COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 80, nº 3 (1991), p. 337-354

<http://www.numdam.org/item?id=CM_1991__80_3_337_0>

Compositio Mathematica **80**: 337–354, 1991.

On the moderate growth of generalized Dirichlet series for hypoelliptic polynomials

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Received 22 February 1990; accepted 13 March 1991

Introduction

This article finds a new analytic use for the functional equation and b-function, which are fundamental objects in the theory of holonomic modules. Recent well known applications of various constructions from D-modules have been primarily to representation theory and Hodge theory. On the other hand the historic motivation of the subject came from number theory (Sato's theory of prehomogeneous vector spaces) and the solution (by Bernstein) of the Division problem in the theory of distributions. In these early works the sheaf theoretic and cohomological/geometric aspects of the theory are largely absent. One simply worked algebraically with modules over the Weyl algebra and exploited the basic notion of a holonomic module to obtain a functional equation, involving a differential operator with polynomial coefficients. This led to important functional equations for the analytic objects of interest to Sato and Bernstein.

The application in this article is to a general counting problem that emerges from additive number theory. Given a polynomial $P \in \mathbb{R}[z_1, \ldots, z_n]$, define

$$N(x) = \operatorname{card}\{m \in \mathbb{N}^n : |P(m)| \leqslant x\}$$

$$\mathcal{N}(x) = \operatorname{card}\{m \in \mathbb{Z}^n : |P(m)| \leqslant x\}.$$

Assuming the functions are finite for each $x > 0$, the problem is to describe their asymptotic behavior as $x \to \infty$.

When P is a positive definite quadratic form, the classical work of Epstein [Ep], Landau [La], among others, described $\mathcal{N}(x)$ using the modularity of the quadratic theta function. This method breaks down however once P is positive definite and homogeneous of degree at least 3. In this case, An-Stein [A-S] have given an asymptotic description for $\mathcal{N}(x)$. Replacing $\mathbb{Z}$ by $\mathbb{N}$, Sargos [Sa-1, 2] has described the behavior of $N(x)$ when P has positive coefficients.

The class of polynomials used in this paper are hypoelliptic and defined over

$\mathbb{R}$. Hypoellipticity is characterized by a simple growth condition on P at infinity. If P is elliptic (in particular, homogeneous and positive definite), then it is hypoelliptic. More significantly, no assumptions about the sign of the coefficients need to be satisfied. Hypoelliptic polynomial can have both positive and negative coefficients.

This paper will assume P is hypoelliptic on $[a, \infty)^n$ for some $a \in (0, 1)$. Thus, $N(x)$ will be of particular interest. It is, however, a trivial exercise to modify the arguments given here, and thereby extend the main results to the asymptotic description of $\mathcal{N}(x)$, when P is hypoelliptic on $\mathbb{R}^n$. For this the discussion in [Li-2] is useful. Thus, the results of this paper extend those of both [A-S] and [Sa-2] in a unified manner. One should also note that the proof of Theorem 4 in [Bo] can be used to describe $\mathcal{N}(x)$, when P is hypoelliptic on $\mathbb{R}^n$, with precision approximately that obtainable by Theorem 1 of this paper. However, the method used by Bochner does not extend to treat $N(x)$.

More generally, a large class of counting functions will be studied. Let φ be any polynomial satisfying the condition

$$\text{Sign } \varphi(m) \text{ is constant for all but at most finitely many } m \in \mathbb{N}^n. \tag{0.1}$$

Then one defines

$$N_\varphi(x) = \sum_{\{m \in \mathbb{N}^n : |P(m)| \leq x\}} \varphi(m).$$

The description of $N_\varphi(x)$ for large x is a solution for a type of weighted lattice point problem, each point m weighted by $\varphi(m)$. Thus, an analysis of $N_\varphi(x)/N(x)$ for large x allows one to study the behavior of the expected value of $\varphi(m)$ when restricted to $\{|P| \leq x\} \cap \mathbb{N}^n$, for all large x. Still more general classes of weight functions can also be used (cf. Concluding Remarks).

The basic observation of this paper is that under the above assumptions upon P, φ, one can combine the b-function and functional equation with classical Tauberian methods to give a general description of $N_\varphi(x)$ as follows:

There exist $\rho_0(\varphi)$, $\mu'(\varphi) > 0$, and non-zero polynomial $A(t)$ such that for any $\varepsilon > 0$,

$$N_\varphi(x) = x^{\rho_0(\varphi)} A(\log x) + O_\varepsilon(x^{\rho_0(\varphi) - \mu'(\varphi) + \varepsilon}) \quad \text{as } x \to \infty. \tag{0.2}$$

This is proved in Section 3.

The significance of (0.2) is felt to lie in its proof, which is conceptually simple and extendable, so far at least, partially to Dirichlet series in $k \geq 2$ variables, cf. [Li-3]. Indeed, the moral of this paper would seem to be that the functional equation from D-modules is an effective and general tool for the analysis of lattice point problems involving polynomials on $\mathbb{R}^n$. This is presumably of most interest for problems in which the structure of automorphic functions is absent.

To prove (0.2) one introduces the generating function

$$D_P(s, \varphi) = \sum_{\{m \in \mathbb{N}^n : P(m) \neq 0\}} \frac{\varphi(m)}{P(m)^s},$$

called a generalized Dirichlet series, and proceeds classically. This means one proves the following assertions:

(0.3)

(1) $D_P(s, \varphi)$ is absolutely convergent and analytic in a halfplane $\sigma = \text{Re}(s) > B(\varphi)$;

(2) $D_P(s, \varphi)$ admits an analytic continuation to the entire $s(=\sigma + it)$ plane as a meromorphic function with at most finitely many poles in any vertical band $\sigma \in [a, b]$;

(3) There exists a first pole $\rho_0(\varphi)$ and a positive constant $\mu(\varphi)$ so that the following "moderate growth" condition holds to the left of the line $\sigma = \rho_0(\varphi)$:

For each $\varepsilon > 0$ and $\sigma_1 < \sigma_2 \leqslant \rho_0(\varphi)$, there exists a constant $C = C(\varepsilon, \sigma_1, \sigma_2)$ such that

$$|D_P(s, \varphi)| < C(1 + |t|^{\mu(\varphi)(\rho_0(\varphi) - \sigma) + \varepsilon}) \tag{0.4}$$

whenever $\sigma \in [\sigma_1, \sigma_2]$ and $|t| \geqslant B$, for some B independent of σ_1, σ_2.

One can then take $\mu'(\varphi) = 1/\mu(\varphi)$ in (0.2).

In the first article on this subject [Li-1], (0.3)(1,2) were proved using the functional equation and b-function. In order to keep the length of this article reasonably modest, the reader will be assumed familiar with [ibid], to which reference will often be made. Thus, the main result, Theorem 1, of this paper is the moderate growth condition (0.4). Here too the functional equation and b-function are used. This paper therefore gives an example demonstrating how these modern algebraic tools can be used to do classical analysis at a level of generality not possible using standard techniques.

On the other hand, this method is not strong enough to detect actual poles of the meromorphic extension of $D_P(s, \varphi)$. One must rely, so far, on a well-known theorem of Landau [H-R, p. 10], showing that assumption (0.1) implies the existence of a first (and necessarily rational, cf. [Li-4]) pole. In addition, if the determination of the precise value of the first pole is of interest, then one needs to carry out the type of geometric analysis at infinity that is described in [Li-1, §5].

Section 1 recalls hypoellipticity and an important property for the proof of Theorem 1. Section 2 states results, needed for the proof of Theorem 1, and which are proved in [Li-1]. Section 3 proves Theorem 1. The derivation of (0.2) from (0.4) is a standard modification of an argument due to Landau [La].

Discussions with Profs. J. Hoffstein and A. Nachman have been very helpful and are appreciated.

Notation

For ease in reading, compiled below is a list of notations used in the paper.

(1) $z_j = x_j + iy_j, j = 1, \ldots, n$.
(2) For $z \in \mathbb{C}^n$, one sets $\|z\| = \max\{|z_i|: i = 1, \ldots, n\}$.
(3) If $A = (A_1, \ldots, A_n)$ is an n-tuple of nonnegative integers, set $|A| = \Sigma_{i=1}^n A_i$, $A! = A_1! \cdots A_n!$, and $z^A = z_1^{A_1} \cdots z_n^{A_n}$.
(4) For A as above, define the differential monomial $D_x^A = D_{x_1}^{A_1} \cdots D_{x_n}^{A_n}$, and similarly for D_y^A.
(5) For any polynomial P, defined over $\mathbb{C}$, one writes

$$P(x, y) = \mathrm{Re}(P)(x, y) + i\,\mathrm{Im}(P)(x, y) = u(x, y) + iv(x, y).$$

Also, one writes $d_P = \deg P$.
(6) For $\theta > 0$, define a closed neighborhood of $[a, \infty)^n$ by

$$\Gamma(\theta) = \{z = (x, y) \in \mathbb{C}^n : x_j \geqslant a \text{ and } |y_j| \leqslant (x_j - a)^\theta, j = 1, \ldots, n\}.$$

For $\delta \geqslant a$, define $\Gamma(\theta, \delta) = \Gamma(\theta) \cap \{z \in \mathbb{C}^n : \|x\| \geqslant \delta\}$.

(7) For P, φ as above, define the δ tail of $D_P(s, \varphi)$ as

$$D^{(\delta)}(s, \varphi) = \sum_{\|m\| > \delta} \frac{\varphi(m)}{P(m)^s}.$$

(8) Set $\tilde{\delta} = [\![\delta]\!] + 1/2$.
(9) *Define the oriented* (*by increasing* x) arcs

$$\gamma_\pm(\theta, \delta) = \{z = x + iy : x \geqslant \tilde{\delta} \text{ and } y = \pm(x - \tilde{\delta})^\theta\}.$$

Assuming $\delta > 1$, let $i \in \{1, \ldots, [\![\delta]\!]\}$, and set $\gamma(i)$ to be a circle centered at i, of radius $r < 1/2$, and oriented counterclockwise. Define

$$\mathscr{C}(\theta, \delta) = \{\gamma(1), \ldots, \gamma([\![\delta]\!]), -\gamma_+(\theta, \delta), \gamma_-(\theta, \delta)\}$$

$$\mathscr{T}(\theta, \delta) = \{\sigma: \{1, \ldots, n\} \to \mathscr{C}(\theta, \delta): \sigma(u) \text{ is unbounded for at least one } u\}.$$

(10) For each $\tau \in \mathscr{T}(\theta, \delta)$ define

$$\Delta_\tau(\theta, \delta) = \tau(1) \times \cdots \times \tau(n),$$

$$\mathscr{E}_\tau(z_1, \ldots, z_n) = \prod_{j=1}^{n} \frac{1}{e^{2\pi i z_j} - 1}\bigg|_{\Delta_\tau(\theta,\delta)},$$

$$\mathscr{I}_\tau(s, \varphi) = \int_{\Delta_\tau(\theta,\delta)} (1/P)^s \varphi \mathscr{E}_\tau \, dz_1 \cdots dz_n.$$

1. Properties of hypoellipticity

Define

$$\widehat{[a, \infty)^n} = \{(x_1, 0, x_2, 0, \ldots, x_n, 0) \in \mathbb{C}^n: (x_1, \ldots, x_n) \in [a, \infty)^n\}.$$

One first recalls the

DEFINITION 1.1. $u(x, y) = \mathrm{Re}(P)$ is hypoelliptic on $\widehat{[a, \infty)^n}$ if for any differential monomial $D^A = D_{x_1}^{A_1} \cdots D_{y_n}^{A_{2n}}$, one has

$$\lim_{\substack{\|x\| \to \infty \\ x \in \widehat{[a,\infty)^n}}} \frac{D^A u}{u}(x, 0) = 0.$$

When P is defined over $\mathbb{R}$ one also says that

DEFINITION 1.2. $P(x_1, \ldots, x_n)$ is hypoelliptic on $[a, \infty)^n$ if for any differential monomial D_x^B,

$$\lim_{\substack{\|x\| \to \infty \\ x \in [a,\infty)^n}} \frac{D_x^B P}{P}(x) = 0.$$

This paper assumes that P is defined over $\mathbb{R}$. A simple argument now shows

PROPOSITION 1.3. *If P is hypoelliptic on $[a, \infty)^n$, then* Re(P) *is hypoelliptic on* $\widehat{[a, \infty)^n}$.

Proof. For an integral vector $I=(i_1, \ldots, i_n)$, define the notation $2|I$ to mean $2|i_j$ for each j.

One observes that $u(x, y)$ has the form

$$u(x, y) = u(x, 0) + \sum_{2|I} a_I(x) y^I.$$

Cauchy–Riemann equations then show that

$$D_y^I u(x, 0) = \pm D_x^I u(x, 0)$$

whenever $2|I$. One then reduces the defining property of Definition 1.1 to that in Definition 1.2. □

Hörmander showed the following property is equivalent to hypoellipticity of P on $[a, \infty)^n$ [Hö, p. 62].

(1.4) There exist positive constants c, C, D such that $|D_x^A P(x)| \leqslant C\|x\|^{-c|A|}|P(x)|$ when $x \in [a, \infty)^n \cap \{\|x\| \geqslant D\}$. □

PROPOSITION 1.5. *If P is hypoelliptic on $[a, \infty)^n$ then there exist $\theta > 0$, $\gamma < 0$ such that*

(i) $\lim_{\substack{\|z\| \to \infty \\ z \in \Gamma(\theta)}} |u(x, y)| = \infty,$

and

(ii) *For all* $(x, y) \in \Gamma(\theta)$,

$$\frac{u(x, y)}{u(x, 0)} = 1 + O(\|(x, y)\|^{\gamma}).$$

Proof. Write

$$u(x, y) = u(x, 0) + \sum_{|A|=2}^{d} \frac{D_y^A u(x, 0)}{A!} y^A.$$

The proof of Proposition 1.3 shows that for each n tuple A,

$$|D_y^A u(x, 0)| = |D_x^A u(x, 0)|.$$

Thus, for D defined in (1.4), $x \in [a, \infty)^n \cap \{\|x\| \geqslant D\}$, implies

$$|u(x, y) - u(x, 0)| \leqslant \sum_{|A|=2}^{d} \frac{|D_x^A u(x, 0)|}{A!} |y^A|$$
$$\leqslant C|u(x, 0)| \sum_{|A|=2}^{d} \|x\|^{-c|A|} |y^A|.$$

For each A, $|y^A| \leqslant \|y\|^{|A|}$. Thus, if $\theta < c$, one concludes by (1.4) that $(x, y) \in \Gamma(\theta)$ implies there exists $\gamma < 0$ such that

$$\frac{u(x, y)}{u(x, 0)} = 1 + O(\|(x, y)\|^{\gamma}).$$

It is now clear that (1.4) also implies (i). □

(1.6) REMARK. Tarski-Seidenberg implies the existence of $\alpha, B, D_0 > 0$ such that

$$|P(z)| \geqslant |u(z)| \geqslant B\|z\|^{n\alpha} \geqslant B|z_1 \cdots z_n|^{\alpha} \quad \forall z \in \Gamma(\theta, D_0). \quad \square$$

(1.6) suffices as a starting point for the description of the meromorphic extension of $D_P(s, \varphi)$. Of interest in this article is the behavior of $|D_P(s, \varphi)|$ in punctured vertical strips of the s-plane of the form

$$\{s = \sigma + it : a \leqslant \sigma \leqslant b,\ s \neq \text{pole of } D_P(s, \varphi)\}.$$

This requires a good estimate for $\operatorname{Arg} P(z)$, for $z \in \Gamma(\theta)$.

PROPOSITION 1.7. *Given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $\theta \in (0, c/2)$ and $z \in \Gamma(\theta, \delta(\varepsilon))$, then $|\operatorname{Arg} P(z)| < \varepsilon$.*

Proof. The Cauchy–Riemann equations in v and u are satisfied in each pair (x_j, y_j) of coordinates. Thus, for each index vector α with $|\alpha|$ odd there exists an index $\gamma(\alpha)$ with $|\alpha| = |\gamma(\alpha)|$ such that

$$D_y^{\alpha} v(x, 0) = D_x^{\gamma(\alpha)} u(x, 0).$$

Moreover, if $|\alpha|$ is even, then it is easy to see that because P is defined over $\mathbb{R}$ one has $D_y^{\alpha} v(x, 0) = 0$ for all x. One can then write

$$v(x, y) = \sum_{\substack{|\alpha| \in [1, q] \\ |\alpha| \text{ odd}}} \frac{D_y^{\alpha} v(x, 0)}{\alpha!} y^{\alpha}.$$

So,

$$\frac{v(x,y)}{u(x,y)} = \sum_{\substack{|\alpha|\in[1,q] \\ |\alpha| \text{ odd}}} \frac{1}{\alpha!} \frac{D_x^{\gamma(\alpha)}u(x,0)}{u(x,0)} \cdot \frac{u(x,0)}{u(x,y)} y^{\alpha}.$$

By (1.4), there exist $c, C, D > 0$ such that if $\|x\| \geqslant D$ then

$$\left|\frac{D_x^{\gamma(\alpha)}u}{u}(x,0)\right| \leqslant C\|x\|^{-c|\alpha|}.$$

By Proposition 1.5, there exists $K > 0$ so that if $\theta \in (0, c)$, $D' \geqslant D$, and $|y_j| \leqslant (x_j - a)^{\theta}$ for each $j = 1, \ldots, n$, then $\|x\| \geqslant D'$ implies

$$\left|\frac{v(x,y)}{u(x,y)}\right| \leqslant K\|x\|^{\theta - c}$$

It is now clear that for each $\varepsilon > 0$, there exists $\delta > 0$ so that $\|x\| \geqslant \delta$ implies $|v(x,y)/u(x,y)| < \varepsilon$ whenever $(x,y) \in \Gamma(\theta, \delta)$, for any $\theta < c/2$. This proves Proposition 1.7. □

(1.8) REMARK. It is useful to make precise the dependence of δ upon ε. If $|v/u| < 1$ then

$$\left|\operatorname{Tan}^{-1}\left(\frac{v}{u}\right)\right| < \left|\frac{v}{u}\right|.$$

Given $\theta \in (0, c/2)$, let $\eta \in (\theta - c, 0)$. One now sets $\delta(\varepsilon) = [\![(\varepsilon/K)^{1/\eta}]\!] + 1$. A simple exercise shows that if $z \in \Gamma(\theta, \delta(\varepsilon))$ then $|\operatorname{Arg} P(z)| < \varepsilon$. □

2. Analytic continuation and the functional equation

Let $N = \deg \varphi$, and α the exponent from (1.6). Set

$$B(\varphi) = \left[\!\!\left[\frac{(N+2)}{\alpha}\right]\!\!\right] + 1.$$

PROPOSITION 2.1. *Assume $\theta \in (0, c)$ and D_0 is chosen so that (1.6) is satisfied. Then, for any $\delta \geqslant D_0$ the following identity holds between analytic functions in the halfplane* $\operatorname{Re}(s) > B(\varphi)$:

$$D^{(\delta)}(s, \varphi) = \sum_{\tau \in \mathscr{T}(\theta,\delta)} I_{\tau}(s, \varphi). \tag{2.2}$$

Proof. Cf. [Li-1, 2.10]. □

One next recalls the basic analytic use of the functional equation. First one compactifies $\mathbb{C}^n \hookrightarrow (P^1\mathbb{C})^n$. In the chart $(\mathbb{C}^n, (w_1, \ldots, w_n))$ at infinity define,

$$R(w_1, \ldots, w_n) = 1/P(1/w_1, \ldots, 1/w_n),$$

$$\frac{\psi(w_1, \ldots, w_n)}{(w_1 \cdots w_n)^{N+2}} \mathrm{d}w_1 \cdots \mathrm{d}w_n = \varphi(1/w_1, \ldots, 1/w_n) d(1/w_1) \cdots d(1/w_n),$$

$$U(N) = 1/(w_1 \cdots w_n)^{N+2},$$

$$E_\tau(w_1, \ldots, w_n) = \mathscr{E}_\tau(1/w_1, \ldots, 1/w_n).$$

Adapting the argument of Bernstein [Be], one shows

PROPOSITION 2.3. *There exist a monic, non-zero polynomial of least degree, $b_N(s)$, and differential operators $\mathscr{P}_0, \ldots, \mathscr{P}_r$ in the ring $D_n(\mathbb{C}[s]) = \mathbb{C}[s]\langle w_1, \ldots, w_n, D_{w_1}, \ldots, D_{w_n}\rangle$ such that*

$$\sum_{i=0}^{r} \mathscr{P}_i(R^{s+i+i} U(N)) = b_N(s) R^s U(N). \tag{2.4}$$

Proof. Cf. [i-1, Prop. 3]. □

The main result of [ibid] was

THEOREM 2.5. *Let θ, D_0 be chosen to satisfy* (2.2). *Then there exists $\tilde{B}(\varphi) \geqslant B(\varphi)$ such that* $\mathrm{Re}(s) > \tilde{B}(\varphi)$ *and $\delta \geqslant D_0$ imply*

$$D^{(\delta)}(s, \varphi) = \frac{1}{b_N(s)} \sum_{\tau \in \mathscr{T}(\theta,\delta)} \sum_{i=0}^{r} \int_{\Delta_\tau(\theta,\delta)} R^{s+i+1} U(N) \mathscr{P}_i^*(\psi E_\tau) \, \mathrm{d}w_1 \cdots \mathrm{d}w_n,$$

where $\mathscr{P}_i^*$ is the adjoint of $\mathscr{P}_i$.

Proof. Cf. [ibid, Theorem 1]. □

The following indices are needed in Section 3. From (2.4) define

$$\kappa_i = \deg_s \mathscr{P}_i(s, w, D_w) \quad i = 0, 1, \ldots, r, \tag{2.7}$$

$$\nu_i = \operatorname{ord} \mathscr{P}_i(s, w, D_w) \quad i = 0, 1, \ldots, r,$$

$$\nu = \max_i \{\nu_i\}$$

$$\kappa = \deg_s b_N(s) \quad \text{and} \quad \beta = \max_i \{\kappa_i - \kappa\}.$$

3. Proof of Theorem 1

The idea of the proof of Theorem 1 is to split $D_P(s, \varphi)$ into a sum of two terms. One summand is the δ tail of the series while the second is a finite sum. One estimates the finite sum crudely (cf. Lemma 3.10) and the δ tail using the functional equation. The basic problem is to estimate $|R^s(w)||_{\Delta_\tau(\theta,\delta)}$, or equivalently $|P^{-s}(z)||_{\Delta_\tau(\theta,\delta)}$, for each τ. This involves both $|P|^{-\sigma}$, which (1.6) bounds, as well as $e^{t \operatorname{Arg} P(z)}$. Proposition 1.7 is now used to control the argument. For given ε one chooses $\delta(\varepsilon)$ according to (1.8). It follows that for any $\delta \geqslant \delta(\varepsilon)$,

$$e^{t \operatorname{Arg} P(z)}|_{\Delta_\tau(\theta,\delta)} < e^{\varepsilon|t|}. \tag{3.1}$$

Next, one views ε, resp. δ, as parameters that depend upon $|t|$ (cf. (3.11)) and which can assume arbitrarily small resp. large values. This allows one to kill off the growth coming from the argument of P. It then becomes necessary to estimate the dependence of the right side of (2.6) upon δ. Here the functional equation is crucial.

REMARK 3.2. Given c satisying (1.4), one assumes θ resp. D_0 is chosen to be any number in $(0, \min\{1, c/2\})$ resp. in $[a, \infty)$ for which (2.2) holds. One also assumes that $\delta \geqslant D_0$. Frequently, one interchanges δ and $\tilde{\delta}(=[\![\delta]\!]+\frac{1}{2})$. This should not cause any confusion. □

To state Theorem 1 it is necessary to define the growth rate $\mu(\varphi)$, cf. (0.4). Using (2.7), define

$$\mu(\varphi) = \max\left\{\frac{nd_P}{c}, \frac{2v}{c} + \beta\right\}. \tag{3.3}$$

THEOREM 1. *Assume P is a real hypoelliptic polynomial on $[a, \infty)^n$, for some $a \in (0, 1)$. Assume φ is a polynomial satisfying the sign condition in* (0.1). *Let $\rho_0(\varphi)$ be the first pole of $D_P(s, \varphi)$. Then, for each $\xi > 0$, $\sigma_1 < \sigma_2 \leqslant \rho_0(\varphi)$, there exists a constant $C = C(\xi, \sigma_1, \sigma_2)$ such that*

$$|D_P(s, \varphi)| < C(1 + |t|^{\mu(\varphi)[\rho_0(\varphi) - \sigma] + \xi})$$

for all $\sigma \in [\sigma_1, \sigma_2]$ and $|t| \geqslant B_1$, where B_1 is independent of σ_1, σ_2.

Proof of Theorem 1. The proof is based upon five lemmas.

LEMMA 3.4. *If $p(w_1, \ldots, w_n)$ is any polynomial, then*

$$|p(w_1, \ldots, w_n)||_{\Delta_\tau(\theta,\delta)} = O(1) \quad \textit{as } \delta \to \infty.$$

Proof of Lemma 3.4. This follows easily by first parametrizing each noncompact arc $\tau(j)$ in the z_j plane as

$$w_j = \frac{x_j \pm i(x_j - \tilde{\delta})^\theta}{x_j^2 + (x_j - \tilde{\delta})^{2\theta}}.$$

Then, setting $x_j = x_j'\tilde{\delta}$ shows that

$$w_j|_{\tau(j)} = \frac{1 \pm i(x_j'\tilde{\delta})^{\theta-1}(1-1/x_j')^{\theta}}{(x_j'\tilde{\delta})[1+(x_j'\tilde{\delta})^{2(\theta-1)}(1-1/x_j')^{2\theta}]}$$
$$= O(1/\delta) \quad \text{as } \delta \to \infty.$$

When $\tau(j)$ is compact, it is clear that $w_j|_{\tau(j)} = O(1)$ as $\delta \to \infty$. This shows Lemma 3.4. □

LEMMA 3.5. *For each* i, τ

$$\mathscr{P}_i^*(\psi E_\tau) = O(\delta^{2\nu_i})O(|s|^{\kappa_i}). \tag{3.6}$$

Proof of Lemma 3.5. By Lemma 3.4, it suffices to show (3.6) with $\psi \equiv 1$. Let $A = (A_1, \ldots, A_n)$ with $|A| \leqslant \nu_i$ and D_w^A be any differential monomial appearing in $\mathscr{P}_i$, one of the operators in (2.6). It is easy to see that $D_w^A(E_\tau)|_{\Delta_\tau(\theta,\delta)}$ equals a linear combination of terms of the form (neglecting irrelevant constants) $T(b,c,c') = \Pi_{j=1}^n T_j(b_j, c_j, c_j')$ where

$$T_j(b_j, c_j, c_j') = \frac{1}{w_j^{b_j}}\left(\frac{e^{2\pi i/w_j}}{e^{2\pi i/w_j}-1}\right)^{c_j}\left(\frac{1}{e^{2\pi i/w_j}-1}\right)^{c_j'}\Bigg|_{\tau(j)}.$$

Moreover, one has

$$b_j \leqslant 2A_j,\ c_j' \geqslant 1 \quad \text{and} \quad c_j \leqslant A_j.$$

The only case presenting any difficulty occurs when $\tau(j)$ is not compact in the z_j plane. It suffices then to assume $\tau(j) = -\gamma_+(\theta,\delta)$. The case $\tau(j) = \gamma_-(\theta,\delta)$ is treated similarly. Thus,

$$w_j|_{\tau(j)} = \frac{x_j}{x_j^2 + (x_j - \tilde{\delta})^{2\theta}} - \frac{i(x_j - \tilde{\delta})^{\theta}}{x_j^2 + (x_j - \tilde{\delta})^{2\theta}}.$$

Substituting this expression for w_j in $T_j(b_j, c_j, c_j')$, setting $x_j = x_j'\tilde{\delta}$, and simplifying yields

$$T_j(b_j, c_j, c_j')$$
$$= (x_j'\tilde{\delta})^{b_j}\left(\frac{e^{2\pi i x_j'\tilde{\delta}}}{e^{2\pi i x_j'\tilde{\delta}} - e^{2\pi(\tilde{\delta}(x_j'-1))^{\theta}}}\right)^{c_j}\left(\frac{e^{2\pi i x_j'\tilde{\delta}}}{e^{2\pi i x_j'\tilde{\delta}} - e^{2\pi(\tilde{\delta}(x_j'-1))^{\theta}}} - 1\right)^{c_j'} O(1). \tag{3.7}$$

Set $X_j = (x_j' - 1)\tilde{\delta}$. As x_j' assumes values in $[1, \infty)$, X_j assumes values in $[0, \infty)$.

Expressing (3.7) in terms of X_j and expanding out $(X_j+\tilde{\delta})^{b_j}$, one sees that $T_j(b_j, c_j, c_j')$ is a linear combination of terms of the form

$$\tilde{\delta}^k X_j^{b_j-k}\left(\frac{e^{2\pi i(X_j+\tilde{\delta})}}{e^{2\pi i(X_j+\tilde{\delta})}-e^{2\pi X_j^\theta}}\right)^{c_j}\left(\frac{e^{2\pi i(X_j+\tilde{\delta})}}{e^{2\pi i(X_j+\tilde{\delta})}-e^{2\pi X_j^\theta}}-1\right)^{c_j'} O(1).$$

One now observes that for any nonnegative integer v there exists a positive number C_v such that for all $X_j \in [0, \infty)$ and any $\tilde{\delta}$ one has

$$\left|X_j^v\left(\frac{e^{2\pi i(X_j+\tilde{\delta})}}{e^{2\pi i(X_j+\tilde{\delta})}-e^{2\pi X_j^\theta}}\right)\right| \leqslant C_v.$$

Thus, one concludes that $T_j(b_j, c_j, c_j') = O(\tilde{\delta}^{b_j})$, and therefore

$$|T(b, c, c')| = O(\tilde{\delta}^{|b|}), \quad |b| = b_1 + \cdots + b_n.$$

This evidently implies

$$|\mathscr{P}_i^*(E_\tau)| = O(\tilde{\delta}^{2v_i})O(|s|^{\kappa_i})$$

as claimed. ☐

The main estimate is this.

LEMMA 3.8. *Assume* $\mathrm{Re}(s) \geqslant \tilde{B}(\varphi)$ *and* $\varepsilon > 0$. *Let* $\delta \geqslant \delta(\varepsilon)$, $\delta(\varepsilon)$ *chosen according to Proposition* 1.7. *Then for each* $i=0, 1, \ldots, r$

$$\left|\sum_{\tau\in\mathscr{T}(\theta,\delta)}\int_{\Delta_\tau(\theta,\delta)} R^{s+i+1}U(N)\mathscr{P}_i^*(\psi E_\tau)\,\mathrm{d}w_1\cdots\mathrm{d}w_n\right|$$
$$= e^{\varepsilon|t|}O(\delta^{2v_i+n[N+2-\alpha(\sigma+i+1)]})O(|s|^{\kappa_i}).$$

Proof of Lemma 3.8. Pick one $\tau \in \mathscr{T}(\theta, \delta)$. Set

$$J_1 = \{j: \tau(j) \text{ is compact in the } z_j \text{ plane}\}, \quad a_1 = \mathrm{card}\, J_1,$$
$$J_2 = \{j: \tau(j) \text{ is not compact in the } z_j \text{ plane}\}, \quad a_2 = \mathrm{card}\, J_2.$$

The primary estimation problem occurs when $a_2 = n$. The discussion below therefore assumes $a_2 = n$. It is left to the reader to check that the estimate obtained in this case determines the maximal order in δ of the sum over all τ appearing in the statement of the lemma.

Write $w_j = u_j + iv_j$. Then x_j is a coordinate for $\tau(j)$ in the w_j plane. Define

$$|\mathrm{d}w_1 \cdots \mathrm{d}w_n|_{\Delta_\tau(\theta,\delta)}\Big| = \prod_{i=1}^{n} \sqrt{\left(\frac{\mathrm{d}u_j}{\mathrm{d}x_j}\right)^2 + \left(\frac{\mathrm{d}v_j}{\mathrm{d}x_j}\right)^2}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \Big|_{[\tilde{\delta},\infty)^n}.$$

One notes first that

$$\left| \int_{\Delta_\tau(\theta,\delta)} R^{s+i+1} U(N) \mathscr{P}_i^*(\psi E_\tau)\, \mathrm{d}w_1 \cdots \mathrm{d}w_n \right|$$
$$\leqslant \int_{\Delta_\tau(\theta,\delta)} |R^{s+i+1} U(N) \mathscr{P}_i^*(\psi E_\tau)|\, |\mathrm{d}w_1 \cdots \mathrm{d}w_n|.$$

Using the notations from the preceding lemmas, it is straightforward, and left to the reader, to show that

$$|\mathrm{d}w_1 \cdots \mathrm{d}w_n|_{\Delta_\tau(\theta,\delta)}| = [\tilde{\delta}^{-n}(x_1' \cdots x_n')^{-2}] u(\tilde{\delta}, x_1', \ldots, x_n')\, \mathrm{d}x_1' \cdots \mathrm{d}x_n' |_{[1,\infty)^n},$$

where

$$u(\tilde{\delta}, x_1', \ldots, x_n') = \prod_{i=1}^{n} u_j(\tilde{\delta}, x_j'),$$

and each $u_j(\tilde{\delta}, x_j')$ satisfies these conditions:

(1) It is integrable in a neighborhood of $x_j' = 1$.
(2) It is bounded in a neighborhood of $x_j' = \infty$.
(3) It is $O(1)$ as $\tilde{\delta} \to \infty$.

By (1.6) and (3.1), there exist constants C, C' independent of σ (when $\sigma \geqslant \tilde{B}(\varphi)$), τ, and δ such that

$$|R^{s+i+1} U(N)|\, |_{\Delta_\tau(\theta,\delta)} \leqslant C\, \mathrm{e}^{\varepsilon|t|}\, |w_1 \cdots w_n|^{\alpha(\sigma+i+1)-(N+2)}|_{\Delta_\tau(\theta,\delta)}$$
$$\leqslant C'\, \mathrm{e}^{\varepsilon|\tau|} \tilde{\delta}^{n[N+2-\alpha(\sigma+i+1)]} (x_1' \cdots x_n')^{N+2-\alpha(\sigma+i+1)}.$$

Combining these two estimates with that from Lemma 3.5, one sees that there exists a constant C'', independent of σ, τ, and δ such that

$$\int_{\Delta_\tau(\theta,\delta)} |R^{s+i+1} U(N) \mathscr{P}_i^*(\psi E_\tau)|\, |\mathrm{d}w_1 \cdots \mathrm{d}w_n|$$
$$\leqslant C'' |s|^{\kappa_i}\, \mathrm{e}^{\varepsilon|t|} \tilde{\delta}^{2\nu_i + n[N+2-\alpha(\sigma+i+1)]-n} I(\sigma),$$

where

$$I(\sigma) = \int_1^\infty \cdots \int_1^\infty (x_1' \cdots x_n')^{N-\alpha(\sigma+i+1)} \mathrm{d}x_1' \cdots \mathrm{d}x_n'.$$

By hypothesis, $N-\alpha(\sigma+i+1) \leqslant -2$ for each i. One concludes there exists a constant C''' independent of σ, τ, and δ such that

$$\left| \int_{\Delta_\tau(\theta,\delta)} R^{s+i+1} U(N) \mathscr{P}_i^*(\psi E_\tau) \mathrm{d}w_1 \cdots \mathrm{d}w_n \right|$$
$$\leqslant C''' |s|^{\kappa_i} \mathrm{e}^{\varepsilon|t|} \tilde{\delta}^{2\nu_i + n[N+1-\alpha(\sigma+i+1)]}. \tag{3.9}$$

Because one has

$$\text{card } \mathscr{T}(\theta, \delta) = O(\tilde{\delta}^n),$$

multiplying the upper bound in (3.9) by $\tilde{\delta}^n$ yields the upper estimate for the sum over τ as asserted in Lemma 3.8. □

As the first consequence of Lemma 3.8, one obtains

COROLLARY 1. *Let* $\varepsilon > 0$ *and* $\delta \geqslant \delta(\varepsilon)$, $\delta(\varepsilon)$ *chosen by Proposition* 1.7. *Assume* $\mathrm{Re}(s) > \tilde{B}(\varphi) - 1$. *Then, one has*

$$|D^{(\delta)}(s, \varphi)| = \mathrm{e}^{\varepsilon|t|} O(|s|^\beta \delta^{2\nu}).$$

Proof. Use (2.6) to give the meromorphic extension into $\mathrm{Re}(s) > \tilde{B}(\varphi) - 1$. In the estimation of $|D^{(\delta)}(s, \varphi)|$ one must then incorporate an $O(|s|^\kappa)$ factor coming from $b_N(s)$. This gives the $O(|s|^\beta)$ in the above statement.

Note too that $\mathrm{Re}(s) > \tilde{B}(\varphi) - 1 \geqslant B(\varphi) - 1$ implies that $N + 2 - \alpha(\sigma + i + 1) \leqslant 0$ for $i = 0, 1, \ldots, r$. Thus, the largest power of δ appearing in the estimate for $|D^{(\delta)}(s, \varphi)|$ should be 2ν. This shows the corollary. □

For $l = 0, 1, 2, \ldots$ and $B > 0$, define

$$\mathscr{S}_l(B) = \{s = \sigma + it \in \mathbb{C}: \tilde{B}(\varphi) - l + 1 \geqslant \sigma > \tilde{B}(\varphi) - l \text{ and } |t| \geqslant B\}.$$

Theorem 1 of [Li-1] implies there exists B_1 such that $\mathscr{S}_l(B_1)$ contains no poles of $D_P(s, \varphi)$ for each $l = 0, 1, 2, \ldots$. Iterating (2.6) immediately shows

COROLLARY 2. *Let* $\varepsilon > 0$ *and* $\delta \geqslant \delta(\varepsilon)$, $\delta(\varepsilon)$ *chosen by Proposition* 1.7. *Assume* $s \in \mathscr{S}_l(B_1)$. *Then*

$$|D^{(\delta)}(s, \varphi)| = \mathrm{e}^{\varepsilon|t|} O(|s|^{l\beta} \delta^{2l\nu}).$$

Now, one can write for any s not equal to a pole of the series,

$$D_P(s, \varphi) = \sum_{\substack{||m|| \in [1,\delta] \\ P(m) \neq 0}} \frac{\varphi(m)}{P(m)^s} + D^{(\delta)}(s, \varphi),$$

where the first term is given the trivial analytic continuation as the finite sum of the entire functions $\varphi(m)/P(m)^s$ and the second term denotes the analytically continued δ tail. One estimates the absolute value of the first term in an elementary way by estimating the absolute value of each summand. The verification of the next lemma is left to the reader.

LEMMA 3.10

(1) *There exists* $D_1 > 0$ *such that if* $\sigma \in [N/\alpha, \tilde{B}(\varphi)]$ *then*

$$\left| \sum_{\substack{||m|| \in [1,\delta] \\ P(m) \neq 0}} \frac{\varphi(m)}{P(m)^s} \right| \leqslant D_1 \delta^n \quad \textit{when } \delta \gg 1.$$

(2) *There exists* $D_2 > 0$ *such that if* $\sigma \in [0, N/\alpha)$ *then*

$$\left| \sum_{\substack{||m|| \in [1,\delta] \\ P(m) \neq 0}} \frac{\varphi(m)}{P(m)^s} \right| \leqslant D_2 \delta^{n(N+1-\sigma\alpha)} \quad \textit{when } \delta \gg 1.$$

(3) *There exists* $D_3 > 0$ *such that if* $\sigma < 0$ *then*

$$\left| \sum_{\substack{||m|| \in [1,\delta] \\ P(m) \neq 0}} \frac{\varphi(m)}{P(m)^s} \right| \leqslant D_3^{-\sigma} \delta^{n(N+1-\sigma d_P)} \quad \textit{when } \delta \gg 1.$$

Combining Lemmas 3.8, 3.10 is now useful by choosing $|t|$ as the underlying parameter upon which both ε and δ will depend. In addition, one specifies the form of $\delta(\varepsilon)$ via (1.8). Thus, given any $\theta \in (0, c/2)$ let $\eta \in (\theta - c, 0)$ be arbitrary and set

$$\varepsilon = \frac{1}{|t|}, \qquad \delta(\varepsilon) = [\![(|t|K)^{-1/\eta}]\!] + 1. \tag{3.11}$$

One concludes immediately

LEMMA 3.12. *For each* $l = 0, 1, 2, \ldots,$ *there exist* K_1, K_2, K_3 *depending only* *upon l and φ such that for any $s \in \mathscr{S}_l(B_1)$ one has*

$$|D_P(s, \varphi)| \leqslant K_1|t|^{-n(N+1-\sigma\alpha)/\eta} + K_2|t|^{-n(N+1-\sigma d_P)/\eta} + K_3|t|^{l(\beta - 2\nu/\eta)}.$$

Define the *positive* number

$$\mu(\eta, \varphi) = \max\left\{\frac{-n\,d_P}{\eta}, \frac{-2\nu}{\eta} + \beta\right\}.$$

One now proceeds classically. For any σ define

$$\pi(\sigma) = \inf\{\gamma > 0 \colon |D_P(\sigma + it, \varphi)| \ll |t|^\gamma, |t| \geqslant 1\}.$$

One knows that

(1) $\pi(\rho_0(\varphi)) = 0$ (continuity of π and $\pi(\sigma) = 0$, $\sigma > \rho_0(\varphi)$).
(2) π is a convex function on $(-\infty, \rho_0(\varphi)]$ (Phragman–Lindelöf).

Combining the five lemmas with these two properties of $\pi(\sigma)$ and a standard argument (cf. [Sa-1, p. 116]), one concludes:

For each $\tau > 0$, $\sigma_1 < \sigma_2 \leqslant \rho_0(\varphi)$ there exists $C = C(\tau, \sigma_1, \sigma_2)$ so that

$$|D_P(\sigma + it, \varphi)| < C(1 + |t|^{\mu(\eta,\varphi)[\rho_0(\varphi) - \sigma] + \tau}) \tag{3.13}$$

for all $\sigma \in [\sigma_1, \sigma_2]$ and $|t| \geqslant B_1$. One now observes that the parameter θ can be chosen arbitrarily small. The estimates in the lemmas are unaffected by the choice of θ, as long as (1.6), (1.7) are satisfied. Thus, for given τ, σ_1 in (3.13), one can choose θ so small that if $\eta \in (\theta - c, 0)$ and $\mu(\varphi)$ is defined by (3.3), then one has

$$|\mu(\eta, \varphi) - \mu(\varphi)| < \frac{\tau}{2(\rho_0(\varphi) - \sigma_1)}.$$

This implies one can replace $\mu(\eta, \varphi)$ by $\mu(\varphi)$ in (3.13), for any choice of τ, σ_1, σ_2, at the expense of replacing τ in the exponent of $|t|$ in (3.13) by $3\tau/2$. This completes the proof of Theorem 1. □

(3.14) REMARK. Theorem 1 is an extension of [Sa-1, Theorem 1.4] to a large class of real polynomials with coefficients not all of the same sign. In [Sa-2], an optimal estimate was given for the growth rate over the set of all polynomials with positive coefficients. An interesting question seems to be how one can estimate the smallest value of $\mu(\varphi)$ with reasonable precision. It would then be interesting to know whether for some polynomials with positive coefficients, the best value of $\mu(\varphi)$ is smaller than that obtained in [ibid]. However, what is important for this paper is the fact that $\mu(\varphi)$ is positive. □

Information given by Theorem 1 converts into asymptotic information about $N_\varphi(x)$ (cf. (0.2)) following a standard procedure, due to Landau [La], also cf. [Ch-Na, §4]. Although Landau assumes the series possesses a "reflection type" functional equation, it is clear from his argument that this assumption is not needed to derive the asymptotic below.

Assume the conditions in Theorem 1 hold for P and φ. Let

$$\rho_0 = \rho(\varphi) \geqslant \rho_1 \geqslant \cdots \rho_k \geqslant \rho_0 - 1/\mu(\varphi) \geqslant \rho_{k+1} \geqslant \cdots$$

be the poles of $D_P(s, \varphi)$, ordered by their real parts. At each ρ_j let

$$\mathrm{Pol}_{s=\rho_j} \frac{D_P(s, \varphi)}{s} = \sum_{i=1}^{n'} \frac{A_{j,i}(\varphi)}{(s-\rho_j(\varphi))^i}$$

be the principal part at ρ_j. Define $N_j(x) = x^{\rho_j} \Sigma_{i=1}^{n'} A_{j,i} \log^{i-1} x$. The corollary of interest from Theorem 1 is

THEOREM 2

$$N_\varphi(x) = \sum_{j=0}^{k} N_j(x) + O_\varepsilon(x^{\rho_0(\varphi) - 1/\mu(\varphi) + \varepsilon}) \quad as\ x \to \infty.$$

CONCLUDING REMARKS. (1) The technique used to prove Theorem 1 for φ a polynomial function on $\mathbb{R}^n$ extend to allow φ to equal a rational function defined in $\Gamma(\theta)$ for some θ. The arguments apply even more generally to any $\varphi \in \mathbb{R}[z_1^{\alpha_1}, \ldots, z_n^{\alpha_n}, \log z_1, \ldots, \log z_n]$ satisfying (0.1), where each $\alpha_i \in \mathbb{R}$.

(2) Using local techniques from $\mathscr{D}$ modules, it is possible to show [Li-4] that the poles of any $D_P(s, \varphi)$ are rational. This is done by showing that all roots of a local b-function at any point on the divisor $\{w_1 \cdots w_n = 0\}$ must be rational.

References

[A-S] C. An and A. Stein: Representations of integers by positive definite forms over arithmetic progressions, *Il. J. of Math.* 24 (1980), 612–618.

[Be] J. Bernstein: The analytic continuation of generalized functions with respect to a parameter, *Functional Analysis and Applications* 6 (1972), 26–40.

[Bo] S. Bochner: Zeta functions and Green's functions for linear partial differential operators of elliptic type with constant coefficients, *Ann. of Math.* 57 (1953), 32–56.

[Ch-Na] K. Chandrasekharan and R. Narasimhan: Functional equations with multiple gamma factors and the average order of arithmetical functions, *Ann. of Math.* 76 (1962), 93–136.

[Ep] P. Epstein: Zur Theorie Allgemeiner Zetafunctionen, *Math. Annalen* 56 (1903), 615–644.

[H-R] G.H. Hardy and M. Riesz: The general theory of Dirichlet's series, Hafner Publishing Co., 1972.

[Ho] L. Hörmander: Analysis of linear partial differential operators II, *Grundlehren*, vol. 257, Springer-Verlag, 1983.

[La] E. Landau: Über die Anzahl der Gitterpunkte in gewissen Bereichen (Zweite Abhandlung), Kgl. Ges. d. Wiss. Nachrichten. *Math-Phys. Klasse.* (Göttingen) 2 (1915), 209–243.

[Li-1] B. Lichtin: Generalized Dirichlet series and *b*-functions, *Comp. Math.* 65 (1988), 81–120.

[Li-2] B. Lichtin: Poles of Dirichlet series and *D*-modules, Théorie des Nombres, Comptes rendus de la Conférence international de Théorie des Nombres tenue à l'Université Laval, 1987, pp. 579–594.

[Li-3] B. Lichtin: The asymptotics of a lattice point problem associated to finitely many polynomials I, *Duke J. of Math.* (to appear).

[Li-4] B. Lichtin: The asymptotics of a lattice point problem determined by a hypoelliptic polynomial (submitted to Proceedings of Conference on *D*-modules and Microlocal Geometry, held in Lisbon, Portugal 1990).

[Sa-1] P. Sargos: Prolongement meromorphe des séries de Dirichlet associées á des fractions rationelles de plusieurs variables, *Ann. Inst. Fourier* 33 (1984), 82–123.

[Sa-2] P. Sargos: Croissance de certaines séries de Dirichlet et applications, *J. reine und ang. Math.* 367 (1986), 139–154.