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On sections of commutative group schemes

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Introduction

Many groups in algebraic geometry admit additional structure. Suppose, for simplicity, that the ground field is $\mathbb{C}$. Suppose given a group G, and suppose given the additional data of a functor

$$\mathbf{G}: \langle\langle \text{nonsingular varieties over } \mathbb{C} \rangle\rangle^{\circ} \to \langle\langle \text{groups} \rangle\rangle$$

such that $\mathbf{G}(\cdot) = G$. Then $\mathbf{G}$ defines the structure of a *topological group* on G, which has the strongest topology such that for every nonsingular variety T, and for every $\eta \in \mathbf{G}(T)$, the induced map $T \to G$ is continuous with respect to the usual topology. In practice, the functor $\mathbf{G}$ will often be defined on the larger category of all $\mathbb{C}$-schemes, but this additional information is not needed to topologize G.

For example, if X is a proper variety, then $G = \mathrm{Pic}(X)$ inherits a Lie group structure from $\mathbf{G}(S) = \mathrm{Pic}(X \times S)/\mathrm{Pic}(S)$. (It may have infinitely many components.) But for many groups G, the induced topology is either infinite-dimensional, non-Hausdorff, or both. Nevertheless, it may be tractable.

Shafarevich [23] has shown that the group $\mathrm{Aut}(\mathbb{A}^n)$ may be thought of as an infinite-dimensional algebraic group. We are not aware of any other structure theorems for non-representable groups in algebraic geometry. There do exist negative characterizations, such as Mumford's argument ([19]; [6] lecture 1) that if X is a complex projective surface with $p_g(X) > 0$, then $\mathrm{CH}_0(X)$ is not finite dimensional. As Bloch writes [5], "algebraic geometers have dealt with non-representable objects before (stacks, algebraic spaces) but these have always been in some sense very close to algebraic varieties. With the cycle groups, one encounters for the first time objects which are geometric in content and yet joyously non-representable."

We would like to find structure in such non-representable groups. To this end, we consider in this paper the following example. Let X be a smooth commutative group scheme over a complex variety S. Identify X with the induced

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representable functor from ⟨⟨S-schemes⟩⟩° to ⟨⟨groups⟩⟩. Let $f: S \to \operatorname{Spec} \mathbb{C}$ be the canonical map. Let $\mathbf{G} = f_* X$, which is the functor from ⟨⟨$\mathbb{C}$-schemes⟩⟩° to ⟨⟨groups⟩⟩ given by

$$\mathbf{G}(T) = \{\text{sections of } X \times T \text{ over } S \times T\}.$$

Our main theorem (6.3) characterizes the restriction of $\mathbf{G}$ to the category of reduced $\mathbb{C}$-schemes.

We find that either $\mathbf{G}$ is representable by a complex Lie group H, or else it is of the form $H \times (\oplus_{i=1}^{\infty} \mathbb{G}_a)$. For example, if $S = \mathbb{A}^1$, and $X = \mathbb{G}_a \times \mathbb{A}^1$, then $G = \oplus_{i=1}^{\infty} \mathbb{G}_a$. If the generic fiber of X/S has finitely generated component group, then so does H. If the generic fiber of X/S has no unipotent part, then the infinite-dimensional form does not occur. As a consequence (6.7) of the main theorem, we are able to describe the topology which is induced by $\mathbf{G}$ on the group G of sections of X/S.

The behavior of $\mathbf{G}$ on the category of all $\mathbb{C}$-schemes does not admit such a simple description. However, in order to prove the main theorem, it is necessary to keep track of the discrepancy between the behavior of $\mathbf{G}$ on ⟨⟨reduced $\mathbb{C}$-schemes⟩⟩, and its behavior on ⟨⟨$\mathbb{C}$-schemes⟩⟩. We do this by constructing a functor $\mathbf{G}_0$, which is representable (in the above generalized sense), and a nilimmersion of functors $\mathbf{G}_0 \to \mathbf{G}$.

In the important case where S is proper, and X is quasiprojective over $\mathbb{C}$, our main theorem is a special case of a representability theorem of Grothendieck ([9] 195-13, 221-20). In this situation, Grothendieck's theorem implies that $\mathbf{G}$ is representable as a functor on ⟨⟨$\mathbb{C}$-schemes⟩⟩. In this sense, Grothendieck's result is much stronger than our main theorem. On the other hand, the finite generation of the component group of H cannot be deduced from Grothendieck's result.

It would be interesting to look for an analog of the main theorem when $\mathbb{C}$ is replaced by an algebraically closed field of positive characteristic. It would also be interesting to study the sections of non-commutative group schemes.

In a later paper, we hope to apply the main theorem to give a description of Pic(X), where X is an complex variety, not necessarily proper. We expect that Pic(X) will have the form H/D or $H/D \times (\oplus_{i=1}^{\infty} \mathbb{C})$, where H is as above and D is a finitely generated subgroup.

1. Generalities and conventions about functors

All rings in this paper are commutative. Let k be a field.

DEFINITION. A k-functor is a functor

$$F: \langle\langle k\text{-schemes}\rangle\rangle^\circ \to \langle\langle \text{sets}\rangle\rangle.$$

A morphism of k-functors is a natural transformation of functors. An fppf k-functor is a k-functor which is also a sheaf with respect to the fppf topology. An étale k-functor is a k-functor which is also a sheaf with respect to the étale topology.

We have:

$$\langle\langle \text{fppf } k\text{-functors} \rangle\rangle \subset \langle\langle \text{étale } k\text{-functors} \rangle\rangle \subset \langle\langle k\text{-functors} \rangle\rangle.$$

These three categories have the same monomorphisms, but different epimorphisms. The monomorphisms are functors F such that $F(X)$ is injective for all k-schemes X. Depending on context, the k-functors under consideration may take values in ⟨⟨sets⟩⟩, ⟨⟨groups⟩⟩, or ⟨⟨abelian groups⟩⟩.

Modulo set-theoretic difficulties, if F is a k-functor, there exists an fppf k-functor F_{fppf} and a map $F \to F_{\mathrm{fppf}}$ which is universal for such maps (cf. [17] p. 57). This associated sheaf construction also works for the étale topology.

Modulo the same difficulties, the categories of (abelian group)-valued k-functors, (abelian group)-valued étale k-functors, and (abelian group)-valued fppf k-functors are abelian. These categories have different exact sequences.

If X is a k-scheme, then there is an induced k-functor, also denoted by X and given by $X(S) = \mathrm{Hom}_k(S, X)$. *In general, we do not distinguish between a scheme and the corresponding representable functor.* Representable functors are fppf sheaves.

DEFINITION. A k-functor is *discrete* if it is representable by a disjoint union of copies of Spec(k).

One can define an fppf $\mathbb{C}$-functor by giving its value on all affine $\mathbb{C}$-schemes. For a $\mathbb{C}$-algebra R, the notation $N(R)$ means $N(\mathrm{Spec}(R))$. Also, the notation $N(\cdot)$ means $N(\mathrm{Spec}(\mathbb{C}))$.

DEFINITION. Let X and Y be k-schemes. Then $\mathbf{Hom}(X, Y)$ is the k-functor given by $T \mapsto \mathrm{Hom}_k(X \times T, Y)$.

DEFINITION. Let S be a k-scheme. Let X be an S-scheme. Then $\mathbf{Sect}(X/S)$ is the k-functor given by

$$T \mapsto \{\text{sections of } X \times T \to S \times T\}.$$

We have $\mathbf{Sect}(X/S) = f_*X$, where $f: S \to \mathrm{Spec}\,\mathbb{C}$ is the canonical map. Observe that $\mathbf{Hom}$ is a special case of $\mathbf{Sect}$:

$$\mathbf{Hom}(X, Y) = \mathbf{Sect}([X \times Y]/X).$$

The k-functors $\mathbf{Hom}(X, Y)$ and $\mathbf{Sect}(X/S)$ are fppf sheaves.

DEFINITION. A morphism $\phi: F \to G$ of k-functors is a *closed immersion* if for every k-scheme T, and for every morphism $T \to G$, the induced morphism of functors $F \times_G T \to T$ "is" a closed immersion of schemes.

REMARK 1.1. Let F, G be k-functors. Let $\phi: F \to G$ be a morphism. Then ϕ is a closed immersion if and only if it is a monomorphism, and if for every k-scheme T, and every $\eta \in G(T)$, there exists a closed subscheme $T_0 \subset T$, such that if Y is a k-scheme and $f: Y \to T$ is any k-morphism, then $f(\eta) \in F(Y)$ if and only if f factors through T_0.

One can also consider *closed subfunctors* of a given k-functor. Let F be an (abelian group)-valued fppf k-functor. It is not true in general that every direct summand of F is a closed subfunctor. Indeed, this will be true if and only if the inclusion $0 \hookrightarrow F$ is a closed immersion. This condition holds for every functor we shall be interested in, but not e.g. for the fppf sheaf associated to the functor defined on k-algebras by $R \mapsto R/N(R)$, where $N(R)$ denotes the set of nilpotent elements of R.

Miscellaneous facts

- The property of being a closed immersion of functors is stable under base extension.
- A closed subfunctor of an fppf [resp. étale] sheaf is an fppf [resp. étale] sheaf.
- If $(F_i)_{i \in I}$ are closed subfunctors of a k-functor G, then so is $\bigcap_{i \in I} F_i$.
- A fiber product in ⟨⟨k-functors⟩⟩ of fppf sheaves is an fppf sheaf.

2. Additive × algebraic functors

DEFINITION. An (abelian group)-valued $\mathbb{C}$-functor is *additive* if it is isomorphic to the fppf sheaf given on $\mathbb{C}$-algebras by $R \mapsto \bigoplus_{i \in I} R$, for some set I. We shall always make the additional requirement that I is countable.

DEFINITION. An (abelian group)-valued $\mathbb{C}$-functor is *algebraic* if it is representable by some $\mathbb{C}$-scheme X, locally of finite type over $\mathbb{C}$.

To be consistent with the literature, we should probably call such functors *locally algebraic*, but for brevity we do not. We recall the following theorem:

THEOREM 2.1. *Let*

$$0 \to F' \to F \to F'' \to 0$$

be an fppf-exact sequence of (abelian group)-valued fppf $\mathbb{C}$-functors. Assume that F' and F'' are algebraic. Then so is F.

Proof. See ([2] 3.5, 7.3(ii)), ([3] 6.3), and ([1] 4.2). □

DEFINITION. An (abelian group)-valued $\mathbb{C}$-functor is *additive* × *algebraic* if it is isomorphic to a functor of the form $A \times R$ where A is additive and R is algebraic.

The two main results of this section are (2.5), which asserts that any extension of additive × algebraic functors is additive × algebraic, and (2.6), which asserts that any closed sub-group-functor of an additive × algebraic functor is additive × algebraic.

There are two kinds of additive × algebraic functors. Firstly there are those which are algebraic. Secondly there are those of the form $A \times \mathbb{G}_a^\infty$, where $\mathbb{G}_a^\infty$ denotes the fppf sheaf associated to the $\mathbb{C}$-functor $X \mapsto \oplus_{i=1}^{\infty} \Gamma(X, \mathcal{O}_X)$. Note that if X is quasicompact then:

$$\mathbb{G}_a^\infty(X) = \bigoplus_{i=1}^{\infty} \Gamma(X, \mathcal{O}_X).$$

Also, $\mathbb{G}_a^\infty$ may be viewed as the coproduct in ⟨⟨fppf $\mathbb{C}$-functors⟩⟩ of countably infinitely many copies of $\mathbb{G}_a$. Any additive × algebraic functor is an fppf $\mathbb{C}$-functor. The following notions make sense for an additive × algebraic functor F:

- whether or not F is *connected*;
- the *connected component of the identity* of F;
- the *component group* of F.

If G is an algebraic functor, one knows that there exist fppf-exact sequences:

$$0 \to G^0 \to G \to D \to 0 \tag{*}$$

and

$$0 \to \mathbb{G}_m^k \times \mathbb{G}_a^n \to G^0 \to A \to 0 \tag{**}$$

where G^0 is connected, D is discrete, A is an abelian variety, and k, n are integers. Furthermore (see 2.2, below), the sequence (*) splits. For proofs relating to (**), see e.g. ([20] 7.2.1, 4.1.3, 3.9 (p. 553)) and ([21] Theorem 16). We use these decompositions repeatedly.

LEMMA 2.2. *Let*

$$0 \to F_0 \to F \to D \to 0 \tag{†}$$

be an fppf-exact sequence of fppf $\mathbb{C}$-functors, where F_0 is additive × algebraic and connected, and D is discrete. Then (†) *splits.*

Proof. The map $F \to D$ is split surjective as a map of set-valued fppf $\mathbb{C}$-functors. Hence $\mathrm{Ext}(D, F_0) \cong \mathrm{Ext}(D(\cdot), F_0(\cdot))$, where the latter Ext is computed in ⟨⟨abelian groups⟩⟩. From (**), above, one concludes that $F_0(\cdot)$ is divisible. Hence $\mathrm{Ext}(D(\cdot), F_0(\cdot)) = 0$. □

LEMMA 2.3. *Let*

$$0 \to V \to F \xrightarrow{\pi} A \to 0$$

be an fppf-exact sequence of (abelian group)-valued fppf $\mathbb{C}$-functors, in which $V \cong \mathbb{G}_a^\infty$ and A is connected algebraic. Then there exists an (abelian group)-valued fppf $\mathbb{C}$-functor $M \subset F$ such that the induced map $M \to A$ is fppf-onto and such that $M \cap V$ is a representable closed subfunctor of V, necessarily of the form $\mathbb{G}_a^k$ for some integer k.

Proof. There exists a scheme X and morphisms $p: X \to A$, $\sigma: X \to F$ such that p is fppf and $\pi \circ \sigma = p$. Let $X^3 = X \times X \times X$. There is an induced map $\sigma^3: X^3 \to F$ given by

$$\sigma^3(x_1, x_2, x_3) = \sigma(x_1) - \sigma(x_2) - \sigma(x_3).$$

Let $\overline{X^3}$ denote the fppf image of σ^3. Then $\overline{X^3}$ is a set-valued fppf $\mathbb{C}$-functor. Define a $\mathbb{C}$-functor P by the fiber product diagram:

$$\begin{array}{ccc} P & \to & X^3 \\ \downarrow & & \downarrow \\ V & \to & F. \end{array}$$

Since $0 \to A$ is a closed immersion, so is $V \to F$. Hence the map $P \to X^3$ is a closed immersion. In particular, P is a scheme of finite type over $\mathbb{C}$. Hence $V(P) = \oplus_{i=1}^\infty \Gamma(P, \mathcal{O}_P)$, so the map $P \to V$ factors through a finite dimensional group-subfunctor $V_0 \subset V$, say $V_0 \cong \mathbb{G}_a^k$. It follows that $\overline{X^3} \cap V \subset V_0$. Let $M = \overline{X^3} + V_0$, i.e. the set-valued fppf subfunctor of F associated to the $\mathbb{C}$-functor given on $\mathbb{C}$-algebras by

$$R \mapsto \{s + v : s \in \overline{X^3}(R),\ v \in V_0(R)\}.$$

We will show that M is in fact a group-subfunctor. For this, it suffices to show that if

$$s, s' \in \overline{X^3}(R), \quad \text{then } s - s' \in M(R).$$

Write

$$s = x_1 - x_2 - x_3, \quad s' = x_1' - x_2' - x_3'.$$

(We omit references to σ.) We must show that

$$x_1 - x_2 - x_3 - x'_1 + x'_2 + x'_3 \in M(R).$$

In the course of the argument, we may replace R by any fppf R-algebra S. We shall do this repeatedly without adjusting the notation. Observe that the map $X \to A$ is onto as a map of fppf sheaves. Replacing R by some fppf R-algebra S, we may find an element $y_1 \in X(R)$ such that $\pi(y_1 - x_1 - x'_2) = 0$. Then $y_1 - x_1 - x'_2 \in V_0(R)$. Therefore, it suffices to show that

$$y_1 - x_2 - x_3 - x'_1 + x'_3 \in M(R).$$

Similarly, we may find an element $y_2 \in X(R)$ such that $\pi(y_2 - x_2 - x_3) = 0$. It suffices to show that $y_1 - y_2 - x'_1 + x'_3 \in M(R)$. Similarly, we may find an element $y_3 \in X(R)$ such that $\pi(y_3 - y_1 - x'_3) = 0$. It suffices to show that $y_3 - y_2 - x'_1 \in M(R)$. This follows from the definition of $M(R)$. Hence M is a group-valued fppf $\mathbb{C}$-functor. Moreover, $M \cap V = V_0$. □

LEMMA 2.4. *Let G be an additive × algebraic $\mathbb{C}$-functor. Let H be an fppf $\mathbb{C}$-functor. Let $k \in \{0, 1, 2, \ldots, \infty\}$. Then any fppf-exact sequence:*

$$0 \to G \to H \to \mathbb{G}_a^k \to 0$$

splits.

Proof. In any abelian category $\mathscr{A}$, one can define abelian groups Ext(X, Y) which classify equivalence classes of extensions of X by Y, see [7]. Furthermore, if arbitrary products and coproducts exist in $\mathscr{A}$, then

$$\mathrm{Ext}\left(\coprod X_i, M\right) \cong \prod \mathrm{Ext}(X_i, M),$$

and

$$\mathrm{Ext}\left(X, \prod M_j\right) \cong \prod \mathrm{Ext}(X, M_j).$$

Taking $\mathscr{A}$ to be the category of fppf $\mathbb{C}$-functors (with values in abelian groups), we may thus reduce to showing that $\mathrm{Ext}(\mathbb{G}_a, G) = 0$. Write $G = G_0 \times D \times V$ where G_0 is a connected group scheme of finite type over $\mathbb{C}$, D is discrete, and $V \in \{0, \mathbb{G}_a^\infty\}$. It suffices to show that $\mathrm{Ext}(\mathbb{G}_a, L) = 0$ whenever $L \in \{G_0, D, \mathbb{G}_a^\infty\}$.

First suppose that $L \in \{G_0, D\}$. By (2.1) this is equivalent to showing that any group scheme extension of $\mathbb{G}_a$ by such an L splits. In case $L = G_0$ this follows from ([22] §8.2, Prop. 2). The case $L = D$ is left to the reader.

Finally suppose that $L = \mathbb{G}_a^\infty$. We must show that any fppf-exact sequence of fppf sheaves

$$0 \to L \to F \xrightarrow{\pi} \mathbb{G}_a \to 0$$

splits. By (2.3) there exists an fppf $\mathbb{C}$-functor $M \subset F$ such that M maps onto $\mathbb{G}_a$ and $M \cap L \cong \mathbb{G}_a^k$ for some integer k. Thus we have an fppf-exact sequence of fppf $\mathbb{C}$-functors:

$$0 \to \mathbb{G}_a^k \to M \to \mathbb{G}_a \to 0.$$

This sequence splits. Hence the exact sequence

$$0 \to L \to F \to \mathbb{G}_a \to 0$$

splits. □

THEOREM 2.5. *Let*

$$0 \to F' \xrightarrow{i} F \xrightarrow{\pi} F'' \to 0$$

be an fppf-exact sequence of (abelian group)-valued fppf $\mathbb{C}$-functors. Assume that F' and F'' are additive × algebraic. Then so is F. Furthermore, if F' and F'' have finitely generated component groups, then so does F.

Proof. Given any list of pieces from which F'' may be built up from via extensions, we may reduce to the cases where F'' is one of those pieces. Thus we may assume that $F'' \in \{\mathbb{G}_a^\infty, G, D\}$ where G is connected algebraic, and D is discrete.

Case I: $F'' = \mathbb{G}_a^\infty$. This case follows immediately from (2.4).

Case II: $F'' = G$. Write $F' = I' \times A'$ where $I' \in \{0, \mathbb{G}_a^\infty\}$ and A' is algebraic. We have fppf-exact sequences:

$$0 \to I' \to (F/A')_{\mathrm{fppf}} \to F'' \to 0$$

and

$$0 \to A' \to F \to (F/A')_{\mathrm{fppf}} \to 0.$$

Assume for the moment that $(F/A')_{\mathrm{fppf}}$ is additive × algebraic. Write

$$(F/A')_{\mathrm{fppf}} = I^* \times A^*,$$

where $I^* \in \{0, \mathbb{G}_a^\infty\}$ and A^* is algebraic. We obtain fppf-exact sequences:

$$0 \to H \to F \to I^* \to 0 \qquad (\dagger)$$

and

$$0 \to A' \to H \to A^* \to 0.$$

By (2.1), H is algebraic. By (2.4), (†) splits, so F is additive × algebraic. Therefore (to prove Case II) it suffices to prove the theorem when $F' \cong \mathbb{G}_a^\infty$ and F'' is connected algebraic.

In that case, by (2.3), there is an fppf $\mathbb{C}$-functor $M \subset F$ such that M maps onto F'' and such that $M \cap F' \cong \mathbb{G}_a^k$ for some integer k. We have an fppf-exact sequence:

$$0 \to \mathbb{G}_a^k \to M \to F'' \to 0.$$

By (2.1), we conclude that M is algebraic. Furthermore,

$$(F/M)_{\mathrm{fppf}} \cong V/\mathbb{G}_a^k \cong \mathbb{G}_a^\infty.$$

We have an fppf-exact sequence:

$$0 \to M \to F \to (F/M)_{\mathrm{fppf}} \to 0.$$

By (2.4), this sequence splits. Hence F is additive × algebraic.

Case III: $F'' = D$. Then the map $F \to F''$ admits a splitting in the category of fppf $\mathbb{C}$-functors (with values in *sets*). If F' has no discrete part, then (by 2.2), any extension of D by F' splits, so F is additive × algebraic. In any case one has an fppf-exact sequence of fppf $\mathbb{C}$-functors:

$$0 \to F'/D' \to (F/D')_{\mathrm{fppf}} \to D \to 0,$$

where D' is the discrete part of F'. Hence $(F/D')_{\mathrm{fppf}}$ is additive × algebraic. Let $E = (F/D')_{\mathrm{fppf}}$. There exists a subfunctor $I \subset E$ such that $I \in \{0, \mathbb{G}_a^\infty\}$ and E/I is connected algebraic. Let F_I be the preimage of I in F. Then one has an fppf-exact sequence:

$$0 \to D' \to F_I \to I \to 0.$$

By (2.4), this exact sequence splits. Hence F_I is additive × algebraic. Since $(F/F_I)_{\mathrm{fppf}}$ is connected algebraic, Case III follows from the fppf-exact sequence:

$$0 \to F_I \to F \to (F/F_I)_{\mathrm{fppf}} \to 0$$

and from Case II. The assertion regarding finite generation of component groups is left to the reader. □

PROPOSITION 2.6. *Let F be a closed sub-group-functor of an additive × algebraic functor G. Then F is additive × algebraic. Furthermore, if G has finitely generated component group, then so does F.*

Proof. The statement regarding finite generation of component groups is left to the reader. We may assume that G is not algebraic, so $G \cong H \times I$ where H is algebraic and $I \cong \mathbb{G}_a^\infty$. Filter $I: I_1 \subset I_2 \subset \cdots \subset I$ where $I_n \cong \mathbb{G}_a^n$. Let $F_n = F \cap (H \times I_n)$. Then F_n is a closed sub-group-functor of $H \times I_n$. Let H_n be the fppf image of the map $F_n \to H$. Then H_n is a closed sub-group-scheme of H, and $H_1 \subset H_2 \subset \cdots$. Since there is a monomorphism $F_{n+1}/F_n \to \mathbb{G}_a$, and an epimorphism $F_{n+1}/F_n \to H_{n+1}/H_n$, we have $H_{n+1}/H_n \in \{0, \mathbb{G}_a\}$. Hence the sequence $H_1 \subset H_2 \subset \cdots$ must stabilize. It follows that the fppf image $\bar{F}$ of the map $F \to H$ is a closed sub-group-functor of H.

We have an fppf-exact sequence:

$$0 \to F \cap I \to F \to \bar{F} \to 0.$$

As $F \cap I$ is a closed sub-group-functor of I, it is clear that $F \cap I$ is additive. By (2.5), F is additive × algebraic. □

COROLLARY 2.7. *Let F be an additive × algebraic functor. Let F_1 and F_2 be closed sub-group-functors of F. Then $F_1 \cap F_2$ is additive × algebraic.*

Proof. The property of being a closed immersion of functors is stable under base extension, so $F_1 \cap F_2 \to F_1$ is a closed immersion of functors. Since $F_1 \to F$ is also a closed immersion of functors, so is $F_1 \cap F_2 \to F$. By (2.6), $F_1 \cap F_2$ is additive × algebraic. □

3. Nilpotent functors

DEFINITION. A (abelian group)-valued k-functor F is *nilpotent* if it is an fppf sheaf and if $F(T) = 0$ for every reduced k-scheme T.

It would be better to call these functors *infinitesimal*, but that would make the subsequent terminology excessively bulky.

DEFINITION. A morphism $\varphi: F \to G$ of k-functors is a *nilimmersion* if it is a closed immersion and if $\varphi(T)$ is an isomorphism for every reduced k-scheme T.

EXAMPLE 3.1. The fppf $\mathbb{C}$-functor N given on affine $\mathbb{C}$-schemes by

$$N(R) = \{\text{nilpotent elements of } R\}$$

is nilpotent. The functor N is not representable, but it may be viewed as a direct limit of representable $\mathbb{C}$-functors, where the direct limit is taken in ⟨⟨fppf $\mathbb{C}$-functors⟩⟩. A suitable directed system is

$$\operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2) \to \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^4) \to \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^8) \to \cdots.$$

For any $\mathbb{C}$-algebra R, one may think of $N(R)$ as the set of solutions (in R) of the infinite system of equations

$$x_1^2 = x_2, \qquad x_2^2 = x_3, \qquad x_3^2 = x_4, \ldots$$

where almost all of the variables are required to be zero. In this way N may be viewed as a closed subfunctor of $\mathbb{G}_a^\infty$, but the inclusion $N \to \mathbb{G}_a^\infty$ is not a group homomorphism.

EXAMPLE 3.2. Let M be the fppf $\mathbb{C}$-functor whose behavior on affine $\mathbb{C}$-schemes is given by: $M(R) = (R[x])^*/R^*$. Then $M(R)$ may be identified with

$$\{1 + n_1 x^1 + n_2 x^2 + \cdots + n_k x^k \colon k \in \mathbb{N},\, n_1, \ldots, n_k \in N(R)\}.$$

Therefore, as a functor to *sets*, M may be identified with $\bigoplus_{i=1}^\infty N$. One may also describe M as the solution functor of a certain infinite system of equations, in which almost all of the variables are required to be zero. Thus, as in the case of N, M may be embedded as a closed subfunctor of $\mathbb{G}_a^\infty$, but not as a closed subgroup-functor. The functor M arises in our analysis of group scheme sections: we have $\mathbf{Hom}(\mathbb{A}^1, \mathbb{G}_m) = \mathbb{G}_m \times M$.

DEFINITION. A (abelian group)-valued $\mathbb{C}$-functor F is *nilpotent* $\times$ *additive* $\times$ *algebraic* if it is of the form $N \times U \times R$ where N is nilpotent, U is additive, and R is algebraic.

LEMMA 3.3. *Let*

$$0 \to B \to F \to L \to 0 \tag{†}$$

be an exact sequence of $\mathbb{C}$-*functors. Assume that* L *is additive* $\times$ *algebraic and that* B *is nilpotent. Then* (†) *admits a unique splitting.*

Proof. If L is representable, it is reduced, so $F(L) \to L(L)$ is an isomorphism, and hence (†) admits a unique splitting. If L is not representable, there exists a filtration $L_1 \subset L_2 \subset \cdots \subset L$ by representable subfunctors, such that $L = (\bigcup_{i=1}^\infty L_i)_{\text{fppf}}$. For each i, base extend by $L_i \to L$ to obtain an exact sequence of $\mathbb{C}$-functors:

$$0 \to B \to F_i \to L_i \to 0. \tag{$†_i$}$$

As L_i is representable, this sequence splits uniquely. We thus obtain a compatible system of maps $L_i \to F$. These define a map $\bigcup_{i=1}^\infty L_i \to F$. Since L and B are fppf sheaves, so is F. Hence we obtain a map $L \to F$ which splits (†). The uniqueness of the splitting of (†) follows from the uniqueness of the splitting of the ($†_i$). □

COROLLARY 3.4. *Let*

$$0 \to F' \to F \to F'' \to 0$$

be an fppf-exact sequence of fppf $\mathbb{C}$*-functors. Assume that* F' *is nilpotent* × *additive* × *algebraic and that* F'' *is additive* × *algebraic. Then* F *is nilpotent* × *additive* × *algebraic. Furthermore, if we are given a decomposition* $F' = N' \times A'$ *where* N' *is nilpotent and* A' *is additive* × *algebraic, then we obtain (canonically) a decomposition* $F = N' \times A$ *for some additive* × *algebraic functor* A.

Proof. Write $F' = N' \times A'$ where N' is nilpotent and A' is additive × algebraic. We have an exact sequence of $\mathbb{C}$-functors:

$$0 \to A' \to F/N' \to F'' \to 0.$$

Since A' and F'' are fppf sheaves, it follows that F/N' is an fppf sheaf and that this exact sequence is fppf-exact. By (2.5), F/N' is additive × algebraic. We have an exact sequence of $\mathbb{C}$-functors:

$$0 \to N' \to F \to F/N' \to 0.$$

By (3.3), this sequence splits uniquely. □

REMARK 3.5. Suppose that one has an exact sequence:

$$0 \to L \to F \to B \to 0 \qquad (\dagger)$$

of $\mathbb{C}$-functors. Assume that L is additive × algebraic and that B is nilpotent. Does (†) split? An affirmative answer to this would allow one to conclude that any closed sub-group-functor of a nilpotent × additive × algebraic functor is nilpotent × additive × algebraic.

4. Sections of tori

The purpose of this section is to prove (4.5), which describes $\mathbf{Hom}(X, \mathbb{G}_m)$, where X is a variety. This reduces to a purely ring-theoretic problem, which we now consider. The problem is to describe the units in the ring $A \otimes_k B$ where k is an algebraically closed field, A is a domain containing k, and B is any ring containing k. First we consider the special case where B is a domain (4.2), then the case where Spec(B) is connected (4.3), and finally the case where B is arbitrary (4.4).

LEMMA 4.1. *Let* V *be an open subvariety of a normal projective variety* $\bar{V}$, *over an algebraically closed field k. Let* $D_1, \ldots, D_r$ *denote the irreducible components of*

$\bar{V} - V$ having codimension one in $\bar{V}$. Then there is a canonical isomorphism of the group $\Gamma(V, \mathcal{O}_V)^*/k^*$ with the group $\mathrm{Div}_0(\bar{V})$ consisting of those formal $\mathbb{Z}$-linear combinations $D = a_1D_1 + \cdots + a_rD_r$ with the properties that D is Cartier and that $\mathcal{O}_{\bar{V}}(D) \cong \mathcal{O}_{\bar{V}}$.

Proof. When $\bar{V}$ is nonsingular, the reader will probably agree that the lemma is geometrically obvious. In any case, we give a formal proof. Let $\mathrm{Div}(V)$ denote the group of Cartier divisors on V. Let $\mathrm{Weil}(V)$ denote the free abelian group on the set of codimension one subvarieties of V. Since V is normal, one knows by ([10] 21.6.9) that the canonical map $\mathrm{Div}(V) \to \mathrm{Weil}(V)$ is injective. Let K_V be the function field of V. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \to & \Gamma(\bar{V}, \mathcal{O}_{\bar{V}})^* & \to & K_{\bar{V}}^* & \to & \mathrm{Div}(\bar{V}) & \xrightarrow{c} & \mathrm{Pic}(\bar{V}) & \to 0 \\ & \downarrow a & & \downarrow = & & \downarrow b & & \downarrow & \\ 0 \to & \Gamma(V, \mathcal{O}_V)^* & \to & K_V^* & \to & \mathrm{Div}(V) & \to & \mathrm{Pic}(V) & \to 0 \end{array}$$

A diagram chase shows that $\mathrm{Coker}(a) \cong \mathrm{Ker}(b) \cap \mathrm{Ker}(c)$. □

LEMMA 4.2. *Let k be an algebraically closed field. Let A and B be rings containing k. Assume that A is a domain, and that B is a domain, or more generally a reduced ring such that $\mathrm{Spec}(B)$ is connected. Let $u \in A \otimes_k B$ be a unit. Then $u = a \otimes b$ for some units $a \in A$ and $b \in B$.*

Proof. We may assume that A and B are finitely generated over k.

Suppose that B is a domain. By localizing at nonzero elements, we may assume that A and B are regular. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Embed X [resp. Y] as an open subvariety of a projective normal variety $\bar{X}$ [resp. $\bar{Y}$]. Note that if resolution of singularities were available, we could assume that $\bar{X}$ and $\bar{Y}$ are nonsingular.

Apply (4.1) in three separate cases:

(i) $V = X$, $\bar{V} = \bar{X}$;
(ii) $V = Y$, $\bar{V} = \bar{Y}$;
(iii) $V = X \times Y$, $\bar{V} = \bar{X} \times \bar{Y}$.

We have an injective map $\phi: \mathrm{Div}_0(\bar{X}) \times \mathrm{Div}_0(\bar{Y}) \to \mathrm{Div}_0(\bar{X} \times \bar{Y})$. To complete the proof, it suffices to show that ϕ is surjective.

We first assume resolution of singularities, as this makes the proof simpler. We may assume that $\bar{X}$ and $\bar{Y}$ are nonsingular. Consider an element of $\mathrm{Div}_0(\bar{X} \times \bar{Y})$. It may be expressed in the form $D+E$ where D is a Cartier divisor supported on $(\bar{X} - X) \times \bar{Y}$ and E is a Cartier divisor supported on $\bar{X} \times (\bar{Y} - Y)$. We know that the image $[D+E]$ of $D+E$ in $\mathrm{Pic}(\bar{X} \times \bar{Y})$ is zero. To complete the proof it suffices to show that $[D] = 0$ and $[E] = 0$. Thus the composite:

$$\mathrm{Div}_0(\bar{X} \times \bar{Y}) \to \mathrm{Pic}(\bar{X}) \times \mathrm{Pic}(\bar{Y}) \to \mathrm{Pic}(\bar{X} \times \bar{Y})$$

is zero and we have to show that the first arrow is zero. It suffices to show that the second arrow is injective. This is clear. The proof (when B is a domain) is now morally complete.

We explain how to bypass the use of resolution of singularities. Consider an element of $\mathrm{Div}_0(\bar{X} \times \bar{Y})$. It may be expressed in the form $D+E$ where D is a Weil divisor supported on $(\bar{X}-X)\times\bar{Y}$ and E is a Weil divisor supported on $\bar{X}\times(\bar{Y}-Y)$. In order to apply the method which we used when $\bar{X}$ and $\bar{Y}$ were nonsingular, it suffices to show that D and E are both Cartier. This will complete the proof (when B is a domain). Let $S \subset \bar{X}\times\bar{Y}$ be $(\bar{X}-X)\times(\bar{Y}-Y)$. Let $U=(\bar{X}\times\bar{Y})-S$. Since $D|_U$ does not meet $E|_U$, we see that $D|_U$ and $E|_U$ are Cartier.

We show that $D|_U$ is induced from a Cartier divisor on $\bar{X}$ via the canonical map $U \to \bar{X}$. Certainly D is the pullback of a Weil divisor D_0 from $\bar{X}$. It suffices to show that D_0 is Cartier. Let $y \in Y$ be any closed point. The composite:

$$\bar{X} = \bar{X} \times \{y\} \to U \to \bar{X}$$

is an isomorphism. Hence $D_0 = D|_{\bar{X}\times\{y\}}$ is Cartier. Thus $D|_U$ is induced from a Cartier divisor on $\bar{X}$. Hence $D|_U$ extends to a Cartier divisor D' on $\bar{X}\times\bar{Y}$. Since $(\bar{X}\times\bar{Y})-U$ has codimension two in $\bar{X}\times\bar{Y}$, it follows that $D=D'$. Hence D is Cartier. Similarly E is Cartier. This completes the proof when B is a domain.

Now suppose that B is not necessarily a domain, but that B is reduced and that $\mathrm{Spec}(B)$ is connected. Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ be the minimal primes of B. Let $x \in (A \otimes B)^*$. For each i, let x_i be its image in $A \otimes B/\mathfrak{p}_i$. Since $B/\mathfrak{p}_i$ is a domain, we may write $x_i = a_i \otimes b_i$ for some units $a_i \in A$ and $b_i \in B/\mathfrak{p}_i$. Let $\bar{B}=B/\mathfrak{p}_1\times\cdots\times B/\mathfrak{p}_n$. Since B is reduced, we have an injective map $B \to \bar{B}$. Identify b_i with the element $(0,\ldots,b_i,\ldots,0)\in\bar{B}$, where b_i is in the ith spot. Identify x with its image in $A\otimes\bar{B}$. Then (in $A\otimes\bar{B}$):

$$x = (a_1 \otimes b_1) + \cdots + (a_n \otimes b_n).$$

Let $\mathfrak{m} \subset A$ be a maximal ideal. Let $\bar{x}$ be the image of x in $B=(A/\mathfrak{m})\otimes B$. In $(A/\mathfrak{m})\otimes\bar{B}$, we have $\bar{x}=(c_1\cdot b_1)+\cdots+(c_n\cdot b_n)$, for suitable constants $c_1,\ldots,c_n\in k$, all nonzero since the a_i are units. Adjusting the original a_i's and b_i's slightly, we may assume that the c_i's are all equal to one: $\bar{x}=b_1+\cdots+b_n$. Hence there exists some $b\in B$ which lifts $b_1,\ldots,b_n$.

Suppose that $V(\mathfrak{p}_i)$ meets $V(\mathfrak{p}_j)$. The existence of b implies that $b_i=b_j$ on $V(\mathfrak{p}_i)\cap V(\mathfrak{p}_j)$. Hence $a_i=a_j$. Since $\mathrm{Spec}(B)$ is connected, $a_1=\cdots=a_n$. Hence $x=a_1\otimes b$. □

We now generalize the lemma to the case where B is not necessarily reduced, still assuming that $\mathrm{Spec}(B)$ is connected.

COROLLARY 4.3. *Let k be an algebraically closed field. Let A and B be rings containing k. Assume that A is a domain and that* Spec(B) *is connected. Let $\mathfrak{m} \subset A$*

be a maximal ideal. Then $(A \otimes_k B)^*$ *is the direct sum of the two subgroups* A^*B^* *and* $1+(\mathfrak{m} \otimes \mathrm{Nil}(B))$, *where* $\mathrm{Nil}(B)$ *denotes the nilradical of* B.

Proof. First we show (I) that A^*B^* and $1+(\mathfrak{m} \otimes \mathrm{Nil}(B))$ together generate $(A \otimes_k B)^*$. Since (by 4.2) A^*B^* surjects onto $(A \otimes B_{\mathrm{red}})^*$, it suffices to show that

$$\mathrm{Ker}[(A \otimes B)^* \to (A \otimes B_{\mathrm{red}})^*] \subset A^*B^* \cdot [1+(\mathfrak{m} \otimes \mathrm{Nil}(B))].$$

Evidently this kernel is $1+[A \otimes \mathrm{Nil}(B)]$, so it suffices to show that

$$1 + [A \otimes \mathrm{Nil}(B)] \subset B^* \cdot [1 + \mathfrak{m} \otimes \mathrm{Nil}(B)].$$

Let

$$x \in 1 + [A \otimes \mathrm{Nil}(B)], \qquad x = 1 + (a_1 \otimes b_1) + \cdots + (a_n \otimes b_n).$$

Write

$$a_i = c_i + y_i \quad \text{where } c_i \in k \quad \text{and} \quad y_i \in \mathfrak{m}.$$

Let

$$u = 1 + (c_1 \otimes b_1) + \cdots + (c_n \otimes b_n).$$

Then

$$u \in B^* \quad \text{and} \quad x = u + (y_1 \otimes b_1) + \cdots + (y_n \otimes b_n).$$

Hence

$$x = u[1 + (y_1 \otimes b_1 u^{-1}) + \cdots + (y_n \otimes b_n u^{-1})].$$

This proves (I).

Now we show (II) that:

$$A^*B^* \cap [1 + (\mathfrak{m} \otimes \mathrm{Nil}(B))] = \{1\}.$$

Indeed, let x lie in the intersection. Write

$$x = uv = 1 + (a_1 \otimes b_1) + \cdots + (a_n \otimes b_n)$$

where $u \in A^*$, $v \in B^*$, $a_i \in \mathfrak{m}$, $b_i \in \mathrm{Nil}(B)$. Passing to $A/\mathfrak{m}$, we conclude that $v \in k$. Similarly, we conclude that $u \in k$. Then $x-1$ is a nilpotent element of k, so $x = 1$. This proves (II). □

As a final generalization, we allow B to be arbitrary.

COROLLARY 4.4. *Let k be an algebraically closed field. Let A and B be rings containing k. Assume that A is a domain. Let $\mathfrak{m} \subset A$ be a maximal ideal.*

- *For any decomposition $B = B_1 \times \cdots \times B_n$, there is a subgroup:*

$$\mu(B_1, \ldots, B_n) = \bigoplus_{i=1}^{n} [A^* B_i^* \oplus (1 + (\mathfrak{m} \otimes \mathrm{Nil}(B_i))]$$

of $(A \otimes_k B)^*$, where $\mathrm{Nil}(B_i)$ denotes the nilradical of B_i.
- *For any $x \in (A \otimes_k B)^*$, there exists a decomposition $B = B_1 \times \cdots \times B_n$ such that $x \in \mu(B_1, \ldots, B_n)$.*
- *If B has only finitely many idempotent elements (e.g. if B is noetherian), we can write $B = B_1 \times \cdots \times B_n$ for rings B_i having connected spectra. Then $\mu(B_1, \ldots, B_n) = (A \otimes_k B)^*$.*

COROLLARY 4.5. *Let S be a variety over an algebraically closed field k. Then for each choice of closed point $x \in S$, one obtains a canonical decomposition of k-functors:*

$$\mathbf{Hom}(S, \mathbb{G}_m) = F \times I$$

where F is representable by a group scheme of the form $\mathbb{G}_m \times \mathbb{Z}^n$ (for some $n \geqslant 0$) and I is nilpotent. The inclusion $F \to \mathbf{Hom}(S, \mathbb{G}_m)$ is independent of x.

Proof. Let F be the fppf k-functor associated to the k-functor F_0 given by $T \mapsto \Gamma(S, \mathcal{O}_S)^* \Gamma(T, \mathcal{O}_T)^*$. The latter group is a subgroup of

$$\Gamma(S \times T, \mathcal{O}_{S \times T})^*,$$

so F is a subfunctor of $\mathbf{Hom}(S, \mathbb{G}_m)$. It follows from (4.1) that $D = \Gamma(S, \mathcal{O}_S)^*/k^*$ is free abelian of finite rank. Since k^* is divisible, the exact sequence:

$$1 \to k^* \to \Gamma(S, \mathcal{O}_S)^* \to D \to 1$$

splits. Choose a splitting. This choice determines an isomorphism of F_0 with the functor given by $T \mapsto \Gamma(T, \mathcal{O}_T)^* \times D$. Hence F is representable as claimed.

Suppose now that T is reduced. We wish to show that the inclusion $F(T) \to \mathbf{Hom}(S, \mathbb{G}_m)(T)$ is surjective. For this we may suppose that T is connected and affine. Then (see e.g. [11] 9.3.13(i)) one knows that $\Gamma(S \times T, \mathcal{O}_{S \times T}) = \Gamma(S, \mathcal{O}_S) \otimes \Gamma(T, \mathcal{O}_T)$. Surjectivity now follows from (4.2).

Finally, let $\mathfrak{m} \subset \Gamma(S, \mathcal{O}_S)$ be the maximal ideal corresponding to the closed point $x \in S$ via the map $S \to \operatorname{Spec} \Gamma(S, \mathcal{O}_S)$. Let I be the fppf subsheaf of

Hom$(S, \mathbb{G}_m)$ associated to the k-functor given by

$$T \mapsto 1 + (\mathfrak{m} \otimes \mathrm{Nil}(\Gamma(T, \mathcal{O}_T))).$$

From (4.4), it follows that the canonical map $F \oplus I \to \mathbf{Hom}(S, \mathbb{G}_m)$ is an isomorphism. □

REMARK 4.6. The k-functor I, viewed as a functor to sets, admits the decomposition $I = \bigoplus_{i=1}^{\infty} N$ (or $I = 0$), where $N(R)$ is the set of nilpotent elements in a ring R (cf. 3.2). Thus I may be embedded as a closed subfunctor of $\mathbb{G}_a^{\infty}$, but not as a closed sub-group-functor (unless $I = 0$).

5. Sections of abelian schemes

LEMMA 5.1. *Let A be a complex abelian variety. Let S be a variety. Then for each choice of a closed point $x \in S$, one obtains a canonical decomposition of $\mathbb{C}$-functors* **Hom**$(S, A) = A \times D \times N$ *where $D \cong \mathbb{Z}^n$ for some $n \geqslant 0$ and N is nilpotent.*

REMARK 5.2. Let A be an abelian variety, and let S be a variety. The functor **Hom**(S, A) is in general not representable. For instance, **Hom**$(\mathbb{A}^1, A)$ is never representable. Certainly the natural map $A \to \mathbf{Hom}(\mathbb{A}^1, A)$ is bijective on T-valued points, for any reduced T. But it is not surjective on T-valued points when $T = \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)$, because $A(T) = \mathbf{T}(A)(\cdot)$ but

$$\mathbf{Hom}(\mathbb{A}^1, A)(T) = \mathrm{Hom}(\mathbb{A}^1, \mathbf{T}(A)).$$

(Here $\mathbf{T}(A)$ stands for the tangent bundle of A.)

Proof of 5.1. There is a map **Hom**$(S, A) \to A$ of $\mathbb{C}$-functors given by evaluation at x. There is also a map:

$$\mathrm{Hom}(S, A) \to \mathrm{Hom}[H_1(S, \mathbb{Z}), H_1(A, \mathbb{Z})]$$

of abelian groups. Let D be its image. Identify D with the corresponding representable functor. We construct a map of $\mathbb{C}$-functors **Hom**$(S, A) \to D$, as follows. It suffices to construct a map λ: Hom$(S \times T, A) \to D$, for each connected $\mathbb{C}$-scheme T. If T is of finite type over $\mathbb{C}$, we could pick a closed point $t \in T$, and let $\lambda(f) = H_1(f|_{S \times \{t\}}, \mathbb{Z})$. Then λ would be independent of the choice $t \in T$.

Consider the general case, where T is an arbitrary $\mathbb{C}$-scheme. Let $f: S \times T \to A$. Let $U \subset T$ be a connected open affine subscheme, say $U = \operatorname{Spec}(R)$. There exists a subring $R_0 \subset R$ which is finitely generated over $\mathbb{C}$, and a morphism $f_0: S \times U_0 \to A$, compatible with f. (Here $U_0 = \operatorname{Spec}(R_0)$.) Let $u \in U_0$ be a closed point, and define $\lambda(f) = H_1(f_0|_{S \times \{u\}}, \mathbb{Z})$. It is not difficult to verify that λ is independent of all choices. Thus we have a map of $\mathbb{C}$-functors $\mathbf{Hom}(S, A) \to D$.

We obtain a map $\phi: \mathbf{Hom}(S, A) \to A \times D$ of $\mathbb{C}$-functors. It is clear that $\phi(T)$ is surjective for every $\mathbb{C}$-scheme T. Let $N = \mathrm{Ker}(\phi)$, so we have an exact sequence:

$$0 \to N \to \mathbf{Hom}(S, A) \to A \times D \to 0.$$

To complete the proof, we must show:

(a) D is free abelian of finite rank;
(b) $\phi(T)$ is injective for every reduced $\mathbb{C}$-scheme T.

It will follow that the above exact sequence splits uniquely, thereby proving the lemma.

We prove (a). By [12], one knows that S is homeomorphic to $|K| - |L|$ for some finite simplicial complex K and some subcomplex $L \subset K$. In particular, S itself has the homotopy type of a finite simplicial complex, so $H_1(S, \mathbb{Z})$ is finitely generated. Then (a) follows from the fact that $H_1(A, \mathbb{Z})$ is a free abelian group of finite rank.

We prove (b). Let T be a reduced $\mathbb{C}$-scheme and $f: S \times T \to A$ be a morphism for which $\phi(f) = 0$. We may assume that T is connected, affine, and of finite type over $\mathbb{C}$. Then $H_1(f|_{S \times \{t\}}, \mathbb{Z}) = 0$ for each closed point $t \in T$.

Let $h: S \to A$ be any morphism for which $H_1(h, \mathbb{Z}) = 0$. Since A has abelian fundamental group, it follows that h factors through the universal cover $\mathbb{C}^N \to A$. Let $H: S \to \mathbb{C}^N$ be the lift. We will show that H is constant. For this, we may reduce to the case where S is a nonsingular curve. Let $\tilde{S}$ be a complete nonsingular curve containing S as an open subvariety. Since any rational map from a nonsingular variety to an abelian variety is a morphism ([13] 10.12(i)), h extends to a morphism $\tilde{h}: \tilde{S} \to A$. Since the natural map $H_1(S, \mathbb{Z}) \to H_1(\tilde{S}, \mathbb{Z})$ is surjective, $H_1(\tilde{h}, \mathbb{Z}) = 0$. Hence $\tilde{h}$ lifts to a map $\tilde{H}: \tilde{S} \to \mathbb{C}^N$. Since every holomorphic function on $\tilde{S}$ is constant, $\tilde{H}$ is constant. Hence H is constant. Hence h is constant.

We conclude that $f|_{S \times \{t\}}$ is constant for each closed point $t \in T$. Since $f|_{\{x\} \times T} = 0$, and since T is reduced, we conclude that $f = 0$. This proves (b). □

LEMMA 5.3. *Let S be a complex variety. Let X be an S-scheme. Let X_0 be a closed subscheme of X. Then the canonical map*

$$\mathbf{Sect}(X_0/S) \to \mathbf{Sect}(X/S)$$

is a closed immersion.

Sketch. Let T be a $\mathbb{C}$-scheme and let $\sigma: S \times T \to X$ be a family of sections of X.

Let $P=(S\times T)\times_X X_0$. Then factorizations of σ through X_0 correspond bijectively with sections of the closed immersion $i: P\to S\times T$. Therefore, the problem is to find a universal map i: $T_0\to T$ which makes i_{T_0} an isomorphism, and such that i is a closed immersion. First suppose that S is affine. Cover T by open affines T_i. For each i, construct a closed subscheme $T_{i,0}\subset T_i$ with the appropriate universal property. By uniqueness, these patch together to form a closed subscheme of T with the appropriate universal property. If S is not affine, cover it by open affines S_j, and for each such j construct a closed subscheme ${}_jT_0\subset T$ as above. Let T_0 be the intersection of the ${}_jT_0$. □

PROPOSITION 5.4. *Let S be a complex variety. Let A be an abelian scheme over S. Then there exists an algebraic functor F and a nilimmersion $F\hookrightarrow$* **Sect**(A/S). *Furthermore, F is of the form $A_0\times D$ where A_0 is an abelian variety and D is a finitely generated discrete group.*

REMARK 5.5. The proof shows that there exists an open subvariety $U\subset S$, such that to any closed point $x\in U$, we may canonically associate a subfunctor F as above.

REMARK 5.6. We do not know if the closed immersion $F\to$ **Sect**(A/S) admits a splitting.

Proof of 5.4. Let K be the function field of S. Let A_0 be the $K/\mathbb{C}$-trace of A_K. (See ([18] 20.5) or ([15] p. 213).) It is an abelian variety A_0 over $\mathbb{C}$, together with a morphism $\tau:(A_0)_K\to A_K$ of K-group schemes, having finite kernel, such that given any abelian variety B (over $\mathbb{C}$) and any morphism $\phi: B_K\to A_K$, there is a unique morphism $\alpha: B\to A_0$ such that $\tau\circ\alpha_K=\phi$. According to ([15] VIII, §3, Corollary 2), the map τ is purely inseparable, and hence (since we are in characteristic zero), τ is a closed immersion.

There exists an open subvariety $U\subset S$ and a closed immersion $A_0\times U\to A_U$ of group schemes which extends τ. By ([14] 4.2), we may reduce to the case where $U=S$, and so assume that we have a closed immersion $A_0\times S\to A$ of S-group schemes. (This step is more carefully explained in the proof of (6.3).)

According to a theorem of Lang and Néron (see [16] Chapter 6, Theorem 2), the group $A(K)/A_0(\cdot)$ is finitely generated. Hence the group $D_0=A(S)/A_0(\cdot)$ is finitely generated.

For each $d\in D_0$, choose some section $\sigma_d: S\to A$ which maps to d. Define $A_d=(A_0\times S)+\sigma_d$, which is a closed subscheme of A. Let P be the scheme-theoretic disjoint union of the A_d, as $d\in D_0$ varies. We have a canonical morphism $h: P\to A$ of S-group schemes. Then $P\cong D_0\times S$. Hence

$$\begin{aligned}\mathbf{Sect}(P/S)&\cong\mathbf{Sect}\left(\frac{A_0\times S}{S}\right)\times\mathbf{Sect}\left(\frac{D_0\times S}{S}\right)\\&\cong\mathbf{Hom}(S,A_0)\times D_0.\end{aligned}$$

By (5.1), we know that **Hom**(S, A_0) is nilpotent × additive × algebraic. Hence so is **Sect**(P/S). Write **Sect**$(P/S) = N \times F$ where N is nilpotent and F is additive × algebraic. We have monomorphisms:

$$F \overset{i}{\hookrightarrow} \mathbf{Sect}(P/S) \overset{j}{\hookrightarrow} \mathbf{Sect}(A/S).$$

Certainly i is a nilimmersion. To complete the proof, we must show that the same holds for j.

We show that $j(T)$ is an isomorphism for any reduced $\mathbb{C}$-scheme T. For this, we may suppose that T is connected and affine. It suffices to show that

$$\mathbf{Sect}(A/S)(\cdot) + \mathbf{Sect}\left(\frac{A_0 \times S}{S}\right)(T) = \mathbf{Sect}(A/S)(T).$$

By construction, any section $\sigma: S \to A$ factors through A_d for some $d \in D_0$. We need to show that any family of sections $f: S \times T \to A$ factors through some A_d. Write $T = \mathrm{Spec}(B)$. Given such an f, there exists a subring $B_0 \subset B$ which is finitely generated over $\mathbb{C}$ and a morphism $f_0: S \times T_0 \to A$ which makes the obvious diagram commute. (Here $T_0 = \mathrm{Spec}(B_0)$.) Hence we may assume that T is of finite type over $\mathbb{C}$. (Also T is affine, connected, and reduced.) Hence $f(S \times T)$ is connected and constructible. For any closed point $t \in T$, $f(S \times \{t\})$ is contained in some A_d. Hence $f(S \times T)$ is contained in the union of the A_d. Since there are only countably many of them, it follows that $f(S \times T)$ is contained in a finite union of them, say $f(S \times T) \subset A_{d_1} \cup \cdots \cup A_{d_n}$. For any closed point $t \in T$, there is a unique i such that $f(S \times \{t\}) \subset A_{d_i}$. These conditions imply that for some $i \in [1, n]$, $f(S \times T) \subset A_{d_i}$. Hence $j(T)$ is an isomorphism for any reduced $\mathbb{C}$-scheme T.

We show that j is a closed immersion. Let T be a $\mathbb{C}$-scheme, and let $\sigma: S \times T \to A$ be a family of sections. We may assume that T is connected. Then $\sigma|_{S \times T_{\mathrm{red}}}$ factors through P. Since $S \times T_{\mathrm{red}}$ is connected, $\sigma|_{S \times T_{\mathrm{red}}}$ factors through A_d for some $d \in D_0$. Therefore, if Y is any T-scheme, and $\sigma|_{S \times Y}$ factors through P, it follows that $\sigma|_{S \times Y}$ factors through A_d. Since A_d is a closed subscheme of A, it follows from (5.3) that j is a closed immersion.

This completes the proof, except for checking that F is representable by a scheme of the desired type. This is easy to check. □

6. Main theorem on group scheme sections

The main result of this section is (6.3). Let S be a scheme. By a *group space* over S, we mean a group object in the category of algebraic spaces over S.

PROPOSITION 6.1. *Let S be a complex variety. Let G be an étale group space over S. Then* **Sect**(G/S) *is discrete.*

Proof. Consider the group **Sect**$(G/S)(\cdot)$. This group generates a discrete subfunctor $D \subset$ **Sect**(G/S). We have to show that the inclusion is an isomorphism. To do this, we must show that if T is any connected $\mathbb{C}$-scheme, and $\sigma: S \times T \to G$ is a family of sections, then there exists a section $\sigma': S \to G$ such that $\sigma = \sigma' \circ \pi_1$, where $\pi_1: S \times T \to S$ is the first projection.

First one may reduce to the case where T is affine, say $T = \mathrm{Spec}(A)$. Having done so, one may find a sub-$\mathbb{C}$-algebra $A_0 \subset A$ which is finitely generated over $\mathbb{C}$ and a family of sections $S \times T_0 \to G$ which is compatible with σ. (Here $T_0 = \mathrm{Spec}(A_0)$.) In this way we may reduce to the case where T is of finite type over $\mathbb{C}$. Since G/S is étale, the map $\mathrm{Hom}_S(S \times T, G) \to \mathrm{Hom}_S(S \times T_{\mathrm{red}}, G)$ is bijective. Hence we may assume that T is reduced.

Pick any closed point $t \in T$. Let $\sigma': S \to G$ be $\sigma|_{S \times \{t\}}$, and let $\sigma_t: S \times T \to G$ be $\sigma' \circ \pi_1$. It suffices to show that $\sigma = \sigma_t$. Let $s \in S$ be a closed point. Then G_s is an étale group space over $\mathrm{Spec}(\mathbb{C})$. By ([1] 4.2), G_s is in fact an étale group scheme. By ([8] 0.3), G_s is separated. The maps $T \to G_s$ induced by σ and σ_t agree at the point t. Since $G_s \times T \to T$ is étale and separated, it follows that $\sigma|_{\{s\} \times T} = \sigma_t|_{\{s\} \times T}$. Hence $\sigma = \sigma_t$. □

LEMMA 6.2. *Let S be a complex variety. Let G be a smooth commutative group scheme over S, with connected fibers. Then there exists a variety U, an étale morphism $\pi: U \to S$, integers $k, n \geqslant 0$, an abelian scheme A over U, and an exact sequence:*

$$0 \to \mathbb{G}_m^k \times \mathbb{G}_a^n \times U \to G_U \to A \to 0$$

of commutative group schemes over U.

Proof. Let K be the function field of S. Since K is perfect, there exists a torus M_0 over K, an abelian variety A_0 over K, an integer $n \geqslant 0$, and an exact sequence:

$$0 \to M_0 \times \mathbb{G}_a^n \to G_K \xrightarrow{p} A_0 \to 0$$

of commutative group schemes over K. (See ([20] 3.9, 4.1.3, 7.2.1) and ([21] Theorem 16).)

By standard arguments (see e.g. [18] 20.9), one may find a nonempty open subset U of S, and an abelian scheme A' over U which extends A_0. By making U sufficiently small, we may also assume that p extends to a surjective morphism $p': G_U \to A'$ of group schemes over U. Consider the morphism $\phi: G_U \times G_U \to A'$ given (in essence) by $\phi(a, b) = p'(a+b) - p'(a) - p'(b)$. Since $\phi = 0$ over the generic point of U, it follows that in fact $\phi = 0$. Hence p' is a group homomorphism.

Let $R = \mathrm{Ker}(p')$. Shrinking U if necessary, we may assume that $R = R_1 \times R_2$, extending the generic decomposition $M_0 \times \mathbb{G}_a^n$. In general, if Y_1 and Y_2 are schemes of finite type over S, and $(Y_1)_K \cong (Y_2)_K$, then there exists a Zariski neighborhood U of the generic point of S such that $(Y_1)_U \cong (Y_2)_U$. Similarly, if $(Y_1)_{K^a} \cong (Y_2)_{K^a}$, then there exists an étale neighborhood U of the generic point of S such that $(Y_1)_U \cong (Y_2)_U$. These statements are also valid for group schemes. We are done since $(R_1)_{K^a} \cong \mathbb{G}_m^k$ (for some k) and $(R_2)_K \cong \mathbb{G}_a^n$. □

THEOREM 6.3. *Let S be a complex variety. Let G be a commutative, flat group scheme, locally of finite type over S. Then there exists an additive × algebraic functor F and a nilimmersion* $F \hookrightarrow$ **Sect**(G/S). *If the generic fiber of G has no unipotent part then F is algebraic and has no unipotent part. If G/S has connected fibers or even if the generic fiber of G has finitely generated component group, then the component group of F is finitely generated.*

REMARK 6.4. Ditto remark (5.5).

REMARK 6.5. We do not know if the nilimmersion $F \to$ **Sect**(G/S) splits. It would also be interesting to analyze the structure of **Sect**$(G/S)/F$. We do not know much about this. (However, see 4.6.)

REMARK 6.6. We do not know if the theorem remains valid when G is replaced by a smooth commutative group space. The proof breaks down because we do not know whether or not a smooth commutative group space G with connected fibers (over a complex variety S) is a scheme. Therefore we are unable to apply ([14] 4.2). In fact, it does not seem to be known if there exist such objects G/S which are not schemes, especially when S is normal. We do not even know if such an object G is necessarily separated.

Proof of 6.3. Since G is a flat group scheme, locally of finite type, and since we are in characteristic zero, G is smooth. We first suppose that G/S has connected fibers.

By ([4] 5.5), we know that G is separated. Let U be as in (6.2). By ([14] 4.2), the map **Sect**$(G/S) \to$ **Sect**(G_U/U) is a closed immersion.

Suppose that we find a subfunctor $F_U \subset$ **Sect**(G_U/U) such that F_U is additive × algebraic and such that the inclusion is a nilimmersion. Then we can define a functor F by the fiber product diagram:

$$\begin{array}{ccc} F & \to & \mathbf{Sect}(G/S) \\ \downarrow & & \downarrow \\ F_U & \to & \mathbf{Sect}(G_U/U). \end{array}$$

Then the map $F \to F_U$ will be a closed immersion. By (2.6), it will follow that F is additive × algebraic. Also, $F \to$ **Sect**(G/S) will be a nilimmersion. In this way we may reduce to the case $S=U$, and so assume that there is an exact sequence:

$$0 \to \mathbb{G}_m^k \times \mathbb{G}_a^n \times S \to G \to A \to 0$$

of commutative group schemes over S.

Let A_0, $h: P \to A$ be as in the proof of (5.4). Then

- h is a disjoint union (on the source) of closed immersions;
- **Sect**(h) is a nilimmersion.

Let $G_P = G \times_A P$. Let $j: G_P \to G$ be the natural map. We note that **Sect**($-/S$), viewed as a functor:

$$\langle\langle S\text{-schemes}\rangle\rangle \to \langle\langle \mathbb{C}\text{-functors}\rangle\rangle$$

preserves fiber products. Hence j has all of the above properties. We have an exact sequence:

$$0 \to \mathbb{G}_m^k \times \mathbb{G}_a^n \times S \to G_P \to P \to 0$$

and hence an exact sequence:

$$0 \to \mathbf{Hom}(S, \mathbb{G}_m^k \times \mathbb{G}_a^n) \to \mathbf{Sect}(G_P/S) \to \mathbf{Sect}(P/S)$$

of $\mathbb{C}$-functors.

Let I be the fppf image of the map $\mathbf{Sect}(G_P/S) \to \mathbf{Sect}(P/S)$. We need to construct an algebraic functor I_0 and a nilimmersion $I_0 \hookrightarrow I$. From the proof of (5.4), we know that **Sect**(P/S) is nilpotent × additive × algebraic. Write

$$\mathbf{Sect}(P/S) = N \times R$$

where N is nilpotent and R is additive × algebraic. We have a natural inclusion $A_0 \subset R$. Clearly

$$A_0 \subset I \subset N \times R.$$

Then $(I/A_0)_{\mathrm{fppf}} \subset N \times (R/A_0)$. We know that R/A_0 is discrete. It follows (exercise) that $(I/A_0)_{\mathrm{fppf}}$ is of the form $N_0 \times D_0$ for some nilpotent $\mathbb{C}$-functor N_0 and some discrete $\mathbb{C}$-functor D_0. Define I_0 by the fiber product diagram

$$\begin{array}{ccc} I & \to & (I/A_0)_{\mathrm{fppf}} \\ \uparrow & & \uparrow \\ I_0 & \to & D_0. \end{array}$$

We have an fppf-exact sequence:

$$0 \to A_0 \to I_0 \to D_0 \to 0$$

of fppf $\mathbb{C}$-functors. It follows by (2.1) that I_0 is algebraic. Furthermore, the map $I_0 \to I$ is as claimed.

We have an fppf-exact sequence:

$$0 \to \mathbf{Hom}(S, \mathbb{G}_m^k \times \mathbb{G}_a^n) \to \mathbf{Sect}(G_P/S) \to I \to 0$$

of fppf $\mathbb{C}$-functors. Let $F_0 = \mathbf{Sect}(G_P/S) \times_I I_0$, the fiber product in the category of $\mathbb{C}$-functors. Since the map $I_0 \to I$ is a nilimmersion, so is the map $F_0 \to \mathbf{Sect}(G_P/S)$. We obtain an fppf-exact sequence:

$$0 \to \mathbf{Hom}(S, \mathbb{G}_m^k \times \mathbb{G}_a^n) \to F_0 \to I_0 \to 0$$

of fppf $\mathbb{C}$-functors.

Observe that

$$\mathbf{Hom}(S, \mathbb{G}_m^k \times \mathbb{G}_a^n) = \mathbf{Hom}(S, \mathbb{G}_m)^k \times \mathbf{Hom}(S, \mathbb{G}_a^n).$$

By (4.5), $\mathbf{Hom}(S, \mathbb{G}_m)$ is nilpotent × additive × algebraic. Clearly

$$\mathbf{Hom}(S, \mathbb{G}_a^n) \in \{0, \mathbb{G}_a^\infty\}.$$

Hence $\mathbf{Hom}(S, \mathbb{G}_m^k \times \mathbb{G}_a^n)$ is nilpotent × additive × algebraic. By (3.4), F_0 is nilpotent × additive × algebraic. In fact, F_0/B is additive × algebraic and the map $F_0 \to F_0/B$ splits uniquely. Let $F = F_0/B$. We have maps:

$$F \hookrightarrow F_0 \hookrightarrow \mathbf{Sect}(G_P/S) \hookrightarrow \mathbf{Sect}(G/S).$$

All of these maps are nilimmersions. This completes the proof when G/S has connected fibers.

We now return to the general case. Let G^0 denote the connected component of the identity of G. Since G is smooth, G^0 is representable, see ([4] 3.10). By ([2] 7.3(i)) and ([3] 6.3), the fppf quotient G/G^0 is representable by an algebraic space, locally of finite type over S. Then G/G^0 is flat over S. (See e.g. [4] 9.2(xi), at least when G/G^0 is a scheme.) It follows that G/G^0 is étale over S.

We have an exact sequence:

$$0 \to \mathbf{Sect}(G^0/S) \to \mathbf{Sect}(G/S) \to \mathbf{Sect}([G/G^0]/S)$$

of $\mathbb{C}$-functors. By (6.1), we know that $\mathbf{Sect}([G/G^0]/S)$ is discrete. Therefore the fppf image of $\mathbf{Sect}(G/S) \to \mathbf{Sect}([G/G^0]/S)$ is representable by some discrete group scheme D. Thus we obtain an fppf-exact sequence of fppf $\mathbb{C}$-functors:

$$0 \to \mathbf{Sect}(G^0/S) \to \mathbf{Sect}(G/S) \to D \to 0.$$

The map $\pi\colon \mathbf{Sect}(G/S) \to D$ splits as a map of fppf $\mathbb{C}$-functors of *sets*. For each $d \in D(\cdot)$, we let $\pi^{-1}(d)$ denote the corresponding subfunctor of $\mathbf{Sect}(G/S)$. Then $\pi^{-1}(d)$ is a torsor under $\mathbf{Sect}(G^0/S)$. Via the first part of the proof, let F^0 be an additive $\times$ algebraic functor with finitely generated component group and let $F^0 \hookrightarrow \mathbf{Sect}(G^0/S)$ be a nilimmersion.

For each $d \in D(\cdot)$, pick some $x_d \in \pi^{-1}(d)(\cdot)$. Then x_d defines an isomorphism $\psi_{x_d}\colon \mathbf{Sect}(G^0/S) \to \pi^{-1}(d)$. Since $F^0(\cdot) = \mathbf{Sect}(G^0/S)(\cdot)$, it follows that for fixed $d \in D$, the subfunctor $\psi_{x_d}(F^0)$ of $\pi^{-1}(d)$ is independent of x_d. Taking the union of the $\psi_{x_d}(F^0)$ as $d \in D(\cdot)$ varies, one obtains a closed subfunctor $F \subset \mathbf{Sect}(G/S)$ which has the needed properties. $\square$

COROLLARY 6.7. *Let S be a complex variety. Let X be a smooth commutative group scheme over S. Let $G = X(S)$ be the group of sections of X/S. Let $\mathbf{G} = \mathbf{Sect}(X/S)$. Give G the strongest topology such that for every nonsingular variety T, and for every $\eta \in \mathbf{G}(T)$, the induced map $T \to G$ is continuous with respect to the usual topology. Then G is a topological group, and as such is isomorphic to*

$$(S^1)^k \times \mathbb{R}^n \times D,$$

where $0 \leqslant k < \infty$, $0 \leqslant n \leqslant \infty$, and D is a discrete group. Furthermore, if X/S has connected fibers, or even if the generic fiber of X/S has finitely generated component group, it follows that D is finitely generated.

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