have

$$M_p(s)\Phi_R^\varepsilon(s) = \alpha_p(s)\Phi_R^\varepsilon(-s) + \beta_p(s)[\Phi_K(-s) + \varepsilon\Phi_{K'}(-s)], \tag{5.5}$$

for some meromorphic functions $\alpha_p(s)$ and $\beta_p(s)$. Note that by Lemma 4.1

$$Z(-s, W^0, M_p(s)\Phi_R^\varepsilon) = \alpha_p(s)Z(-s, W^0, \Phi_R^\varepsilon(-s)). \tag{5.6}$$

We will now determine the function $\alpha_p(s)$.

PROPOSITION 5.1.

$$\alpha_p(s) = \varepsilon p^{-3s-2}\frac{(1-\varepsilon p^{-s-1})^2(1+\varepsilon p^{1-s})(1+p^{1-2s})}{(1-\varepsilon p^{1-s})(1-p^{1-4s})}.$$

First we have

PROPOSITION 5.2.

$$M_p(s)\tilde{\Phi}^0 = \sum_{i=0}^{3} \xi_i(s)\tilde{\Phi}^i(-s),$$

where

$$\xi_0(s) = p^{-6s}\frac{(1-p^{-1})(1-p^{-3})}{(1-p^{1-4s})(1-p^{1-2s})}$$

$$\xi_1(s) = p^{-5s-1}\frac{(1-p^{-1})(1-p^{-2s-1})}{(1-p^{1-4s})(1-p^{1-2s})}$$

$$\xi_2(s) = p^{-4s-2}\frac{(1-p^{-1})}{(1-p^{1-2s})}$$

and

$$\xi_3(s) = p^{-3s-3}$$

Proof. We use a result of Igusa [18].

LEMMA 5.3. (Igusa)

$$\int_{\mathrm{Sym}_n(\mathbb{Z}_p)} |\det x|^s \, dx = \prod_{i=1}^{[n/2]} \frac{(1-p^{1-2i})}{(1-p^{-1-2i-2s})} \cdot \begin{cases} \dfrac{(1-p^{-n-1-s})}{(1-p^{-1-s})} & \textit{if } n \textit{ is even} \\[2ex] \dfrac{(1-p^{-n})}{(1-p^{-1-s})} & \textit{if } n \textit{ is odd.} \end{cases}$$

Now for $0 \leqslant j \leqslant 3$, we compute

$$M_p(s)\Phi^0(w_j) = \int_{\mathrm{Sym}_3(\mathbb{Q}_p)} \Phi^0(w_3 n(b) w_j,\, s)\, \mathrm{d}b.$$

Write

$$b = \begin{pmatrix} x & y \\ {}^t y & z \end{pmatrix} \quad \text{with } x \in \mathrm{Sym}_{3-j}(\mathbb{Q}_p) \quad y \in M_{3-j,j}(\mathbb{Q}_p) \quad z \in \mathrm{Sym}_j(\mathbb{Q}_p).$$

Then, as in (4.11),

$$\begin{aligned}(0, 1_3) w_3 n(b) w_j &= {}^t\alpha^{-1}(c, 1_3) \\ &= -\left(\begin{array}{cc|cc} 1 & -y & x & 0 \\ 0 & -z & {}^t y & 1 \end{array}\right),\end{aligned}$$

and so

$${}^t\alpha^{-1} = -\begin{pmatrix} x & 0 \\ {}^t y & 1 \end{pmatrix},$$

and

$$c = \begin{pmatrix} x^{-1} & -x^{-1} y \\ -{}^t y x^{-1} & -z + {}^t y x^{-1} y \end{pmatrix} \equiv 0 \ (\mathrm{mod}\ p)$$

Thus our integral becomes

$$\begin{aligned}&\int_{\substack{x^{-1} \equiv 0 \bmod p \\ x^{-1} y \equiv 0 \bmod p \\ z - {}^t y x^{-1} y \equiv 0 \bmod p}} |\det x|^{-2s-2}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\ &= \int_{\substack{x^{-1} \equiv 0 \bmod p \\ y \equiv 0 \bmod p \\ z \equiv 0 \bmod p}} |\det x|^{j-2s-2}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\ &= p^{-j(3-j)} p^{-j(j+1)/2} \int_{x^{-1} \equiv 0 \bmod p} |\det x|^{j-2s-2}\, \mathrm{d}x.\end{aligned}$$

Since the measure $|\det x|^{j/2-2}\, \mathrm{d}x$ on $\mathrm{Sym}_{3-j}(\mathbb{Q}_p) \cap \mathrm{GL}_{3-j}(\mathbb{Q}_p)$ is invariant under inversion, the last integral becomes

$$\int_{x \equiv 0 \bmod p} |\det x|^{2s-2}\, \mathrm{d}x$$

which is just

$$p^{-(3-j)(2s-2)-[(3-j)(4-j)/2]}$$

times the integral whose values are given in Lemma 5.3. Since

$$\xi_j(s) = p^{j(-s+1)} M_p(s) \Phi^0(w_j),$$

we obtain the claimed expressions. $\square$

Note that, for any s, the ε-eigenspace for u_p is spanned by the vectors

$$\Psi^0(s) = \tilde{\Phi}^0(s)|(1 + \varepsilon u_p) \quad \text{and} \quad \Psi^1(s) = \tilde{\Phi}^1(s)|(1 + \varepsilon u_p). \tag{5.7}$$

It follows from Proposition 5.2 that

$$M_p(s)\Psi^0 = (\xi_0 + \varepsilon\xi_3)\Psi^0(-s) + (\xi_1 + \varepsilon\xi_2)\Psi^1(-s). \tag{5.8}$$

On the other hand, since

$$\Phi_K(s)|(1 + \varepsilon u_p) = (1 + \varepsilon p^{3s+3})\Psi^0(s) + p^{s+1}(1 + \varepsilon p^{s+1})\Psi^1(s), \tag{5.9}$$

we may use (5.3) and (5.8) to determine $M_p(s)\Psi^1$. It is then a routine, though tedious, matter to solve for $\alpha_p(s)$. This was done using *Mathematica*; we omit the details. $\square$

For use in the proof of the functional equation in the next section, we now let

$$\Phi_p^\natural(s) = \begin{cases} \Phi_K(s) & \text{if } p \nmid N \\ \xi_p(s)^{-1} \cdot \Phi_R^\varepsilon(s) & \text{if } p \mid N, \text{ with } \varepsilon = \varepsilon_p = -a_p b_p c_p, \end{cases} \tag{5.10}$$

where, as before,

$$\xi_p(s) = 2\varepsilon_p p^{-s-1}(p + 1)^{-3}(1 - \varepsilon_p p^{-s-1})^2 b_p(s), \tag{5.11}$$

in the second case. Note that

$$\xi_p(0) = 2\varepsilon_p p^{-1}(p + 1)^{-3}(1 + \varepsilon_p p^{-1})^{-2}. \tag{5.12}$$

We may then summarize our calculations:

$$Z(s, W_p^0, \Phi_p^\natural(s)) = \frac{1}{b_p(s)} L_p(\pi, s + \tfrac{1}{2}), \tag{5.13}$$

and

$$Z(-s,\, W_p^0,\, M_p(s)\Phi_p^\natural) = \delta_p(s)\cdot\frac{a_p(s)}{b_p(s)}\, Z(-s,\, W_p^0,\, \Phi_p^\natural(-s)). \tag{5.14}$$

Here $a_p(s)$ and $b_p(s)$ are given by (5.2) and (5.4), respectively, and

$$\delta_p(s) = \begin{cases} 1 & \text{if } p \nmid N \\ \dfrac{\xi_p(-s)}{\xi_p(s)}\cdot\dfrac{b_p(s)}{a_p(s)}\cdot\alpha_p(s) = \varepsilon_p p^{-5s} & \text{if } p \mid N. \end{cases} \tag{5.15}$$

6. The archimedean case

Next we review the archimedean case. Since we have assumed that the newforms f, g and h all have weight 2, the necessary local factors were computed by Garrett [9]. The groups G, P, H, etc. are defined as in the p-adic case, with $\mathbb{R}$ replacing $\mathbb{Q}_p$. We let G^+ denote the subgroup of G for which the scale $\nu(g)$ takes values in $\mathbb{R}_+^\times$, and we fix the maximal compact subgroup

$$K = \left\{ \mathbf{k} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{Sp}_3(\mathbb{R}) \,\middle|\, k = a + ib \in U(3) \right\}. \tag{6.1}$$

of G^+. The archimedean local components of the triple of automorphic cuspidal representations determined by our newforms are all the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of 'weight' 2. For the discrete series representation of even 'weight' $2n$ the 'holomorphic' vector is given in the Whittaker model, by [12]

$$W_0^{2n}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} k_\theta\right) = \begin{cases} e(x)|y|^n \mathrm{e}^{-2\pi y}\mathrm{e}^{2in\theta} & \text{if } y > 0 \\ 0 & \text{if } y < 0. \end{cases} \tag{6.2}$$

Here $e(x) = \mathrm{e}^{2\pi i x}$ and $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Define a function on H by

$$W^{2n}(h) = W_0^{2n}(h_1)W_0^{2n}(h_2)W_0^{2n}(h_3).$$

Note that the support of W lies in $H^+ = H \cap (\mathrm{GL}_2(\mathbb{R})^+)^3 = H \cap G^+$.

As before, a section $\Phi(s) \in I(s)$ is determined by its restriction to K; and so, for any even integer $2n$, we let $\Phi^{2n}(s)$ denote the section whose restriction to K is given by

$$\Phi^{2n}(\mathbf{k}, s) = (\det k)^{2n}. \tag{6.3}$$

We want to compute the local zeta integral $Z(s, W^{2n}, \Phi^{-2n}(s))$, again given by (2.3), with this choice of W and $\Phi(s)$. Noting that $H^+ = Z_G \cdot \mathrm{SL}_2(\mathbb{R})^3$ we fix the invariant measure $d\dot{h}$ on the coset space $Z_G U_0 \backslash H^+$ by taking coset representatives

$$\dot{h} = u(x)t(a)\mathbf{k}_\theta$$

with $u(x)$ and $t(a)$ given by (4.8) and (4.7), respectively, and with

$$\mathbf{k}_\theta = (k_{\theta_1}, k_{\theta_2}, k_{\theta_3}).$$

Then let

$$d\dot{h} = |a|^{-2}\,\mathrm{d}x\,\mathrm{d}^\times a\,\mathrm{d}\mathbf{k}_\theta,$$

with $\mathrm{d}\mathbf{k}_\theta = \mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3$ and $\mathrm{d}^\times a = \mathrm{d}^\times a_1\,\mathrm{d}^\times a_2\,\mathrm{d}^\times a_3$, for $\mathrm{d}^\times a_i = |a_i|^{-1}\,\mathrm{d}a_i$, and for $\mathrm{d}\theta_i$ and $\mathrm{d}a_i$ the standard Lebesque measure on $\mathbb{R}$. We then obtain

$$\begin{aligned} Z(s, W^{2n}, \Phi^{-2n}(s)) &= \int_{Z_G U_0 \backslash H} \Phi^{-2n}(\delta\dot{h}, s) W^{2n}(\dot{h})\,\mathrm{d}\dot{h} \\ &= \frac{1}{2} 8\pi^3 \int_{\mathbb{R}} \int_{(\mathbb{R}_+^\times)^3} \Phi^{-2n}(\delta u(x)t(a), s) W^{2n}(u(x)t(a))|a|^{-2}\,\mathrm{d}x\,\mathrm{d}^\times a. \end{aligned} \tag{6.4}$$

Here observe that $8\pi^3 = \mathrm{vol}((\mathrm{SO}(2)^3)$ and that the intersection of Z_G with $\mathrm{SL}_2(\mathbb{R})^3$ has order 2.

The calculation which follows is substantially that of Garrett and Harris [10]; we include the details for the sake of completeness and because we need precise information for our functional equation.

First we determine the Iwasawa decomposition of $\delta u(x)t(a)$. Write $\delta u(x)t(a) = nm(\alpha, 1)\mathbf{k}$ with $\mathbf{k} \in K$. Then

$$(0, 1_3) \cdot \delta u(x)t(a) \begin{pmatrix} 1_3 \\ i \cdot 1_3 \end{pmatrix} = i\,{}^t\alpha^{-1} k, \tag{6.5}$$

where $k \in U(3)$ is the element associated to $\mathbf{k}$. On the other hand,

$$(0, 1_3) \cdot \delta u(x)t(a) \begin{pmatrix} 1_3 \\ i \cdot 1_3 \end{pmatrix} = i \cdot \begin{pmatrix} \frac{x}{3} - ia_1^2 & \frac{x}{3} - ia_2^2 & \frac{x}{3} - ia_3^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1^{-1} & & \\ & a_2^{-1} & \\ & & a_3^{-1} \end{pmatrix}. \tag{6.6}$$

Setting

$$z = x + i(a_1^2 + a_2^2 + a_3^2),$$

we obtain

$$|\det \alpha| = |a||z|^{-1}, \tag{6.7}$$

with $|a| = |a_1 a_2 a_3|$, and

$$\det k = \frac{\bar{z}}{|z|}. \tag{6.8}$$

Thus

$$\Phi^{-2n}(\delta u(x)t(a), s) = |a|^{2s+2}|z|^{-2s-2}\left(\frac{\bar{z}}{|z|}\right)^{-2n}. \tag{6.9}$$

Substituting this expression into the zeta integral and using the formula for W^{2n}, we obtain

$$Z(s, W^{2n}, \Phi^{-2n}(s))$$
$$= 4\pi^3 \int_{\mathbb{R}} \int_{(\mathbb{R}_+^\times)^3} |a|^{2s+2n}|z|^{-2s-2}\left(\frac{\bar{z}}{|z|}\right)^{-2n} \mathrm{e}(x)\,\mathrm{e}^{-2\pi(a_1^2+a_2^2+a_3^2)}\,\mathrm{d}x\,\mathrm{d}^\times a. \tag{6.10}$$

Setting $a_i^2 = y_i$, and substituting $-x$ for x, this becomes

$$\frac{\pi^3}{2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} |y|^{s+n}|z|^{-2s-2+2n}z^{-2n}\mathrm{e}(-x)\,\mathrm{e}^{-2\pi(y_1+y_2+y_3)}\,\mathrm{d}x\,\mathrm{d}^\times y.$$

Recalling that [10]

$$\int_{-\infty}^{\infty} \mathrm{e}(-x)(x+iy)^{-\alpha}(x-iy)^{-\beta}\,\mathrm{d}x$$
$$= \frac{(-2\pi i)^\alpha(2\pi i)^\beta}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\infty} (t+1)^{\alpha-1}t^{\beta-1}\,\mathrm{e}^{-2\pi y(1+2t)}\,\mathrm{d}t, \tag{6.11}$$

and setting $\alpha = s+1+n$ and $\beta = s+1-n$, we obtain

$$\frac{\pi^3}{2}\frac{(-1)^n(2\pi)^{2s+2}}{\Gamma(s+1+n)\Gamma(s+1-n)}$$

times the integral

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |y|^{s+n} e^{-4\pi(y_1+y_2+y_3)(t+1)}(t+1)^{\alpha-1} t^{\beta-1}\, dt\, d^\times y,$$

$$= (4\pi)^{-3s-3n}\Gamma(s+n)^3 \int_0^\infty (t+1)^{-3s-3n+\alpha-1} t^{\beta-1}\, dt$$

$$= (4\pi)^{-3s-3n}\Gamma(s+n)^3 \frac{\Gamma(\beta)\Gamma(3s+3n-\alpha-\beta+1)}{\Gamma(3s+3n-\alpha+1)}.$$

Collecting terms and simplifying we obtain

$$Z(s, W^{2n}, \Phi^{-2n}(s)) = (-1)^n 2^{-4s-6n+1} \pi^{-s-3n+5} \frac{\Gamma(s+n)^3\Gamma(s+3n-1)}{\Gamma(s+n+1)\Gamma(2s+2n)}. \tag{6.12}$$

Now, as in Section 1, since the Hodge numbers associated to F are

$$h^{6n-3,0} = h^{0,6n-3} = 1 \quad \text{and} \quad h^{4n-2,2n-1} = h^{2n-1,4n-2} = 3,$$

the predicted Γ factor for a triple of forms of weight $2n$ is

$$L_\infty(F, s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-2n+1)^3,$$

and thus

$$L_\infty(F, s+3n-1) = L_\infty(\pi, s+\tfrac{1}{2}) = (2\pi)^{-4s-6n+1}\Gamma(s+n)^3\Gamma(s+3n-1), \tag{6.13}$$

while

$$b_\infty(s) = \pi^{-3s-2}\Gamma(s+1)\Gamma(2s+1).$$

Thus

$$Z(s, W^{2n}, \Phi^{-2n}(s)) = \xi_\infty(s) \frac{1}{b_\infty(s)} L_\infty(\pi, s+\tfrac{1}{2}), \tag{6.14}$$

with

$$\begin{aligned}\xi_\infty(s) &= (-1)^n \pi^{3n+2} \frac{\Gamma(s+1)\Gamma(2s+1)}{\Gamma(s+n+1)\Gamma(2s+2n)} \\ &= (-1)^n \pi^{3n+2}[(2s+2n-1)(2s+2n-2)\cdots \\ &\qquad \cdots (2s+1)(s+n)(s+n-1)\cdots(s+1)]^{-1}.\end{aligned} \tag{6.15}$$

On the other hand, we know by [24, Lemma 4.6] that

$$M(s)\Phi^{-2n} = \alpha_\infty(s, 2n)\Phi^{-2n}(-s), \tag{6.16}$$

with

$$\alpha_\infty(s, 2n) = (-1)^n 2^{-6s+3}\pi^{9/2} \times \frac{\Gamma(2s)\Gamma(2s-\frac{1}{2})\Gamma(2s-1)}{\Gamma(s+n+1)\Gamma(s+n+\frac{1}{2})\Gamma(s+n)\Gamma(s-n+1)\Gamma(s-n+\frac{1}{2})\Gamma(s-n)}. \tag{6.17}$$

(Note that, because of our normalization, we must replace s by $2s$ in Lemma 4.6 of [24]. Also, in that Lemma a factor of i^l was omitted from the expression for $d_{n,v}(s, l)$ and the second Γ factor in the denominator of that expression should be $\Gamma_n[(s+\rho_n-l)/2]$.) Again we let

$$\Phi^\natural_\infty(s) = \xi_\infty(s)^{-1}\Phi^{-2n}_\infty(s), \tag{6.18}$$

and, as in (5.13), we write

$$Z(-s, W^{2n}, M(s)\Phi^\natural_\infty) = \delta_\infty(s)\frac{a_\infty(s)}{b_\infty(s)}Z(-s, W^{2n}, \Phi^\natural_\infty(-s)). \tag{6.19}$$

We obtain, after a short calculation,

$$\delta_\infty(s) = \frac{\xi_\infty(-s)}{\xi_\infty(s)}\frac{b_\infty(s)}{a_\infty(s)}\alpha_\infty(s, 2n) = -1. \tag{6.20}$$

Note that for $n = 1$ (the case of ultimate interest for us),

$$\xi_\infty(s) = -\frac{\pi^5}{(s+1)(2s+1)}. \tag{6.21}$$

and so $\xi_\infty(0) = -\pi^5$.

7. The global functional equation

In this section we will assemble the local results of Sections 2–6 to obtain the functional equation of $L(s, F)$.

Let G, and Z_G, $P = MN$, H, U, U_0, etc., be the global analogues of the groups considered in Section 2, and for each place v of $\mathbb{Q}$, denote by G_v, P_v, etc., the corresponding completions. Let

$$K = \prod_v K_v,$$

where K_v is the maximal compact subgroup of G_v fixed in Section 2 and Section 6. For $s \in \mathbb{C}$, let $I(s) = \otimes_v I_v(s)$ be the global induced representation, consisting of smooth, right K-finite functions on $G(\mathbb{A})$ which satisfy the global analogue of (2.1) for all $a \in \mathrm{GL}_3(\mathbb{A})$, $v \in \mathbb{A}^\times$ and $n \in N(\mathbb{A})$. For an entire section $\Phi(s) \in I(s)$, we have the Eisenstein series

$$E(g, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(\gamma g, s), \tag{7.1}$$

which is absolutely convergent for $\mathrm{Re}(s) > 1$. As is well known, this series has a meromorphic analytic continuation to the whole s plane and has a functional equation

$$E(g, s, \Phi) = E(g, -s, M(s)\Phi), \tag{7.2}$$

where $M(s)\colon I(s) \to I(-s)$ is the global intertwining operator which, for $\mathrm{Re}(s) > 1$, is given by the integral

$$M(s)\Phi(g) = \int_{N(\mathbb{A})} \Phi(w_3 n g, s)\, \mathrm{d}n. \tag{7.3}$$

Now let $\tilde{F}$ denote the function on $(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))^3$ associated to $F = f \otimes g \otimes h$ of (1.4). Explicitly, we define a compact open subgroup

$$\hat{\Gamma}_0(N) = \prod_{p|N} \Gamma_0(p) \times \prod_{p\nmid N} \mathrm{GL}_2(\mathbb{Z}_p) \tag{7.4}$$

of $\mathrm{GL}_2(\mathbb{A})$, and for any $g \in \mathrm{GL}_2(\mathbb{A})^3$ we write $g = \gamma \cdot g_\infty \cdot k$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q})^3$, $g_\infty \in \mathrm{GL}_2^+(\mathbb{R})^3$, and $k \in \hat{\Gamma}_0(N)^3$. Then we set

$$\tilde{F}(g) = j(g_\infty, i)^{-2} F(g_\infty(i)), \tag{7.5}$$

where, for $g_\infty = (g_1, g_2, g_3)$ with $g_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$,

$$j(g_\infty, i) = (c_1 i + d_1)(c_2 i + d_2)(c_3 i + d_3)(\det(g_1 g_2 g_3))^{-1/2}. \tag{7.6}$$

For a factorizable section $\Phi(s) = \otimes_v \Phi_v(s) \in I(s)$, the basic formula due to Garrett [9] and to Piatetski-Shapiro and Rallis [31] is

$$Z(s, F, \Phi) = \int_{Z_G(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \tilde{F}(h)E(h, s, \Phi)\,\mathrm{d}h$$
$$= \prod_v Z_v(s, W_{F,v}, \Phi_v), \tag{7.7}$$

where $W_F = \otimes_v W_{F,v}$ is the global Whittaker function determined by $\tilde{F}$ and ψ [12], and where $Z_v(s, W_{F,v}, \Phi_v)$ is the local zeta integral considered in section 2. Note that, by the functional equation of $E(g, s, \Phi)$,

$$Z(s, F, \Phi) = Z(-s, F, M(s)\Phi). \tag{7.8}$$

We now choose the local components of $\Phi(s)$ as in Section 2; let

$$\Phi^\natural(s) = \otimes_v \Phi_v^\natural(s), \tag{7.9}$$

where the function $\Phi_v^\natural(s)$ is given by (5.10) for the finite places and by (6.18) for the archimedean place. By (5.13) and (6.14) we have

PROPOSITION 7.1.

$$Z(s, F, \Phi^\natural) = \int_{Z_G(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \tilde{F}(h)E(h, s, \Phi^\natural)\,\mathrm{d}h = \frac{1}{b(s)}\Lambda(F, s+2), \tag{7.10}$$

where $\Lambda(F, s)$ is as in Section 1, *and $b(s) = \zeta^*(2s+2)\zeta^*(4s+2)$.*

Here $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, so by the functional equation $\zeta^*(s) = \zeta^*(1-s)$ we have $a(s) = b(-s)$. On the other hand, by (5.14) and (6.19),

$$Z(-s, F, M(s)\Phi^\natural) = \left(\prod_v \delta_v(s)\right)\cdot\frac{a(s)}{b(s)}\cdot Z(-s, F, \Phi^\natural(-s))$$
$$= \left(\prod_v \delta_v(s)\right)\cdot\frac{a(s)}{b(s)}\frac{1}{b(-s)}\Lambda(F, -s+2). \tag{7.11}$$

This yields

$$\Lambda(F, s+2) = \left(\prod_v \delta_v(s)\right)\cdot\Lambda(F, -s+2). \tag{7.12}$$

Shifting s by 2 and using the values of $\delta_v(s)$ given in (5.15) and (6.20), we obtain the functional equation

$$\Lambda(F, s) = -\left(\prod_{p|N} \varepsilon_p\right) \cdot N^{-5s+10}\Lambda(F, 4-s), \tag{7.13}$$

claimed in Section 1.

We will now show that $\Lambda(F, s)$ is entire. This fact should follow immediately from the results of Ikeda [20], provided we were to check that our 'bad' Euler factors agree with his. Instead of doing this, we will sketch a proof that there are no poles based on the results of [26] and [25], [34].

Consider again the integral representation

$$\Lambda(F, s+2) = \int_{Z_G(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \tilde{F}(h)b(s)E(h, s, \Phi^{\natural})\, dh,$$

of (7.10). By the functional equation, it suffices to show that the right hand side of this expression has no poles in the half plane $\mathrm{Re}(s) \geqslant 0$. Let $S = \{\infty\} \cup \{p : p \mid N\}$ and observe that we have

$$b(s)E(g, s, \Phi^{\natural}) = \left(\prod_{v \in S} \xi_v(s)^{-1} b_v(s)\right) \cdot b_S(s)E(g, s, \Phi^{\varepsilon}),$$

where $b_S(s) = \Pi_{p \notin S}\, b_p(s)$, and

$$\Phi^{\varepsilon}(s) = \Phi_{\infty}^{-2}(s) \otimes (\otimes_{p|N} \Phi_{R_p}^{\varepsilon_p}(s))(\otimes_{p\nmid N} \Phi_{K_p}(s)).$$

By (5.11) and (6.21)

$$\xi_p(s)^{-1}b_p(s) = \varepsilon_p p^{s+1}(p+1)^3(1-\varepsilon_p p^{-s-1})^{-2},$$

and

$$\xi_{\infty}(s)^{-1}b_{\infty}(s) = -\pi^{-3s-7}\Gamma(s+2)\Gamma(2s+2).$$

These factors are holomorphic in the half plane of interest. On the other hand, it is proved in [26] that for any standard section $\Phi(s)$ (the restriction of such a section to K is independent of s) which is K_p invariant for $p \notin S$, the normalized Eisenstein series $b_S(s)E(g, s, \Phi)$ has at most simple poles at $s = 1, \frac{1}{2}, -\frac{1}{2}, -1$. (Note that the variable s of [26] must be replaced by $2s$ to obtain the Eisenstein series of our present paper.)

Since the central character of our Eisenstein series is trivial, the residue at the point $s = 1$ is a constant function of g, while the residue at $s = \frac{1}{2}$ is a (regularized) theta series associated to a split binary quadratic form [25]. These results hold for the series $b_S(s)E(g, s, \Phi^\varepsilon(s))$ as well, since $\Phi^\varepsilon(s)$ is a finite linear combination of standard sections with entire coefficients. But since $E(g, s, \Phi^\varepsilon(s))$ is an eigenfunction with non-trivial eigencharacter for K_∞, its residue at $s = 1$ must vanish. Thus the only possible pole of $\Lambda(F, s + 2)$ in the half plane $\mathrm{Re}(s) \geqslant 0$ is at $s = \frac{1}{2}$.

To exclude this pole we observe that the restriction to $H(\mathbb{A})$ of the regularized binary theta series can be shown to be orthogonal to all cusp forms on this group.

Thus we have:

COROLLARY 7.2. *$\Lambda(F, s)$ is entire.*

Note that the key ingredients in the proof – the fact that the residue of the regularized Eisenstein series at $s = \frac{1}{2}$ is a theta series series associated to a split binary quadratic form and the fact that the restriction of such a series to $H(\mathbb{A})$ is orthogonal to all cusp forms – require only that the product of the central characters of the triple of cusp forms be trivial. Thus the same argument shows that the triple product L-function is entire under this condition.

Actually, Ikeda [20] shows that, up to twisting each of the given triple of cuspidal automorphic representations π_i of $\mathrm{GL}(2, \mathbb{A})$ by a character of the form $|\det g|^{s_i}$ for some $s_i \in \mathbb{C}$, the triple product L function can only have poles when the product of the central characters is a *non-trivial* quadratic character ω. In this case, if K is the quadratic extension corresponding to ω, there exist quasicharacters χ_1, χ_2 and χ_3 of K with $\chi_1\chi_2\chi_3 = 1$ such that $\pi_i = \pi(\chi_i)$. The poles of the Langlands L-function (with functional equation relating s to $1 - s$) then occur at $s = 0$ and 1.

It is perhaps enlightening to describe the Eisenstein series which occurs here in classical language. We will do this only in the case $N = p$ is a prime; the general case goes along the same lines.

For convenience we write $K = \prod_q K_q$ where $K_q = \mathrm{GSp}_3(\mathbb{Z}_q)$ is our fixed maximal compact subgroup of G_q and let

$$\Gamma = G(\mathbb{Q}) \cap (G(\mathbb{R})^+ \cdot K) = \mathrm{GSp}_3(\mathbb{Z}). \tag{7.14}$$

Similarly, we let $K' = K_0(p) \times \prod_{q \neq p} K_q$ and let

$$\Gamma' = G(\mathbb{Q}) \cap (G(\mathbb{R})^+ \cdot K') = \Gamma_0(p), \tag{7.15}$$

be the corresponding congruence subgroup of Γ. Note that $\nu(K) = \nu(K') = \hat{\mathbb{Z}}^\times$ so that, by the strong approximation theorem,

$$G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+K = G(\mathbb{Q})G(\mathbb{R})^+K'.$$

Since $E(g, s, \Phi^{\natural})$ is left $G(\mathbb{Q})$ and right K' invariant, it suffices to evaluate it for $g \in G(\mathbb{R})^{+}$, which we view as an element of $G(\mathbb{A})$ with trivial components at all finite places. Moreover, since $\Phi_{\infty}^{\natural}$ is an eigenfunction of K_{∞}, we may assume that $g = g_{\infty}$ has the form

$$g_{\infty} = n(x)m(v) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} v & \\ & {}^{t}v^{-1} \end{pmatrix}, \tag{7.16}$$

with $v \in \mathrm{GL}_3(\mathbb{R})^{+}$, and we set

$$Z = x + iy = x + i\,{}^{t}vv = g(i \cdot 1_3) \in \mathfrak{H}_3. \tag{7.17}$$

We also observe that

LEMMA 7.3.

$$P(\mathbb{Q})\backslash G(\mathbb{Q}) \simeq (P(\mathbb{Q}) \cap \Gamma)\backslash\Gamma,$$

In particular, a set of coset representatives for $P(\mathbb{Q})\backslash G(\mathbb{Q})$ *may be taken to lie in* $\mathrm{Sp}_3(\mathbb{Z})$.

Thus we obtain

$$E(g, s, \Phi^{\natural}) = \sum_{\gamma \in P(\mathbb{Q}) \cap \Gamma\backslash\Gamma} \Phi_{\infty}^{\natural}(\gamma g, s)\Phi_{p}^{\natural}(\gamma), \tag{7.18}$$

and it only remains to evaluate the two factors of each term.

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Q})$ with $\nu(\gamma) = 1$, we write $\gamma g = nm(\alpha, 1)k$ in the Iwasawa decomposition, with $\alpha \in \mathrm{GL}_3(\mathbb{R})^{+}$ and have

$$(0, 1_3)\cdot \gamma g \cdot \begin{pmatrix} 1_3 \\ i \cdot 1_3 \end{pmatrix} = i \cdot {}^{t}\alpha^{-1}k \tag{7.19}$$

as in (6.5). On the other hand, we find that

$$(0, 1_3)\cdot \gamma g \cdot \begin{pmatrix} 1_3 \\ i \cdot 1_3 \end{pmatrix} = i \cdot (c\bar{Z} + d)\,{}^{t}v^{-1}, \tag{7.20}$$

and thus,

$$\det(\alpha) = |\det(\alpha)| = \det(v) \cdot |\det(cZ + d)|^{-1}, \tag{7.21}$$

and

$$\det(k) = \det(c\bar{Z} + d)|\det(cZ + d)|^{-1}. \tag{7.22}$$

Thus, recalling (6.18) and the fact that $\det(y) = \det(v)^2$, we have

LEMMA 7.4.

$$\Phi_\infty^\natural(\gamma g, s) = \xi_\infty(s)^{-1}\det(c\bar{Z} + d)^{-2}|\det(cZ + d)|^{2-2s-2}\cdot\det(y)^{s+1}$$

where

$$\xi_\infty(s)^{-1} = -\pi^{-5}(s+1)(2s+1).$$

Next note that we must have $\varepsilon_p = -a_p b_p c_p = -1$, since $N = p$. Thus, recalling (5.10), (5.11), (5.4),

$$\Phi_p^\natural(s) = \xi_p(s)^{-1}\Phi_R^{\varepsilon_p}(s),$$

with

$$\xi_p(s)^{-1} = -\frac{1}{2}p^{s+1}(p+1)^3\,\frac{(1-p^{-2s-2})(1-p^{-4s-2})}{(1+p^{-s-1})^2}.$$

Now by (3.8) and the definition of $\Phi_p^i(s)$ given after (3.2), we have, for $\gamma \in \Gamma$ as above,

$$\Phi_R^{\varepsilon_p}(\gamma, s) = (-1)^{r_p(c)}p^{-r_p(c)(s+1)}, \tag{7.23}$$

where $r_p(c)$ is the rank (mod p) of the matrix $c \in M_3(\mathbb{Z})$. Collecting these facts we obtain the expression

PROPOSITION 7.5. *If $N = p$ is a prime, then*

$$\begin{aligned} E(\bar{Z}, s) &= \det(v)^{-2}E(g, s, \Phi^\natural) \\ &= \frac{1}{2}\pi^{-5}(s+1)(2s+1)p^{s+1}(p+1)^3\,\frac{(1-p^{-s-1})(1-p^{-4s-2})}{(1+p^{-s-1})} \\ &\quad\times \sum_{\gamma\in P\cap\Gamma\backslash\Gamma}(-1)^{r_p(c)}p^{-r_p(c)(s+1)}\det(c\bar{Z}+d)^{-2}|\det(cZ+d)|^{-2s}\cdot\det(y)^s. \end{aligned}$$

Here, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_3(\mathbb{Z})$ *and $r_p(c)$ denotes the rank*(mod p) *of c.*

Note that $E(\bar{Z}, s)$ is a (non-holomorphic) Siegel modular form of weight -2 for $\Gamma' = \Gamma_0(p)$, while $\overline{E(\bar{Z}, \bar{s})}$ is a (non-holomorphic) Siegel modular form of weight 2 for $\Gamma' = \Gamma_0(p)$.

8. The Weil-Siegel formula and consequences

In this section we will recall the relation between the central value of the Eisenstein series $E(g, s, \Phi)$ of Section 7 and a certain type of theta series associated to a definite quaternion algebra. This relation is the Weil-Siegel formula as extended in [24, 33], and in our present situation, it provides an explicit formula for the central value of the triple product L-function. If, for our given F, the sign $\varepsilon = -\prod_{p|N}(-a_p b_p c_p)$ in the functional equation of $L(F, s)$ is -1, then $L(F, 2) = 0$. Thus we will restrict our attention to the case in which $\varepsilon = +1$. In fact, it can be shown that when $\varepsilon = -1$, we have $E(g, 0, \Phi^{\natural}) = 0$ as well, where $E(g, s, \Phi^{\natural})$ is the Eisenstein series which occurs in the integral representation (7.10).

Let $S = \{p : p \mid N \text{ and } \varepsilon_p = -a_p b_p c_p = -1\} \cup \{\infty\}$ be the even set of places of $\mathbb{Q}$ determined by F and let B be the unique definite quaternion algebra ramified at the places in S. Let $R \subset B$ be an Eichler order with reduced discriminant N. Note that R is unique up to local conjugacy. Let ν: $B \to \mathbb{Q}$ be the reduced norm, and let V, (,) be the rational vector space $V = B$ with quadratic form $\nu(x) = \frac{1}{2}(x, x)$. Here the associated bilinear form is given by $(x, y) = tr_B(xy^{\iota})$ where $x \mapsto x^{\iota}$ is the main involution of B, and tr_B: $B \to \mathbb{Q}$ is the reduced trace.

Define a Schwartz function $\varphi = \otimes_v \varphi_v \in S(V(\mathbb{A}))$ by

$$\varphi_p = \text{characteristic function of } R_p \qquad (R_p = R \otimes_{\mathbb{Z}} \mathbb{Z}_p) \tag{8.1}$$

and

$$\varphi_\infty(x) = e^{-2\pi\nu(x)}. \tag{8.2}$$

Then let

$$\tilde{\varphi} = \varphi \otimes \varphi \otimes \varphi \in S(V(\mathbb{A})^3). \tag{8.3}$$

Let $G = \mathrm{GSp}_3$ and let $G^1 = \mathrm{Sp}_3$ be the kernel of the scale map. Also let $O(V)$ be the orthogonal group of V, (,). We will describe $O(V)$ more explicitly in terms of $B^{\times}$ in a moment. Recall that for our fixed additive character ψ there is a Weil representation $\omega = \omega_\psi$ of $G^1(\mathbb{A})$ on $S(V(\mathbb{A})^3)$, which commutes with the natural action of $O(V)(\mathbb{A})$ [42]. The theta function

$$\theta(g, r; \tilde{\varphi}) = \sum_{x \in V(\mathbb{Q})^3} (\omega(g)\tilde{\varphi})(r^{-1}x) \tag{8.4}$$

is left $G^1(\mathbb{Q})$ invariant as a function of $g \in G^1(\mathbb{A})$ and left $O(V)(\mathbb{Q})$ invariant as a function of $r \in O(V)(\mathbb{A})$. Moreover, since B is a division algebra so that the quadratic space V, (,) is anisotropic, the space $O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})$ is compact and the integral

$$I(g, \tilde{\varphi}) = \int_{O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})} \theta(g, r; \tilde{\varphi})\, \mathrm{d}r \tag{8.5}$$

is absolutely convergent. Here we normalize the invariant measure $\mathrm{d}r$ so that

$$\mathrm{vol}(O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A}), \mathrm{d}r) = 1.$$

For $P = MN$ and K as in Section 7, we have an Iwasawa decomposition

$$G(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})K, \qquad g = nm(a, v)k,$$

with $n \in N(\mathbb{A})$, $k \in K$, and where, for $a \in \mathrm{GL}_3(\mathbb{A})$ and $v \in \mathbb{A}^\times$, $m(a, v)$ is given by (2.0). Although a and v are not uniquely determined by g, the quantities

$$|a(g)| = |\det a|_{\mathbb{A}} \quad \text{and} \quad |v(g)| = |v|_{\mathbb{A}}$$

are well defined.

For $s \in \mathbb{C}$, $g_1 \in G^1(\mathbb{A})$ and $\tilde{\varphi} \in S(V(\mathbb{A})^3)$ as above, we let

$$\Phi_{\mathrm{flat}}(g_1, s) = (\omega(g_1)\tilde{\varphi})(0) \cdot |a(g_1)|^{2s}. \tag{8.6}$$

Here the subscript 'flat' refers to the fact that the restriction of this section to K is independent of s. Note that by the formulas for the action of $M(\mathbb{A})$ and $N(\mathbb{A})$ in the Weil representation, the function $g_1 \mapsto (\omega(g_1)\tilde{\varphi})(0)$ lies in $I(0)$, and, hence, the function $g_1 \mapsto \Phi(g_1, s)$ lies in $I(s)$. For any $g \in G(\mathbb{A})$, we let

$$g_1 = \begin{pmatrix} 1 & \\ & v(g)^{-1} \end{pmatrix} \cdot g,$$

and set

$$\Phi_{\mathrm{flat}}(g, s) = |v(g)|^{-3s-3} \Phi_{\mathrm{flat}}(g_1, s). \tag{8.7}$$

It is not difficult to check that $\Phi_{\text{flat}}(s)$ lies in the space $I(s)$, the global induced representation defined in Section 7. As in that section, we may then form the Eisenstein series $E(g, s, \Phi_{\text{flat}})$, which extends by analytic continuation to a meromorphic function on the whole s plane. The function $E(g, s, \Phi_{\text{flat}})$ is holomorphic at $s = 0$ and the Weil-Siegel formula [24, 33] asserts that

$$E(g, 0, \Phi_{\text{flat}}) = 2I(g, \tilde{\varphi}). \tag{8.8}$$

Note that the factor of 2 occurs because we are on the unitary axis [24].

We next must determine the local components of $\Phi_{\text{flat}}(s) = \otimes_v \Phi_{\text{flat},v}(s)$.

First we consider φ_p, the characteristic function of $R_p \subset B_p$. Let

$$\hat{R}_p = \{x \in B_p \mid \text{tr}_B(xy^\iota) \in \mathbb{Z}_p \text{ for all } y \in R_p\}, \tag{8.9}$$

and let

$$\tilde{\varphi}_p(x) = \int_{B_p} \psi(\text{tr}_B(xy^\iota))\varphi_p(y)\, dy, \tag{8.10}$$

be the Fourier transform of φ_p, where dy is the self dual measure on B_p for the pairing $\langle x, y\rangle = \psi(\text{tr}_B(xy^\iota))$. Then it is easy to check that

LEMMA 8.1.

$$\tilde{\varphi}_p = \kappa_p \cdot \textit{the characteristic function of } \hat{R}_p,$$

where

$$\kappa_p = \text{vol}(R_p, dx) = \begin{cases} p^{-1} & \textit{if } p \mid N \\ 1 & \textit{if } p \nmid N. \end{cases}$$

Also recall that the group $SL_2(\mathbb{Q}_p)$ acts on $S(V_p)$ via the Weil representation $\omega^{(1)}$ determined by our fixed ψ, and that, for this action [21]

$$\omega^{(1)}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right)\varphi(x) = \varepsilon_p \int_{B_p} \psi(\text{tr}_B(xy^\iota))\varphi(y)\, dy = \varepsilon_p \hat{\varphi}_p(x), \tag{8.11}$$

where we recall that

$$\varepsilon_p = \begin{cases} -1 & \text{if } B_p \text{ is a division algebra} \\ 1 & \text{otherwise.} \end{cases} \tag{8.12}$$

Similarly, the group $\mathrm{Sp}_3(\mathbb{Q}_p)$ acts on $S(V_p^3) \simeq S(V_p) \otimes S(V_p) \otimes S(V_p)$ via the Weil representation $\omega = \omega^{(3)}$, and the restriction of this representation to the subgroup $\mathrm{SL}_2(\mathbb{Q}_p)^3$ is the outer tensor product of the representations $\omega^{(1)}$ on the three factors. In particular, the element w_i of (3.2), for $0 \leqslant i \leqslant 3$, acts by (8.11) on the last i components of an element $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$, and acts trivially on the other components.

PROPOSITION 8.2. *For any prime p, let* $\tilde{\varphi}_p = \varphi_p \otimes \varphi_p \otimes \varphi_p \in S(V_p^3)$ *with* φ_p *given by* (8.1).

(i) *if* $p \nmid N$, *then for all* $g \in G^1(\mathbb{Q}_p)$,

$$\Phi_{\mathrm{flat}}(g, 0) = (\omega(g)\tilde{\varphi}_p)(0) = \Phi_{K_p}(g, 0),$$

where $\Phi_{K_p}(s)$ *is the normalized* K_p *invariant section of* $I_p(s)$, *as in Section* 3.

(ii) *If* $p \mid N$, *then for all* $g \in G^1(\mathbb{Q}_p)$,

$$\Phi_{\mathrm{flat}}(g, 0) = (\omega(g)\tilde{\varphi}_p)(0) = \Phi_{R_p}^{\varepsilon_p}(g, 0),$$

where ε_p *is as above and* $\Phi_{R_p}^{\varepsilon_p}(s)$ *is the section defined in Section* 3, (3.8).

Note that (ii) is the main reason for our choice of the functions $\Phi_{R_p}^{\varepsilon_p}(s)$ in Section 3, and also that the section $\Phi_{R_p}^{\varepsilon_p}(s)$ 'twists away' from $\Phi_{\mathrm{flat}}(s)$ as we move away from the point $s = 0$.

Proof. For convenience, we will write φ for $\tilde{\varphi}_p$. First note that it suffices to prove these identities for $g = k \in K_p \cap G^1(\mathbb{Q}_p)$. Moreover, for any element $k \in K_0(p) \cap G^1(\mathbb{Q}_p)$ we have a decomposition

$$k = \begin{pmatrix} a & b \\ pc & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} {}^t d^{-1} & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ pd^{-1}c & 1 \end{pmatrix}. \tag{8.13}$$

Note that the element d here lies in $\mathrm{GL}_3(\mathbb{Z}_p)$. Now we have

LEMMA 8.3. $\varphi = \tilde{\varphi}_p$ *is* $K_0(p)$ *invariant.*

Proof. By (8.13) and the standard formulas for the action of $\mathrm{Sp}_3(\mathbb{Q}_p)$ in the Weil representation,

$$\omega(k)\varphi(x) = \psi\left(\frac{1}{2}\mathrm{tr}(bd^{-1}(x, x))\right) \cdot \left(\omega\left(\begin{pmatrix} 1 & 0 \\ pd^{-1}c & 1 \end{pmatrix}\right)\varphi\right)(x{}^t d^{-1}).$$

Since

$$\begin{pmatrix} 1 & 0 \\ pd^{-1}c & 1 \end{pmatrix} = w_3 \begin{pmatrix} 1 & -pd^{-1}c \\ 0 & 1 \end{pmatrix} w_3^{-1},$$

we have

$$\omega\left(\begin{pmatrix} 1 & 0 \\ pd^{-1}c & 1 \end{pmatrix}\right)\varphi(x^t d^{-1}) = \int_{V^3} \psi\left(-\frac{1}{2}p\,\mathrm{tr}(d^{-1}c(y,y))\right)\psi(\mathrm{tr}(d^{-1}(x,y)))\hat{\varphi}(-y)\,\mathrm{d}y,$$

where, by Lemma 8.1, $\hat{\varphi}$ is κ_p^3 times the characteristic function of $(\hat{R}_p)^3 \subset V_p^3$. Thus this integral becomes

$$\kappa_p^3 \int_{\hat{R}_p^3} \psi(-\tfrac{1}{2}p\,\mathrm{tr}(d^{-1}c(y,y)))\psi(\mathrm{tr}(d^{-1}(x,y)))\,\mathrm{d}y.$$

The following fact is easily checked

LEMMA 8.4. *If y_1 and y_2 are in $\hat{R}_p^3$, then $\mathrm{tr}_B(y_1 y_2^\iota) \in \kappa_p \cdot \mathbb{Z}_p$, where κ_p is as in Lemma* 8.1.

Now let $d^{-1}c = T = {}^tT \in M_3(\mathbb{Q}_p)$. Then

$$\mathrm{tr}(T(y,y)) = \sum_i T_{i,i} 2\nu(y_i) + \sum_{i<j} 2T_{i,j}\,\mathrm{tr}_B(y_i y_j^\iota) \in 2\kappa_p \cdot \mathbb{Z}_p,$$

and so

$$\psi(-\tfrac{1}{2}p\,\mathrm{tr}(d^{-1}c(y,y))) = 1$$

for $y \in \hat{R}_p^3$. Thus our integral is just

$$\kappa_p^3 \cdot \int_{\hat{R}_p^3} \psi(\mathrm{tr}(d^{-1}(x,y)))\,\mathrm{d}y = \begin{cases} 1 & \text{if } x \in R_p^3 \\ 0 & \text{otherwise.} \end{cases}$$

Since, for $x \in R_p^3$, $\psi(\tfrac{1}{2}\mathrm{tr}(bd^{-1}(x,x))) = 1$, we obtain $\omega(k)\varphi = \varphi$, as claimed. □

Since, by the Lemma just proved, the function $g \mapsto (\omega(g)\varphi)(0)$ lies in the space $I_p(0)^{K_0(p)}$, it suffices to compute the values of this function on the w_j's for $0 \leqslant j \leqslant 3$. But, as remarked above, $\omega(w_j)$ acts componentwise, so that

$$(\omega(w_j)\varphi)(0) = (\varepsilon_p \kappa_p)^j.$$

Thus

$$(\omega(g)\varphi)(0) = \sum_j (\varepsilon_p \kappa_p)^j \Phi_j(g, 0),$$

as claimed. □

In the archimedean case, since B is definite, it is not difficult to check that if $\tilde{\varphi}_\infty = \varphi_\infty \otimes \varphi_\infty \otimes \varphi_\infty$, with $\varphi_\infty(x) = e^{-2\pi\nu(x)}$ as in (8.2), then

$$(\omega(g)\tilde{\varphi}_\infty)(0) = \Phi^2_\infty(g, 0), \tag{8.14}$$

for all $g \in G^1(\mathbb{R})$. Here $\Phi^{2n}_\infty(\mathbf{k}, s) = (\det k)^{2n}$ as in (6.3). In fact, since $V_\mathbb{R}$ is positive definite of dimension 4, it is well known (see, for example, [23]) that the Gaussian $\tilde{\varphi}_\infty$ is an eigenvector for the action of $K_\infty = U(3)$ with character $\det(k)^2$. On the other hand, the K_∞-types in the induced representation $I_\infty(0)$ occur with multiplicity 1 [27], so that the function $g \mapsto (\omega(g)\tilde{\varphi}_\infty)(0)$ must be a multiple of $\Phi^2_\infty(\cdot, 0)$. Since both functions take the value 1 at $g = e$, we obtain (8.14). Also, taking the complex conjugate, we have

$$\Phi^{-2}_\infty(g, 0) = \overline{\Phi^2_\infty(g, 0)} = \overline{\omega(g)\tilde{\varphi}_\infty(0)}. \tag{8.15}$$

Now let $\Phi^\natural(s) = \otimes_v \Phi^\natural_v(s) \in I(s)$ be the section chosen in Section 7, with local components given by (5.10) and (6.18). Then we define

$$\begin{aligned}\Psi(g, s) &= \Phi^{-2}_\infty(g_\infty, s) \cdot \prod_{p|N} \Phi^{\varepsilon_p}_{R_p}(g_p, s) \cdot \prod_{p\nmid N} \Phi_{K_p}(g_p, s)\\ &= \xi_\infty(s) \cdot \prod_{p|N} \xi_p(s) \cdot \Phi^\natural(g, s).\end{aligned} \tag{8.16}$$

Recall that $\xi_p(s)$ and $\xi_\infty(s)$ are given by (5.11) and (6.21), respectively. Note that the section

$$\Phi(g, s) = \overline{\Psi(g, \bar{s})} = \Phi^2_\infty(g_\infty, s) \cdot \prod_{p|N} \Phi^{\varepsilon_p}_{R_p}(g_p, s) \cdot \prod_{p\nmid N} \Phi_{K_p}(g_p, s)$$

is not standard, i.e., its restriction to K is not independent of s, but that the difference

$$\Phi_{\text{diff}}(g, s) = \Phi(g, s) - \Phi_{\text{flat}}(g, s)$$

has at least a simple zero at $s = 0$. Here $\Phi_{\text{flat}}(s)$ is the standard section associated to our fixed Schwartz function $\tilde{\varphi}$ of (8.3). Now

$$E(g, s, \Phi^\natural(s)) = \xi(s)^{-1} \cdot E(g, s, \Psi), \tag{8.17}$$

where

$$\xi(s) = \xi_\infty(s) \cdot \prod_{p|N} \xi_p(s). \tag{8.18}$$

Then

$$E(g, \bar{s}, \Psi) = E(g, \bar{s}, \bar{\Phi}) = \overline{E(g, s, \Phi)} = \overline{E(g, s, \Phi_{\text{flat}})} + \overline{E(g, s, \Phi_{\text{diff}})}.$$

Thus by the Weil-Siegel formula (8.8)

$$\begin{aligned} E(g, s, \Phi^{\natural}(s))|_{s=0} &= \xi(0)^{-1} E(g, s, \Psi)|_{s=0} \\ &= \xi(0)^{-1}[\overline{E(g, s, \Phi_{\text{flat}})}|_{s=0} + \overline{E(g, s, \Phi_{\text{diff}})}|_{s=0}] \\ &= \xi(0)^{-1} 2\overline{I(g, \tilde{\varphi})}. \end{aligned} \tag{8.19}$$

9. A formula for $L(F, 2)$

Using the Weil-Siegel formula of the last section together with the integral representation (7.7) and (7.10), we obtain the following formula:

$$L(F, 2) = 2b(0)\xi(0)^{-1} L_\infty(F, 2)^{-1} \cdot \int_{Z_G(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \tilde{F}(g)\overline{I(g, \tilde{\varphi})}\, \mathrm{d}g. \tag{9.1}$$

Note that

$$b(0) = \pi^{-2}\zeta(2)^2,$$
$$L_\infty(F, 2) = (2\pi)^{-5}$$
$$\xi_\infty(0) = -\pi^5$$

and

$$\xi_p(0) = 2\varepsilon_p p^{-1}(p+1)^{-3}(1 + \varepsilon_p p^{-1})^{-2}. \tag{9.2}$$

Using these values and the fact that $\Pi_{p|N}\,\varepsilon_p = -1$, this becomes

$$\begin{aligned} L(F, 2) = 2^{10-t}\pi^2 \cdot \frac{N}{24^2} \prod_{p|N} (p+1)^3(1 + \varepsilon_p p^{-1})^2 \\ \cdot \int_{Z_G(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \tilde{F}(g)\overline{I(g, \tilde{\varphi})}\, \mathrm{d}g, \end{aligned} \tag{9.3}$$

where $t = \#\{p : p \mid N\}$.

Next we want to reduce this integral to a more classical form. Suppose that Ω is a function on $\mathfrak{h}^3$ which is left invariant under the action on $\Gamma_0(N)^3$. For

$g \in H(\mathbb{A})$, write $g = \gamma \cdot g_\infty \cdot k$ with $\gamma \in H(\mathbb{Q})$, $g_\infty \in H(\mathbb{R})^+ = (\mathrm{GL}_2^+(\mathbb{R}))^3 \cap H(\mathbb{R})$, and $k \in K'_f = \widehat{\Gamma_0(N)^3} \cap H(\mathbb{A}_f)$; and let

$$\tilde{\Omega}(g) = \Omega(g_\infty(i)) = \Omega(z_1, z_2, z_3), \tag{9.4}$$

where $g_\infty(i) = (z_1, z_2, z_3)$. Note that $\tilde{\Omega}$ is left $Z_G(\mathbb{A})H(\mathbb{Q})$-invariant and right K' invariant, where $K' = K'_\infty \cdot K'_f$ with $K'_\infty = \mathrm{SO}(2)^3$. Let $Z' = Z_G(\mathbb{A}_f) \cap K'_f$ and note that

$$Z_G(\mathbb{A}) = Z_G(\mathbb{Q})Z_G(\mathbb{R})^+ Z'.$$

Also observe that

$$\begin{aligned} H(\mathbb{Q})Z_G(\mathbb{A}) \cap H(\mathbb{R})^+ K'_f &= H(\mathbb{Q})Z_G(\mathbb{R})^+ Z' \cap H(\mathbb{R})^+ K'_f \\ &= Z_G(\mathbb{R})^+ Z' \cdot \Gamma_0(N)^3. \end{aligned}$$

Recall that in section 4 and section 6 we have fixed measures dg_v on each of the groups H_v and that the measure dg on $H(\mathbb{A})$ is the corresponding product measure. Then, noting that $\mathrm{vol}(\mathrm{SO}(2)^3/(\pm 1)^3) = \pi^3$ and that $|a|^{-2}\, dx\, d^\times a = \frac{1}{2} y^{-2}\, dx\, dy$, and using the observations above, we have

$$\begin{aligned} \int_{Z_G(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \tilde{\Omega}(g)\, dg &= \int_{H(\mathbb{Q})Z_G(\mathbb{R})^+ Z' \backslash H(\mathbb{Q})H(\mathbb{R})^+ K'_f} \tilde{\Omega}(g)\, dg \\ &= \int_{Z_G(\mathbb{R})^+ Z' \cdot \Gamma_0(N)^3 \backslash H(\mathbb{R})^+ K'_f} \tilde{\Omega}(g)\, dg \\ &= \int_{\Gamma_0(N)^3 \backslash (\mathrm{SL}_2(\mathbb{R})^3 \times Z' \backslash K'_f)} \tilde{\Omega}(g_\infty)\, dg_\infty \\ &= \left(\prod_{p|N} (p+1)^{-3} \right) \cdot \int_{Z\Gamma_0(N)^3 \backslash \mathrm{SL}_2(\mathbb{R})^3} \tilde{\Omega}(g_\infty)\, dg_\infty \\ &= \left(\prod_{p|N} (p+1)^{-3} \right) \frac{\pi^3}{8} \cdot \int_{\Gamma_0(N)^3 \backslash \mathfrak{H}^3} \Omega(z)\, d\mu(z), \end{aligned} \tag{9.5}$$

with

$$d\mu(z) = \prod_{j=1}^{3} \frac{dx_j\, dy_j}{y_j^2} \qquad \text{with } z_j = x_j + i y_j. \tag{9.6}$$

Here the factor $\Pi_{p|N}(p+1)^{-3}$ is the index of K'_f in $K_{H,f}$ and is equal to the volume of the group $Z' \backslash K'_f$. Taking $\tilde{\Omega}(g) = \tilde{F}(g)\overline{I(g, \tilde{\varphi})}$, with $\tilde{F}(g) =$

$F(g_\infty(i))j(g_\infty, i)^{-2}$ (see (7.5)), we obtain

$$L(F, 2) = 2^{7-t}\pi^5 \frac{N}{24^2} \prod_{p|N} (1 + \varepsilon_p p^{-1})^2 \cdot \int_{\Gamma_0(N)^3\backslash\mathfrak{H}^3} F(z)j(g_\infty, i)^{-2}\overline{I(g_\infty, \tilde{\varphi})}\, d\mu(z). \tag{9.7}$$

Note that we may take

$$g_\infty = (g_1, g_2, g_3)$$

with

$$g_j = \begin{pmatrix} 1 & x_j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_j & 0 \\ 0 & a_j^{-1} \end{pmatrix}, \qquad a_j = y_j^{-1/2}, \tag{9.8}$$

so that $j(g_\infty, i)^{-2} = y_1 y_2 y_3$.

To compute

$$I(g_\infty, \tilde{\varphi}) = \int_{O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})} \theta(g_\infty, r; \tilde{\varphi})\, dr,$$

we need a classical expression for the theta function $\theta(g_\infty, r; \tilde{\varphi})$. Let

$$M = (B^\times \times B^\times)_0 = \{(b_1, b_2) \in B^\times \times B^\times \mid \nu(b_1) = \nu(b_2)\}, \tag{9.9}$$

and define an involution $\tau: M \to M$ by

$$\tau: (b_1, b_2) \mapsto (b_2 \cdot \nu^{-1}, b_1 \cdot \nu^{-1}), \tag{9.10}$$

where $\nu = \nu(b_1) = \nu(b_2)$. There is then a surjective homomorphism

$$\rho: M \rtimes \langle \tau \rangle \to O(V) \tag{9.11}$$

where, for $x \in V = B$,

$$\rho(b_1, b_2)x = b_1 x b_2^{-1}, \tag{9.12}$$

and

$$\rho(\tau)x = x^\iota. \tag{9.13}$$

This yields an exact sequence

$$1 \to Z_M \to M \rtimes \langle \tau \rangle \to O(V) \to 1, \tag{9.14}$$

with

$$Z_M = \{(z, z) \in B^\times \times B^\times \mid z \in \text{center of } B^\times\}. \tag{9.15}$$

Note that Z_M has index 2 in the center of M. Here we are extending the classical isomorphism $M/Z_M \xrightarrow{\sim} \mathrm{SO}(V)$ by the orthogonal involution $x \mapsto x'$, which has determinant -1. For the Eichler order R as above, let $\hat{R} = R \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ and let

$$K_M = (\hat{R}^\times \times \hat{R}^\times)_0, \tag{9.16}$$

so that K_M is a compact open subgroup of $M(\mathbb{A}_f)$. For each place v, let $\tau_v\colon M_v \to M_v$ be the involution which extends τ, and let

$$C = \prod_v \langle \tau_v \rangle. \tag{9.17}$$

We note that ρ maps C isomorphically to a compact subgroup of $O(V)(\mathbb{A})$, and give C the corresponding topology. Note that for each p the group $K_{M_p} = (R_p^\times \times R_p^\times)_0$ is preserved by the action of τ_p and that the image of $K_M \rtimes \Pi_p \langle \tau_p \rangle$ under ρ is a compact open subgroup of $O(V)(\mathbb{A}_f)$.

There is a double coset decomposition

$$B_{\mathbb{A}}^\times = \coprod_{i=1}^{n} B^\times B_{\mathbb{R}}^\times b_i \hat{R}^\times, \tag{9.18}$$

where we may choose the representatives b_i to have $b_{i,\infty} = 1$ and $\nu(b_i) = 1$. Recall that $\nu(B^\times) = \mathbb{Q}_+^\times$ [43, p. 206].

LEMMA 9.1. (i)

$$M(\mathbb{A}) = \coprod_{i,j=1}^{n} M(\mathbb{Q}) M(\mathbb{R}) m_{i,j} K_M,$$

where $m_{i,j} = (b_i, b_j) \in (B_{\mathbb{A}}^\times \times B_{\mathbb{A}}^\times)_0 = M(\mathbb{A})$.

(ii) *Let*

$$\Gamma_{i,j} = M(\mathbb{Q}) \cap M(\mathbb{R}) m_{i,j} K_M m_{i,j}^{-1}$$

and

$$\Gamma_i = B_{\mathbb{Q}}^{\times} \cap B_{\mathbb{R}}^{\times} b_i \hat{R}^{\times} b_i^{-1}.$$

Then $\Gamma_{i,j}$ and Γ_i are finite groups and

$$\Gamma_{i,j} \simeq \Gamma_i \times \Gamma_j.$$

Proof. If $m = (m_1, m_2) \in M(\mathbb{A})$ with $m_i \in B_{\mathbb{A}}^{\times}$, write

$$m_i = \gamma_i m_{i,\infty} b_{r_i} k_i$$

with $\gamma_i \in B_{\mathbb{Q}}^{\times}$, $m_{i,\infty} \in B_{\mathbb{R}}^{\times}$, $k_i \in \hat{R}^{\times}$ and b_{r_i} one of the coset representatives of (9.18). Then, since $\nu(m_1) = \nu(m_2)$, we have

$$\nu(\gamma_1 \gamma_2^{-1}) = \nu(m_{1,\infty}^{-1} m_{2,\infty}) \cdot \nu(k_1^{-1} k_2) \in \mathbb{R}_+^{\times} \cdot \hat{\mathbb{Z}}^{\times} \cap \mathbb{Q}^{\times},$$

which implies that $\nu(\gamma_1) = \nu(\gamma_2)$, and hence that $\nu(m_{1,\infty}) = \nu(m_{2,\infty})$ and $\nu(k_1) = \nu(k_2)$. Thus we may write

$$m = (\gamma_1, \gamma_2)(m_{1,\infty}, m_{2,\infty})(b_{r_1}, b_{r_2})(k_1, k_2)$$

with each factor in $M(\mathbb{A})$, as required in (i). We omit the proof of (ii), which is similar. □

We fix a measure on $M(\mathbb{A})$ as follows. First, on $M(\mathbb{A}_f)$ fix the Haar measure for which the compact open subgroup K_M has measure 1. Note that $Z_M(\mathbb{R}) \simeq \mathbb{R}^{\times}$ and that $M(\mathbb{R}) \simeq Z_M(\mathbb{R})^+ \times M(\mathbb{R})^1$ where $Z_M(\mathbb{R})^+$ is the identity component of $Z_M(\mathbb{R})$ and $M(\mathbb{R})^1$ is the subgroup of $M(\mathbb{R})$ consisting of elements (b_1, b_2) with $\nu(b_1) = \nu(b_2) = 1$. We choose Haar measure on $M(\mathbb{R})$ to be the product of $d^{\times}z = z^{-1}\,dz$ (dz the Lebesque measure on $\mathbb{R}$) on $Z_M(\mathbb{R})^+$ and the measure on the compact group $M(\mathbb{R})^1$ with $\mathrm{vol}(M(\mathbb{R})^1) = 1$. Finally we normalize the measure on the compact group C to have $\mathrm{vol}(C) = 1$.

Now ρ induces a map

$$(Z_M(\mathbb{R})^+ M(\mathbb{Q}) \backslash M(\mathbb{A})) \times C \to O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A}) \tag{9.19}$$

which is surjective and proper. Pushing forward the measure defined above, we obtain a measure $d_0 r$ on $O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})$ characterized by the identity

$$\int_{O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})} f(r)\, d_0 r = \int_{(Z_M(\mathbb{R})^+ M(\mathbb{Q}) \backslash M(\mathbb{A})) \times C} f(\rho(m, c))\, dm\, dc. \tag{9.20}$$

In particular,

$$\begin{aligned}\operatorname{vol}(O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A}), \mathrm{d}_0 r) &= \operatorname{vol}(Z_M(\mathbb{R})^+ M(\mathbb{Q})\backslash M(\mathbb{A}), \mathrm{d}m)\\ &= \frac{1}{4}\sum_{i,j=1}^{n}\frac{1}{w_{i,j}},\end{aligned} \tag{9.21}$$

by the decomposition of Lemma 9.5, and with

$$4w_{i,j} = |\Gamma_{i,j}| = |M(\mathbb{Q}) \cap (M(\mathbb{R})\cdot m_{i,j}K_M m_{i,j}^{-1})|. \tag{9.22}$$

By (ii) of Lemma 9.1, $w_{i,j} = w_i w_j$, where

$$2w_i = |\Gamma_i| = |B_{\mathbb{Q}}^{\times} \cap (B_{\mathbb{R}}^{\times}\cdot(b_i \hat{R}^{\times} b_i^{-1}))|. \tag{9.23}$$

Thus the measure $\mathrm{d}r$ of the Weil-Siegel formula, which is normalized to have

$$\operatorname{vol}(O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A}), \mathrm{d}r) = 1$$

is

$$\mathrm{d}r = \left(\sum_i \frac{1}{2w_i}\right)^{-2} \mathrm{d}_0 r. \tag{9.24}$$

Thus we obtain

$$\begin{aligned}I(g_\infty, \tilde{\varphi}) &= \int_{O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})} \theta(g_\infty, r; \tilde{\varphi})\,\mathrm{d}r\\ &= \left(\sum_i \frac{1}{2w_i}\right)^{-2}\cdot \int_{Z_M(\mathbb{R})^+ M(\mathbb{Q})\backslash M(\mathbb{A})\times C} \theta(g_\infty, \rho(m, c); \tilde{\varphi})\,\mathrm{d}m\,\mathrm{d}c.\end{aligned} \tag{9.25}$$

Note that the function $\tilde{\varphi}$ is invariant under K_M, C, and $M(\mathbb{R})^1$, so that the integral here is just

$$\begin{aligned}&\int_{Z_M(\mathbb{R})^+ M(\mathbb{Q})\backslash M(\mathbb{A})\times C} \theta(g_\infty, \rho(m, c); \tilde{\varphi})\,\mathrm{d}m\,\mathrm{d}c\\ &\quad= \int_{Z_M(\mathbb{R})^+ M(\mathbb{Q})\backslash M(\mathbb{A})} \theta(g_\infty, \rho(m); \tilde{\varphi})\,\mathrm{d}m\\ &\quad= \sum_{i,j} \int_{Z_M(\mathbb{R})^+ \Gamma_{i,j}\backslash M(\mathbb{R}) m_{i,j} K_M m_{i,j}^{-1}} \theta(g_\infty, \rho(m m_{i,j}); \tilde{\varphi})\,\mathrm{d}m\end{aligned}$$

$$= \sum_{i,j} \frac{1}{4w_{i,j}} \int_{M(\mathbb{R})^1 \cdot m_{i,j} K_M m_{i,j}^{-1}} \theta(g_\infty, \rho(m m_{i,j}); \tilde{\varphi})\, \mathrm{d}m$$

$$= \frac{1}{4} \sum_{i,j} \frac{1}{w_i w_j} \cdot \theta(g_\infty, \rho(m_{i,j}); \tilde{\varphi}). \tag{9.26}$$

But for g_∞ as above, and for $x = (x_1, x_2, x_3) \in V(\mathbb{R})^3$,

$$(\omega(g_\infty)\tilde{\varphi})(x) = y_1 y_2 y_3 e(\mathrm{tr}(zv(x))), \tag{9.27}$$

with $\mathrm{tr}(zv(x)) = z_1 v(x_1) + z_2 v(x_2) + z_3 v(x_3)$. Thus,

$$\theta(g_\infty, \rho(m_{i,j}); \tilde{\varphi}) = y_1 y_2 y_3 \sum_{x \in (B_{\mathbb{Q}} \cap b_i \hat{R} b_j^{-1})^3} e(\mathrm{tr}(zv(x)))$$

$$= 8 y_1 y_2 y_3 \theta_{i,j}(z_1) \theta_{i,j}(z_2) \theta_{i,j}(z_3), \tag{9.28}$$

where, for $z \in \mathfrak{H}$,

$$\theta_{i,j}(z) = \frac{1}{2} \sum_{x \in I_{i,j}} e(zv(x)), \tag{9.29}$$

with

$$I_{i,j} = B_{\mathbb{Q}} \cap b_i \hat{R} b_j^{-1}. \tag{9.30}$$

For $z \in \mathfrak{H}^3$, let

$$\Theta(z) = \sum_{i,j} \frac{1}{w_i w_j} \theta_{i,j}(z_1) \theta_{i,j}(z_2) \theta_{i,j}(z_3). \tag{9.31}$$

so that the preceeding analysis yields

$$\overline{I(g_\infty, \tilde{\varphi})} = 2 \cdot \left(\sum_i \frac{1}{2w_i} \right)^{-2} \cdot y_1 y_2 y_3 \cdot \overline{\Theta(z)}. \tag{9.32}$$

Combining the pieces we finally obtain

THEOREM 9.2.

$$L(F, 2) = \frac{2^{8-t} \pi^5}{N} \int_{\Gamma_0(N)^3 \backslash \mathfrak{H}^3} F(z) \cdot \overline{\Theta(z)} (y_1 y_2 y_3)^2 \, \mathrm{d}\mu(z).$$

with $\mathrm{d}\mu(z)$ defined in (9.6) and $\Theta(z)$ defined in (9.30).

Proof. It only remains to check the value of the constant. Combining (9.7), and (9.25)–(9.30), we find

$$2^{7-t}\pi^5 \frac{N}{24^2} \prod_{p|N} (1+\varepsilon_p p^{-1})^2 \cdot 2 \left(\sum_i \frac{1}{2w_i}\right)^{-2}. \tag{9.33}$$

But by Eichler's class number formula [41],

$$\left(\sum_i \frac{1}{2w_i}\right) = \frac{N}{24} \prod_{p|N} (1+\varepsilon_p p^{-1}). \tag{9.34}$$

Thus the constant is

$$2^{8-t}\pi^5/N$$

as claimed.

10. The diagonal cycle

We recall that $N \geqslant 1$ is a square free integer and for each prime $p|N$ we have a sign $\varepsilon_p = \pm 1$ determined by $F = f \otimes g \otimes h$. We assume that $\varepsilon = -\Pi_{p|N}\varepsilon_p = +1$, so that there is a definite quaternion algebra B over $\mathbb{Q}$ (unique up to isomorphism) which is ramified at the even set of places $S = \{p : \varepsilon_p = -1\} \cup \{\infty\}$. Let R be an order of reduced discriminant N in B. At places $p \in S$, $R_p = R \otimes \mathbb{Z}_p$ is the unique maximal order in the local division algebra $B_p = B \otimes \mathbb{Q}_p$. At places $p \notin S$, R_p is conjugate to the Eichler order

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) | c \equiv 0 \pmod{N\mathbb{Z}_p}\right\}$$

in the matrix algebra $B_p \simeq M_2(\mathbb{Q}_p)$. In any case, R exists and is unique up to local conjugacy.

Let $\hat{R} = R \otimes \hat{\mathbb{Z}}$ and let $\hat{B} = B \otimes \hat{Z} = B \otimes_{\mathbb{Q}} \mathbb{A}_f$. Let D be the curve of genus 0 over $\mathbb{Q}$ associated to the quaternion algebra B, with a right action of the group $B^\times/\mathbb{Q}^\times$, and let X be the curve defined by [13, section 3, 4]

$$X = ((\hat{R}^\times \backslash \hat{B}^\times) \times D)/B^\times. \tag{10.1}$$

In (10.1) the group $B^\times$ acts simultaneously on the set $\hat{R}^\times \backslash \hat{B}^\times$ and the curve D. Since the double coset space $\hat{R}^\times \backslash \hat{B}^\times / B^\times$ is finite, X is a disjoint union of curves of genus 0 over $\mathbb{Q}$. Indeed, let $\{g_1, \ldots, g_n\}$ be a set of representatives for the double cosets and put $R_i = B \cap g_i^{-1}\hat{R}g_i$. Note that we may as well take

$g_i = b_i^{-1} = b_i^\iota$ as in section 4. Then R_i is another order of reduced discriminant N in B and the group $\Gamma_i = R_i^\times/\langle \pm 1\rangle$ is finite of order w_i. We have an isomorphism [13, sections 3, 5]

$$X \simeq \coprod_{i=1}^{n} D/\Gamma_i \tag{10.2}$$

taking the coset $\hat{R}g_i \times y \pmod{B^\times}$ to the coset $y \pmod{\Gamma_i}$ on the ith component of X.

In classical terms, the choice of coset representatives $\{g_1, \ldots, g_n\}$ in $\hat{B}^\times$ corresponds to a choice of left ideals $I_1, \ldots, I_n$ for R which represent the distinct left ideal classes ($I_i = \hat{R}g_i \cap B$) and R_i is the right order of I_i. The lattice

$$M_{i,j} = I_j^{-1}I_i = g_j^{-1}\hat{R}g_i \cap B = (b_i\hat{R}b_j^{-1} \cap B)^\iota \tag{10.3}$$

in B has left order R_j and right order R_i. Its theta function

$$\theta_{i,j} = \theta_{M_{i,j}}(q) = \frac{1}{2}\sum_{b \in M_{i,j}} q^{\nu(b)/\nu(M_{i,j})} \tag{10.4}$$

is a modular form of weight 2 for $\Gamma_0(N)$ [7]. We have $\theta_{i,j} = \theta_{j,i}$ as the canonical anti-involution of B identifies the lattice $M_{i,j}$ with a multiple of the lattice $M_{j,i}$.

The weighted sum, for any j, of the theta series $\theta_{i,j}$:

$$E = \sum_{i=1}^{n} \frac{1}{w_i}\theta_{i,j} = \sum_{n \geqslant 0} c_n q^n \tag{10.5}$$

is the unique Eisenstein series of weight 2 for $\Gamma_0(N)$ which has first Fourier coefficient $c_1 = 1$ and satisfies $E \mid w_p = \varepsilon_p \cdot E$ for all primes $p \mid N$ (where w_p is the Atkin-Lehner involution). The constant term c_0 of E is given by Eichler's mass formula [41, Ch. V, Cor. 2.3]:

$$c_0 = \sum_{i=1}^{n} \frac{1}{2w_i} = \frac{1}{24}\prod_{p \mid N}(p + \varepsilon_p), \tag{10.6}$$

and the L-function $L(E, s) = \Sigma_{n \geqslant 1} c_n n^{-s}$ is given by

$$L(E, s) = \zeta(s)\zeta(s-1)\prod_{p \mid N}(1 + \varepsilon_p p^{1-s}). \tag{10.7}$$

We note that E is *not* an eigenfunction for the Hecke operators T_p, when $p \mid N$ and $\varepsilon_p = +1$.

Let $\mathrm{Pic}(X)$ be the free abelian group of rank n of isomorphism classes of line bundles on X. This has, as basis, the elements $\{e_1, \ldots, e_n\}$ where e_i has degree 1 on the component $X_i = D/\Gamma_i$ and degree 0 on X_j for $j \neq i$. Since we will not be particularly concerned with questions of integrality here, we let $P = \mathrm{Pic}(X) \otimes \mathbb{Q} = \otimes_{i=1}^{n} \mathbb{Q}e_i$. There is a non-degenerate positive-definite height pairing [13, 4.5]

$$\langle\ ,\ \rangle: P \times P \to \mathbb{Q} \tag{10.8}$$

defined by the formula $\langle e_i, e_j \rangle = \delta_{i,j} w_i$ on basis elements. The linear form $\deg: P \to \mathbb{Q}$ defined by $\deg(\Sigma\, a_i e_i) = \Sigma\, a_i$ is given by the inner product with the element $a_E = \Sigma(1/w_i)e_i$:

$$\deg b = \langle a_E, b \rangle. \tag{10.9}$$

Let $M_2(N)$ be the space of modular forms of weight 2 for $\Gamma_0(N)$ with rational Fourier coefficients, and define the pairing

$$\phi: P \times P \to M_2(N) \tag{10.10}$$

by the formula $\phi(e_i, e_j) = \theta_{i,j}$ on basis elements. Then

$$\phi(a_E, b) = \deg b \cdot E \tag{10.11}$$

for all $b \in P$.

LEMMA 10.1. *Let $a, b \in P$. Then the first Fourier coefficient $a_1(\phi(a, b))$ is equal to $\langle a, b \rangle$.*

Proof. The composite $a_1 \circ \phi$ is a bilinear form on P, so to verify that it is equal to $\langle\ ,\ \rangle$ we must check that they agree on a basis. But $a_1(\theta_{i,j}) = \delta_{i,j} \cdot w_i$. □

If l is a prime with $l \nmid N$ we define the Hecke correspondence t_l on X as in [13, section 4]; this correspondence is self dual, of bidegree $l+1$. The operators t_l act linearly on P, commute with each other, and are self-adjoint with respect to the height pairing. They may therefore be simultaneously diagonalized on $P_{\mathbb{R}} = \mathrm{Pic}(X) \otimes \mathbb{R}$.

PROPOSITION 10.2. *If $f = \Sigma_{n=1}^{\infty} c_n q^n$ is a cuspidal newform of weight 2 for $\Gamma_0(N)$, there is a unique line $\langle a_f \rangle$ in $P_{\mathbb{R}}$ such that*

$$t_l(a_f) = c_l \cdot a_f$$

for all primes $l \nmid N$.

Proof. This follows from Eichler's trace formula and the theorem of multiplicity 1, [13, 21]. □

PROPOSITION 10.3. *The pairing* $\phi: P \times P \to M_2(N)$ *satisfies*

$$\phi(t_l a, b) = \phi(a, t_l b) = \phi(a, b) | T_l$$

for all primes $l \nmid N$, *where* T_l *is the lth Hecke operator on forms of weight* 2 *for* $\Gamma_0(N)$.

Proof. It suffices to check this for $a = e_i$ and $b = e_j$. We do this via the theory of Brandt matrices, as in [13]. □

COROLLARY 10.4. *Let* f *be a cuspidal newform for* $\Gamma_0(N)$ *and* a_f *a corresponding eigenvector in* $P_{\mathbb{R}}$. *Then*

$$\phi(a_f, b) = \langle a_f, b \rangle \cdot f$$

for all $b \in P_{\mathbb{R}}$.

Proof. By Proposition 10.3 and the multiplicity one theorem for $M_2(N)$, $\phi(a_f, b)$ is a multiple of f. We identify that multiple by equating first Fourier coefficients, using Lemma 10.1. □

Note. If $p | N$ one can define involutions u_p of X as in [13, section 4]. The formula analogous to that of Proposition 10.14 is:

$$\phi(u_p a, b) = \phi(a, u_p b) = \varepsilon_p \cdot \phi(a, b) | w_p \tag{10.12}$$

where w_p is the Atkin-Lehner involution. Since a_E is fixed by all u_p's, this shows that the Eisenstein series E is an eigenvector for all w_p's, with eigenvalue ε_p.

Now let $d \geqslant 1$ be an integer; we have induced pairings

$$\begin{cases} \langle\, , \rangle^{\otimes d}\colon P^{\otimes d} \times P^{\otimes d} \to \mathbb{Q} \\ \phi^{\otimes d}\colon P^{\otimes d} \times P^{\otimes d} \to M_2(N)^{\otimes d}. \end{cases} \tag{10.13}$$

If $A = a_1 \otimes \cdots \otimes a_d$ and $B = b_1 \otimes \cdots \otimes b_d$ we have, by definition of the tensor product,

$$\langle A, B \rangle^{\otimes d} = \prod_{i=1}^{d} \langle a_i, b_i \rangle$$

and

$$\phi^{\otimes d}(A, B) = \phi(a_1, b_1)(q_1)\phi(a_2, b_2)(q_2) \cdots \phi(a_d, b_d)(q_d),$$

with $q_j = e^{2\pi i \tau_j}$. From our results when $d = 1$, we immediately deduce

COROLLARY 10.5. (i) *The Fourier coefficient* $c_{1,1,\ldots,1}$ *of* $q_1 q_2 \cdots q_d$ *in* $\phi^{\otimes d}(A, B)$ *is equal to* $\langle A, B\rangle^{\otimes d}$.

(ii) *If* $F = f_1 \otimes \cdots \otimes f_d$ *is a product of newforms and* $A_F = a_{f_1} \otimes \cdots \otimes a_{f_d}$ *is the corresponding eigenvector (unique up to scalars) in* $P_{\mathbb{R}}^{\otimes d}$, *then*

$$\phi^{\otimes d}(A_F, B) = \langle A_F, B\rangle^{\otimes d} \cdot F$$

for all $B \in P_{\mathbb{R}}^{\otimes d}$.

Henceforth we take $d = 3$ and define the diagonal element

$$\Delta = \sum_{i=1}^{n} \frac{1}{w_i} e_i^{\otimes 3} \tag{10.14}$$

in $P_{\mathbb{R}}^{\otimes 3}$. This can be viewed as the codimension 2 cycle X, embedded diagonally in the 3-fold $Y = X^3$.

PROPOSITION 10.6. *We have*

$$\begin{aligned}\phi^{\otimes 3}(\Delta, \Delta) &= \sum_{i,j=1}^{n} \frac{1}{w_i w_j} \theta_{i,j}(q_1)\theta_{i,j}(q_2)\theta_{i,j}(q_3)\\ &= \Theta(q_1, q_2, q_3)\end{aligned}$$

in $M_2(N)^{\otimes 3}$.

Proof. By definition

$$\phi^{\otimes 3}(\Delta, \Delta) = \sum_{i,j} \frac{1}{w_i w_j} \phi^{\otimes 3}(e_i \otimes e_i \otimes e_i, e_j \otimes e_j \otimes e_j)$$

and $\phi(e_i, e_j) = \theta_{i,j}$. □

Let $F = f \otimes g \otimes h$ be our given cuspidal eigenform in $M_2(N)_{\mathbb{R}}^{\otimes 3}$. We then have an orthogonal decomposition

$$P_{\mathbb{R}}^{\otimes 3} = \langle A_F\rangle \oplus \langle A_F\rangle^{\perp}$$

with respect to the height pairing $\langle\ ,\ \rangle^{\otimes 3}$, where $\langle A_F\rangle$ is 1-dimensional. Similarly, we have an orthogonal decomposition

$$M_2(N)_{\mathbb{R}}^{\otimes 3} = \langle F\rangle \oplus \langle F\rangle^{\perp}$$

with respect to the Petersson inner product. If B is in $P_{\mathbb{R}}^{\otimes 3}$, we let B_F be its component in the space $\langle A_F\rangle$; if Ψ is in $M_2(N)_{\mathbb{R}}^{\otimes 3}$, we let Ψ_F be its component in the space $\langle F\rangle$.

COROLLARY 10.7. *In* $M_2(N)_{\mathbb{R}}^{\otimes 3}$ we have

$$\Theta_F = \langle \Delta_F, \Delta_F \rangle^{\otimes 3} \cdot F.$$

Proof. We have $\phi^{\otimes 3}(\Delta, \Delta) = \Theta$ by Proposition 10.6. Applying a projector in the Hecke algebra to the F-eigenspace, we find that $\phi^{\otimes 3}(\Delta_F, \Delta) = \Theta_F$. But by Corollary 10.5, $\phi^{\otimes 3}(\Delta_F, \Delta) = \langle \Delta_F, \Delta \rangle^{\otimes 3} \cdot F = \langle \Delta_F, \Delta_F \rangle^{\otimes 3} \cdot F$. $\square$

Note. Since $\theta_{i,j} = \frac{1}{2} + \Sigma_{n \geq 1} c_n q^n$, the constant Fourier coefficient of

$$\Theta = \sum_{i,j} \frac{1}{w_i w_j} \theta_{i,j}^{\otimes 3}$$

is given by

$$c_{0,0,0}(\Theta) = \frac{1}{2} \left(\frac{1}{24} \prod_{p|N} (p + \varepsilon_p) \right)^2. \tag{10.15}$$

Indeed

$$\sum_{i,j} \frac{1}{w_i w_j} \left(\frac{1}{2} \right)^3 = \frac{1}{2} \sum_i \frac{1}{2w_i} \sum_j \frac{1}{2w_j},$$

and the sums are equal to $\frac{1}{24} \Pi_{p|N} (p + \varepsilon_p)$ by Eichler's mass formula (10.6).

Finally, we give an elementary expression for the height pairing $\langle \Delta_F, \Delta_F \rangle^{\otimes 3}$ which appears in Corollary 10.7. Recall that $F = f \otimes g \otimes h$ and let a_f, a_g, and a_h be corresponding eigenvectors in $P_{\mathbb{R}}$. We may write

$$\begin{cases} a_f = \sum \lambda_i(f) e_i \\ a_g = \sum \lambda_i(g) e_i \\ a_h = \sum \lambda_i(h) e_i, \end{cases} \tag{10.16}$$

in terms of our canonical basis $\langle e_i \rangle$ of P, where the coefficients $\lambda_i(f)$ lie in the totally real field $\mathbb{Q}(f)$ and are uniquely determined up to a scalar (and similarly for $\lambda_i(g)$ and $\lambda_i(h)$).

PROPOSITION 10.8.

$$\langle \Delta_F, \Delta_F \rangle^{\otimes 3} = \frac{(\Sigma_i w_i^2 \lambda_i(f) \lambda_i(g) \lambda_i(h))^2}{(\Sigma_i w_i \lambda_i(f)^2)(\Sigma_i w_i \lambda_i(g)^2)(\Sigma_i w_i \lambda_i(h)^2)}.$$

Proof. Write $\Delta_F = \beta \cdot (a_f \otimes a_g \otimes a_h)$ in $P_{\mathbb{R}}^{\otimes 3}$, with $\beta \in \mathbb{R}^\times$. Then

$$\beta = \frac{\langle \Delta_F, a_f \otimes a_g \otimes a_h \rangle^{\otimes 3}}{\langle a_f, a_f \rangle \langle a_g, a_g \rangle \langle a_h, a_h \rangle}$$
$$= \frac{\Sigma_i w_i^2 \lambda_i(f) \lambda_i(g) \lambda_i(h)}{(\Sigma_i w_i \lambda_i(f)^2)(\Sigma_i w_i \lambda_i(g)^2)(\Sigma_i w_i \lambda_i(h)^2)}.$$

Since

$$\langle \Delta_F, \Delta_F \rangle^{\otimes 3} = \beta^2 \langle a_f, a_f \rangle \langle a_g, a_g \rangle \langle a_h, a_h \rangle,$$

this gives the stated result. □

COROLLARY 10.9. *The projection* $\Delta_F = 0$ *in* $P_{\mathbb{R}}^{\otimes 3}$ *if and only if*

$$\sum_i w_i^2 \lambda_i(f) \lambda_i(g) \lambda_i(h) = 0.$$

11. The main formula

In Theorem 9.2 we established the following analytic identity for the special values of $L(F, s)$ at $s = 2$

$$L(F, 2) = \frac{2^{8-t}\pi^5}{N} \int_{\Gamma_0(N)^3 \backslash \mathfrak{H}^3} F(z) \overline{\Theta(z)} (y_1 y_2 y_3)^2 \, d\mu(z), \tag{11.1}$$

where $F(z_1, z_2, z_3) = f(z_1) g(z_2) h(z_3)$ and $\Theta(z_1, z_2, z_3)$ is the sum of the genus 3 theta series of certain quaternary quadratic forms, restricted to the diagonal in Siegel space. The precise definition of $\Theta(z)$ is given in (9.31).

In Proposition 10.6, we showed that $\Theta(z)$ was equal to the value $\phi^{\otimes 3}(\Delta, \Delta)$, where Δ is the diagonal cycle on the 3-fold X^3 defined in (10.14). As a corollary (10.7), we deduced that the projection Θ_F of $\Theta(z)$ to the F-component of $M_2(N)_{\mathbb{R}}^{\otimes 3}$ was given by the formula

$$\Theta_F = \langle \Delta_F, \Delta_F \rangle^{\otimes 3} \cdot F, \tag{11.2}$$

where $\langle \; , \; \rangle^{\otimes 3}$ is the height pairing on $P^{\otimes 3}$.

Let us normalize the Petersson product on $M_2(N)^{\otimes 3}$ by defining

$$(F, G) = 2^9 \pi^6 \int_{\Gamma_0(N)^3 \backslash \mathfrak{H}^3} F(z) \overline{G(z)} (y_1 y_2 y_3)^2 \, d\mu(z). \tag{11.3}$$

The integral converges provided F is cuspidal. Since the eigencomponents of Θ

other than Θ_F are orthogonal to F, we may combine (11.1), (11.2) and Theorem 9.2 to obtain

THEOREM 11.1.

$$L(F, 2) = \frac{(F, F)}{2\pi N 2^t} \langle \Delta_F, \Delta_F \rangle^{\otimes 3}. \tag{11.4}$$

In some sense, this is the main formula, although we will make its dependence on the triple $f \otimes g \otimes h$ a bit more explicit later. It expresses the special value as the product of a period

$$\Omega(F) \underset{\text{defn}}{=} \frac{(F, F)}{2\pi N 2^t} \tag{11.5}$$

with the algebraic height pairing $\langle \Delta_F, \Delta_F \rangle^{\otimes 3}$. This leads to the useful

COROLLARY 11.2. (a) *We have $L(F, 2) \geqslant 0$, with equality if and only if $\Delta_F = 0$ in $P_{\mathbb{R}}^{\otimes 3}$.*

(b) *The ratio $L(F, 2)/\Omega(F) = A(F)$ lies in the subfield $\mathbb{Q}(F)$ of $\mathbb{C}$ generated by the coefficients of the Dirichlet series $L(F, s)$. For all automorphisms σ of $\mathbb{C}$, we have $A(F)^\sigma = A(F^\sigma)$.*

Proof. (a) We have $\Omega(F) = (F, F)/2\pi N 2^t > 0$ and $\langle \Delta_F, \Delta_F \rangle^{\otimes 3} \geqslant 0$. Since $\langle\ ,\ \rangle^{\otimes 3}$ is positive definite on $P_R^{\otimes 3}$, $\langle \Delta_F, \Delta_F \rangle^{\otimes 3} = 0$ if and only if $\Delta_F \equiv 0$.

(b) These statements are clear for $\langle \Delta_F, \Delta_F \rangle^{\otimes 3}$, as the height pairing is defined on the rational vector space $P^{\otimes 3}$ and the eigencomponent Δ_F is defined in $P^{\otimes 3} \otimes \mathbb{Q}(F)$. □

Now recall that $F(z) = f(z_1)g(z_2)h(z_3)$. Hence, the Petersson product (F, F), which is given by the integral (11.3), can be written as the product of 3 integrals

$$\begin{aligned}(F, F) &= 8\pi^2 \int_{\Gamma_0(N)\backslash\mathfrak{H}} |f(z_1)|^2 \, dx_1 \, dy_1 \cdot 8\pi^2 \int_{\Gamma_0(N)\backslash\mathfrak{H}} |g(z_2)|^2 \, dx_2 \, dy_2 \\ &\quad \times 8\pi^2 \int_{\Gamma_0(N)\backslash\mathfrak{H}} |h(z_3)|^2 \, dx_3 \, dy_3 \\ &= \|\omega_f\|^2 \cdot \|\omega_g\|^2 \cdot \|\omega_h\|^2 \end{aligned} \tag{11.6}$$

with $\omega_f = 2\pi i\, f(z_1)\, dz_1 = f(q_1)\, dq_1/q_1$ the normalized eigendifferential on $X_0(N)$ (and similarly for ω_g and ω_h) [16]. Using the elementary expression for $\langle \Delta_F, \Delta_F \rangle^{\otimes 3}$ given in Proposition 10.8, we obtain a "factored" form of our main formula:

COROLLARY 11.3.

$$L(f \otimes g \otimes h, 2) = \frac{\|\omega_f\|^2 \cdot \|\omega_g\|^2 \cdot \|\omega_h\|^2}{2\pi N 2^t} \frac{(\Sigma\, w_i^2 \lambda_i(f)\lambda_i(g)\lambda_i(h))^2}{\Sigma\, w_i \lambda_i(f)^2\, \Sigma\, w_i \lambda_i(g)^2\, \Sigma\, w_i \lambda_i(h)^2},$$

where the algebraic numbers $\lambda_i(f)$, $\lambda_i(g)$ *and* $\lambda_i(h)$ *are defined in* (10.16). *In particular,* $L(f \otimes g \otimes h, 2) = 0$ *if and only if*

$$\sum_i w_i^2 \lambda_i(f)\lambda_i(g)\lambda_i(h) = 0.$$

We now consider the implications of Corollary 11.3 in the degenerate case when $g = h$. In that case, the 4-dimensional l-adic representation

$$\sigma_g \otimes \sigma_h = \sigma_g^{\otimes 2} = \mathrm{Sym}^2 \sigma_g \oplus \textstyle\bigwedge^2 \sigma_g = \mathrm{Sym}^2\, \sigma_g \oplus \mathbb{Q}_l(-1)$$

is decomposable, and we obtain a corresponding factorization of the triple product L-function

$$L(f \otimes g \otimes g, s) = L(f \otimes \mathrm{Sym}^2\, g, s)L(f, s-1) \tag{11.7}$$

in the right half plane $\mathrm{Re}(s) > \frac{5}{2}$ of convergence of the Euler product. Our results on the analytic continuation of $L(f \otimes g \otimes g, s)$, along with the classical results of Hecke on the analytic continuation of $L(f, s-1)$, show that $L(f \otimes \mathrm{Sym}^2\, g, s)$ has a meromorphic continuation to the entire s-plane, and satisfies a functional equation when s is replaced by $4 - s$. By results of Gelbart-Jacquet [11], Jacquet-Piatetski-Shapiro-Shalika [22], and Waldspurger [29] (cf. Shahidi [38] p. 256 Theorem 4.3 for details), $L(f \otimes \mathrm{Sym}^2\, g, s)$ is entire, i.e., the function $L(f \otimes g \otimes g, s)$ is divisible by $L(f, s-1)$.

PROPOSITION 11.4. *Assume that* $L(f, 1) = 0$. *Then the eigenvector* $\Sigma\lambda_i(f)e_i = a_f$ *is orthogonal in* $P_{\mathbb{R}}$ *to all vectors of the form*

$$b_g = \sum_i w_i \lambda_i(g)^2 e_i.$$

Proof. We have

$$\langle a_f, b_g \rangle = \sum_i w_i^2 \lambda_i(f)\lambda_i(g)\lambda_i(g),$$

which, by Corollary 11.3, is zero if and only if $L(f \otimes g \otimes g, 2) = 0$. Since we are assuming that $L(f, 1) = 0$ and since $L(f \otimes \mathrm{Sym}^2 g, s)$ does not have a pole at $s = 2$, this follows from (11.7).

In the case $f = g$ we have the further factorization

$$L(f \otimes f \otimes f, s) = L(\mathrm{Sym}^3 f, s) L(f, s-1)^2. \tag{11.8}$$

By the Corollary on page 264 of [38], $L(\mathrm{Sym}^3 f, s)$ is regular at $s = 2$.

COROLLARY 11.5. *If* $L(f, 1) = 0$ *we have*

$$\sum_i w_i^2 \lambda_i(f)^3 = 0,$$

where $a_f = \Sigma_i \lambda_i(f) e_i$ *in* $P_{\mathbb{R}}$.

The formula in Theorem 11.1 continues to hold in certain cases where f, g, and h are not cusp forms. For example, assume that $N = p$ is a prime and that $f = g = h = E$, the Eisenstein series of weight 2 on $\Gamma_0(p)$. Then

$$L(F, s) = \zeta(s)\zeta(s-1)^3\zeta(s-2)^3\zeta(s-3)(1 - p^{1-s})(1 - p^{2-s})^3(1 - p^{3-s}). \tag{11.9}$$

At $s = 2$ we have

$$\begin{aligned} L(F, 2) &= \zeta(2)\zeta(0)^3\zeta(-1)\left(1 - \frac{1}{p}\right)(1 - p) \cdot \lim_{s \to 2} (\zeta(s-1)(1 - p^{2-s}))^3 \\ &= \left(\frac{\pi^2}{6}\right) \cdot \left(-\frac{1}{2}\right)^3 \left(-\frac{1}{12}\right)\left(1 - \frac{1}{p}\right)(1 - p) \cdot (\log p)^3 \\ &= -\frac{\pi^2 (p-1)^2 (\log p)^3}{4p \cdot (12)^2}. \end{aligned} \tag{11.10}$$

On the other side, since [13, p. 168]:

$$\|\omega_E\|^2 = -\frac{\pi \cdot \log p \cdot (p-1)}{12} \tag{11.11}$$

by evaluation of a residue in a Rankin L-function, we have

$$(F, F) = -\frac{\pi^3 (\log p)^3 \cdot (p-1)^3}{(12)^3}. \tag{11.12}$$

Since $a_E = \Sigma(1/w_i)e_i$, we find that

$$A(F) = \langle \Delta_F, \Delta_F \rangle^{\otimes 3} = \frac{\left(\Sigma \dfrac{1}{w_i}\right)^2}{\left(\Sigma \dfrac{1}{w_i}\right)^3} = \frac{12}{p-1}. \tag{11.13}$$

Hence,

$$L(F, 2) = \frac{(F, F)}{4\pi p} \cdot A(F)$$

as in Theorem 11.1. We view this degenerate case as a good check on our constants.

12. Examples

For small level N, we tabulate in Table 12.5 the coefficients $\lambda_i(f)$ of the eigenfunction

$$a_f = \sum_{i=1}^{n} \lambda_i(f) e_i \tag{12.1}$$

associated to f in $\mathrm{Pic}(X) \otimes \mathbb{R}$. We will only consider the case when $f = \Sigma\, a_n q^n$ has integral Fourier coefficients, where the calculations are due to Birch [2] and to Mestre and Oesterlé [28]. In this case we may take a_f to be an indivisible element in $\mathrm{Pic}(X)$, so the coefficients $\lambda_i(f)$ are integers with total gcd $= 1$. This normalizes them up to sign; we note that $\Sigma_{i=1}^n \lambda_i(f) = 0$, as f is a cusp form.

We also tabulate the algebraic part of the special value of the L-function of the triple product $F = f \otimes f \otimes f$. We have the factorization

$$L(F, s) = L(\mathrm{Sym}^3 f, s) L(f, s-1)^2 \tag{12.2}$$

and the formula

$$\varepsilon_p(F) = -a_p(f)^3 = -a_p(f) = \text{the eigenvalue of } w_p \text{ on } f. \tag{12.3}$$

We only consider f where $\varepsilon(F) = -\Pi_{p|N}\, \varepsilon_p(F) = +1$. Then $A(F)$ is given by

$$A(F) = \frac{(\Sigma_{i=1}^n w_i^2 \lambda_i(f)^3)^2}{(\Sigma_{i=1}^n w_i \lambda_i(f)^2)^3}. \tag{12.4}$$

For some small square-free levels N we give the eigenvalues ε_p (for ascending factors $p \mid N$) of the rational newforms f, the rank of the $\mathbb{Z}$-module Pic(X), the values $\lambda_i(f)$ (with a subscript w_i if $w_i \neq 1$), and the value of $A(F)$ with numerator and denominator separated as in (12.4). In all cases listed, f is the unique newform of level N with the given eigenvalues ε_p. We note that $A(F) = 0$ for f of levels $N = 55$ and 73; some other small levels where $A(F) = 0$ are $N = 85$, 109, 139. In all of these cases $L(\mathrm{Sym}^3 f, 2) = 0$, as one knows that $L(f, 1) \neq 0$.

As a supplement to this table, we record in Table 12.6 a few approximate numerical values of the period $\Omega(F)$ and of $L(F, 2)$, which were computed directly by Joe Buhler. Combined with the values of $A(F)$ from Table 12.5, these provide a numerical check on the identity of Theorem 11.1. Here we should remark that in all cases $X_0(N)$ has genus 1, and that, as in Corollary 11.3,

$$\Omega(F) = \frac{(\|\omega_f\|^2)^3}{2\pi N 2^t},$$

where f is the unique normalized newform and $t = \#\{p : p \mid N\}$. Now $\|\omega_f\|^2$ is the volume of the period lattice of a Neron differential, with respect to the measure $\mathrm{d}z\,\mathrm{d}\bar{z} = 2\,\mathrm{d}x\,\mathrm{d}y$. In each case, we let c^+ be the smallest real period and c^- the imaginary part of the smallest imaginary period. Then

$$\|\omega_f\|^2 = \begin{cases} c^+ \cdot c^- & \text{if there is one component on the real locus} \\ 2c^+ \cdot c^- & \text{if there are two components in the real locus.} \end{cases}$$

We have one component for $N = 11, 14, 17, 19$ and two components for $N = 15, 21$.

Table 12.5

N	ε_p	$n = \text{rank } P$	the values $\lambda_i(f)_{w_i}$	$A(F)$
11	-1	2	$1_3 - 1_2$	$5^2/5^3 = \frac{1}{5}$
14	$+1, -1$	2	$1_2 - 1$	$3^2/3^3 = \frac{1}{3}$
15	$+1, -1$	2	$1_3 - 1$	$8^2/4^3 = 1$
17	-1	2	$1_3 - 1$	$8^2/4^3 = 1$
19	-1	2	$1_2 - 1$	$3^2/3^3 = \frac{1}{3}$
21	$-1, +1$	2	$1_3 - 1$	$8^2/4^3 = 1$
33	$+1, -1$	4	$1_3 \quad 1 \; -1 \; -1$	$8^2/6^3 = \frac{8}{27}$
37	-1	3	$2 \; -1 \; -1$	$6^2/6^3 = \frac{1}{6}$
55	$-1, +1$	4	$1 \quad 1 \; -1 \; -1$	$0^2/4^3 = 0$
67	-1	6	$1_2 \quad 1 \quad 0 \quad 0 \; -1 \; -1$	$3^2/5^3 = \frac{9}{125}$
73	-1	6	$1 \quad 1 \quad 1 \; -1 \; -1 \; -1$	$0^2/6^3 = 0$
89	-1	8	$1_3 \quad 1 \quad 1 \quad 1 \; -1 \; -1 \; -1 \; -1$	$8^2/10^3 = \frac{8}{125}$

Table 12.6

N	c^+	c^-	$\Omega(F)$	$A(F)$
11	1.269209304279553	2.917633233876991	0.36735781565218	$\frac{1}{5}$
14	1.981341956066883	2.650982479364973	0.41184233277353	$\frac{1}{3}$
15	1.400603042332602	1.596242222131783	0.23713761450161	1
17	1.547079753551120	2.745739118089753	0.35880388406686	1
19	1.359759733488310	4.127092391717245	0.74021117609074	$\frac{1}{3}$
21	1.804461621553968	1.910989780751829	0.62151107987297	1

Table 12.7

N	$\Omega(F)\cdot A(F)$	$L(2, F)$
11	0.07347156313043	0.07347156313043
14	0.13728077759118	0.13728077759118
15	0.23713761450161	0.23713761450162
17	0.35880388406686	0.35880388406684
19	0.24673705869691	0.24673705869692
21	0.62151107987297	0.62151107987296

Finally, in Table 12.8, we tabulate the values of $A(F)$ for f of prime conductor N satisfying $100 < N < 1000$. We write $A(F) = M_3^2/M_2^3$ as in (12.4), with

$$M_3 = \sum_{i=1}^{n} w_i^2 \lambda_i(f)^3$$

Table 12.8

N	M_2	M_3	N	M_2	M_3
109	4	0	503	38	8
113	12	8	557	182	2
139	6	0	563	52	0
179	9	3	571	48	0
233	54	40	571	120	24
307	13	3	593	116	8
307	10	0	643	32	0
307	11	3	659	141	47
307	15	3	701	42	30
353	48	40	709	44	0
389	40	0	733	48	12
431	40	8	739	33	3
433	28	0	997	96	0
443	62	30	997	48	0
503	100	16			

and

$$M_2 = \sum_{i=1}^{n} w_i \lambda_i(f)^2,$$

and give the values of M_2 and M_3.

All cases when $N \geqslant 389$ and $M_3 = 0$ are caused by the vanishing of $L(f, 1)$, i.e., by the fact that the associated elliptic curve has positive rank over $\mathbb{Q}$.

13. The first derivative

We now will present a conjectural formula for the derivative $L'(F, 2)$, in the case when the sign $\varepsilon = -\prod_{p|N} \varepsilon_p$ in the functional equation for $L(F, s)$ is equal to -1. In this case, $L(F, 2) = 0$ by (1.14). Our conjectural formula has the same shape as Theorem 11.1, but the three-fold X^3 will be replaced by the triple product of a Shimura curve X over $\mathbb{Q}$ and the pairing $\langle\ ,\ \rangle^{\otimes 3}$ by the Beilinson-Bloch height on codimension 2 cycles which are homologically trivial.

Fix $F = f \otimes g \otimes h$ in $S_2(N)_{\mathbb{R}}^{\otimes 3}$ with $\varepsilon(F) = -1$, and let B be the indefinite quaternion algebra over $\mathbb{Q}$ (unique up to isomorphism) which is ramified at the even set of finite places $p \mid N$ where $\varepsilon_p(F) = -a_p b_p c_p = -1$. Let R be an order in B of reduced discriminant N; the order R is unique up to local conjugacy – at places $p \in S$, R_p is the unique maximal order in B_p, and at places $p \notin S$, R_p is conjugate to the Eichler order of $M_2(\mathbb{Z}_p)$ with $c \equiv 0 \pmod{N\mathbb{Z}_p}$. Let D be the curve of genus zero over $\mathbb{Q}$ associated to B, and let $X(\mathbb{C})$ be the Riemann surface

$$X(\mathbb{C}) = (\hat{R}^\times \backslash \hat{B}^\times) \times (D(\mathbb{C}) - D(\mathbb{R}))/B^\times, \tag{13.1}$$

where $B^\times$ acts simultaneously on the right of the left coset space $\hat{R}^\times \backslash \hat{B}^\times$ and on the Riemann surface $D(\mathbb{C}) - D(\mathbb{R}) \simeq \mathfrak{H}^\pm$. Since $\hat{B}^\times = \hat{R}^\times \cdot B_+^\times$ by the strong approximation theorem [7] – here $B_+^\times$ consists of the elements with $\nu(b) > 0$ – and since every projective $\mathbb{Z}$-module of rank 1 is free, we have an isomorphism

$$X(\mathbb{C}) \simeq \mathfrak{H}^+/\Gamma, \tag{13.2}$$

where $\Gamma = \hat{R}^\times \cap B_+^\times = R_+^\times$ is a discrete subgroup of $(B \otimes \mathbb{R})_+^\times \simeq \mathrm{GL}_2(\mathbb{R})_+$, which acts properly discontinuously on $\mathfrak{H}^+$. The quotient $\mathfrak{H}^+/\Gamma$ is compact, except in the case when S is empty. In that case, Γ is conjugate to the congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ and the quotient in (13.2) can be naturally compactified by the addition of the finite set

$$D(\mathbb{Q})/\Gamma \xrightarrow{\sim} \mathbb{P}^1(\mathbb{Q})/\Gamma_0(N) \tag{13.3}$$

of cusps, which has cardinality $2^{\#\{p|N\}}$. We will henceforth use X to denote the complete non-singular algebraic curve over $\mathbb{C}$ whose complex points form the compactification of (13.2).

Shimura [39] proved that the curve X has a canonical model over $\mathbb{Q}$, which is the classifying space for polarized abelian varieties with endomorphisms by R. For any integer $m \geqslant 1$ which is prime to N, we have a Hecke correspondence T_m on X which is self-dual and defined over $\mathbb{Q}$[39]. There is also a canonical class d_1 in $\mathrm{Pic}^1(X) \otimes \mathbb{Q} = \mathrm{Pic}^1(X)_{\mathbb{Q}}$ which is rational over $\mathbb{Q}$ and satisfies

$$T_m(d_1) = \sigma_1(m) \cdot d_1 \tag{13.4}$$

for all $(m, N) = 1$. When $X = X_0(N)$, d_1 is represented by the class of any cusp [15].

Let Y be the projective, non-singular 3-fold X^3 over $\mathbb{Q}$ and let ΔX be the 1-cycle of X diagonally embedded in Y. Let x_1 be a rational point of X, and define the partial diagonal cycles

$$\begin{aligned}
\Delta_{12}(x_1) &= \{(x, x, x_1) \mid x \in X\} \\
\Delta_{23}(x_1) &= \{(x_1, x, x) \mid x \in X\} \\
\Delta_{13}(x_1) &= \{(x, x_1, x) \mid x \in X\} \\
\Delta_{1}(x_1) &= \{(x, x_1, x_1) \mid x \in X\} \\
\Delta_{2}(x_1) &= \{(x_1, x, x_1) \mid x \in X\} \\
\Delta_{3}(x_1) &= \{(x_1, x_1, x) \mid x \in X\}
\end{aligned} \tag{13.5}$$

Finally, define the 1-cycle on Y

$$\begin{aligned}
\Delta X(x_1) = \Delta X &- \Delta_{12}(x_1) - \Delta_{13}(x_1) - \Delta_{23}(x_1) \\
&+ \Delta_1(x_1) + \Delta_2(x_1) + \Delta_3(x_1)
\end{aligned} \tag{13.6}$$

A short computation shows that $\Delta X(x_1)$ has trivial image in the cohomology of Y (the integral of any closed 2-form over $\Delta X(x_1)$ is zero).

Let $CH^2(Y)_{\mathbb{Q}}$ be the rational vector space of codimension 2 cycles on Y up to linear (=rational) equivalence over $\bar{\mathbb{Q}}$, which are fixed by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Let $CH^2(Y)^0_{\mathbb{Q}}$ be the kernel of the cycle class mapping

$$CH^2(Y)_{\mathbb{Q}} \to H^4(Y, \mathbb{Q}(2)). \tag{13.7}$$

By the above remarks, the class of $\Delta X(x_1)$ lies in $CH^2(Y)^0_{\mathbb{Q}}$. One shows easily that this class depends only on the class of x_1 in $\mathrm{Pic}^1(X)_{\mathbb{Q}}$. Hence we have a natural diagonal class:

$$\Delta = \Delta X(d_1) \tag{13.8}$$

in $CH^2(Y)^0_{\mathbb{Q}}$.

Under mild technical restrictions (which are satisfied for our 3-fold Y), Bloch [4] and Beilinson [1] have defined a symmetric, bilinear height pairing

$$\langle \ , \ \rangle^{BB}: CH^2(Y)^0_{\mathbb{Q}} \times CH^2(Y)^0_{\mathbb{Q}} \to \mathbb{R} \tag{13.9}$$

as the sum of local terms $\langle \ , \ \rangle^{BB}_p$. They conjecture that this pairing is non-degenerate, but we do not wish to assume this here. Let Ker^{BB} denote the left (and right) kernel of this pairing, and define the rational vector space

$$P = P(Y) = CH^2(Y)^0_{\mathbb{Q}}/\mathrm{Ker}^{BB}. \tag{13.10}$$

We then have a non-degenerate pairing (by definition):

$$\langle \ , \ \rangle^{BB}: P \times P \to \mathbb{R}. \tag{13.11}$$

The Hecke correspondences T_m of X give rise to symmetric correspondences $T_{m_1} \times T_{m_2} \times T_{m_3}$ of $Y = X^3$. They give endomorphisms of $CH^2(Y)^0_{\mathbb{Q}}$ which are self-adjoint with respect to the pairing $\langle \ , \ \rangle^{BB}$ [15], and so preserve the subspace Ker^{BB} and act on P.

PROPOSITION 13.1. *If t is a $\mathbb{Q}$-linear combination of Hecke operators $T_{m_1} \times T_{m_2} \times T_{m_3}$ which annihilates the space $S_2(N)^{\otimes 3}$, then $t \cdot \Delta \equiv 0$ in P.*

The proof is not difficult, and will be given in a forthcoming paper [14].

Using Proposition 13.1, we can define the F-isotypic component Δ_F of the class Δ in $P_{\mathbb{R}}$. Namely, let t_F be an $\mathbb{R}$-linear combination of Hecke operators $T_{m_1} \times T_{m_2} \times T_{m_3}$ which projects to the (1-dimensional) F-isotypic component in $S_2(N)^{\otimes 3}$. We then define

$$\Delta_F = t_F \cdot \Delta \in P_{\mathbb{R}}. \tag{13.12}$$

Although t_F is not unique, the difference $(t_F - t'_F) \cdot \Delta$ is zero in $P_{\mathbb{R}}$, for two different projectors t_F and t'_F, by Proposition 13.1. Since $\langle \ , \ \rangle^{BB}$ is defined on $P_{\mathbb{R}}$, we may calculate the pairing of Δ_F with itself.

CONJECTURE 13.2. *We have the formula*

$$L'(F, 2) = \frac{(F, F)}{2\pi N 2^t} \langle \Delta_F, \Delta_F \rangle^{BB},$$

where (F, F) *is the Petersson product defined in* (11.3) *and* (11.6).

Some evidence for Conjecture 13.2 will be presented in a forthcoming paper with Zagier [14].

14. References

1. A. Beilinson, Higher regulators and values of L-functions, *J. Soviet Math.* 30 (1985), 2036–2070.
2. B. J. Birch, Hecke actions and classes of ternary quadratic forms, preprint.
3. D. Blasius, Critical values of certain tensor product L-functions, *Inv. Math.* 90 (1987), 181–188.
4. S. Bloch, Height pairings for algebraic cycles, *J. Pure and Applied Algebra* 34 (1984), 119–145.
5. W. Casselman, On representations of GL(2) and arithmetic of modular curves, in *Modular Forms of One Variable II*, Lect. Notes in Math., vol. 349, Springer-Verlag, New York, Heidelberg, Berlin, 1973, 107–141.
6. P. Deligne, Formes modulaires et representations de GL(2), *Lect. Notes in Math.* 349, Springer-Verlag, New York, Heidelberg, Berlin, 1973.
7. M. Eichler, Zur Zahlentheorie der Quaternion-Algebren, *Crelle J.* 195 (1955), 127–151.
8. M. Fichler, Quaternaire quadratische Formen und die Riemannsche Vermutung für die Kongruenzzetatfunction, *Arch. Math.* 5 (1955), 355–366.
9. P. B. Garrett, Decomposition of Eisenstein series: Rankin triple products, *Ann. of Math.* 125 (1987), 209–235.
10. P. B. Garrett and M. Harris, Special values of triple product L-functions, preprint (1989).
11. S. Gelbart and H. Jacquet, A relation between automorphic forms on GL(2) and GL(3), *Ann. Sci. E. N. S.* 11 (1978), 471–542.
12. R. Godement, *Notes on Jacquet-Langlands Theory*, Institute for Advanced Study, Princeton, N.J., 1970.
13. B. H. Gross, Heights and special values of L-series, *CMS Conf. Proc.* 7 (1987), 115–187.
14. B. H. Gross, S. Kudla and D. Zagier, in preparation (1991).
15. B. H. Gross and C. Schoen, The diagonal on the 3 fold product of a curve, in preparation (1990).
16. B. H. Gross and D. Zagier, Heegner points and the derivatives of L-series, *Inv. Math.* 84 (1986).
17. M. Harris and S. Kudla, The central critical value of a triple product L-function, *Annals of Math.* 133 (1991), 605–672.
18. J. Igusa, Some results on p-adic complex powers, *Amer. J. Math.* 106 (1984), 1013–1032.
19. T. Ikeda, On the functional equations of the triple L-functions, *J. Math. Kyoto Univ.* 29-2 (1989), 175–219.
20. T. Ikeda, On the poles of the triple product L-function, preprint (1990).
21. H. Jacquet and R. P. Langlands, in Automorphic Forms on GL(2), *Lect. Notes in Math.* 114, Springer-Verlag, New York, Heidelberg, Berlin, 1970.
22. H. Jacquet, I. I. Piatetski-Shapiro and J. Shalika, On Euler products and the classification of automorphic representations I, II, *Amer. J. Math.* 103 (1981), 499–558 and 777–815.
23. M. Kashiwara and M. Vergne, On the Segal-Shale-Weil representations and harmonic polynomials, *Invent. Math.* 44 (1978), 1–47.
24. S. S. Kudla and S. Rallis, On the Weil-Siegel formula, *J. Reine Angew. Math.* 391 (1988), 65–84.
25. S. S. Kudla and S. Rallis, A regularized Siegel-Weil formula: the first term identity, in preparation (1990).

26. S. S. Kudla and S. Rallis, Poles of Eisenstein series and *L*-functions, Festschrift in honor of I. Piatetski-Shapiro, *Israel Math. Conf. Proc.*, vol. 3, Weizmann Sci. Press, Israel (1990), 81–110.
27. S. S. Kudla and S. Rallis, Degenerate principal series and invariant distributions, *Isr. J. Math.* 69 (1990), 25–45.
28. J.-F. Mestre and J. Oesterle, Elliptic Curves of Prime Conductor, preprint.
29. C. Moeglin and J.-L. Waldspurger, Le spectre résiduel de GL(n), *Ann. Sci. E. N. S.* 22 (1989), 605–674.
30. T. Orloff, Special values of mixed weight triple products, *Inv. Math.* 90 (1987), 169–180.
31. I. I. Piatetski-Shapiro and S. Rallis, Rankin triple *L*-functions, *Compositio Math.* 64 (1987), 333–399.
32. D. Prasad, Trilinear forms for GL(2) and local epsilon factors, *Compositio Math.* 75 (1990), 1–46.
33. S. Rallis, *L*-Functions and the Oscillator Representation, *Lect. Notes in Math.* 1245, Springer-Verlag, New York, Heidelberg, Berlin, 1987.
34. S. Rallis, Poles of Standard L *functions, Proc. ICM, Kyoto* (1990).
35. J.-P. Serre, Facteurs locaux des fonctions zeta des varietes algebriques (definitions et conjectures), *Sem. DPP* 19 (1969).
36. F. Shahidi, On certain *L*-functions, *Amer. J. Math.* 103 (1981), 297–355.
37. F. Shahidi, On the Ramanunjan conjecture and finiteness of poles for certain *L*-functions, *Ann. Math.* 127 (1988), 547–584.
38. F. Shahidi, Third symmetric power *L*-functions on GL(2), *Compositio Math.* 70 (1989), 245–273.
39. G. Shimura, Construction of Class Fields and Zeta Functions of Algebraic Curves, *Ann. Math.* 85 (1967), 58–159.
40. G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press, 1971.
41. M. F. Vigneras, Arithmétique des Algèbres de Quaternions, *Lect. Notes in Math.* 800, Springer-Verlag, New York, Heidelberg, Berlin, 1980.
42. *A. Weil, Sur certains groupes d'opérateurs unitaires, Acta Math.* 111 (1964), 143–211.
43. A. Weil, *Basic Number Theory*, Springer-Verlag, New York, Heidelberg, Berlin, 1974.