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Lorentz invariant distributions supported on the forward light cone

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1. Introduction

From the physical point of view, for instance in the quantum theory of the electromagnetic field, it is of interest to give, for any finite-dimensional module U for the connected Lorentz group G, a description of the space $\bar{J}(U)$ of all the U-valued distributions on (the dual of the) Minkowski space-time that are invariant under G and supported on the closed forward light cone, see [2]. Lorentz invariant distributions have of course been studied in depth, not only on physical space-time but on the more general spaces $\mathbf{R}^{m,n}$ with a quadratic form of signature (m, n), see [10], [12], [13], the work of Gårding and J.E. Roos in [3], [11], [16]. However the case of the *vector-valued* distributions as well as the situation when their supports are required to be in the *forward* (as opposed to the *full*) light cone have not received the emphasis they deserve in the mathematical literature. Our aim here is to supplement these papers with a consideration of these two aspects. We restrict ourselves to the case of signature $(1, n)$.

We shall now briefly explain some of the main ideas of the paper. For this purpose it is enough to consider scalar distributions. In his seminal paper [14] M. Riesz studies the wave operator $\square$ and its complex powers. For Riesz, as well as for Gelfand and Shilov who took this up later, the study of invariant distributions associated to the quadratic form ω was essentially a question of the analytic continuation of the powers ω^s ($s \in \mathbf{C}$). The description of *all* invariant distributions with or without support conditions, did not emerge as an objective until the works of Methée and others referred to above focussed attention on this goal. Finally, it was Harish-Chandra who realized (in a different context) the fruitfulness of regarding the space of invariant distributions as a *module* for the algebra of polynomial differential operators, especially for the Lie algebra $\mathfrak{a}$

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isomorphic to $\mathfrak{sl}(2, \mathbf{C})$, with basis $\square$, ω, and their commutator $[\square, \omega]$, which is essentially the Euler vector field. This idea is also our starting point and one of the main results is a complete description of this module structure. Rallis and Schiffmann in [11] have studied related modules, but as modules for $\square$ only.

The closed forward light cone supports two invariant measures: δ, the Dirac measure at the origin, and α_0^+, the invariant measure on the open forward light cone. One may naively expect that all invariant distributions supported by the closed forward light cone may be obtained as linear combinations of the $\square^k\delta$ and $\square^k\alpha_0^+$ $(k \geqslant 0)$. This is in fact true if d, the dimension of the underlying space, is odd. But when d is even this is no longer the case. Then the distribution $\square^{d-2/2}\alpha_0^+$ becomes a multiple of δ – a circumstance compatible with the Huygens principle – and so one cannot gain access to those invariant distributions which are nonzero away from the origin and have a transversal order at the points of the forward cone that is $\geqslant \frac{d-2}{2}$. Thus, to generate all invariant distributions one has to start with an invariant distribution τ of transversal order $\frac{d-2}{2}$; the ones with higher transversal order are then obtained from the $\square^k\tau$, and the ones with lower transversal order are obtained from the $\omega^k\tau$. One also sees from this description that the module of invariant distributions will have a much more complicated structure when d is even, and that in particular it will not be cyclic for $\square$ alone.

The construction of τ is therefore one of the central concerns of this paper. For use in applications in physics it is also important to obtain an explicit space-time expression for τ. This is done in Section 7.2 where the formula for τ on the space $\mathscr{S}_0$ of Schwartz functions vanishing at the origin is calculated. The extension of τ to the full Schwartz space $\mathscr{S}$ is not unique; however, all extensions are invariant and they are determined by the value at one element of $\mathscr{S}$ outside $\mathscr{S}_0$.

The method we use for constructing τ is not the only one possible. The theory of Riesz distributions can be used for this purpose, as we discuss in greater detail in [8]. For instance, if (R_s) is the Riesz family (R_s is the restriction to the open solid forward light cone of the power ω^s with a suitable normalization), it turns out that, up to a constant depending on d, the distribution τ is equal to

$$\left.\frac{d}{ds}\right|_{s=0} R_s \quad \text{on } \mathscr{S}_0.$$

Also the principal results on $\bar{J}(\mathbf{C})$ can now be obtained, although at various stages in the arguments, one has to make use of a generalization to the one-sided context of the theory of Methée. Therefore we have preferred to develop in this paper the entire theory in a direct and elementary manner, independent of the rather sophisticated Methée calculus.

This work was inspired by questions posed to VSV by a group of physicists at the Istituto Nazionale di Fisica Nucleare in Genova, consisting of Professors G. Cassinelli, G. Olivieri and P. Truini. We are grateful to Professor J.J. Duistermaat for stimulating discussions on the results of this paper.

2. Notations and statement of main results

We set

$$d = n + 1, \qquad \mathbf{R}^{1,n} \simeq \mathbf{R}^d.$$

We work in $\mathbf{R}^{1,n}$, for $n \geqslant 3$, with coordinates $(p_0, p_1, \ldots, p_n)$ and fundamental quadratic form

$$\omega = p_0^2 - p_1^2 - \cdots - p_n^2 = p_\mu p^\mu,$$

where $p^\mu = \pm p_\mu$ according as μ is 0 or >0. We write G for $\mathbf{SO}(1,n)^\circ$, the connected component containing the identity of the subgroup of $\mathbf{GL}(\mathbf{R}^{1,n})$ of elements fixing ω. We have: $g=(a_{ij})\in G$ if and only if g fixes ω, $\det(g)=1$, and $a_{00} > 0$ ($\geqslant 1$ actually). Moreover $\mathrm{Lie}(G) = \mathfrak{g}$ is its Lie algebra, acting on $\mathbf{R}^{1,n}$ via the vector fields

$$M_{\mu\nu} = -p_\nu \partial_\mu + p_\mu \partial_\nu, \qquad M_{0\mu} = p_0 \partial_\mu + p_\mu \partial_0.$$

Here $\mu, \nu = 1, \ldots, n$, for $\mu \neq \nu$, and $\partial_\mu = \partial/\partial p_\mu$. The forward light cone is

$$X_0^+ = \{(p_\mu) \mid p_0 > 0,\ p_\mu p^\mu = 0\},$$

and its closure $\mathrm{Cl}(X_0^+) = X_0^+ \cup \{0\}$.

G operates naturally on the usual space of test functions $C_c^\infty(\mathbf{R}^{1,n})$ (resp. $\mathscr{S}(\mathbf{R}^{1,n})$) and hence on the dual space of distributions (resp. tempered distributions). More generally let U be a finite-dimensional G-module. A U-valued distribution (resp. tempered distribution) is a continuous map $T\colon C_c^\infty(\mathbf{R}^{1,n}) \to U$ (resp. $\mathscr{S}(\mathbf{R}^{1,n}) \to U$). G acts on these by

$$(g \cdot T)(f) = g \cdot T(f^{g^{-1}}), \quad \text{where } f^{g^{-1}}(p) = f(g \cdot p).$$

T is *invariant if* $g \cdot T = T$, for all $g \in G$. For any T, $\mathrm{supp}(T)$ denotes its support. Our main concern is with

$$\bar{J}(U) = \{T \mid T \text{ an invariant distribution with } \mathrm{supp}(T) \subset \mathrm{Cl}(X_0^+)\}.$$

Many computations involving the distributions in $\bar{J}(U)$ only exploit their behavior under multiplication by the quadratic form ω and application of the wave operator

$$\Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial p_0^2} - \frac{\partial^2}{\partial p_1^2} - \cdots - \frac{\partial^2}{\partial p_n^2}.$$

Clearly it is also natural to consider along with these the commutator of ω and $\Box$, cf. [5]; up to constants it is equal to the radial or Euler vector field

$$\mathscr{E} = \sum p_\mu \partial_\mu = p_0 \frac{\partial}{\partial p_0} + \cdots + p_n \frac{\partial}{\partial p_n},$$

which detects the homogeneity properties of the distributions. These three operators generate a 3-dimensional simple subalgebra $\mathfrak{a}$ of the algebra of polynomial differential operators. In fact, for any polynomial h let $M(h)$ be the operator of multiplication by h. Then, if we write

$$H = \mathscr{E} + \frac{d}{2}, \qquad X = \tfrac{1}{2}M(\omega), \qquad Y = -\tfrac{1}{2}\Box,$$

we have the commutation rules

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H;$$

so that

$$\mathfrak{a} = \mathbf{C}\cdot H + \mathbf{C}\cdot X + \mathbf{C}\cdot Y$$

is a three-dimensional simple subalgebra of the algebra of polynomial differential operators on $\mathbf{R}^{1,n}$. It is clear that $\mathfrak{a}$ operates on $\bar{J}(U)$ so that $\bar{J}(U)$ is an $\mathfrak{a}$-module; and we shall give a complete description of $\bar{J}(U)$ as a module for $\mathfrak{a}$.

First we come to the modules U that actually do occur in this set-up. Let $\mathscr{P}$ be the algebra of polynomial functions on $\mathbf{R}^{1,n}$. We have a natural action of G on $\mathscr{P}$. An element $u \in \mathscr{P}$ is said to be *harmonic* if $\Box u = 0$. We write $\mathscr{H}$ for the (graded) G-submodule of $\mathscr{P}$ of harmonic polynomials and $\mathscr{H}^j$ for its subspace of homogeneous elements of degree j. These are stable and irreducible under G. If U is an irreducible finite-dimensional G-module, then $\bar{J}(U)$ is nonzero if and only if $U \simeq \mathscr{H}^j$.

The closed forward cone is stratified by two orbits: the vertex and the open forward cone. Accordingly we have an injection of the $\mathfrak{a}$-module $K(U)$ of the Lorentz invariant distributions supported by the vertex into $\bar{J}(U)$. Let $J(U)$ be the space of germs of Lorentz invariant distributions defined on invariant open neighborhoods of X_0^+ with supports contained in X_0^+. Then restriction of distributions supported on $\mathrm{Cl}(X_0^+)$ to open invariant neighborhoods of X_0^+ gives a mapping $\bar{J}(U) \to J(U)$. These maps lead to the sequence

$$0 \to K(U) \hookrightarrow \bar{J}(U) \to J(U) \to 0. \tag{*}$$

It is important to establish at the outset that (*) is exact. Although $K(U)$ and $J(U)$ are relatively simple to describe as $\mathfrak{a}$-modules, the exact sequence (*) does not split when d is even (it does, if d is odd). The analysis of (*) is thus a basic issue when d is even. We shall now proceed to a more detailed discussion of this point.

We recall the *Verma modules* $V(\lambda)$ ($\lambda \in \mathbf{C}$), cf. §3 infra. If $\lambda = i$ is an integer $\geqslant 0$, then $F(i)$ denotes the irreducible finite-dimensional module of dimension $i + 1$, and we have the exact sequence

$$0 \to V(-i-2) \to V(i) \to F(i) \to 0.$$

Next we consider the module $M(i)$ which may be characterized as the module, unique up to isomorphism, for which there is a *nonsplitting* exact sequence

$$0 \to F(i) \to M(i) \to V(-i-2) \to 0.$$

A precise formulation of the results above is now that always

$$K(U) \simeq V\left(-\frac{d}{2} - j\right) \quad (U \simeq \mathscr{H}^j),$$

and that

$$J(U) \simeq \begin{cases} V\left(\dfrac{d}{2} - 2 + j\right), & \text{for } d \text{ odd;} \\ M\left(\dfrac{d}{2} - 2 + j\right), & \text{for } d \text{ even.} \end{cases}$$

When d is even, the submodule $F\left(\dfrac{d}{2} - 2\right)$ corresponds of course to the linear span of the elements $\square^k \alpha_0^+ \left(0 \leqslant k \leqslant \dfrac{d}{2} - 2\right)$.

In order to study (*), we consider, for any integer $i \geqslant 0$, modules W for which there is a *nonsplitting* exact sequence

$$0 \to V(-i-2) \to W \to M(i) \to 0. \tag{W, i}$$

The moduli space for all such modules W is $\mathbf{P}^1(\mathbf{C})$, and for every value of the modulus we construct W as a quotient of the universal enveloping algebra of $\mathfrak{a}$ by an explicitly given ideal.

It therefore remains to verify the exactness of (*) as well as to determine the exact parameter that corresponds to the module under consideration. For both questions the essential case is that of $U \simeq \mathbf{C}$. Note that $\omega: \mathbf{R}^{1,n}\backslash\{0\} \to \mathbf{R}$ is a surjective submersion, therefore the pullback $\omega^*(\delta_0^{(d-2/2)})$ of the $\frac{d-2}{2}$-derivative of the Dirac measure δ_0 at 0 in $\mathbf{R}$ gives a distribution on $\mathbf{R}^{1,n}\backslash\{0\}$ homogeneous of degree $-d$. The surjectivity of $\bar{J}(\mathbf{C}) \to J(\mathbf{C})$ now comes down to proving that $\omega^*(\delta_0^{(d-2/2)})$ can be extended to a Lorentz invariant tempered distribution supported on $\mathrm{Cl}(X_0^+)$. Such tempered extensions exist and they are automatically Lorentz invariant. The determination of the modulus for the general case $\bar{J}(U)$ is a delicate calculation.

Our main theorem, which summarizes the results above, is as follows.

2.1. THEOREM. *As usual* $d = n + 1$.

(a) *For any integer* $i \geqslant 0$ *the set of isomorphism classes of* $\mathfrak{a}$*-modules* W *admitting a nonsplitting exact sequence* (W, i) *is in natural bijection with* $\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$. *Let* $W(i:\gamma)$ *be the module corresponding to* $\gamma \in \mathbf{P}^1(\mathbf{C})$.

(b) *If* U *is an irreducible finite-dimensional* G*-module, then* $\bar{J}(U)$ *is nonzero if and only if* $U \simeq \mathscr{H}^j$. *In this case,*

$$\bar{J}(U) \simeq \begin{cases} V\left(\frac{d}{2} - 2 + j\right) \oplus V\left(-\frac{d}{2} - j\right), & \textit{for } d \textit{ odd}; \\ W\left(\frac{d}{2} - 2 + j : \frac{1}{2}\left(\frac{d-2}{2} + j\right)\right), & \textit{for } d \textit{ even}. \end{cases}$$

The invariant γ is a subtle one. However there is a qualitative difference between the module $W(i:\infty)$ and the modules $W(i:\gamma)$ with γ finite, namely, $W(i:\infty)$ is a weight module while the $W(i:\gamma)$ with γ finite are not so; the action of H on these involves a two-step nilpotent corresponding to each eigenvalue of multiplicity 2.

In the description of the $\bar{J}(U)$ there is a noticeable contrast between the case of odd or even dimension d; this is directly related to the Huygens principle. If d is odd there exist no invariant distributions T supported by the forward light cone that are fundamental solutions of the wave operator, i.e. with $\square T = \delta$. In fact, in this case the Dirac measure at 0 is a highest weight vector in the Verma module occurring as the second summand; and therefore it cannot be in the image of $\square$.

It would be interesting to know whether there exist geometric realizations for the modules $W(i:\gamma)$ for finite values of γ different from $\frac{1}{2}(i+1)$, as spaces of vector-valued distributions or variations thereof. And furthermore, can one explain the isomorphisms between the modules having the same value of $\frac{d}{2} - 2 + j$?

3. Construction of the modules $W(j:\gamma)$

In this section we shall give the construction of the modules $W(j:\gamma)$. We consider modules for $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$ with the commutation rules

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

For any module V and any $c \in \mathbf{C}$, $V[c]$ denotes the generalized eigenspace of H for the eigenvalue c, namely the space of all $v \in V$ such that $(H - c)^k v = 0$, for some k.

Let us write $V(\lambda)$ for the Verma module with basis $(v_\lambda, v_{\lambda-2}, \ldots)$ such that

$$Xv_\lambda = 0, \qquad Xv_{\lambda-2k} = k(\lambda + 1 - k)v_{\lambda-2(k-1)},$$
$$Yv_{\lambda-2k} = v_{\lambda-2(k+1)}, \qquad Hv_{\lambda-2k} = (\lambda - 2k)v_{\lambda-2k}.$$

We next introduce the modules $M(j)$, with j an integer $\geqslant 0$. Let us consider modules M for which there is a nonsplitting exact sequence

$$0 \to F(j) \to M \to V(-j-2) \to 0. \tag{E}$$

Then H acts semisimply on M with simple spectrum $\{j, j-2, \ldots\}$. It is obvious that

$$\text{(E) does not split} \Leftrightarrow XM[-j-2] \neq 0 \Leftrightarrow \dim \ker_M(X) = 1.$$

For any integer $j \geqslant 0$ one can construct such a module $M(j)$ as follows. The module $M(j)$ has a basis $\{m(k)\}_{k=j,j-2,\ldots}$ such that

$$Xm(k) = m(k+2), \qquad (m(j+2) = 0); \qquad Hm(k) = km(k);$$
$$Ym(k) = -\tfrac{1}{4}(k+j)(k-j-2)m(k-2).$$

Conversely any M satisfying (E) and nonsplit is isomorphic to $M(j)$; to see this we observe that X is surjective on M and so we can find $0 \neq m(k) \in M[k]$, $Xm(k) = m(k+2)$ $(m(j+2) = 0)$; the commutation rules then lead to the above action of Y.

3.1. We shall next consider modules W admitting an exact sequence

$$0 \to V(-j-2) \to W \to M(j) \to 0. \tag{3.1}$$

PROPOSITION. *Let $w \in W[-j-2]$ map to a nonzero element of $M(j)$. Then YXw and $(H+j+2)w$ are independent of the choice of w; and*

$$(3.1)\ \textit{splits} \Leftrightarrow YXw = 0 \quad \textit{and} \quad (H+j+2)w = 0.$$

Proof. The first assertion is clear since, for any $u \in V(-j-2)[-j-2]$, $Xu = 0$ and $(H+j+2)u = 0$. Also the implication $\Rightarrow$ is obvious. Suppose conversely that $YXw = 0$ and $(H+j+2)w = 0$. Let $w_k = X^k w$, $0 \leqslant k \leqslant j+1$. Since w_{j+1} maps to $X^{j+1}m \neq 0$ where m is the image of w, it must be nonzero. Let

$$L = \langle w_{j+1}, w_j, \ldots, w, Yw, Y^2w, \ldots \rangle,$$

where $\langle \cdots \rangle$ denotes linear span. We leave it to the reader to verify that L is a submodule. It is obvious that $L \simeq M(j)$ and $W = V(-j-2) \oplus L$. ■

3.2. DEFINITION. Suppose that (3.1) does not split. Since $(H+j+2)w$ and YXw map to 0 in $M(j)$, they both lie in $V(-j-2)[-j-2]$ and so satisfy a nontrivial relation $\alpha YXw - \beta(H+j+2)w = 0$. Hence we have a unique point $\gamma = \dfrac{\beta}{\alpha} \in \mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$. It is obvious that $\gamma = \gamma(W)$ is an invariant of W. Our aim is to show that all points of $\mathbf{P}^1(\mathbf{C})$ arise in this manner and that γ determines W uniquely. We write $W(j:\gamma)$ for any W for which $\gamma(W) = \gamma$.

It is clear that the spectrum of H in W is $\{j, j-2, \ldots\}$; the eigenvalues $j, j-2, \ldots, -j$ are simple and the others are double.

LEMMA. *Let $\gamma = \gamma(W)$. If γ is finite, W is not a weight module; and for any $r \geqslant 1$, $H+j+2r$ is a nonzero nilpotent on $W[-j-2r]$. If $\gamma = \infty$, then W is a weight module. In either case, w generates W. Finally, if γ is finite, and $1 \leqslant r \leqslant j+1$,*

$$Y^r X^r w = (r-1)!^2 \binom{j}{r-1} \gamma (H+j+2)w.$$

Proof. If γ is finite, we have $w \in W[-j-2]$, and $(H+j+2)w \neq 0$, proving that $H+j+2$ is a nonzero nilpotent on $W[-j-2]$. Furthermore, for any $k \geqslant 1$, X is a bijection of $W[-j-2(k+1)]$ with $W[-j-2k]$ taking $H-2$ to H. This implies, by induction on k, that $H+j+2k$ is nonzero nilpotent on $W[-j-2k]$ for all k. In either case let $W' = U(\mathfrak{g}) \cdot w$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. It is clear that W' maps onto $M(j)$. So to prove that $W' = W$ we need to verify that W' contains $\ker(W \to M(j))$. If γ is finite and

$v = (H + j + 2)w$, then $v \neq 0$, $(H + j + 2)v = 0$, and v maps to zero in $M(j)$, so that v generates $\ker(W \to M(j))$. If $\gamma = \infty$, we take $v = YXw$; then again $v \neq 0$ and maps to 0 in $M(j)$ so that it generates $\ker(W \to M(j))$. We prove the last assertion of the lemma by induction on r. This is clear for $r = 1$. Let $r > 1$ and assume it for lower values of r. Now

$$Y^{r+1}X^{r+1} = Y^r Y X^{r+1} = Y^r X^r (YX - rH - r(r+1));$$

therefore we get, if we remember that $(H + j + 2)^2 w = 0$,

$$\begin{aligned} &Y^{r+1}X^{r+1}w \\ &\quad = (\gamma(H + j + 2) - rH - r(r+1))(r-1)!^2 \binom{j}{r-1} \gamma(H + j + 2)w \\ &\quad = r!^2 \binom{j}{r} \gamma(H + j + 2)w. \end{aligned}$$

3.3. We write $\langle\!\langle \cdots \rangle\!\rangle$ for the ideal generated by $\cdots$.

LEMMA. *Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then*

$$M(j) \simeq U(\mathfrak{g})/K(j),$$

where

$$K(j) = \langle\!\langle H + j + 2,\ X^{j+2},\ YX \rangle\!\rangle.$$

Proof. If $0 \neq u \in M(j)[-j-2]$, the map $a \mapsto a \cdot u$ $(a \in U(\mathfrak{g}))$ gives an exact sequence $U(\mathfrak{g})/K(j) \to M(j) \to 0$. The elements $Y^r X^s$, for $r \geqslant 0$, $s \leqslant j + 1$ span $U(\mathfrak{g})$ mod $K(j)$. On the other hand, if $r \geqslant 1$ and $s \geqslant 1$,

$$\begin{aligned} Y^r X^s &= Y^{r-1}(X^{s-1}Y - (s-1)X^{s-2}(H + s - 2))X \\ &\equiv (s-1)(j + 2 - s)Y^{r-1}X^{s-1} \bmod K(j). \end{aligned}$$

So Y^r, for $r \geqslant 0$, and X^s, for $1 \leqslant s \leqslant j + 1$, span $U(\mathfrak{g})$ mod $K(j)$. The weights of Y^r and X^s are respectively $-j-2-2r$ and $-j-2+2s$. This shows that the spectrum of H on $U(\mathfrak{g})/K(j)$ is simple and is contained in $\{j, j-2 \ldots\}$, and, hence, that $U(\mathfrak{g})/K(j) \simeq M(j)$. ■

3.4. We now consider the $W(j:\gamma)$. We treat first the case of finite γ. The vector w is as in Section 3.1. For any integer $j \geqslant 0$ and $\gamma \in \mathbf{C}$ let

$$I(j:\gamma) = \langle\!\langle (H+j+2)^2,\quad X^{j+2},\quad X(H+j+2),\quad YX - \gamma(H+j+2) \rangle\!\rangle.$$

Let

$$\Gamma = H(H+2) + 4YX, \quad \Gamma_j = \Gamma - j(j+2).$$

LEMMA. *$I(j:\gamma)w = 0$ and $\Gamma_j^2 \in I(j:\gamma)$. In particular $\Gamma_j^2 = 0$ on $W(j:\gamma)$.*

Proof. The first relation is obvious. Since $\Gamma_j \equiv (H - j + 4\gamma)(H + j + 2)$ mod $I(j:\gamma)$ we get $\Gamma_j^2 \equiv 0$ mod $I(j:\gamma)$, proving the second relation. ∎

3.5. LEMMA. $\dim(U(\mathfrak{g})/I(j:\gamma))[-j-2] = 2$.

Proof. Let us write I for $I(j:\gamma)$, W for $U(\mathfrak{g})/I$, and define

$$I' = \langle\!\langle (H+j+2)^2, \quad X^{j+2}, \quad X(H+j+2) \rangle\!\rangle.$$

Now $Y^r X^s (H+j+2)^t$, for $r, s, t \geqslant 0$, form a basis for $U(\mathfrak{g})$ and it is clear that the subfamily with $r \geqslant 0$ and (s,t) satisfying one of: (a) $s \geqslant j+2$; (b) $t \geqslant 2$; (c) $s \geqslant 1$, $t \geqslant 1$, span I'. Hence $Y^r X^s$, for $r \geqslant 0$, $0 \leqslant s \leqslant j+1$, and $Y^r(H+j+2)$, for $r \geqslant 0$, form a basis for $U(\mathfrak{g})$ mod I'. The weights of these are, respectively, $-j-2+2s-2r$ and $-j-2-2r$, so that if $W' = U(\mathfrak{g})/I'$, and $a \mapsto a'$ is the natural map $U(\mathfrak{g}) \to W'$, then $W'[-j-2]$ has the basis u', v_s', for $0 \leqslant s \leqslant j+1$, where $u = H+j+2$ and $v_s = Y^s X^s$. Now $I' \subset I$ and $I = I' + U(\mathfrak{g})(YX - \gamma(H+j+2))$. Hence $W = W'/W''$ where W'' is the submodule of W' generated by $(YX - \gamma(H+j+2))'$ which lies in $W'[-j-2]$. But $W''[-j-2]$ is spanned by the elements $a \cdot (YX - \gamma(H+j+2))'$ where a is of the weight 0 for the adjoint representation, i.e., a commutes with H. Now the centralizer of H in $U(\mathfrak{g})$ is the algebra generated by H and YX and is therefore abelian. Hence, $W''[-j-2]$ is $L(W'[-j-2])$ where L is the endomorphism of W' which is induced by left multiplication by $YX - \gamma(H+j+2)$ on $U(\mathfrak{g})$. Hence,

$$W[-j-2] = W'[-j-2]/L(W'[-j-2]).$$

But a simple calculation shows that

$$Lu' = 0, \quad Lv_s' = -s(j+1-s)v_s' + v_{s+1}' (1 \leqslant s \leqslant j+1), \quad Lv_0' = v_1' - \gamma u'.$$

So for the kernel κ and the range ρ of L on $W'[-j-2]$ we have

$$\kappa \supset \langle u', v_{j+1}' \rangle, \quad \rho = \langle v_1' - \gamma u', (v_{s+1}' - s(j+1-s)v_s')_{1 \leqslant s \leqslant j} \rangle,$$

where $\langle \cdots \rangle$ denotes linear span. The vectors determining ρ are linearly independent, so that $\kappa = \langle u', v_{j+1}' \rangle$ and the dimension of ρ is $j+1$. This proves that $W[-j-2] = W'[-j-2]/\rho \simeq \mathbf{C}^2$. ∎

3.6. Since $I(j:\gamma) \subset K(j)$, we now may introduce the module $V' = K(j)/I(j:\gamma)$.

LEMMA. *We have $V' \simeq V(-j-2)$ and the sequence*

$$0 \to V' \to U(\mathfrak{g})/I(j:\gamma) \to U(\mathfrak{g})/K(j) \to 0$$

is exact and nonsplitting. In particular, for $W = U(\mathfrak{g})/I(j:\gamma)$ we have $\gamma(W) = \gamma$.

Proof. We know Y^rX^s $(r \geqslant 0,\ s \leqslant j+1)$, $Y^r(H+j+2)$ $(r \geqslant 0)$ span $U(\mathfrak{g})$ mod $I(j:\gamma)$. But if $r \geqslant 1$, $s \geqslant 1$,

$$\begin{aligned} Y^rX^s = Y^{r-1}YXX^{s-1} &\equiv \gamma Y^{r-1}(H+j+2)X^{s-1} \\ &\equiv \text{const } Y^{r-1}X^{s-1} \bmod I(j:\gamma). \end{aligned}$$

Hence,

$$X^s\ (1 \leqslant s \leqslant j+1) \quad \text{and} \quad Y^r,\ Y^r(H+j+2)\ (r \geqslant 0) \text{ span } U(\mathfrak{g}) \bmod I(j:\gamma).$$

These have weights $-j-2+2s$, $-j-2-2r$, respectively. So the spectrum of H on W is contained in $\{j, j-2, \ldots\}$ with the eigenvalues of multiplicity $\leqslant 1$ or $\leqslant 2$ according as they are $\geqslant -j$ or $\leqslant -j-2$. Hence, the spectrum of H on V' is simple and contained in $\{-j-2, -j-4, \ldots\}$. By 3.5 we know that $-j-2$ is an eigenvalue. Hence, $-j-2$ is the highest weight of V', showing that $V' \simeq V(-j-2)$. On the other hand, $H+j+2 \notin I(j:\gamma)$ as otherwise we would have $K(j) \subset I(j:\gamma) \subset K(j)$, giving $V' = 0$. We may therefore take w to be the image of $H+j+2$ in W and find that $\gamma(W) = \gamma$. ■

3.7. We now take up the case $\gamma = \infty$. We set

$$I(j:\infty) = \langle\langle H+j+2,\ X^{j+2},\ \Gamma_j^2 \rangle\rangle.$$

LEMMA. $\dim(U(\mathfrak{g})/I(j:\infty))[-j-2] = 2.$

Proof. Write

$$I = I(j:\infty), \quad W = U(\mathfrak{g})/I, \quad I' = \langle\langle H+j+2,\ X^{j+2} \rangle\rangle, \quad W' = U(\mathfrak{g})/I'$$

and a' for the image in W' of $a \in U(\mathfrak{g})$. Let $W'' = U(\mathfrak{g}) \cdot (\Gamma_j^2)'$. Then $Y^rX^s(H+j+2)^t$, for $r, s, t \geqslant 0$, form a basis for $U(\mathfrak{g})$ and the subfamily with either $s \geqslant j+2$ or $t \geqslant 1$ span I'. Hence, Y^rX^s $(r \geqslant 0,\ 0 \leqslant s \leqslant j+1)$ form a basis of $U(\mathfrak{g})$ mod I'. In particular, $v_s = (Y^sX^s)'$ $(0 \leqslant s \leqslant j+1)$ form a basis of $W'[-j-2]$. Since Γ_j^2 is in the center of $U(\mathfrak{g})$ we have $W'' = \Gamma_j^2 \cdot W'$ so that

$W[-j-2] = W'[-j-2]/\Gamma_j^2(W'[-j-2])$. A simple calculation shows that

$$\Gamma_j v_s = -4s(j+1-s)v_s + 4v_{s+1},$$

from which it follows easily that (restricted to $W'[-j-2]$) $\ker(\Gamma_j^2)$ has dimension 2 and that $W'[-j-2] = \ker(\Gamma_j^2) \oplus \operatorname{im}(\Gamma_j^2)$. ∎

3.8. As $I(j:\infty) \subset K(j)$ we are allowed to introduce the module $V' = K(j)/I(j:\infty)$.

LEMMA. *Let $W = U(\mathfrak{g})/I(j:\infty)$. Then $V' \simeq V(-j-2)$ and*

$$0 \to V' \to W \to U(\mathfrak{g})/K(j) \to 0$$

is a nonsplitting exact sequence. In particular, $\gamma(W) = \infty$.

Proof. Write $I = I(j:\infty)$. The images v_{rs} of $Y^r X^s$ in $U(\mathfrak{g})/I$ for $r \geqslant 0$ and $0 \leqslant s \leqslant j+1$ span W. An easy calculation shows that, with $\gamma_j = \Gamma_j/4$,

$$\gamma_j \cdot v_{rs} = -s(j+1-s)v_{rs} + v_{(r+1)(s+1)},$$

and, hence, that for suitable constants a_{rs}, b_{rs},

$$0 = \gamma_j^2 \cdot v_{rs} = s^2(j+1-s)^2 v_{rs} + a_{rs} v_{(r+1)(s+1)} + b_{rs} v_{(r+2)(s+2)}.$$

It follows that for $1 \leqslant s \leqslant j$ and constants c_{rs} we have

$$v_{rs} = c_{rs} v_{(r+j+1-s)(j+1)}.$$

Thus v_{r0} and $v_{r(j+1)}$ span W, and their weights are respectively $-j-2-2r$ and $j-2r$. Consequently, the spectrum of H in V' is simple and contained in $\{-j-2, -j-4, \ldots\}$. As $-j-2$ is an eigenvalue by 3.7, it follows as before that $V' \simeq V(-j-2)$. If w is the image of 1 in W, we must have $YXw \neq 0$; for, otherwise, $YX \in I$ which would imply that $K(j) \subset I \subset K(j)$ so that V' would be 0. Since $(H+j+2)w = 0$, we have $\gamma(W) = \infty$. ∎

3.9. THEOREM. *For each $\gamma \in \mathbf{P}^1(\mathbf{C})$ there is exactly one $W(j:\gamma)$ up to isomorphism. It is a weight module for $\gamma = \infty$ while for finite γ the action of $H+j+2k$ on $W(j:\gamma)[-j-2k]$ for each $k \geqslant 1$ is a nonzero nilpotent. In either case, if $w \in W(j:\gamma)[-j-2]$ maps to a nonzero element in $M(j)$, there is a unique isomorphism of $W(j:\gamma)$ with $U(\mathfrak{g})/I(j:\gamma)$ that takes w to the image of* 1.

Proof. Let W be such that (3.1) is exact and nonsplitting and let $\gamma(W) = \gamma$. If J is the annihilator of w in $U(\mathfrak{g})$, it is direct that $I(j:\gamma) \subset J$. Hence there is a unique surjective homomorphism $U(\mathfrak{g})/I(j:\gamma) \to W$ that takes the image of 1 to w. This must be an isomorphism since the multiplicities of the eigenvalues of H are the same for both modules. The remaining statements are clear. Note that this proves Theorem 2.1(a). ■

3.10. The modules $W(j:\gamma)$ do not seem to have been encountered before in representation theory; however, the modules $W(j:\infty)$ are the same as the modules T_j discussed in [18, p. 185].

3.11. We shall now give another description of $W(j:\gamma)$. Let $\mathscr{C}$ be the category of modules V such that:

(1) $V = \bigoplus_{c \in \mathbf{C}} V[c]$, where the $V[c]$ are the generalized weight spaces for H;
(2) $V[c] \neq 0$, only for $c \in \mathbf{Z}$, and finite-dimensional for all c.

As before let

$$\Gamma = H(H+2) + 4YX;$$

and for any $V \in \mathscr{C}$ and integer $c \geqslant 0$, let V_c be the maximal subspace of V on which $\Gamma - c(c+2)$ is nilpotent. The V_c are submodules and $V = \bigoplus V_c$. We write E_c for the projections $V \to V_c$. The functor β_j, for integers $j \geqslant 0$, is then defined by

$$\beta_j(V) = E_j(V \otimes F(j)).$$

It is clear that this is a covariant functor which is exact, that is, takes exact sequences to exact sequences. Our aim is to prove the following theorem whose proof requires some preparation.

THEOREM. *We have*

$$\beta_j(W(0:\gamma)) = W(j:(j+1)\gamma).$$

3.12. We write $\{f_j, f_{j-2}, \ldots, f_{-j}\}$ for a basis of $F(j)$ with f_k of weight k and $Yf_k = f_{k-2}$ $(Yf_{-k} = 0)$. The following proposition is the special case, for $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$, of [1, Lemma 5].

PROPOSITION. *Let $c \in \mathbf{C}$. Then $V(c) \otimes F(j)$ has a flag of submodules whose successive quotients are*

$$V(c+j),\ V(c+j-2), \ldots,\ V(c-j).$$

More precisely, let $v(c)$ be a nonzero highest weight vector of $V(c)$ and let

$$L_r = \mathfrak{Y}(v(c) \otimes \langle f_j, f_{j-2}, \ldots, f_{j-2r} \rangle),$$

where $\mathfrak{Y} = \mathbf{C}[Y] \subset U(\mathfrak{g})$. *Then*:

(1) *the* L_r *are submodules and* $0 \subset L_0 \subset \cdots \subset L_j = V(c) \otimes F(j)$;
(2) $L_r/L_{r-1} \simeq V(c+j-2r)$;
(3) *The image of* $v(c) \otimes f_{j-2r} \in L_r$ *lies in* $V(c+j-2r)[c+j-2r]$ *and is not zero.*

3.13. DEFINITION. Suppose $W \in \mathscr{C}$ has a flag of submanifolds

$$0 = W_{-1} \subset W_0 \subset \cdots \subset W_p = W.$$

Assume that for all $m = 0, \dots, p$, we have that $(W_m/W_{m-1})_{c(m)} = W_m/W_{m-1}$, for some $c(m) \in \mathbf{C}$; that is, Γ has a single eigenvalue $c(m)(c(m)+2)$ on W_m/W_{m-1}. For a fixed r, $0 \leqslant r \leqslant p$, the number $c(r)$ or W_r/W_{r-1} is said to be *isolated* if $c(m)(c(m)+2) \neq c(r)(c(r)+2)$ for $m \neq r$.

3.14. Let the context be as above and let π be the natural map

$$W_m \to W_m/W_{m-1}.$$

LEMMA

(1) *Suppose* $x \in W_m$ *and* $\pi x \neq 0$, *then* $E_{c(m)}x \neq 0$; *in fact,*

$$\pi E_{c(m)} x = \pi x.$$

(2) *Suppose further that* W_r/W_{r-1} *is isolated. Then* $W' := E_{c(r)}W$ *is contained in* W_r *and* $W_r = W' \oplus W_{r-1}$.

Proof. The assertion (1) follows from the commutativity of the diagram

$$\begin{array}{ccc} W_m & \xrightarrow{\pi} & W_m/W_{m-1} \\ {\scriptstyle E_{c(m)}}\downarrow & & \Vert \\ (W_m)_{(m)} & \xrightarrow{\pi} & (W_m/W_{m-1})_{c(m)} \end{array}$$

For (2) we write $W'' = (I - E_{c(r)})(W)$. Then $W = W' \oplus W''$ so that $W''_{c(r)} = 0$. For any m,

$$W_m = (W_m \cap W') \oplus (W_m \cap W''),$$

$$(W_m \cap W'')/(W_{m-1} \cap W'') \hookrightarrow W_m/W_{m-1}.$$

Taking $m = r$ we see that $W_r \cap W'' = W_{r-1} \cap W''$. Hence,

$$W_r/W_{r-1} \simeq (W_r \cap W')/(W_{r-1} \cap W').$$

On the other hand, if $m \neq r$, the imbedding

$$(W_m \cap W')/(W_{m-1} \cap W') \hookrightarrow W_m/W_{m-1}$$

shows that $W_{m-1} \cap W' = W_m \cap W'$. Hence,

$$W_m \cap W' = \begin{cases} 0, & m < r; \\ W', & m \geqslant r. \end{cases}$$

This proves that $W_r = W' \oplus W_{r-1}$.

3.15. Let $c \in \mathbf{C}$, L_r as in Proposition 3.12, and $c(r) = c + j - 2r$.

LEMMA. *We have*

$$E_{c(r)}(v(c) \otimes f_{j-2r}) \neq 0,$$

and its image in L_r/L_{r-1} *generates* L_r/L_{r-1}.

Proof. Follows from Proposition 3.12(2) and Lemma 3.14(1). ∎

3.16. We continue in the above context but with $V(-2)$ in place of $V(c)$.

LEMMA. *Let* $E = E_j$, *then*:

(1) $E(v(-2) \otimes f_{-j}) \neq 0$;
(2) $\beta_j(V(-2)) \simeq V(-j-2)$;
(3) $\beta_j(M(0)) \simeq M(j)$.

Proof. Assertion (1) is immediate from Lemma 3.15. For (2), notice that $V(-2) \otimes F_j$ has the flag of submodules L_r with $L_r/L_{r-1} \simeq V(-2+j-2r)$ with $r = 0, 1, \ldots, j$. Then L_j/L_{j-1} is isolated and so by Lemma 3.14(2) we find that $\beta_j(V(-2)) \simeq V(-j-2)$. We now prove (3). From (2) and the exact sequence

$$0 \to F(0) \to M(0) \to V(-2) \to 0,$$

we get the exact sequence

$$0 \to F(j) \to \beta_j(M(0)) \to V(-j-2) \to 0.$$

It is thus enough to prove that this does not split. If it does, then $V(-j-2)$ is a direct summand of $\beta_j(M(0))$ and, hence, of $M(0) \otimes F(j)$. So there exists $v \neq 0$ in $M(0) \otimes F(j)$ of weight $-j-2$ with $Xv = 0$. Write

$$v = v_{-j+2r} \otimes f_{-j+2r} + v_{-j+2r+2} \otimes f_{-j+2r+2} + \cdots,$$

where $r \geqslant 0$, $v_{-j+2r} \neq 0$, and v_{-j+2r} is of weight $-2r-2$. Then

$$0 = Xv = (Xv_{-j+2r}) \otimes f_{-j+2r} + \cdots \Rightarrow Xv_{-j+2r} = 0.$$

This means that v_{-j+2r} is of weight 0 so that $r = -1$, a contradiction. ∎

Let us write $W(\gamma)$ for $W(0:\gamma)$. From the above lemma we obtain the exact sequence

$$0 \to V(-j-2) \to \beta_j(W(\gamma)) \to M(j) \to 0.$$

We are thus reduced to verifying that this does not split and computing the γ-invariant of $\beta_j(W(\gamma))$. Consider

$$0 \to V(-2) \to W(\gamma) \to M(0) \to 0,$$

and let $w \in W(\gamma)[-2]$ be as in 3.1. Let

$$u = YXw, \qquad v = (H+2)w.$$

3.17. LEMMA. *With notation as above we have the following*:

(1) $E(Y^{j+1}X^{j+1}(w \otimes f_{-j})) = (j+1)!j!E(u \otimes f_{-j})$;
(2) $E((H+j+2)(w \otimes f_{-j})) = E(v \otimes f_{-j})$.

Proof. A standard $\mathfrak{sl}(2)$ calculation shows that $X^j f_{-j} = j!^2 f_j$. Also we know that $X^{j+1} f_{-j} = 0$, $X^2 w = 0$. Hence,

$$\begin{aligned} Y^{j+1}X^{j+1}(w \otimes f_{-j}) &= Y^{j+1}((j+1)(Xw) \otimes (X^j f_{-j})) \\ &= (j+1)!j!Y^{j+1}((Xw) \otimes f_j) \\ &= (j+1)!j!Y^j((YXw) \otimes f_j + (Xw) \otimes f_{j-2}). \end{aligned}$$

But $YXw = u \in V(-2)[-2]$ and $Y^j(u \otimes f_j) \in L_0 \simeq V(j-2)$, according to Proposition 3.12. Hence, $EL_0 = 0$, so that

$$\begin{aligned} E(Y^{j+1}X^{j+1}(w \otimes f_{-j})) &= (j+1)!j!\,E(Y^j((Xw) \otimes f_{j-2})) \\ &= (j+1)!j!\,E(Y^{j-1}(u \otimes f_{j-2} + (Xw) \otimes f_{j-4})). \end{aligned}$$

Again $Y^{j-1}(u \otimes f_{j-2}) \in L_1$, and $EL_1 = 0$, so that

$$E(Y^{j+1}X^{j+1}(w \otimes f_{-j})) = (j+1)!j!\,E(Y^{j-1}((Xw) \otimes f_{j-4})).$$

This argument can be continued until we get

$$\begin{aligned} E(Y^{j+1}X^{j+1}(w \otimes f_{-j})) &= (j+1)!j!\,E(Y((Xw) \otimes f_{-j})) \\ &= (j+1)!j!\,E(u \otimes f_{-j}). \end{aligned}$$

For (2) we simply note

$$(H + j + 2)(w \otimes f_{-j}) = ((H + 2)w \otimes f_{-j}). \qquad \blacksquare$$

3.18. We recall the exact sequence

$$0 \to V(-j-2) \to \beta_j(W(\gamma)) \to M(j) \to 0.$$

LEMMA. *Let* $w' = E(w \otimes f_{-j})$. *Then* $w' \in \beta_j W(\gamma)[-j-2]$ *and maps to a nonzero element of* $M(j)[-j-2]$. *Moreover the above exact sequence does not split, and so,* $\beta_j W(\gamma) = W(j:\gamma')$ *for some* $\gamma' \in \mathbf{P}^1(\mathbf{C})$; *and* γ' *is finite if and only if* γ *is.*

Proof. The image of w' in $M(j)[-j-2]$ is $E(m(-2) \otimes f_{-j})$ where $m(-2) \in M(0)$ is the image of w. Under $M(0) \to V(-2)$ we have $m(-2) \mapsto v' \in V(-2)[-2]$ with $v' \neq 0$, so that under $M(j) \to V(-j-2)$, the element $E(m(-2) \otimes f_{-j})$ goes to $E(v' \otimes f_{-j})$ which is $\neq 0$ by Lemma 3.16(1). Hence, $E(m(-2) \otimes f_{-j}) \neq 0$.

We now have

$$(H + j + 2)w' = E((H + j + 2)(w \otimes f_{-j})) = E(v \otimes f_{-j}).$$

Suppose now that $\gamma \in \mathbf{C}$. Then $v \neq 0$ and therefore $(H + j + 2)w' \neq 0$. So $\gamma' \in \mathbf{C}$. Suppose next that $\gamma = \infty$. Then $v = 0$ and $u \neq 0$. So $(H + j + 2)w' = 0$; moreover

$$Y^{j+1}X^{j+1}w' = (j+1)!j!\,E(u \otimes f_{-j}).$$

This implies that $YXw' \neq 0$. Indeed, if $YXw' = 0$, then as $(H + j + 2)w' = 0$, Proposition 3.1 will apply to give the conclusion that $\beta_j(W(\gamma)) = V(-j-2) \oplus M(j)$, which implies that $Y^{j+1}X^{j+1} = 0$ on $\beta_j(W(\gamma))$. $\blacksquare$

3.19. We can complete the proof of Theorem 3.11 by proving the following lemma.

LEMMA. $\beta_j(W(\gamma)) = W(j:(j+1)\gamma)$.

Proof. We need only consider the case when γ is finite. Then

$$Y^{j+1}X^{j+1}w' = (j+1)!j!\,E(u \otimes f_{-j}).$$

But $u = YXw = \gamma(H+2)w$, so that

$$Y^{j+1}X^{j+1}w' = (j+1)!j!\gamma(H+j+2)w'.$$

Let γ' be such that $YXw' = \gamma'(H+j+2)w'$. Then by Lemma 3.2,

$$Y^{j+1}X^{j+1}w' = j!^2\gamma'(H+j+2)w'.$$

Hence equating the right-hand sides, we find

$$\gamma' = (j+1)\gamma. \qquad \blacksquare$$

4. The modules $\mathscr{H}^j$

We now return to the framework of Section 2. The properties of the modules $\mathscr{H}^j$ are quite classical and have been studied, in a vastly more general context, by Kostant [9, §§0–2]. For our purposes we need a few refinements which, although readily deducible from Kostant's theory, are hard to find in the literature. Let X be a complex vector space of finite dimension on which a connected complex reductive algebraic group L operates. Let $\mathscr{P}$ be the algebra of polynomial functions on X. For any x in X we write $L(x)$ for the orbit of x under L. If $x \in X$, let $F(L(x))$ denote the space of functions on $L(x)$ that are restrictions of the polynomial functions to $L(x)$; moreover, let $R(L(x))$ denote the space of L-finite functions on $L(x)$, namely, the linear span of all finite-dimensional spaces of functions on $L(x)$ that are stable under L and carry a holomorphic (=rational) representation of L. It is obvious that $F(L(x)) \subset R(L(x))$, and it is important to know when there is equality here. It is known (see [9]) that if the variety $\mathrm{Cl}(L(x))$ is normal and $\mathrm{Cl}(L(x))\setminus L(x)$ is of codimension at least 2 in $\mathrm{Cl}(L(x))$, then $F(L(x)) = R(L(x))$. Let $\mathscr{S}$ be the algebra of constant coefficient differential operators on X. We shall naturally identify it with the symmetric algebra over X. We write $I(\mathscr{P})$ (resp. $I(\mathscr{S})$) for the subalgebra of L-invariant elements of $\mathscr{P}$ (resp. $\mathscr{S}$). An element f of $\mathscr{P}$ is called *harmonic* if $\partial(u)f = 0$ for all $u \in I(\mathscr{S})$ whose constant terms are zero. We denote by $\mathscr{H}$ the graded space of harmonic elements and by $\mathscr{H}^j$ its homogeneous components.

Let us now introduce the variety $\mathscr{N}$ of zeros of the ideal $I(\mathscr{P})^+ \cdot \mathscr{P}$ where $I(\mathscr{P})^+$ is the set of all elements of $I(\mathscr{P})$ that vanish at the origin of X. Following Kostant let us make the following assumptions:

(1) $I(\mathscr{P})^+ \cdot \mathscr{P}$ is a prime ideal;
(2) there exists an orbit O dense in $\mathscr{N}$;
(3) there exists a symmetric nondegenerate bilinear form $(.,.)$ for X invariant under L.

Under these conditions, which we shall call (**K**), we have the following results (cf. [9]):

(1) the natural map $I(\mathscr{P}) \otimes \mathscr{H} \to \mathscr{P}$ is an isomorphism; in particular $\mathscr{P}$ is a free $I(\mathscr{P})$-module;
(2) $\mathscr{H}$ is the linear span of the monomials $l(x)^n$, for $n = 0, 1, 2, \ldots,$ where $l(x)$ is the linear function $y \mapsto (x, y)$ on X;
(3) let $x \in X$ be called *quasiregular* if

$$\mathscr{N} \subset \mathrm{Cl}\left(\bigcup_{c \in \mathbf{C}^*} L(cx) \right);$$

then if x is quasiregular, the (surjective) restriction map $\mathscr{P} \to F(L(x))$ is an isomorphism onto when restricted to $\mathscr{H}$.

The structure of $\mathscr{H}$ (hence of $\mathscr{P}$) as an L-module may be obtained under suitable additional assumptions. Let $x \in X$ be any quasiregular element; for instance, we may take x to be such that $L(x)$ is dense in $\mathscr{N}$. Let L^x be the stabilizer of x in L and let $R(L/L^x)$ be the left L-module of all regular functions on L that are right L^x-invariant. If in addition $\mathrm{Cl}(L(x)) \backslash L(x)$ is of codimension $\geqslant 2$ in $\mathrm{Cl}(L(x))$, we have the L-module isomorphism

$$\mathscr{H} \simeq R(L/L^x),$$

which, together with Frobenius reciprocity shows that $\mathscr{H}$ contains precisely those irreducible representations of L whose contragredients contain nonzero vectors invariant under L^x, and that the multiplicity of such a representation is the dimension of the space of L^x-invariant vectors in the contragredient representation. Finally, we have an isomorphism

$$p \mapsto \partial(p)$$

of $\mathscr{P}$ with $\mathscr{S}$ under which the linear function $l(x)\colon y \mapsto (x, y)$ goes over to the directional derivation $\partial(x)$ in the direction of x. The bilinear form for $\mathscr{P}$ defined by

$$(p, q) = (\partial(q)(p))(0), \qquad p, q \in \mathscr{P},$$

is then symmetric, nondegenerate, and L-invariant; and with respect to it $I(\mathscr{P})^+ \cdot \mathscr{P}$ and $\mathscr{H}$ are mutual orthogonal complements. Of course

$$\mathscr{P} = I(\mathscr{P})^+ \cdot \mathscr{P} \oplus \mathscr{H}.$$

We may now specialize to the situation when L is the *full* special orthogonal group of $(.,.)$. The conditions (**K**) are satisfied. The variety $\mathcal{N}$ is the cone of null elements

$$\mathcal{N} = \{x \in X \mid (x, x) = 0\};$$

and it follows from classical results of Seidenberg [9, p. 391] that $\mathcal{N}$ is normal. Let us define

$$Q_t = \{x \in X \mid (x, x) = t\}, \qquad Q_0 = \{x \in X \mid x \neq 0, (x, x) = 0\}.$$

Then the Q_t are all orbits for L; the Q_t for $t \neq 0$ are closed and smooth while Q_0 is smooth and open dense in $\mathcal{N}$. It follows from this that any $x \neq 0$ in X is quasiregular and $F(Q_x) = R(Q_x) \simeq \mathcal{H}$. Of course we shall continue to use the notation ω for the quadratic form $(.,.)$ and $\square$ for the operator which is $\Sigma_k \partial^2/\partial z_k^2$ with respect to any linear system (z_k) of coordinates associated to an orthonormal basis. A polynomial is harmonic if $\square p = 0$. The following result is well-known, and we give a sketch of its proof for completeness.

4.1. PROPOSITION. *We have the following.*

(1) *The modules $\mathcal{H}^j$ are irreducible for all $j \geqslant 0$, self-contragredient, and they are mutually inequivalent.*
(2) *Setting $a_{(n)} = a \cdot (a+1) \cdots (a+n-1)$, we have*

$$\dim(\mathcal{H}^j) = \frac{(j+1)_{(n-1)}}{(n-1)!} + \frac{j_{(n-1)}}{(n-1)!}.$$

In particular, for any $x \neq 0$, the action of L on $F(Q_x) = R(Q_x)$ is multiplicity-free and contains only those representations that contain a nonzero vector fixed by L^x. These representations are self-contragredient and have a one-dimensional space of L^x-invariant vectors.

Proof. Using an orthonormal basis for X we identify X with $\mathbf{C}^{n+1}$ so that $\omega = \Sigma_{0 \leqslant k \leqslant n} z_k^2$ and $\square = \Sigma_{0 \leqslant k \leqslant n} \partial^2/\partial z_k^2$. If we take $x = (1, 0, \ldots, 0)$ and note that the restriction map from $\mathbf{SO}(n+1, \mathbf{C})$ to $\mathbf{SO}(n+1, \mathbf{R})$ induces an isomorphism of $R(\mathbf{SO}(n+1, \mathbf{C})/\mathbf{SO}(n, \mathbf{C}))$ with $R(\mathbf{SO}(n+1, \mathbf{R})/\mathbf{SO}(n, \mathbf{R}))$, we can make use of the compactness to infer the self-contragredient nature of the representations involved. If we knew that the $\mathcal{H}^j$ are irreducible and the dimension formula is true, all the assertions would follow easily.

We now prove the dimension formula. We shall work with the real subspace $\mathcal{P}_\mathbf{R}$ of $\mathcal{P}$ on which $(.,.)$ is positive definite. The isomorphism $p \mapsto \partial(p)$ of $\mathcal{P}$ with $\mathcal{S}$ takes ω to $\square$. Moreover for $p, q \in \mathcal{P}$ we have

$$(\omega \cdot p, q) = \partial(\omega p)(q)(0) = (\partial(p)\square)(q)(0) = (p, \square q),$$

showing that ω and $\square$ are transposes of each other with respect to $(.,.)$. As $M(\omega)$ maps $\mathscr{P}^j_{\mathbf{R}}$ injectively into $\mathscr{P}^{j+2}_{\mathbf{R}}$, the operator $\square$ will map $\mathscr{P}^j_{\mathbf{R}}$ surjectively onto $\mathscr{P}^{j-2}_{\mathbf{R}}$ for all $j \geqslant 0$. The dimension formula now follows from the fact that $\mathscr{H}^j$ is the kernel of $\square$ on $\mathscr{P}^j$.

It remains to prove the irreducibility of $\mathscr{H}^j$ under $\mathbf{SO}(n+1, \mathbf{R})$. This will be done if we show that the space of functions in $\mathscr{H}^j$ that are fixed by $\mathbf{SO}(n, \mathbf{R})$ is of dimension $\leqslant 1$. For this we set up an injection of this space into the space of solutions of an ordinary differential equation of the second order on the open interval $(-1, +1)$. To this end, let Ω be the Casimir operator of $\mathbf{SO}(n+1, \mathbf{R})$. Its action on functions on $\mathbf{R}^{n+1}$ is given by

$$\Omega = \frac{1}{2} \sum_{r \neq s} (-p_r \partial_s + p_s \partial_r)^2.$$

A simple calculation shows that

$$\Omega = r^2 \square - \mathscr{E}^2 - (n-1)\mathscr{E},$$

where $r^2 = \Sigma_j p_j^2$ and $\mathscr{E}$ is the Euler operator. So Ω acts as the scalar $-j(j+n-1)$ on $\mathscr{H}^j$. Let Z be the space of functions z on the unit sphere S^n that are smooth and fixed by $\mathbf{SO}(n, \mathbf{R})$, and satisfy $\Omega z = -j(j+n-1)z$. Let

$$z^*(t) = z(t, 0, \ldots, 0, \sqrt{1-t^2}), \ (-1 < t < 1).$$

The differential equation $\Omega z = -j(j+n-1)z$ then transcribes into the differential equation

$$(1-t^2)\frac{d^2 z^*}{dt^2} - nt\frac{dz^*}{dt} + j(j+n-1)z^* = 0.$$

This is an equation of the Fuchsian type with singularities at $\pm 1, \infty$; and it is an easy consequence of the theory of such equations that the space of solutions bounded at $t = 1$ is one-dimensional; indeed, this follows from the fact that the roots of the indicial equation at $t = 1$ are 0 and $-\frac{1}{2}n$. ■

For our purposes we need to refine these results by bringing in the group G, the component of the identity of $\mathbf{SO}(1, n)$. Let us therefore take X as $\mathbf{C}^{1,n}$, the complexification of $\mathbf{R}^{1,n}$, and L as $G_{\mathbf{C}}$, the complexification of G, namely, the special orthogonal group of the quadratic form

$$\omega = z_0^2 - z_1^2 - \cdots - z_n^2.$$

We then view the forward light cone X_0^+ as the G-orbit of the point $(1,0,\dots,0,1)$, and write G_0 (respectively $G_{\mathbf{C},0}$) for its stabilizer in G (resp. $G_{\mathbf{C}}$). We then have the following

4.2. LEMMA. *The stabilizer $G_{\mathbf{C},0}$ is isomorphic to the semidirect product of $\mathbf{SO}(n-1,\mathbf{C})$ with the additive group of $\mathbf{C}^{n-1}$ viewed as a module for $\mathbf{SO}(n-1,\mathbf{C})$ via the contragredient action. In particular, $G_{\mathbf{G},0}$ is connected, as are the stabilizers of all the nonzero points of X.*

Proof. We select a basis $(f_j)_{0\leqslant j\leqslant n}$ such that $f_0=(1,0,\dots,0,1)$, $f_n=(0,0,\dots,1)$, and $f_1,\dots,f_{n-1}$ span a space S orthogonal to f_0. Then any element $g\in G_{\mathbf{C},0}$ gives rise to a vector $a=(a_1,a_2,\dots,a_{n-1})$ and a $h=(h_{ij})\in\mathbf{SO}(n-1,\mathbf{C})$ such that

$$g\cdot f_j = a_j f_0 + \sum_{1\leqslant i\leqslant n-1} h_{ij} f_i.$$

It is then easy to verify that the map $g\mapsto(h,a)$ gives the isomorphism described. ■

4.3. LEMMA. *Every continuous finite-dimensional representation of G extends to a rational representation of $G_{\mathbf{C}}$.*

Proof. Since $\mathbf{SO}(n)$ is the maximal compact subgroup of G, the fundamental group of G is (± 1). Let $G_{\mathbf{C}}^*$ be the universal covering group of $G_{\mathbf{C}}$ and $\pi\colon G_{\mathbf{C}}^*\to G_{\mathbf{C}}$ be the covering morphism with kernel $K_0\simeq(\pm 1)$. Let H_n be the component of the identity of $H_n'=\pi^{-1}(G)$. It is clearly a question of proving that $K_0\subset H_n$. Now $K_{n+1}=\pi^{-1}(\mathbf{SO}(n+1))$ is a compact subgroup of $G_{\mathbf{C}}^*$ having the same dimension as its maximal compact subgroup and, hence, K_{n+1} is connected. If $K_n=\pi^{-1}(\mathbf{SO}(n))$ and K_n' is its component of identity, the diagram

$$K_{n+1}/K_n' \to K_{n+1}/K_n \to S^n$$

shows, in view of the simple connectedness of S^n, that $K_n=K_n'$. But then $K_0\subset H_n$, proving that $H_n=H_n'$. ■

4.4. PROPOSITION. *For any $x\neq 0$ in $\mathbf{R}^{1,n}$ let $Q_{x,\mathbf{R}}$ be the G-orbit of x and let R_x be the space of G-finite functions on it, namely the span of all spaces of functions on the orbit that are stable under G and induce a continuous representation on it. Then the restriction map from $\mathscr{P}$ maps onto R_x and is an isomorphism of $\mathscr{H}$ onto it.*

Proof. Only the surjectivity of $\mathscr{P}\to R_x$ is not obvious. If G^x is the stabilizer of x in G, then $R_x\simeq R(G/G^x)$. By the previous lemma the elements of $R(G/G^x)$ are restrictions of the rational matrix elements of $G_{\mathbf{C}}$ to G, and these are right invariant under the stabilizer of x in $G_{\mathbf{C}}$ since this stabilizer is connected by Lemma 2. Hence, $R_x\subset R(Q_x)$. ■

4.5. COROLLARY. *For an irreducible finite-dimensional G-module U the following are equivalent*:

(1) *there exists a G-module imbedding* $U \hookrightarrow F(Q_{x,\mathbf{R}})(=R_x)$;
(2) $U \simeq \mathscr{H}^j$ *for some j*;
(3) *The space* U_x *of vectors fixed by* G^x *is nonzero.*

In this case $\dim(U_x) = 1$.
Proof. Obvious. ∎

4.6. PROPOSITION. *Let* $U \simeq \mathscr{H}^j$ *be as above and let* $u \neq 0$ *be in* U_x. *Then there is a unique harmonic G-invariant map* $h_u\colon \mathbf{C}^{1,n} \to U$ *such that* $h_u(x) = u$; *moreover* h_u *is necessarily homogeneous of degree j. The elements*

$$h_u, \omega h_u, \ldots, \omega^r h_u, \ldots$$

form a basis for the space of G-invariant maps $\mathbf{C}^{1,n} \to U$. *Finally, if U is not equivalent to any* $\mathscr{H}^j$, *there is no invariant map* $\mathbf{C}^{1,n} \to U$ *other than zero.*

Proof. There is a unique G-invariant map $h\colon Q_{x,\mathbf{R}} \to U$ with $h(x) = u$; its components are in $R(Q_{x,\mathbf{R}})$ and so there is a unique harmonic map $h_u\colon \mathbf{C}^{1,n} \to U$ that restricts to h. This map is necessarily G-invariant. To prove that this is homogeneous of degree j it is enough to construct such a map independently. Let (h_i) be a basis of $\mathscr{H}^j$ and let (k_i) be the contragredient basis. If

$$h(y) = \sum_j h_j(y)k_j,$$

it is immediate that h is a nonzero G-invariant, harmonic map that is homogeneous of degree j. For the rest of the proof, let I_d be the space of G-invariant maps of $\mathbf{C}^{1,n}$ into U that are homogeneous of degree d. First let $j = 0$. Then h_u is a nonzero constant and the assertion is just that

$$1, \omega, \ldots, \omega^r, \ldots$$

form a basis for the space of G-invariant scalar polynomial functions. Let $j \geqslant 1$ and let e be the smallest degree in which we can find a homogeneous polynomial map g into U of degree e that is G-invariant and nonzero. If $t = (1, 0, \ldots, 0, 1)$, there is a constant c such that $g - ch_u$ vanishes on $Q_{t,\mathbf{R}}$, hence, on the complex orbit Q_t, so that $g - ch_u = \omega g_1$; g_1 is G-invariant. If $e < j$, then g_1 will have a nonzero component in degree $e-2$, contradicting the definition of e, and so, $e \geqslant j$. Let g be a G-invariant polynomial map into U of degree $d \geqslant j$. If $d = j$, g_1 is zero so that $g = ch_u$; if $d > j$, then $g = \omega g_2$ where g_2 is homogeneous of degree $d-2$, and we use the induction hypothesis to complete the proof. The last assertion is obvious. ∎

5. Structure of $J(U)$

Let M, N be smooth manifolds, N being a submanifold of M, regularly imbedded, in particular locally closed. Let $p \in N$ and T a distribution, scalar or vector-valued, defined around p with $\operatorname{supp}(T) \subset N$. If $x_1, \ldots, x_m$, $y_1, \ldots, y_r$ are local coordinates defined around p vanishing at p such that N is locally defined as $y_1 = \cdots = y_r = 0$, then T can be written as $\Sigma_\gamma \partial^\gamma S_\gamma$ where S_γ are distributions defined on N around p, and $\partial^\gamma = (\partial/\partial y_1)^{\gamma_1} \cdots (\partial/\partial y_r)^{\gamma_r}$, the sum being finite. The germs of the S_γ at p are uniquely determined by T. The *transversal order*, $o_p(T)$, of T at p, is then defined as the largest integer r for which there is a γ with $|\gamma| = r$ such that the germ at p of S_γ is nonzero. It is easy to see that this definition is independent of the choice of local coordinates. If N is of codimension-one in M, and $g \in C^\infty(M)$ is such that $\mathrm{d}g_p \neq 0$ and N is locally defined as the set of zeros of g, then

$$o_p(T) = k \Leftrightarrow p \in \operatorname{supp}(g^k T) \quad \text{and} \quad p \notin \operatorname{supp}(g^{k+1} T).$$

If T is invariant under a Lie group G acting on M and N is invariant, $o_p(T) = o_{g \cdot p}(T)$. In particular, if N is an orbit, $o_p(T)$ is constant for $p \in N$ and we denote it by $o_N(T)$ or $o(T)$.

5.1. DEFINITION. Let U be an irreducible G-module. We define $J(U)$ to be the space of germs of G-invariant distributions T defined on invariant open neighborhoods of X_0^+ with $\operatorname{supp}(T) \subset X_0^+$. Similarly we define $\bar{J}(U)$ to be the space of germs of G-invariant distributions T defined on $\mathbf{R}^{1,n}$ with $\operatorname{supp}(T) \subset \operatorname{Cl}(X_0^+)$. These spaces admit filtrations $(J_k(U))$ (resp. $(\bar{J}_k(U))$ defined by the condition $o(T) \leqslant k$, the transversal order being with respect to the orbit X_0^+. When $U = \mathbf{C}$ we omit U. Once and for all we choose and fix a G-invariant harmonic map $h_U \colon \mathbf{C}^{1,n} \to U$, nonzero iff $U \simeq \mathscr{H}^j$ for some j.

5.2. LEMMA. *Let $T \in J(U)$. Then we have*

$$T \in J_k(U) \Leftrightarrow X^{k+1} T = 0.$$

Proof. Immediate since X_0^+ is locally the set where ω is zero; and $\mathrm{d}\omega \neq 0$ everywhere on it. ■

5.3. A standard way to build invariant distributions is by 'pullback' of arbitrary distributions via an invariant map. Let us now make this more precise. A *volume form* on a smooth manifold M is a nowhere zero element of $\bigwedge^* M$ of maximal degree. Let us now consider two smooth manifolds M, N and a smooth map $\pi \colon M \to N$ which is surjective and everywhere submersive. Let vol_M (resp. vol_N) be a volume form on M (resp. N). If $p = \dim M - \dim N$, there exists a unique exterior differential form η on M of degree p, such that $\eta_m = (\mathrm{vol}_M)_m/(\mathrm{vol}_N)_{\pi(m)}$ for all $m \in M$. It is clear that η defines a volume form $\eta(n)$ on the fiber $\pi^{-1}(\{n\})$ and that $\eta(n)$ is smooth with respect to n. If we choose (oriented) local coordinates on M and N so that π has the form $(x, y) \mapsto y$, and if the volume forms are, respectively, $v_M(x, y)\,\mathrm{d}x\,\mathrm{d}y$ and $v_N(y)\,\mathrm{d}y$, then

$$\eta(y) = \frac{v_M(x, y)}{v_N(y)}\,\mathrm{d}x.$$

Let μ, v, and $\mu(n)$ be the measures defined by the forms vol_M, vol_N and $\eta(n)$. For any $f \in C_c^\infty(M)$ we define

$$\pi_* f(n) = \int_{\pi^{-1}(\{n\})} f\,\mathrm{d}\mu(n) \quad (n \in N).$$

It can then be shown that $\pi_* f$ lies in $C_c^\infty(N)$, while $\mathrm{supp}(\pi_* f) \subset \pi(\mathrm{supp}(f))$, and $f \mapsto \pi_* f$ is a continuous linear map of $C_c^\infty(M)$ *onto* $C_c^\infty(N)$. The dual map, denoted by π^*, is the pullback *map* of distributions

$$(\pi^* t)(f) = t(\pi_* f), \qquad f \in C_c^\infty(M).$$

For more details, we refer to [17, §I.2]. See also [6], where it is mentioned that a similar construction goes back to some unpublished work of L. Schwartz, and [4]. We shall need to make use of the following properties:

(1) For any locally integrable function g on N, we have

$$\int_M f(g \circ \pi)\,\mathrm{d}\mu = \int_N (\pi_* f) g\,\mathrm{d}v.$$

(2) Let D be a differential operator on M and let us suppose that there is a differential operator D_π on N such that for all smooth f in N, $D(f \circ \pi) = (D_\pi f) \circ \pi$. If † denotes transposes with respect to the volume forms, then

$$\pi_*(D^\dagger f) = D_\pi^\dagger(\pi_* f), \qquad (D_\pi t) \circ \pi = D(t \circ \pi).$$

(3) $\mathrm{supp}(t \circ \pi) \subset \pi^{-1}(\mathrm{supp}(t))$.

(4) If $N' \subset N$ is a closed submanifold, then $M' = \pi^{-1}(N')$ is a closed submanifold of M; and for any distribution t on N with $\mathrm{supp}(t) \subset N'$ and any $n \in N'$ with $o_n(t) = k$ and $m \in \pi^{-1}(n)$, one has

$$o_m(t \circ \pi) = k.$$

We only need (4) in the special case when $N' = \{n\}$ and this is seen as follows. Going over to the local picture let $t = \partial_y^\beta \delta$ where δ is the Dirac measure at $y = 0$; then

$$t \circ \pi = \sum_{0 \leqslant \gamma \leqslant \beta} \partial_y^\gamma S_\gamma,$$

with

$$S_\gamma(\theta) = (-1)^{|\beta| - |\gamma|} \binom{\beta}{\gamma} \int \theta(x)(\partial_y^{\beta - \gamma}\eta)(x, 0)\,\mathrm{d}x.$$

Here $\eta(x, y)\,\mathrm{d}x$ is the ratio of volume forms. This shows that S_β is the measure $\eta(x, 0)\,\mathrm{d}x$, and so, as $\eta(0, 0) > 0$, we have $0 \in \operatorname{supp}(S_\beta)$, showing that the transversal order of $t \circ \pi$ at $(0, 0)$ is $|\beta|$.

5.4. We now apply these considerations to the case $M = \mathbf{R}^{1,n} \backslash \{0\}$, $N = \mathbf{R}$, and $\pi = \omega$, the volume forms being the standard ones $\mathrm{d}p_0\,\mathrm{d}p_1 \cdots \mathrm{d}p_n$ and $\mathrm{d}t$. We exclude the origin to make ω submersive. Then for $f \in C_c^\infty(\mathbf{R}^{1,n} \backslash \{0\})$,

$$(\omega_* f)(a) = \int_{|p'|^2 > -a} (f(p_0', p') + f(-p_0', p'))\frac{\mathrm{d}p'}{2p_0'},$$

with

$$p' = (p_\mu)_{1 \leqslant \mu \leqslant n},\ |p'| = (p_1^2 + \cdots + p_n^2)^{1/2}, \quad p_0' = (p_1^2 + \cdots + p_n^2 + a)^{1/2}.$$

If δ_a is the delta function on $\mathbf{R}$ at the point $a \in \mathbf{R}$, then $\omega^*(\delta_a)$ is the positive measure α_a on the hyperboloid $\omega = a$, invariant under the full Lorentz group; and the integral above is the expression for $\int f\,\mathrm{d}\alpha_a$. For $a > 0$ the hyperboloid splits into two connected components (sheets) and we write $\alpha_a^\pm$ for the restrictions of α_a to the two components. If $a = 0$, then $\omega^{-1}(\{0\})$ splits into the two halves of the light cone and we define $\alpha_0^\pm$ analogously. Thus

$$\int_{X_0^+} f\,\mathrm{d}\alpha_0^+ = \int_{\mathbf{R}^n \backslash \{0\}} f(|p'|, p')\frac{\mathrm{d}p'}{2|p'|},$$

for all $f \in C_c^\infty(\mathbf{R}^{1,n} \backslash \{0\})$. It is obvious that the integral above converges even when $f \in \mathscr{S}(\mathbf{R}^{1,n})$, so that the $\alpha_0^\pm$ are tempered measures on $\mathbf{R}^{1,n}$.

We remark that the integral defining $\omega_* f$ exists for all $f \in C_c^\infty(\mathbf{R}^{1,n})$ if $a \neq 0$. The function $\omega_* f = Mf$ thus defined is the so-called mean value map and plays a fundamental role in the works [10], [3], [16].

5.5. SOME FORMULAE. Write

$$u_0 = p_0^2 - p_1^2 - \cdots - p_n^2, \qquad u_\mu = p_\mu \ (1 \leqslant \mu \leqslant n).$$

The (u_μ) form a coordinate system on the open set $\mathbf{R}_+^{1,n}$ of $p \in \mathbf{R}^{1,n}$ with $p_0 > 0$. We have

$$\frac{\partial}{\partial u_0} = \frac{1}{2p_0}\frac{\partial}{\partial p_0}; \quad \frac{\partial}{\partial u_\mu} = \frac{1}{p_0} M_{0\mu} \ (1 \leqslant \mu \leqslant n); \quad M_{\mu\nu} = u_\mu \frac{\partial}{\partial u_\nu} - u_\nu \frac{\partial}{\partial u_\mu},$$

$$\left[M_{\mu\nu}, \frac{\partial}{\partial u_0}\right] = 0, \quad \left[M_{0\mu}, \frac{\partial}{\partial u_0}\right] = -\frac{1}{2p_0^2} M_{0\mu} \ (1 \leqslant \mu, \nu \leqslant n).$$

In particular, on $\mathbf{R}_+^{1,n}$, the vector fields $\partial/\partial u_\mu$ $(1 \leqslant \mu \leqslant n)$ generate the same module (over $C_c^\infty(\mathbf{R}_+^{1,n})$) as all the M_{rs} $(0 \leqslant r, s \leqslant n)$. Further,

$$\Box 1 = 0; \quad \Box \omega^k = 4k(k + \tfrac{1}{2}(n-1))\omega^{k-1} \ (k \geqslant 1);$$

$$\Box \omega p_\mu = 2(n+3)p_\mu \ (1 \leqslant \mu \leqslant n); \quad \mathscr{E} u_0 = 2u_0; \quad \mathscr{E} u_\mu = u_\mu.$$

From these we get the following expressions for $\Box$ and $\mathscr{E}$, where as usual $d = n + 1$,

$$\mathscr{E} = 2u_0 \frac{\partial}{\partial u_0} + \sum_{1 \leqslant \mu \leqslant n} u_\mu \frac{\partial}{\partial u_\mu},$$

$$\Box = 4u_0 \frac{\partial^2}{\partial u_0^2} + 2d \frac{\partial}{\partial u_0} + 4 \sum_{1 \leqslant \mu \leqslant n} u_\mu \frac{\partial^2}{\partial u_0 \partial u_\mu} - \sum_{1 \leqslant \mu \leqslant n} \frac{\partial^2}{\partial u_\mu^2}.$$

It is immediate from these that $\Box$ and $\mathscr{E}$ have associated to them differential operators $\Box_\omega$, $\mathscr{E}_\omega$

$$\Box_\omega = 4t \frac{d^2}{dt^2} + 2d \frac{d}{dt}, \qquad \mathscr{E}_\omega = 2t \frac{d}{dt}.$$

5.6. LEMMA. *The measure α_0^+ satisfies on $\mathbf{R}^{1,n}$ the differential equations*

$$X\alpha_0^+ = 0, \qquad H\alpha_0^+ = \left(\frac{d}{2} - 2\right)\alpha_0^+.$$

Proof. The first assertion is obvious. Since $H = \mathscr{E} + \frac{d}{2}$ and $\alpha_0^+ = \omega^*(\delta_0)$, the second assertion on $\mathbf{R}^{1,n}\backslash\{0\}$ follows from the readily verified formula

$$\left(2t\frac{d}{dt}+\frac{d}{2}\right)\delta_0=\left(\frac{d}{2}-2\right)\delta_0$$

on $\mathbf{R}$. To see that this relation holds on all of $\mathbf{R}^{1,n}$ we note that $S=H\alpha_0^+-\left(\frac{d}{2}-2\right)\alpha_0^+$ has support contained in $\{0\}$ and satisfies $XS=0$. Going over to Fourier transforms we get $\square\hat{S}=0$ where $\hat{S}$ is an invariant polynomial. This means that $\hat{S}$ is invariant and harmonic, and, hence, a constant. It follows that $S=c\delta$ for some constant c. So S will be zero if we can find $f\in\mathscr{S}(\mathbf{R}^{1,n})$ with $f(0)\neq 0$ but $S(f)=0$. If we take $f=e^{-g}$ where $g(p)=\frac{1}{2}(p_0^2+\cdots+p_n^2)$, it is a routine calculation that $S(f)=0$. ■

5.7. LEMMA. *We have the following.*

(1) $J(U)=0$, *unless* $U\simeq\mathscr{H}^j$ *for some* j.

(2) *If* $U\simeq\mathscr{H}^j$, $J_0(U)$ *is one-dimensional and spanned by* $h_U\cdot\alpha_0^+$ *where* h_U *is a nonzero invariant map*: $\mathbf{C}^{1,n}\to U$.

(3) $h_U\cdot\alpha_0^+$ *lies in* $\bar{J}_0(U)$ *and satisfies, on all of* $\mathbf{R}^{1,n}$

$$X(h_U\cdot\alpha_0^+)=0,\quad H(h_U\cdot\alpha_0^+)=\left(\frac{d}{2}-2+j\right)h_U\cdot\alpha_0^+.$$

(4) $J_k(U)/J_{k-1}(U)\neq 0$ *for all* $k\geqslant 0$.

Proof. If $J(U)\neq 0$ there is a $t\neq 0$ in $J(U)$ such that $Xt=0$. So t *lives* on $X_0^+\cong G/G^x$ $(x=(1,0,\ldots,0,1))$ and, hence, its components may be viewed as G-finite distributions on G/G^x. By a well-known result they are C^∞ functions on G/G^x, showing that there is a nonzero invariant function with values in U. This proves assertion (1) and establishes at the same time that $J_0(U)=\mathbf{C}\cdot h_U\cdot\alpha_0^+$. Furthermore, as $\mathscr{E}h_U=jh_U$, if h_U is homogeneous of degree j, we have, $Hh_U\cdot\alpha_0^+=\left(\frac{d}{2}-2+j\right)h_U\cdot\alpha_0^+$ on $\mathbf{R}^{1,n}$. For the last assertion note that the pullback of $\delta_0^{(k)}$ is invariant and has transversal order k. ■

5.8. Let us now consider an $\mathfrak{a}$-module V such that $V=\bigcup_{k\geqslant 0}V_k$ where $V_k=\{v\in V\mid X^{k+1}v=0\}$. Each V_k is H-stable as $X^{k+1}H=(H-2k-2)X^{k+1}$.

LEMMA. *Suppose that* $\dim(V_0)=1$ *and let* $\lambda\in\mathbf{C}$ *be such that* $H=\lambda\cdot 1$ *on* V_0. *If* $\lambda\notin\{0,1,\ldots\}$, *then* $V\simeq V(\lambda)$. *If* $\lambda=j$ *is an integer* $\geqslant 0$, *then* $V\simeq F(j)$ *or* $V\simeq M(j)$.

Proof. Let $0\neq v_\lambda\in V_0$ so that $Xv_\lambda=0$, $Hv_\lambda=\lambda v_\lambda$. As the mapping $X\colon V_k/V_{k-1}\to V_{k-1}/V_{k-2}$ is injective, we get $\dim(V_k/V_{k-1})\leqslant\dim V_0=1$, for all k. Hence $\dim V_k\leqslant(k+1)$, for all k. Further $X^k(H-\lambda+2k)=(H-\lambda)X^k$ and so, as

$X^k V_k \subset V_0$, we must have $(H-\lambda+2k)V_k \subset V_{k-1}$. In particular

$$\prod_{0 \leqslant j \leqslant k} (H-(\lambda-2j))(V_k) = 0.$$

So H acts semisimply on V with a simple spectrum $\subset \{\lambda, \lambda-2, \ldots\}$. Let W be the cyclic module generated by v_λ. If λ is not an integer $\geqslant 0$, then $W = V(\lambda)$ and it is clear from the spectra that $W = V$. Suppose that $\lambda = j$ is an integer $\geqslant 0$. Since the kernel of X on $V(j)$ is 2-dimensional, W is not equivalent to $V(j)$ and so $W \simeq F(j)$. If $V_{j+1}/V_j = 0$, then $V_{k+1}/V_k = 0$ for all $k \geqslant j$ and so $V = F(j)$. If $V_{j+1}/V_j \neq 0$ and $0 \neq u \in V_{j+1}/V_j \subset V/V_j$, we see that $Hu = (-j-2)u$, $Xu = 0$, so that u generates $V(-j-2)$. In other words, V has a submodule V' satisfying an exact sequence

$$0 \to F(j) \to V' \to V(-j-2) \to 0;$$

this does not split because $\dim \ker_{V'}(X) = 1$. As $\{j, j-2, \ldots\}$ is the spectrum of H on V', we must have $V = V'$. The results of Section 3 now imply that $V \simeq M(j)$. ∎

5.9. PROPOSITION. *Suppose $U \simeq \mathscr{H}^j$. Then*

$$J(U) \simeq \begin{cases} V\left(\dfrac{d}{2}-2+j\right), & \text{for } d \text{ odd;} \\ M\left(\dfrac{d}{2}-2+j\right), & \text{for } d \text{ even.} \end{cases}$$

In all cases, multiplication by h_U gives an isomorphism of $J = J(\mathbf{C})$ with $J(U)$

$$h_U \colon J \simeq J(U).$$

Proof. Follows immediately from Lemmas 5.7 and 5.8, except for the last statement. Consider the map $m\colon t \mapsto h_U t$ of J into $J(U)$. This is injective because h_U is nowhere zero on X_0^+. On the other hand, it takes H on J to $H+j$ on $J(U)$. So the range of m contains all the $J(U)\left[\dfrac{d}{2}-2+j-2k\right]$, $(k \geqslant 0)$. So m is surjective. ∎

6. Structure of $\bar{J}(U)$

We have the restriction map $\bar{J}(U) \to J(U)$ that maps $\bar{J}_k(U)$ into $J_k(U)$ for all

$k \geqslant 0$. Let $K(U)$, $K_k(U)$ be the respective kernels so that we have the diagrams

$$0 \to K(U) \to \bar{J}(U) \to J(U) \to 0,$$

$$0 \to K_k(U) \to \bar{J}_k(U) \to J_k(U) \to 0.$$

Let δ be the delta function at the origin of $\mathbf{R}^{1,n}$. We shall first prove the following lemma.

6.1. LEMMA. *$K(U)=0$ and, hence, $\bar{J}(U)=0$ unless $U \simeq \mathscr{H}^j$ for some j. Suppose that $U \simeq \mathscr{H}^j$ and that $(h_a)_{1 \leqslant a \leqslant N}$ are the components of the map h_U with respect to some basis of U. Then, if t_r is the U-valued distribution whose components in that basis are $\Box^r \partial(h_a)\delta$, the t_r, for $r = 0, 1, \ldots,$ form a basis of $K(U)$. In particular*

$$K(U) \simeq V\left(-\frac{d}{2} - j\right) \quad (U \simeq \mathscr{H}^j).$$

Proof. We show first that

$$X\delta = 0, \qquad H\delta = -\frac{d}{2}\delta;$$

$$K(\mathbf{C}) = \{P(\Box)\delta \mid P \text{ an arbitrary polynomial}\} \simeq V\left(-\frac{d}{2}\right).$$

Indeed, since $p_k\delta = 0$ and $\mathscr{E}^\dagger = -\mathscr{E} - d$, the equations involving δ are clear. On the other hand, elements of $K(\mathbf{C})$ are supported by $\{0\}$ and so are of the form $P(\Box)\delta$, with P being a polynomial; in fact $P \mapsto P(\Box)\delta$ is a linear isomorphism. In other words $K(\mathbf{C})$ is a cyclic module with δ as a generator and Y acts injectively on it. So

$$K(\mathbf{C}) \simeq V\left(-\frac{d}{2}\right).$$

Let now U be an irreducible G-module with basis (e_a). The elements of $K(U)$ are of the form $(D_a\delta)$ where the D_a are constant-coefficient differential operators; the invariance implies that the linear span of the D_a is either 0 or carries a representation contragredient to U. This already shows that $K(U) = 0$ unless U is $\simeq$ to some $\mathscr{H}^j$. Suppose that $U \simeq \mathscr{H}^j$. Proposition 4.6 now shows that the t_r form a basis of $K(U)$. For the identification of $K(U)$ with the appropriate Verma module it is enough to verify that

$$Xt_0 = 0, \qquad \mathscr{E}t_0 = (-d - j)t_0.$$

It is actually enough to check the second relation; for then $Ht_r = \left(-\frac{d}{2} - j - 2r\right)t_r$ and so t_0 has the highest weight in $K(U)$. If $p \in \mathscr{H}^j$, $\mathscr{E}\partial(p)\delta = -d\partial(p)\delta + [\mathscr{E}, \partial(p)]\delta$, and so we need only check that $[\mathscr{E}, \partial(p)] = -j\partial(p)$. We shall show that this is true for *any* $p \in \mathscr{P}$ that is homogeneous of degree j. This is trivial for $j = 0$ and let us suppose that it is true in degrees $<j$. We may assume that $\partial = \partial^\beta$ where $\beta = (\beta_0, \ldots, \beta_n)$ and $|\beta| = j$. Then $[\partial, p_k\partial_k] = \beta_k\partial^\beta$ for all k; so that $[\mathscr{E}, \partial(p)] = -j\partial(p)$. ∎

The obvious question now is whether the map $\bar{J}(U) \to J(U)$ is surjective. We shall do this by first replacing U by $\mathbf{C}$ and by showing that the elements of $J(\mathbf{C})$ extend to tempered distributions in all of $\mathbf{R}^{1,n}$. We then use a cohomology argument to show the existence of invariant tempered extensions. We begin with a preparatory result.

6.2. LEMMA. *Let s be an integer $\geqslant 0$ and let $\mathscr{S}^{(s)}$ be the subspace of $\mathscr{S}(\mathbf{R}^{1,n})$ of all elements f such that $\partial^\beta f(0) = 0$, for $|\beta| < s$. If r is any integer $<s$ and γ is a multiindex with $|\gamma| = r$, then there are linear continuous maps $L^\alpha: \mathscr{S}^{(s)} \to \mathscr{S}$ such that*

$$\partial^\gamma f = \sum_{|\alpha| = s-r} p^\alpha L^\alpha f, \quad \textit{with } p^\alpha = p_0^{\alpha_0} \cdots p_n^{\alpha_n}.$$

Proof. Analogous to the arguments in the proof of [7, Lemma 7.1.4]. ∎

6.3. LEMMA. *Let r, s be integers $\geqslant 0$. Then there is a tempered distribution t on $\mathbf{R}^{1,n}$ such that*

(1) $\operatorname{supp}(t) \subset \operatorname{Cl}(X_0^+)$,
(2) *for all $f \in \mathscr{S}^{(s+r+1)}$, in particular in $C_c^\infty(\mathbf{R}_+^{1,n})$,*

$$t(f) = \int_{X_0^+} \frac{1}{p_0^r} \frac{\partial^s f}{\partial p_0^s} \, d\alpha_0^+.$$

Proof. By Lemma 6.2 we can find $f^\beta \in \mathscr{S}$ $(|\beta| = r + 1)$ depending linearly and continuously on $f \in \mathscr{S}^{(s+r+1)}$, such that

$$\frac{\partial^s f}{\partial p_0} = \sum_{|\beta| = r+1} p^\beta f^\beta.$$

Then, for $f \in \mathscr{S}^{(s+r+1)}$

$$\int_{X_0^+} \frac{1}{p_0^r} \left|\frac{\partial^s f}{\partial p_0^s}\right| d\alpha_0^+ \leqslant \sum_{|\beta| = r+1} \int_{\mathbf{R}^n \setminus \{0\}} |f^\beta(|q|, q)| \, d^n q,$$

and it is clear that the right side is a continuous seminorm on $\mathscr{S}$. So

$$t: f \mapsto \int_{X_0^+} \frac{1}{p_0^r} \frac{\partial^s f}{\partial p_0^s} \, d\alpha_0^+$$

is a continuous linear functional on $\mathscr{S}^{(s+r+1)}$ and so extends to a tempered distribution on $\mathbf{R}^{1,n}$. If $f \in C_c^\infty(\mathbf{R}^{1,n})$ and $\operatorname{supp}(f) \cap \operatorname{Cl}(X_0^+) = \varnothing$, then f vanishes in a neighborhood of 0, so that $f \in \mathscr{S}^{(s+r+1)}$; and $t(f)$ is 0 by its definition. ■

6.4. PROPOSITION. *Any element of $\bar{J}(U)$ is tempered, and the maps*

$$\bar{J}(U) \to J(U), \quad \bar{J}_k(U) \to J_k(U), \quad k \geqslant 0$$

are surjective.

Proof. The basic assertion to prove is that any element of $J(U)$ is the restriction of a tempered element of $\bar{J}(U)$. Indeed, if this is done, all elements of $\bar{J}(U)$ are tempered since all elements of $K(U)$ are tempered. The surjectivity of the second map also follows. In fact, let $t \in J_k(U)$ and $\bar{t} \in \bar{J}(U)$ be such that $\bar{t}$ restricts to t. Then $X^{k+1}\bar{t}$ lies in $K(U)$. But as $K(U)$ is a Verma module whose highest weight is 0, it is clear that X acts surjectively on it and so we can find $t' \in K(U)$ such that $X^{k+1}t' = X^{k+1}\bar{t}$. So $\bar{t} - t'$ lies in $\bar{J}_k(U)$ and restricts to t. Furthermore it is enough to prove the basic assertion in the case when $U = \mathbf{C}$, in view of the isomorphism $s \mapsto h_U s$ of $J(\mathbf{C})$ with $J(U)$. Finally, since $J(\mathbf{C}) \simeq M\left(\frac{d}{2} - 2\right)$, it is enough to prove that the elements of weights $\frac{d}{2} - 2$ and $-\frac{d}{2}$ are extendable, the latter being necessary only when d is even. The former is α_0^+ which is already defined as an element of $\bar{J}(\mathbf{C})$. Let t be the latter, d being even; t has transversal order $\frac{d-2}{2}$.

We begin by remarking that it is enough to prove that t has a tempered extension $\bar{t}$ to all of $\mathbf{R}^{1,n}$, with $\operatorname{supp}(\bar{t}) \subset \operatorname{Cl}(X_0^+)$, but not necessarily invariant. For, suppose that we have constructed such a $\bar{t}$. Let us write, for any $Z \in \mathfrak{g}$, $\tau(Z)$ for the vector field on $\mathbf{R}^{1,n}$ defined by Z. Since $\tau(Z)t = 0$, for all $Z \in \mathfrak{g}$, the distribution $\tau(Z)\bar{t}$ has support $\subset \{0\}$. As $\tau(Z)\bar{t}$ depends linearly on Z we can find an integer $r \geqslant 0$ such that $\tau(Z)\bar{t} \in L^{(r)}$ for all $Z \in \mathfrak{g}$, with $L^{(r)}$ being the span of the derivatives of δ of order $\leqslant r$. Now $L^{(r)}$ is stable under G, hence under $\tau(Z)$, for $Z \in \mathfrak{g}$; and it is immediate that $Z \mapsto \tau(Z)\bar{t}$ is an 1-cocycle for this $\mathfrak{g}$-module. Since $\mathfrak{g}$ is semisimple, $H^1(\mathfrak{g}, L^{(r)}) = 0$; and so we can find $s \in L^{(r)}$ such that $\tau(Z)\bar{t} = \tau(Z)s$, for all $Z \in \mathfrak{g}$. This means that $\bar{t}_1 = \bar{t} - s$ also extends t, has support $\subset \operatorname{Cl}(X_0^+)$, and is infinitesimally invariant. It is thus a tempered element of $\bar{J}(U)$ that restricts to t.

In order to prove the existence of a suitable $\bar{t}$ it is sufficient to show that for any integer $r \geqslant 0$, the restriction to $\mathbf{R}^{1,n}_{+}$ of the pullback $t_r = \omega^*(\delta_0^{(r)})$ extends to a tempered distribution on $\mathbf{R}^{1,n}$ with support $\subset \mathrm{Cl}(X_0^+)$. If $f \in C_c^\infty(\mathbf{R}^{1,n}_{+})$ and $a > 0$,

$$(\omega_* f)(a) = \int_{\mathbf{R}^n} f(\sqrt{|p|^2 + a}, p) \frac{d^n p}{2\sqrt{|p|^2 + a}}.$$

So

$$t_r(f) = \lim_{a \to 0+} \left((-1)^r \frac{d^r}{da^r} \omega_* f \right)(a).$$

It is easy to see by induction that $(d^r/da^r)(\omega_* f)$ is a linear combination of functions of a of the form

$$\int_{\mathbf{R}^n} \frac{\partial^s}{\partial p_0^s} f(\sqrt{|p|^2 + a}, p) \frac{d^n p}{(|p|^2 + a)^{(k+1)/2}},$$

where k is an integer $\geqslant 0$, and, hence, $t_r(f)$ is a linear combination of linear functionals of the form considered in Lemma 6.3. ∎

6.5. LEMMA. *Suppose d is odd. Then*

$$\bar{J}(U) \simeq K(U) \oplus J(U) \simeq V\left(-\frac{d}{2} - j\right) \oplus V\left(\frac{d}{2} - 2 + j\right).$$

In particular $\square$ acts injectively on $\bar{J}(U)$.

Proof. By (3) of Lemma 5.7 and Proposition 5.9, if W is the cyclic module generated by $\alpha_U^+ = h_U \alpha_0^+$ and R is the restriction map $\bar{J}(U) \to J(U)$, then $R(W) = J(U) \simeq V\left(\frac{d}{2} - 2 + j\right)$, so that $W \simeq V\left(\frac{d}{2} - 2 + j\right)$ and R is an isomorphism on W. ∎

From now on we shall assume that d is even. At this stage we know that $\bar{J}(U)$ is 0 unless $U \simeq \mathscr{H}^j$ for some j, and that when this is so, it admits an exact sequence

$$0 \to V\left(-\frac{d}{2} - j\right) \to \bar{J}(U) \to M\left(\frac{d}{2} - 2 + j\right) \to 0.$$

In Lemma 6.8 below we shall verify that this exact sequence satisfies the criterium of Proposition 3.1 for being nonsplitting. Then it follows from

Theorem 2.1(a) that there exists $\gamma \in \mathbf{P}^1(\mathbf{C})$ such that $\bar{J}(U) \simeq W\left(\frac{d}{2} - 2 + j : \gamma\right)$.
We shall now begin the proof that

$$\bar{J}(U) \simeq W\left(\frac{d}{2} - 2 + j : \frac{1}{2}\left(\frac{d-2}{2} + j\right)\right).$$

This is a delicate calculation, which depends on explicit construction of certain distributions invariant under the *full* Lorentz group. We begin with

6.6. LEMMA. *All elements of $\bar{J}(\mathbf{C})$ are invariant under the space inversion S. For any distribution s in $\bar{J}(\mathbf{C})$ let s° be its transform under the space-time inversion ST. Then the map $s \mapsto s + s^\circ$ is an $\mathfrak{a}$-module isomorphism of $\bar{J}(\mathbf{C})$ with the space $\mathscr{L}$ of $\mathbf{SO}(1, n)$-invariant distributions supported on the full light cone.*

Proof. We start with the exact sequence

$$0 \to K \to \bar{J}(\mathbf{C}) \to J(\mathbf{C}) \to 0, \quad \text{where } K = \{P(\Box)\delta\}.$$

It is clear that the elements of K are invariant under all of $\mathbf{SO}(1, n)$. On the other hand $J(\mathbf{C})$ is spanned by the $u_p = \Box^p \alpha_0^+ \left(0 \leqslant p \leqslant \frac{d}{2} - 2\right)$, and the $v_r = \Box^r \tau$ $(r \geqslant 0)$, where τ is a nonzero element of weight $-\frac{d}{2}$. The u_p are obviously S-invariant. Since S commutes with H, the v_r are either invariant or change sign when transformed by S; but, as X commutes with S and Xv_0 is a nonzero multiple of $u_{d/2-2}$, it must be that v_0 is also S-invariant, so that all the v_r are S-invariant. So all elements of $J(\mathbf{C})$ are S-invariant. But then this must be true of $\bar{J}(\mathbf{C})$ also. We consider next the map $s \mapsto s + s^\circ$, $s \in \bar{J}(\mathbf{C})$. It is clear that it maps $\bar{J}(\mathbf{C})$ into $\mathscr{L}$ and that it commutes with the action of $\mathfrak{a}$. Suppose that for some s, we have $s + s^\circ = 0$. It is then obvious that s has to be zero on $\mathbf{R}_+^{1,n}$, and so $s \in K$, showing that $2s = s + s^\circ = 0$. To prove that it is surjective it is thus enough to verify that the multiplicities of the eigenvalues of H in $\mathscr{L}$ are not greater than the corresponding multiplicities in $\bar{J}(\mathbf{C})$. Let $\mathscr{L}^\times$ be the space of $\mathbf{SO}(1, n)$-invariant distributions defined on invariant neighborhoods of X_0 in $\mathbf{R}^{1,n} \backslash \{0\}$ and supported on X_0. We then have the sequence

$$0 \to K \to \mathscr{L} \to \mathscr{L}^\times \to 0.$$

It is obvious that this is exact at the second place and that the map $s \mapsto s + s^\circ$ is an isomorphism of $J(\mathbf{C})$ with $\mathscr{L}^\times$. This proves the required bounds on the H-multiplicities. ■

COROLLARY. *The above sequence is exact.*

DETERMINATION OF $\gamma(\bar{J}(\mathbf{C}))$. We suppose that $d \geqslant 4$ is even. The starting point is the observation that by Fourier analysis the delta functions on $\mathbf{R}$ and their derivatives can be expressed as superpositions of the exponential functions $e^{i\lambda t}$, and the pullbacks of the latter are just the functions $e^{i\lambda\omega}$. The resulting integrals define formally the pullbacks *on all of* $\mathbf{R}^{1,n}$; however, the integrals do not in general converge, and so have to be regularized using the transformation $\lambda \mapsto \lambda^{-1}$ that is familiar from stationary phase analysis. We now proceed to give the details.

Our starting point is the well-known transformation formula, see [7, (7.6.2)′], valid for all $\lambda \in \mathbf{R}\backslash\{0\}$, and all $f \in \mathscr{S}$:

$$\int e^{i\lambda\omega} f \, dp = \theta(\lambda)|\lambda|^{-d/2} \int e^{-i\omega/(4\lambda)} \hat{f} \, dp,$$

where

$$\theta(\lambda) = 2^{-d}\pi^{-d/2} \exp(-\pi i \operatorname{sgn}(\lambda)(d-2)/4),$$

$$\hat{f}(p) = \int e^{ip\cdot q} f(q) \, dq \quad (p \cdot q = p_k q^k).$$

We shall also choose and fix, once and for all, $u \in C_c^\infty(\mathbf{R})$ such that u is equal to 1 in a neighborhood of $\lambda = 0$; then $v = 1 - u$ is 0 in a neighborhood of $\lambda = 0$, equal to 1 for $|\lambda| \gg 0$. For any distribution T on $\mathbf{R}^{1,n}$ we write $T^\times$ for its restriction to $\mathbf{R}^{1,n}\backslash\{0\}$.

We then define the tempered distribution τ on $\mathbf{R}^{1,n}$ by

$$2\pi\tau(f) = \int \lambda^{d-2/2} u \, d\lambda \int e^{i\lambda\omega} f \, dp + \\ + \int \lambda^{d-2/2} v\theta|\lambda|^{-d/2} \, d\lambda \int (e^{-i\omega/(4\lambda)} - 1)\hat{f} \, dp.$$

To see that there is absolute convergence we use the estimate

$$|e^{-i\omega/(4\lambda)} - 1| \leqslant \frac{|\omega|}{4|\lambda|}.$$

If f lies in $C_c^\infty(\mathbf{R}^{1,n}\backslash\{0\})$, we know that $\int e^{i\lambda\omega} f \, dp$ is rapidly decreasing as $|\lambda| \to \infty$, and so, we have in this case

$$2\pi\tau(f) = \int \lambda^{d-2/2}\, \mathrm{d}\lambda \int \mathrm{e}^{i\lambda\omega} f\, \mathrm{d}p.$$

6.7. LEMMA. *The distribution τ restricts on $\mathbf{R}^{1,n}\backslash\{0\}$ to a nonzero multiple of $\omega^*(\delta_0^{(d-2/2)})$, the pullback under ω of $\delta_0^{(d-2/2)}$. In fact,*

$$\tau^{\times} = (-i)^{d-2/2}\omega^*(\delta_0^{(d-2/2)}).$$

Proof. If $0 \notin \mathrm{supp}(f)$ we have

$$\tau(f) = \frac{1}{2\pi}\int \lambda^{d-2/2}\, \mathrm{d}\lambda \int \mathrm{e}^{i\lambda\omega} f\, \mathrm{d}p = \frac{1}{2\pi}\int \lambda^{d-2/2}\widehat{\pi_* f}(\lambda)\, \mathrm{d}\lambda$$
$$= i^{d-2/2}(\pi_* f)^{(d-2/2)}(0).$$

■

6.8. LEMMA. *We have*

$$\left(H + \frac{d}{2}\right)\tau = 2(-\pi i)^{d-2/2}\delta.$$

Proof. We note that $\mathscr{E}^\dagger = -\mathscr{E} - d$ and $\widehat{\mathscr{E}^\dagger f} = \mathscr{E}\hat{f}$. Then

$$2\pi\tau(\mathscr{E}^\dagger f) = \int \lambda^{d-2/2} u\, \mathrm{d}\lambda \int \mathrm{e}^{i\lambda\omega}\mathscr{E}^\dagger f\, \mathrm{d}p$$
$$= \int \lambda^{d-2/2} u\, \mathrm{d}\lambda \int 2i\lambda\omega\, \mathrm{e}^{i\lambda\omega} f\, \mathrm{d}p + \int \lambda^{d-2/2}\theta|\lambda|^{-d/2} v\, \mathrm{d}\lambda$$
$$\cdot\left(\int \frac{2i\omega}{4\lambda}\mathrm{e}^{-i\omega/(4\lambda)}\hat{f}\, \mathrm{d}p - d\int (\mathrm{e}^{-i\omega/(4\lambda)} - 1)\hat{f}\, \mathrm{d}p\right).$$

Let us now write

$$A(\lambda) = \int \mathrm{e}^{i\lambda\omega} f\, \mathrm{d}p, \qquad B(\lambda) = \int \mathrm{e}^{-i\omega/(4\lambda)}\hat{f}\, \mathrm{d}p.$$

Then

$$2\pi(\tau(\mathscr{E}^\dagger f) + d\tau(f)) = \int 2\lambda^{d/2} A' u\, \mathrm{d}\lambda +$$
$$+ d\int \lambda^{d-2/2} A u\, \mathrm{d}\lambda + \int 2\lambda^{d/2}\theta|\lambda|^{-d/2} B' v\, \mathrm{d}\lambda =: I_A + I_B.$$

Integrating by parts, we get

$$I_A = -2 \int \lambda^{d/2} A u' \, \mathrm{d}\lambda.$$

Moreover, observing that θ and $\lambda^{d/2}|\lambda|^{-d/2} = (\operatorname{sgn} \lambda)^{d/2}$ are locally constant and denoting temporarily by Z the boundary term, we have

$$I_B = -2 \int \lambda^{d/2} \theta |\lambda|^{-d/2} B v' \, \mathrm{d}\lambda + Z = -2 \int \lambda^{d/2} A v' \, \mathrm{d}\lambda + Z.$$

Now $u' + v' = 0$ so that $I_A + I_B = Z$. On the other hand, $B(\pm\infty) = \int \hat{f} \, \mathrm{d}p = (2\pi)^d f(0)$ and, writing $\sigma = 2^{-d}\pi^{-d/2}$, we have

$$\theta \operatorname{sgn}(\pm\infty) = \pm(-1)^l \sigma \quad \text{or} \quad \mp(-1)^l i\sigma,$$

according as $d = 4l + 2$ or $d = 4l + 4$. Hence,

$$Z = [2(\operatorname{sgn} \lambda)^{d/2} \theta v B]_{-\infty}^{\infty} = 4\pi^{d/2}(-i)^{d-2/2}\delta(f). \qquad \blacksquare$$

6.9. LEMMA. *We have*

$$\Box \omega \tau = -(d-2)\left(H + \frac{d}{2}\right)\tau.$$

In particular,

$$\gamma(\bar{J}(\mathbf{C})) = \tfrac{1}{4}(d-2).$$

Proof. First we observe that $\widehat{\omega \Box f} = \Box \omega \hat{f}$. Moreover,

$$\Box g(\omega) = (\Box_\omega g)(\omega),$$

where

$$\Box_\omega = 4t \frac{d^2}{\mathrm{d}t^2} + 2d \frac{d}{\mathrm{d}t},$$

so that

$$\Box(\omega \, \mathrm{e}^{i\lambda\omega}) = (-4\lambda^2\omega^2 + 2i(d+4)\lambda\omega + 2d) \, \mathrm{e}^{i\lambda\omega},$$

$$\Box(\mathrm{e}^{-i\omega/(4\lambda)}) = \left(-\frac{\omega}{4\lambda^2} - \frac{id}{2\lambda}\right) \mathrm{e}^{-i\omega/(4\lambda)}.$$

From these formulae we get

$$
\begin{aligned}
&2\pi\tau(\omega\Box f)\\
&\quad= \int \lambda^{d-2/2} u \, \mathrm{d}\lambda \int (-4\lambda^2\omega^2 + 2(d+4)i\lambda\omega + 2d)\, \mathrm{e}^{i\lambda\omega} f \, \mathrm{d}p \, +\\
&\quad\quad + \int \lambda^{d-2/2} \theta v |\lambda|^{-d/2} \, \mathrm{d}\lambda \int \left(-\frac{\omega^2}{4\lambda^2} - \frac{id\omega}{2\lambda} \right) \mathrm{e}^{-i\omega/(4\lambda)} \hat{f} \, \mathrm{d}p.
\end{aligned}
$$

Let A, B be as before. Then we get

$$2\pi\tau(\omega\Box f) = I_A + I_B,$$

where

$$I_A = \int \lambda^{d-2/2} u (4\lambda^2 A'' + 2(d+4)\lambda A' + 2dA) \, \mathrm{d}\lambda,$$

$$I_B = \int \lambda^{d-2/2} \theta v |\lambda|^{-d/2} (4\lambda^2 B'' - 2(d-4)\lambda B') \, \mathrm{d}\lambda.$$

If we now integrate by parts and use that $A(\lambda) = \theta(\lambda)|\lambda|^{-d/2} B(\lambda)$, we get

$$I_A + I_B = Z,$$

where Z is the boundary term, which turns out to be

$$Z = [-2(d-2)(\operatorname{sgn}\lambda)^{d/2} \theta v B]_{-\infty}^{\infty}.$$

But our earlier calculation led to the formula

$$2\pi\left(\left(H + \frac{d}{2}\right)\tau\right)(f) = [2(\operatorname{sgn}\lambda)^{d/2} \theta v B]_{-\infty}^{\infty}.$$

Hence, we obtain

$$\tau(\omega\Box f) = -(d-2)\left(\left(H + \frac{d}{2}\right)\tau\right)(f). \qquad \blacksquare$$

6.10. DETERMINATION OF $\gamma(\bar{J}(U))$. We shall now sketch the calculation for showing that

$$\gamma(\bar{J}(U)) = \frac{1}{2}\left(\frac{d-2}{2} + j\right), \quad \text{if } U \simeq \mathscr{H}^j.$$

For brevity we put

$$c=\frac{d-2}{2}, \qquad \gamma_0=\tfrac{1}{2}c,$$

so that $W_0=\bar{J}(\mathbf{C})\simeq W(0:\gamma_0)$. We then choose $t_0\in W_0\left[-\frac{d}{2}\right]$ such that $X^c t_0=\alpha_0^+$; then $Y^c\alpha_0^+=\eta\delta$, for some constant $\eta\neq 0$. On the other hand, $YXt_0=\gamma_0\left(H+\frac{d}{2}\right)t_0$, so that by Lemma 3.2 we have $Y^cX^ct_0=\gamma\left(H+\frac{d}{2}\right)t_0$, where $\gamma=(c-1)!^2\gamma_0$. Thus $\left(H+\frac{d}{2}\right)t_0=\gamma^{-1}\eta\delta$. Select a basis (e_a) for U and let (h_a) be the components of the map h_U. We put $t=(\partial(h_a)t_0)$ and view t as an element of $W=\bar{J}(U)$. It is clear that $t\in W\left[-\frac{d}{2}-j\right]$.

LEMMA. *Let $\mathcal{D}$ be the algebra of differential operators on $\mathbf{C}^{1,n}$ with polynomial coefficients, and for any $u\in\mathcal{D}$, let D_u be the derivation $D\mapsto[u,D]$ of $\mathcal{D}$. Then, for any homogeneous polynomial p of degree r (identifying p with the operator of multiplication by p)*

$$D_X^r(\partial(p))=(-1)^r r!\,p, \qquad D_Y^r(p)=(-1)^r r!\,\partial(p).$$

In particular,

$$D_X^{r+1}(\partial(p))=0, \qquad D_Y^{r+1}(p)=0.$$

Proof. This is a well-known lemma of Harish-Chandra [5, p. 99]. If (x_j) is the standard basis for $\mathbf{C}^{1,n}$, then we have $[X,\partial_j]=-\varepsilon_j x_j$, and $[Y,x_j]=-\varepsilon_j\partial_j$; this proves the lemma for $r=1$. The general case is done by induction on the degree. ∎

6.11. LEMMA. *We have*

$$X^{c+j}\partial(h_a)t_0=\binom{c+j}{c}(-1)^j j!\,h_a\alpha_0^+.$$

In particular $\partial(h_a)t_0$ maps to a nonzero element of $J(U)\left[-\frac{d}{2}-j\right]$.

Proof. For any $u\in\mathcal{D}$ let L_u (resp. R_u) be the operator of left (resp. right) multiplication in $\mathcal{D}$ by u; then L_u, R_u, and D_u commute with each other and $D_u=L_u-R_u$. So, by Lemma 6.10,

$$X^{c+j}\partial(h_a)=(D_X+R_X)^{c+j}(\partial(h_a))=\sum_{0\leq s\leq j}\binom{c+j}{s}D_X^s(\partial(h_a))X^{c+j-s}.$$

On the other hand, $X^{c+1}t_0 = 0$, so that

$$X^{c+j}\partial(h_a)t_0 = \binom{c+j}{j} D_X^j(\partial(h_a))X^c t_0 = \binom{c+j}{c}(-1)^j j!\, h_a \alpha_0^+. \qquad \blacksquare$$

6.12. LEMMA. *We have*

$$Y^{c+j}h_a = D_a Y^c,$$

where $D_a \in \mathscr{D}$; moreover, for some $\Omega_a \in \mathscr{D}$,

$$D_a = \binom{c-1+j}{c-1}(-1)^j j!\,\partial(h_a) + Y\Omega_a.$$

Proof. As before we write $L_Y = D_Y + R_Y$ so that, by Lemma 6.10,

$$Y^{c+j}h_a = \sum_{c \leqslant q \leqslant c+j} \binom{c+j}{q} D_Y^{c+j-q}(h_a)Y^q = D_a Y^c,$$

where

$$\begin{aligned}
D_a &= \sum_{0 \leqslant r \leqslant j} \binom{c+j}{c+r} D_Y^{j-r}(h_a)R_Y^r = \sum_{0 \leqslant r \leqslant j} \binom{c+j}{c+r} D_Y^{j-r}(h_a)(L_Y - D_Y)^r \\
&= \sum_{0 \leqslant r \leqslant j}\ \sum_{0 \leqslant p \leqslant r} (-1)^{r-p}\binom{c+j}{c+r}\binom{r}{p} D_Y^{j-p}(h_a)L_Y^p \\
&= \sum_{0 \leqslant p \leqslant j} D_Y^{j-p}(h_a) \sum_{p \leqslant r \leqslant j} (-1)^{r-p}\binom{c+j}{c+r}\binom{r}{p} L_Y^p \\
&= \left(\sum_{0 \leqslant r \leqslant j} (-1)^r \binom{c+j}{c+r}\right) D_Y^j(h_a) + Y\Omega_a = \binom{c-1+j}{c-1} D_Y^j(h_a) + Y\Omega_a. \qquad \blacksquare
\end{aligned}$$

6.13. COMPLETION OF THE CALCULATION OF $\gamma(\bar{J}(U))$. By Lemmas 6.11 and 6.12, we have, for some constant η',

$$\begin{aligned}
Y^{c+j}X^{c+j}\partial(h_a)t_0 &= (-1)^j j! \binom{c+j}{c} D_a Y^c \alpha_0^+ = \eta(-1)^j j! \binom{c+j}{c} D_a \delta \\
&= \eta \binom{c+j}{c}\binom{c-1+j}{c-1} j!^2\, \partial(h_a)\delta + \eta' Y\Omega_a \delta.
\end{aligned}$$

For any $u \in \mathscr{D}$ let u_0 be its local expression at the origin, namely the differential operator with constant coefficients obtained by freezing the coefficients of u at 0.

Then $u\delta = ((u^\dagger)_0)^\dagger\delta$; and so

$$Y\Omega_a\delta = ((\Omega_a^\dagger Y)_0)^\dagger\delta = ((\Omega_a^\dagger)_0 Y)^\dagger\delta = Y((\Omega_a^\dagger)_0)^\dagger\delta.$$

Writing $\Omega_a' = ((\Omega_a^\dagger)_0)^\dagger$ we thus have for some constant b,

$$Y^{c+j}X^{c+j}\partial(h_a)t_0 = b\partial(h_a)\delta + \eta' Y\Omega_a'\delta.$$

On the other hand, we know that the left side is of the form $b'\partial(h_a)\delta$ where b' is a constant. Hence,

$$(b' - b)\partial(h_a) = \eta' Y\Omega_a'.$$

But the restriction map of $\mathscr{H}$ on the light cone is injective; and this, when interpreted in the symmetric algebra, implies that $\partial(\mathscr{H})$ is linearly independent of the ideal generated by Y. Hence, the above relation must imply that $\eta'\Omega_a' = 0$. In other words,

$$Y^{c+j}X^{c+j}\partial(h_a)t_0 = \eta\binom{c+j}{c}\binom{c-1+j}{c-1}j!^2\partial(h_a)\delta.$$

If we write $\gamma' = \gamma(\bar{J}(U))$, we know (by Lemma 3.2 again)

$$\begin{aligned} Y^{c+j}X^{c+j}\partial(h_a)t_0 &= \gamma'(c-1+j)!^2\left(H + \frac{d}{2} + j\right)\partial(h_a)t_0 \\ &= \gamma'(c-1+j)!^2\gamma^{-1}\eta\partial(h_a)\delta, \end{aligned}$$

since $\left(H + \frac{d}{2}\right)t_0 = \gamma^{-1}\eta\delta$. Hence,

$$\gamma' = \binom{c+j}{c}\binom{c-1+j}{c-1}j!^2((c-1+j)!)^{-2}\gamma.$$

If we remember that $\gamma = (c-1)!^2\gamma_0 = (c-1)!^2\tfrac{1}{2}c$, we get

$$\gamma' = \frac{c+j}{c}\frac{1}{2}c = \frac{1}{2}(c+j).$$

This completes the proof of Theorem 2.1(b). ■

6.14. REMARK. The reader will see that the surjectivity of the map $\bar{J}(\mathbf{C}) \to J(\mathbf{C})$ and the determination of $\gamma(\bar{J}(U))$ are the key steps in the arguments above. One can use the Riesz distributions R_s to give an alternative treatment of these two points, cf. [8].

7. Remarks and additional results

We retain the assumption of d being even.

7.1. SOME FUNDAMENTAL SOLUTIONS. We shall now verify that

$$\Box^{d-2/2}\alpha_0^+ = \pi^{d-2/2}2^{d-3}\Gamma\left(\frac{d-2}{2}\right)\delta. \tag{F}$$

In particular, on Minkowski space-time, we have,

$$\Box\alpha_0^+ = 2\pi\delta.$$

As before, let $c = \dfrac{d-2}{2}$. It is obvious that assertion (F) is equivalent to

$$\Box^c\alpha_0 = 2^{2c}\pi^c(c-1)!\,\delta. \tag{F'}$$

According to Lemma 6.8 we have

$$\left(H + \frac{d}{2}\right)\tau = b\delta, \quad \text{with } b = 2(-\pi i)^c.$$

Now there exists $a \neq 0$ such that $X\tau = aY^{c-1}\alpha_0$, while $YX\tau = \gamma b\delta$. So (Lemma 3.2)

$$Y^cX^c\tau = \gamma(c-1)!^2 b\delta, \quad X^c\tau = X^{c-1}(aY^{c-1}\alpha_0) = a(c-1)!^2\alpha_0,$$

which lead to

$$Y^c\alpha_0 = \gamma\frac{b}{a}\delta.$$

It is thus enough to calculate a.

Since both τ and $Y^{c-1}\alpha_0$ are nonzero on $\mathbf{R}^{1,n}\backslash\{0\}$ we can determine a working on $\mathbf{R}^{1,n}\backslash\{0\}$ and hence working entirely with the pullbacks. For any distribution T on $\mathbf{R}^{1,n}$ let $T^\times$ be its restriction to $\mathbf{R}^{1,n}\backslash\{0\}$. We have $\tau^\times = (-i)^c\omega^*(\delta_0^{(c)})$; so that $X\tau^\times = \frac{1}{2}ci^c(-1)^{c-1}\omega^*(\delta_0^{(c-1)})$ while $\alpha_0^\times = \omega^*(\delta_0)$. We shall now compute $Y^{c-1}\alpha_0^\times$. Clearly there exists ε such that $Y^{c-1}\alpha_0^\times = \varepsilon\omega^*(\delta_0^{(c-1)})$. On the other hand, the pullback intertwines the action of Y with that of $-2t\dfrac{d^2}{dt^2} - d\dfrac{d}{dt} =: \Omega$. So if we write $\Omega_1 = ((\Omega^{\dagger c-1})_0)^\dagger$, then $Y^{c-1}\alpha_0^\times = \Omega_1\omega^*(\delta_0)$. Since $\Omega_1 = \varepsilon(d/dt)^{c-1}$ we can determine it by acting on t^{c-1}. As $\Omega^\dagger = -2\left(t\frac{d^2}{dt^2} - (c-1)\frac{d}{dt}\right)$, we find that

$$\varepsilon = (-1)^{c-1}2^{c-1}(c-1)!, \qquad a = \frac{ci^c}{2^c(c-1)!}.$$

From this the desired formulae follow at once. Related formulae occur in [13]; results of this kind go back essentially to [15]. Professor Duistermaat pointed out to us that the result above, as well as the Lemmas 6.8 and 6.9, also can be deduced by using [7, §3.2].

From (F) it is easy to deduce the corresponding formulae in the vector-valued situation. Let $U \simeq \mathscr{H}^j$ and let (h_a) be the components of h_U with respect to some basis of U. Then, using Lemma 6.12 and proceeding as we did in the determination of $\gamma(\bar{J}(U))$, we find

$$\Box^{(d-2/2)+j}h_a\alpha_0^+ = 2^{d-3+j}\pi^{d-2/2}\Gamma\left(\frac{d-2}{2}+j\right)\partial(h_a)\delta.$$

The methods developed in this paper allow one to obtain a host of similar formulae.

7.2. SPACE-TIME EXPRESSIONS FOR τ. The distribution τ of Lemma 6.7 is unique only up to a translation by a multiple of δ; and so one can expect explicit formulae for it only on the subspace $\mathscr{S}_0$ of the Schwartz space consisting of the functions that vanish at 0. Let $\tau_{k,a}$ $(0 \leqslant k \leqslant c)$ be the distribution given by

$$\tau_{k,a}(f) = (-1)^k \frac{d^k}{db^k}\bigg|_{b=a} M^+(f:b) \quad (f \in \mathscr{S}),$$

where $a > 0$ is fixed and $M^+(f:b)$ is the *restricted mean value*

$$M^+(f:b) = \int \frac{f(p_0', p)}{2p_0'}\,d^n p, \qquad p_0' = (|p|^2+b)^{1/2},\ b > 0.$$

For $b = a > 0$ we can differentiate under the integral sign and we obtain

$$\tau_{k,a}(f) = \sum_{0\leqslant s\leqslant k} C_{k,s}\int \frac{1}{(p_0')^{2k+1-s}}\frac{\partial^s f}{\partial p_0^s}(p_0', p)\,d^n p,$$

where $p_0' = (|p|^2+a)^{1/2}$ and $f \in \mathscr{S}$, and the $C_{k,s}$ are uniquely determined by

$$C_{0,0} = \tfrac{1}{2}, \quad 2C_{k+1,s} = (2k+1-s)C_{k,s} - C_{k,s-1}.$$

The limit as $a \to 0+$ of the $\tau_{k,a}$ will now be invariant distributions. There is no difficulty when $k < c$. But when $k = c$, $s = 0$, the integrand is $f(0)(p_0')^{-n} + O((p_0')^{-(n-1)})$ near $p = 0$; and so for the passage to the limit we must suppose $f(0) = 0$. Thus

$$\tau_c(f) = \sum_{0 \leqslant s \leqslant c} C_{c,s} \int \frac{1}{|p|^{n-s}} \frac{\partial^s f}{\partial p_0^s}(|p|, p) d^n p \quad (f \in \mathscr{S}_0).$$

In particular, for $n = 3$ we get

$$\tau_1(f) = \frac{1}{4} \int f(|p|, p) \frac{d^3 p}{|p|^3} - \frac{1}{4} \int \frac{\partial f}{\partial p_0}(|p|, p) \frac{d^3 p}{|p|^2} \quad (f \in \mathscr{S}_0).$$

In the physics literature τ_1 is often written as $\delta'(p_\mu p^\mu)$ and the above expression, quite well-known, plays an important role in the quantum theory of the electromagnetic field in the so-called Landau gauge, see [2, Th. 2.1].

7.3. RELATION TO THE THEORY OF METHÉE. The determination of $\gamma(\bar{J}(\mathbf{C}))$ (cf. Lemma 6.9) may also be carried out with the help of this theory. The central construction in it is the mean value map M that takes $f \in C_c^\infty(\mathbf{R}^{1,n})$ into the function $Mf = \omega_* f$ defined and smooth away from the origin in $\mathbf{R}$, vanishing for large $|t|$. One of the main results of that theory is that the range of M is *precisely* the space of all functions on $\mathbf{R}$ of the form

$$u(t) = h_1(t) + h_2(t) \log \frac{1}{|t|},$$

where h_i belong to $C_c^\infty(\mathbf{R})$, and $h_2^{(k)}(0) = 0$, for $0 \leqslant k \leqslant \frac{d}{2} - 2$. Although the h_i are not uniquely determined by u, the derivatives $h_i^{(k)}(0)$ are determined; and if we write

$$l_k(u) = \frac{h_2^{(k)}(0)}{k!}, \qquad m_k(u) = \frac{h_1^{(k)}(0)}{k!},$$

then

$$\Lambda_k : f \mapsto l_k(Mf), \quad M_k : f \mapsto m_k(Mf)$$

are $\mathbf{SO}(1, n)$-invariant distributions; moreover

$$\Lambda_k = 0, \quad \text{for } 0 \leqslant k \leqslant \frac{d}{2} - 2,$$

and the Λ_k, for $k \geqslant \frac{d-2}{2}$, and the M_k, for $k \geqslant 0$, form a basis for the space of such distributions that are supported by the full light cone. The formulae

$$M\Box f = 4\left(t\frac{d^2}{dt^2} - \left(\frac{d}{2} - 2\right)\frac{d}{dt}\right)Mf, \quad M\mathscr{E}f = 2\left(t\frac{d}{dt} - \frac{d-2}{2}\right)Mf,$$

lead to the following formulae

$$\Box\Lambda_k = 4(k+1)\left(k - \left(\frac{d}{2} - 2\right)\right)\Lambda_{k+1} \quad \left(k \geqslant \frac{d-2}{2}\right),$$

$$\Box M_k = 4(k+1)\left(k - \left(\frac{d}{2} - 2\right)\right)M_{k+1} - 4\left((2k+1) - \left(\frac{d}{2} - 2\right)\right)\Lambda_{k+1} \quad (k \geqslant 0),$$

$$\mathscr{E}\Lambda_k = -2(k+1)\Lambda_k, \quad \omega\Lambda_k = \Lambda_{k-1} \quad \left(k \geqslant \frac{d-2}{2}\right),$$

$$\mathscr{E}M_k = -2(k+1)M_k + 2\Lambda_k, \quad \omega M_k = M_{k-1} \quad (k \geqslant 0).$$

Let

$$\theta = \Lambda_{d-2/2}, \quad \tau = M_{d-2/2}.$$

Since $\Lambda_k = 0$, for $0 \leqslant k \leqslant \frac{d}{2} - 2$, we have $X\theta = 0$. So θ is a linear combination of α_0 and δ. But as $\left(H + \frac{d}{2}\right)\theta = 0$ and as α_0 is of weight $\frac{d}{2} - 2$, we see that θ is a nonzero multiple of δ, so that

$$\left(H + \frac{d}{2}\right)\tau = a\delta \quad (a \neq 0).$$

Moreover

$$YX\tau = -\tfrac{1}{4}\Box\omega M_{d-2/2} = -\tfrac{1}{4}\Box M_{d/2-2} = \frac{d-2}{2}\Lambda_{d-2/2}.$$

Hence,

$$YX\tau = \tfrac{1}{4}(d-2)\left(H + \frac{d}{2}\right)\tau,$$

proving what we want. The determination of $\gamma(\bar{J}(U))$ is as before.

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