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Invariant analysis and contractions of symmetric spaces: Part II

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Abstract. For any symmetric space S, we obtain explicit links between its function $e(X, Y)$, introduced in Part I, and invariant differential operators, or spherical functions, on S. In certain cases, this leads to a 'generalized Duflo's isomorphism' for S.

Introduction

Let $S = G/H$ be a symmetric space. Its tangent space S_0 at the origin can be considered as a (flat) symmetric space too, and the aim of this paper is to study, in some detail, the links between harmonic analysis of H-invariant functions (or distributions) on S and on S_0.

A major role is played in this matter by an interesting function $e(X, Y)$, given by the geometry of S. This numerical function of two tangent vectors X, Y in S_0 was introduced and studied in Part I; however the present paper can be read independently, except for details of certain proofs. The main properties of $e(X, Y)$ are recalled in Section 1, where two new results are also established (Propositions 1.1 and 1.4); the latter implies that e is globally defined when S is a semisimple Riemannian space of the noncompact type.

A natural passage from S_0 to S is given by the exponential mapping Exp; we relate a function u on S_0 and a function $\tilde{u}$ on S according to

$$u(X) = J(X)^{1/2}\tilde{u}(\operatorname{Exp} X),$$

where J is the Jacobian of Exp. This map $\tilde{}$ extends to distributions, in particular to invariant differential operators (distributions supported at the origin), yielding a linear isomorphism $p \to \tilde{p}$ between the algebra $\mathbf{D}(S_0)$ of H-invariant constant coefficients differential operators on S_0, and the algebra $\mathbf{D}(S)$ of G-invariant differential operators on S. In Section 2, we extend to H-invariant distributions the formula

$$\langle \tilde{u} * \tilde{v}, \tilde{f} \rangle = \langle u(X) \otimes v(Y), e(X, Y) f(X + Y) \rangle$$

(where f is a test function), already proved for functions in Part I. Thus e enters when transferring convolution from S to S_0, and this is the key to all other results proved here.

The formula applies to differential operators now, which form the subject-matter of Section 3. Theorem 3.1 provides an explicit expression, in exponential coordinates, of the action of $\mathbf{D}(S)$ on H-invariant distributions: when transferring this action by means of $\tilde{}$, the operator $\tilde{p}$ in $\mathbf{D}(S)$ corresponds to an H-invariant operator $p(X, \partial_X)$ on S_0, with analytic coefficients, and with symbol

$$p(X, \xi) = e(X, \partial_\xi)p(\xi),$$

for X in S_0 and ξ in its dual space. This answers a question by Duflo ([5], Problem 7). In this formula, the variable Y has been replaced by $\partial_\xi = \partial/\partial\xi$ in the Taylor expansion of $e(X, Y)$ at the origin; this gives a differential operator of infinite order, but no problem arises when applying it to polynomials. Besides, Theorem 3.2 shows how the e-function of S determines the structure of the algebra $\mathbf{D}(S)$: the map $p \to \tilde{p}$ becomes an *isomorphism of algebras* of $\mathbf{D}(S_0)$ onto $\mathbf{D}(S)$, when the former is equipped with the product $\times$ defined by

$$(p \times q)(\xi) = e(\partial_\xi, \partial_\eta)(p(\xi)q(\eta))_{\eta=\xi} = p(\xi)q(\xi) + \text{lower order terms.}$$

As above, elements of $\mathbf{D}(S_0)$ have been here (canonically) identified to H-invariant polynomials p, q on the dual space of S_0.

We conjecture that, under a mild assumption on S (see §1.1), the e-function is symmetric: $e(X, Y) = e(Y, X)$. Then, the law $\times$ would be commutative, whence a new 'proof' of Duflo's theorem on the commutativity of $\mathbf{D}(S)$; a partial converse of this theorem is given in Proposition 3.4.

The space $\mathbf{D}(S_0)$ is now endowed with two different structures of algebra, with respective products $p(\xi)q(\xi)$ and $(p \times q)(\xi)$; the latter may be viewed as a deformation of the former. An interesting question is the existence of an isomorphism $\mathscr{A}$ between those two structures. Composition with $\tilde{}$ would then give an isomorphism of algebras between $\mathbf{D}(S_0)$ (with its usual structure) and $\mathbf{D}(S)$, which might be called a generalized Duflo's isomorphism for S. Such an isomorphism can be constructed by means of e, when assuming that $\mathbf{D}(S)$ is commutative and $\mathbf{D}(S_0)$ is a polynomial algebra (Corollary 3.3); but the general case remains an open question.

The above results can be written in more explicit form for second order invariant differential operators, e.g. the Laplace–Beltrami operator of a pseudo-Riemannian symmetric space (Proposition 3.5 and Corollary 3.6). This gives a new proof, and a generalization, of a result proved by Helgason for the Laplacian of a Riemannian space of the noncompact type; it also implies a relation between e and the Jacobian J (Corollary 3.6(iii)).

The rest of the paper studies the links between spherical functions of a Riemannian symmetric space and its e-function. When transferred to the tangent space by means of $\tilde{}$, a spherical function on S becomes an H-invariant analytic function ψ on S_0. The main result of Section 4 (Theorem 4.2) states that its Taylor expansion at the origin may be written as

$$\psi(X) = \sum_{\alpha} a_\alpha p_\alpha^*(X),$$

where the p_α^* are a basis of H-invariant polynomials on S_0, and the coefficients a_α are determined inductively by e and the eigenvalues corresponding to the given spherical function. The proof being a bit technical, the simpler case of isotropic Riemannian spaces is handled before (§4.1); the function ψ is then given by a Schrödinger equation

$$\Delta\psi(X) + \Delta_Y e(X, 0) . \psi(X) = \lambda\psi(X),$$

and its expansion is clearly a perturbation of the classical expansion of Bessel functions.

In Section 5, we investigate a little further the links between spherical functions on the Riemannian space S and on S_0 (i.e. generalized Bessel functions). Assuming that S has 'sufficiently many' spherical functions (e.g. a Riemannian space of the noncompact type), it is easy to obtain a new construction of the isomorphism $\mathscr{A}$ mentioned above; $\mathscr{A}$ can be read off from the coefficients a_α in the expansion of ψ (Proposition 5.1). After suitable extension of $\mathscr{A}$ from polynomials to analytic functions, it follows that

$$\psi_\xi(X) = \mathscr{A}\left(\int_H e^{\langle \xi, h.X\rangle}\, \mathrm{d}h\right),$$

which means that $\mathscr{A}$, acting on the dual variable ξ, transforms generalized Bessel functions into spherical functions of S; then $\mathscr{A}$ also relates the spherical Plancherel measure of S to the Lebesgue measure of S_0.

The same can be done for the 'contracted' symmetric spaces S_t, with $0 \leqslant t \leqslant 1$, intermediate between S_0 and $S_1 = S$. Expansion of the corresponding operator $\mathscr{A}_t$ with respect to the parameter t:

$$\mathscr{A}_t = I + t^4 \mathscr{A}_4 + t^6 \mathscr{A}_6 + \cdots$$

leads to the Stanton–Tomas expansion of spherical functions, and to the Mehler–Heine formula (§5.3). We also give explicit induction formulas, in the case of isotropic Riemannian spaces; it can be shown that they lead to the remarkable form given by Duistermaat [6] to spherical functions, a subject to which we hope to hark back in forthcoming work.

Open questions: Besides the conjectures formulated on the function e, several questions arise in connection with the operator $\mathscr{A}$ leading to a generalized Duflo's isomorphism:

- give a general construction of $\mathscr{A}$ in terms of e,
- as a modest first step, compute the fourth-order aberration $\mathscr{A}_4$,

- relate $\mathscr{A}$ to some integral transform similar to the Abel transform for semisimple spaces of the noncompact type,
- extend $\mathscr{A}$ to 'arbitrary' H-invariant distributions.

This last result would in particular imply local solvability of all elements of $\mathbf{D}(S)$.

Notations

We abbreviate as (I) the (frequent) references to Part I of this paper, and we keep to its notation. In particular $S = G/H$ will denote, throughout the article, a connected and simply connected symmetric coset space; G is a connected real Lie group with identity e, and H is the connected component of e in the fixed point subgroup of σ, an involutive automorphism of G. The topological assumptions on S, G, H are not essential; we shall never repeat them, simply calling S 'a symmetric space'. We set $n = \dim S$.

Let $o = H$ be the origin of S, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ the decomposition of the Lie algebra of G induced by σ, as the sum of the Lie algebra of H and a vector space $\mathfrak{s}$, which can be identified with the tangent space S_0 to S at the origin. The notation S_0 will be used, instead of $\mathfrak{s}$, whenever it seems useful to emphasize its (flat) symmetric space structure $S_0 = G_0/H$, where G_0 is the semi-direct product of $\mathfrak{s}$ with H.

Dots will denote several natural actions, such as $h.X$ (adjoint action of $h \in H$ on $X \in \mathfrak{s}$), or $h.\xi$ (coadjoint action of h on $\xi \in \mathfrak{s}^*$, the dual space of $\mathfrak{s}$).

Let $\mathbf{D}(S)$ be the algebra of G-invariant linear differential operators on S, with complex coefficients. Similarly $\mathbf{D}(S_0)$ is the algebra of H-invariant constant coefficients linear differential operators on the vector space S_0; it is canonically isomorphic to $I(\mathfrak{s})$, the subalgebra of H-invariant elements in the complexified symmetric algebra of $\mathfrak{s}$.

Let $\mathrm{Exp}: \mathfrak{s} \to S$ be the exponential mapping, and $J(X) = \det_{\mathfrak{s}}(\operatorname{sh} \operatorname{ad} X/\operatorname{ad} X)$ its Jacobian; if V is a finite dimensional vector space and u a linear map, $\det_V u$, $\operatorname{tr}_V u$, mean the determinant, trace, of u as an endomorphism of V.

If f is a smooth map between manifolds, its differential at x will be denoted by $D_x f$, as a linear map between tangent spaces. When computing partial derivatives in coordinates, the classical multi-index notations ∂^α, $|\alpha|$, $\alpha!$ will be frequently used. If M is a manifold, $\mathscr{D}(M)$ is Schwartz' notation for the space of C^∞, complex-valued, compactly supported functions on M.

1. The e-function of S

1.1. In this section, we gather the main properties of the function $e(X, Y)$ which will be needed in the sequel; we refer to (I) Section 3 for proofs.

One can construct an H-invariant open neighborhood $\mathfrak{s}'$ of the origin in $\mathfrak{s} = S_0$ such that Exp is a diffeomorphism of $\mathfrak{s}'$ onto $S' = \operatorname{Exp} \mathfrak{s}'$, an open neighborhood Ω_0 of $(0,0)$ in $\mathfrak{s}' \times \mathfrak{s}'$, and an analytic function $e: \Omega_0 \to]0, +\infty[$, endowed with the following properties:

(i) If (X, Y) belongs to Ω_0 and h to H, then $(h.X, h.Y)$ and $(-X, -Y)$ belong to Ω_0 and $e(h.X, h.Y) = e(X, Y) = e(-X, -Y)$,
(ii) $e(X, Y) = 1$ whenever (X, Y) is in Ω_0 and X, Y generate a solvable Lie subalgebra of $\mathfrak{g}$ (in particular when $[X, Y] = 0$),
(iii) Let $B_{\mathfrak{g}}$ and $B_{\mathfrak{h}}$ denote the respective Killing forms of $\mathfrak{g}$ and $\mathfrak{h}$. The Taylor expansion of e at $(0,0)$ begins as:

$$e(X, Y) = 1 + \tfrac{1}{4}\operatorname{tr}_{\mathfrak{h}} \operatorname{ad} T + \tfrac{1}{32}(\operatorname{tr}_{\mathfrak{h}} \operatorname{ad} T)^2 - \tfrac{1}{48}\operatorname{tr}_{\mathfrak{h}} \operatorname{ad}(uT) - \tfrac{1}{240}b(T, T) + \cdots \tag{1}$$

where $T = [X, Y] \in \mathfrak{h}$, $u = (\operatorname{ad} X)^2 + \operatorname{ad} X . \operatorname{ad} Y + (\operatorname{ad} Y)^2$, $b = B_{\mathfrak{g}} - 2B_{\mathfrak{h}}$, and $\cdots$ have order $\geqslant 6$ with respect to (X, Y). Thus significant simplifications occur under the assumption:

(A) *the character* $\operatorname{tr}_{\mathfrak{h}}\operatorname{ad}$ *of* $\mathfrak{h}$ *extends to a character of the Lie algebra* $\mathfrak{g}$.

In fact, since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$, $[\mathfrak{h}, \mathfrak{s}] \subset \mathfrak{s}$, and $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{h}$, assumption (A) is equivalent to: $\operatorname{tr}_{\mathfrak{h}}\operatorname{ad}[X, Y] = 0$ for all X, Y in $\mathfrak{s}$. *Property (A) holds when S carries a G-semi-invariant measure.* It is interesting to note that (A) is precisely the assumption under which Duflo [4] has proved that $\mathbf{D}(S)$ is a commutative algebra. When (A) holds, we have

$$e(X, Y) = 1 - \tfrac{1}{240}b(T, T) + \tfrac{1}{1512}b(T, uT) + \text{order} \geqslant 8. \tag{1'}$$

(iv) The space S is called *special* when its e-function is identically one. This property (which implies (A)) holds, by (ii), when G is a solvable group. Two conjectures were made in (I), both implying that all spaces $G_{\mathbf{C}}/G_{\mathbf{R}}$, or $G \times G/\text{diagonal}$, are special.
(v) We add the following

CONJECTURE. *Assume (A). Then* $e(X, Y) = e(Y, X)$.

Conversely, the second order terms in (1) show that (A) is a necessary condition for the symmetry of e. A brief discussion of this conjecture will be given in Section 3.4.

1.2. The following technical result will be needed when dealing with second order operators in Section 3.5.

PROPOSITION 1.1. *Assume (A). Then* $D_Y e(X,0) = 0$, *and* $D_X e(0, Y) = 0$, *for all* $(X, 0)$ *and* $(0, Y)$ *in* Ω_0.

Again, the second order terms in (1) imply the necessity of the assumption.

Proof. We refer to (I) Sections 2 and 3 for several notations and results needed in the proof. We only recall here that e was defined from the differential equation

$$D_t \log e_t(X, Y) = \mathrm{tr}_{\mathfrak{h}}\, t^{-1}E(tX_t, tY_t), \qquad e_0(X, Y) = 1 \tag{2}$$

(cf. (I) §3.3), where $0 \leqslant t \leqslant 1$, $e_t(X, Y) = e(tX, tY)$ ((i) §3.4), E is a certain endomorphism of $\mathfrak{h}$, and $X_t = X_t(X, Y)$, $Y_t = Y_t(X, Y)$ are elements of $\mathfrak{s}$ satisfying differential equations of the form ((I) §2.3)

$$\begin{matrix} D_t X_t = [F_t(X_t, Y_t), X_t] \\ D_t Y_t = [G_t(X_t, Y_t), Y_t] \end{matrix}, \qquad X_0 = X, \qquad Y_0 = Y. \tag{3}$$

Taking coordinates $Y_1, \ldots, Y_n$ in $\mathfrak{s}$, we wish to prove that

$$D_t(D_{Y_j} \log e_t)(X, 0) = 0 \tag{4}$$

for all $j = 1, \ldots, n$, all $t \in [0, 1]$, and all X in a small neighborhood of 0 in $\mathfrak{s}$; since $(D_{Y_j} \log e_0)(X, 0) = 0$, this will imply the first part of the proposition, by analytic continuation. The proof of the second part is entirely similar.

Differentiating (2) we obtain first

$$D_t(D_{Y_j} \log e_t)(X, Y) = \mathrm{tr}_{\mathfrak{h}}(D_X E\,.\,D_{Y_j} X_t + D_Y E\,.\,D_{Y_j} Y_t),$$

where the derivatives of E are taken at (tX_t, tY_t). The definitions of F, G, E in (I) easily show that $F_t(X, 0) = G_t(X, 0) = 0$ and $E(X, 0) = 0$. It follows that $X_t = X$, $Y_t = 0$ is the (unique) solution of (3) when $Y = 0$; besides $D_X E(tX, 0) = 0$. Thus

$$D_t(D_{Y_j} \log e_t)(X, 0) = \mathrm{tr}_{\mathfrak{h}}(D_Y E(tX, 0)\,.\,D_{Y_j} Y_t(X, 0)).$$

But $D_Y Y_t(X, 0)$ is the identity mapping of $\mathfrak{s}$: in fact this is true for $t = 0$ (since $Y_0(X, Y) = Y$), and we have, by (3),

$$D_t(D_{Y_j} Y_t)(X, Y) = [D_{Y_j}(G_t(X_t, Y_t)), Y_t] + [G_t(X_t, Y_t), D_{Y_j} Y_t],$$

which vanishes for $Y = 0$ in view of the above remarks. We now obtain

$$D_t(D_{Y_j} \log e_t)(X, 0) = D_{Y_j}(\mathrm{tr}_{\mathfrak{h}} E)(tX, 0), \tag{5}$$

and the required equality (4) will follow from two lemmas.

LEMMA 1.2. *There exist real constants a_n such that*

$$\mathrm{tr}_{\mathfrak{h}} E(X, Y) = \sum_0^\infty a_n \,\mathrm{tr}_{\mathfrak{h}}\, \mathrm{ad}(x^{2n+1} Y) + O(Y^2),$$

where $x = \mathrm{ad}\, X$, and the series converges for X, Y near 0 in $\mathfrak{s}$.

Proof. We know from (I) Theorem 3.15 that $E(X, Y)$ is given, near the origin, by a convergent series in the (noncommutative) variables $x = \mathrm{ad}\, X$, $y = \mathrm{ad}\, Y$, which can be written in the following form

$$E(X, Y) = \sum_{j,k} u_j(xy - yx)v_k,$$

where u_j, v_k are (noncommutative) monomials in x and y, of the same parity.

Thus all linear terms with respect to y in this expansion can be written as $a(x^{\alpha+1}yx^{\beta}-x^{\alpha}yx^{\beta+1})$, where a is a constant, and $\alpha+\beta$ is even. Since even monomials in ad X and ad Y define endomorphisms of $\mathfrak{h}$, they can be commuted under $\mathrm{tr}_{\mathfrak{h}}$, and the above term has the same trace on $\mathfrak{h}$ as

$$a(x^{\alpha+\beta+1}y - yx^{\alpha+\beta+1}) \quad \text{if } \alpha, \beta \text{ are even,}$$

or as

$$a(yx^{\alpha+\beta+1} - x^{\alpha+\beta+1}y) \quad \text{if } \alpha, \beta \text{ are odd.}$$

We thus obtain

$$\mathrm{tr}_{\mathfrak{h}} E(X, Y) = \sum_{0}^{\infty} a_n \, \mathrm{tr}_{\mathfrak{h}}(x^{2n+1}y - yx^{2n+1}) + O(Y^2),$$

and Lemma 1.2 is an immediate consequence of the following equality:

LEMMA 1.3. *For any integer $n \geqslant 0$, there exists an even monomial u_n (in x and y) such that*

$$\mathrm{ad}(x^{2n+1}Y) = 2^{2n}(x^{2n+1}y - yx^{2n+1}) + [x^2, u_n].$$

Proof. By induction on n, we have

$$\begin{aligned}\mathrm{ad}(x^{2n+1}Y) &= \sum_{0\leqslant p\leqslant 2n+1} (-1)^p C_{2n+1}^{p} x^{2n+1-p} y x^p \\ &= \sum C_{2n+1}^{2q} x^{2n-2q+1} y x^{2q} - \sum C_{2n+1}^{2q+1} x^{2n-2q} y x^{2q+1},\end{aligned}$$

where the $C_{\cdot}^{\cdot}$ are the binomial coefficients. Repeated commutation of x^2 with even monomials yields

$$\mathrm{ad}(x^{2n+1}Y) = \left(\sum_q C_{2n+1}^{2q}\right) x^{2n+1}y - \left(\sum_q C_{2n+1}^{2q+1}\right) yx^{2n+1} + [x^2, u_n],$$

whence the lemma.

To complete the proof of Proposition 1.1, it is enough to observe that $\mathrm{tr}_{\mathfrak{h}} E(X, Y) = O(Y^2)$, by Lemma 1.2 and assumption (A); then (5) implies (4), and the proposition follows.

REMARK. The function e cannot be written as

$$e(X, Y) = \frac{f(X+Y)}{f(X)f(Y)},$$

with f differentiable near 0, except in the trivial case of a special symmetric space. In fact this equality implies $f(0) = 1$, $e(X, Y) = e(Y, X)$, hence property (A), and Proposition 1.1 gives

$$0 = \frac{D_Y e(X,0)}{e(X,0)} = \frac{Df(X)}{f(X)} - \frac{Df(0)}{f(0)};$$

thus f is the exponential of a linear form, and e must be identically one.

1.3. The following proposition (which will not be used in the sequel) implies that the e-function is globally defined in the case of semi-simple Riemannian symmetric spaces of the noncompact type.

PROPOSITION 1.4. *Assume H is a compact group, and* $\mathrm{Exp}\colon \mathfrak{s} \to S$ *is a global diffeomorphism. Then we may take $\mathfrak{s}' = \mathfrak{s}$ and $\Omega_0 = \mathfrak{s} \times \mathfrak{s}$, so that e is analytic on the whole $\mathfrak{s} \times \mathfrak{s}$.*

Proof. In the notation of (I) Section 2.2–2.3, we have $\Omega = \mathfrak{s} \times \mathfrak{s}$, and the functions $F_t(X, Y)$, $G_t(X, Y)$ used in the construction of e are analytic in (t, X, Y) on $\mathbf{R} \times \mathfrak{s} \times \mathfrak{s}$. Let us recall that the solutions X_t, Y_t of (3) above are obtained as $X_t = a_t . X$, $Y_t = b_t . Y$, where a_t, $b_t \in H$ are given by

$$\begin{aligned} D_t a_t &= D_e R_{a_t} . F_t(a_t . X, b_t . Y) \\ D_t b_t &= D_e R_{b_t} . G_t(a_t . X, b_t . Y) \end{aligned} \tag{6}$$

with R_a meaning right translation by a in H (cf. [9] p. 570). Now $(D_e R_a . F_t(a . X, b . Y),\ D_e R_b . G_t(a . X, b . Y))$ is a smooth time-dependent vector field on the compact manifold $H \times H$, for any given (X, Y) in $\mathfrak{s} \times \mathfrak{s}$. It follows that a_t, b_t, X_t, Y_t, e_t are defined, and analytic, in (t, X, Y) on $\mathbf{R} \times \mathfrak{s} \times \mathfrak{s}$.

2. The e-function and invariant distributions

The purpose of this section is to prove a general formula relating convolution of invariant distributions on S and on its tangent space $S_0 = \mathfrak{s}$, by means of the function $e(X, Y)$ described above. This formula was proved in (I) p. 263 for functions, by a mere change of variables in an integral; however, the proof is more delicate for arbitrary distributions.

In the sequel, 'distribution' means distribution-density, and H-invariance of a distribution means that, for all h in H and all test functions f,

$$\langle u(X), f(h . X) \rangle = \langle u(X), f(X) \rangle,$$

where $\langle\,,\rangle$ is the duality bracket between distributions and functions.

We recall the notations $\mathfrak{s}'$, Ω_0 from Section 1. Let $\mathfrak{s}''$ be the set of all $X \in \mathfrak{s}'$ such that $(X, 0) \in \Omega_0$; this $\mathfrak{s}''$ is an open, star-shaped, H-invariant neighborhood of 0 in $\mathfrak{s}$. Clearly $\mathfrak{s}'' = \mathfrak{s}$ whenever Proposition 1.4 applies.

The function $\tilde{f}$ is defined by $f(X) = J(X)^{1/2} \tilde{f}(\mathrm{Exp}\, X)$ for $X \in \mathfrak{s}'$, and the distribution $\tilde{u}$ (on the open subset $S' = \mathrm{Exp}\, \mathfrak{s}'$ of S) by $\langle \tilde{u}, \tilde{f} \rangle = \langle u, f \rangle$, for all $f \in \mathscr{D}(\mathfrak{s}')$; these definitions obviously extend to the case of some open subset U of $\mathfrak{s}'$ replacing $\mathfrak{s}'$.

THEOREM 2.1. *Let u, v be H-invariant distributions on open subsets of $\mathfrak{s}$, and f a C^∞ function on an open subset of $\mathfrak{s}$, with suitable supports. Then*

$$\langle \tilde{u} * \tilde{v}, \tilde{f} \rangle = \langle u(X) \otimes v(Y), e(X, Y) f(X + Y) \rangle.$$

This holds in particular when U is an H-invariant open subset of $\mathfrak{s}''$, with $u \in \mathscr{D}'(U)^H$, $f \in C^\infty(U)$, $\operatorname{supp} v = \{0\}$, *and* $\operatorname{supp} u \cap \operatorname{supp} f$ *is a compact subset of U.*

Here $*$ means convolution on S, as defined in [9] p. 557. The precise assumption on supports will be specified at the end of the proof below, but the special case mentioned in the theorem will suffice for later use.

Proof. (a) Let u be an H-invariant distribution on $\mathfrak{s}$, and $F: \mathfrak{s} \to \mathfrak{h}$ be a C^∞ map, such that $\operatorname{supp} u \cap \operatorname{supp} F$ is compact. Then

$$\langle u(X), \operatorname{tr}_{\mathfrak{s}}(\operatorname{ad} X \circ D_X F(X))\rangle = 0. \tag{1}$$

In fact, let (A_i) be a fixed basis of $\mathfrak{h}$, whence the decomposition $F(X) = \Sigma_i F_i(X) A_i$; in the space $\operatorname{Hom}(\mathfrak{s}, \mathfrak{s}) = \mathfrak{s}^* \otimes \mathfrak{s}$, we have

$$\operatorname{ad} X \circ D_X F(X) = \sum DF_i(X) \otimes [X, A_i],$$

and

$$\operatorname{tr}_{\mathfrak{s}}(\operatorname{ad} X \circ D_X F) = \sum DF_i(X) . [X, A_i],$$

where . means duality between $\mathfrak{s}^*$ and $\mathfrak{s}$. From the H-invariance of u we get

$$D_{t=0}\langle u(X), F_i(e^{t \operatorname{ad} A_i} X)\rangle = \langle u, DF_i . [X, A_i]\rangle = 0,$$

and (1) follows by summation over i.

(b) Here we must recall some notations from [9] §4 and (I) §2:

$$\Phi_t(X, Y) = (X_t, Y_t) = (a_t . X, b_t . Y),$$

where $0 \leqslant t \leqslant 1$, a_t, b_t belong to H and depend on X, Y; let

$$\Omega_t = \Phi_t(\Omega_0), \qquad Z_t(X, Y) = t^{-1} Z(tX, tY),$$

so that $\operatorname{Exp} tZ_t(X, Y) = \exp tX . \operatorname{Exp} tY$ and $Z_t(X, Y)) = X + Y$.

Let $g: \Omega_0 \to \mathbf{C}$ be a C^∞ function, and let $g_t: \Omega_t \to \mathbf{C}$ be defined by

$$\begin{aligned} g_t(\Phi_t(X, Y)) &= \left(\frac{J(tX)J(tY)}{J(tX + tY)}\right)^{1/2} e(tX, tY)^{-1} g(X, Y) \\ &= (\det D\Phi_t(X, Y))^{-1} \det \operatorname{Ad} a_t \det \operatorname{Ad} b_t \, g(X, Y) \end{aligned}$$

with $0 \leqslant t \leqslant 1$ (cf. (I) Proposition 3.14). From the equalities ([9] p. 570–571):

$$D_t \log \det D\Phi_t(X, Y) = \operatorname{tr}_{\mathfrak{s}}(\operatorname{ad}(F_t + G_t) - \operatorname{ad} X \circ D_X F_t - \operatorname{ad} Y \circ D_Y G_t)(\Phi_t(X, Y))$$

$$D_t \log \det \operatorname{Ad} a_t = \operatorname{tr}_{\mathfrak{s}} \operatorname{ad} F_t(\Phi_t(X, Y))$$

$$D_t \log \det \operatorname{Ad} b_t = \operatorname{tr}_{\mathfrak{s}} \operatorname{ad} G_t(\Phi_t(X, Y)),$$

we obtain

$$D_t(g_t(\Phi_t(X, Y))) = (g_t . \operatorname{tr}_{\mathfrak{s}}(\operatorname{ad} X \circ D_X F_t + \operatorname{ad} Y \circ D_Y G_t))(\Phi_t(X, Y)).$$

The left-hand side is also

$$(D_t g_t)(\Phi_t(X, Y)) + (D_{X,Y} g_t)(\Phi_t(X, Y)) \circ D_t \Phi_t(X, Y).$$

Replacing $D_t \Phi_t$ by its expression (cf. (3) §1), then composing with Φ_t^{-1}, we easily deduce that

$$D_t g_t = \mathrm{tr}_{\mathfrak{s}}(\mathrm{ad}\ X \circ D_X(g_t F_t) + \mathrm{ad}\ Y \circ D_Y(g_t G_t)), \tag{2}$$

where both sides are now taken at $(X, Y) \in \Omega_t$.

(c) We can now apply (1) with $u \otimes v$, $H \times H$, $\mathfrak{s} \times \mathfrak{s}$, (X, Y) and $(g_t F_t, g_t G_t)$ replacing u, H, $\mathfrak{s}$, X and F respectively. From (1) and (2) it follows that

$$\langle u \otimes v, D_t g_t \rangle = 0, \quad \text{whence } \langle u \otimes v, g_1 \rangle = \langle u \otimes v, g_0 \rangle.$$

Replacing g by $e.g$, the resulting equality is, with $\Phi = \Phi_1$,

$$\left\langle u(X)v(Y), \left(\frac{J(X)J(Y)}{J(Z(X, Y))}\right)^{1/2} g(\Phi^{-1}(X, Y)) \right\rangle = \langle u(X)v(Y), e(X, Y)g(X, Y) \rangle \tag{3}$$

for any H-invariant u, v, and any g.

(d) The particular choice $g(X, Y) = f(X + Y)$, where $f : \mathfrak{s} \to \mathbf{C}$ is C^∞, yields

$$\left\langle u(X)v(Y), \left(\frac{J(X)J(Y)}{J(Z(X, Y))}\right)^{1/2} f(Z(X, Y)) \right\rangle = \langle u(X)v(Y), e(X, Y)f(X + Y) \rangle. \tag{4}$$

(e) Finally the left-hand side of (4) is easily transformed, by definitions of Z and $\tilde{}$, into

$$\langle \tilde{u}(\bar{x}), \langle \tilde{v}(\bar{y}), \tilde{f}(x.\bar{y}) \rangle \rangle = \langle \tilde{u} * \tilde{v}, \tilde{f} \rangle$$

where $\bar{x}$, $\bar{y} \in G/H$ are the cosets of x, $y \in G$.

(f) As regards supports, the main problem is to make sure that $\langle u \otimes v, D_t g_t \rangle$ is well defined. The proof works, for instance, whenever u, $v \in \mathscr{D}'(\mathfrak{s})$, $g \in C^\infty(\Omega_0)$, provided that $(\operatorname{supp} u \times \operatorname{supp} v) \cap \Phi_t(\operatorname{supp} g)$ is a compact subset of Ω_0 for all $0 \leqslant t \leqslant 1$. It also works when $\operatorname{supp} v = \{0\}$ and u, f are as stated in the theorem, since $\Phi_t(X, 0) = (X, 0)$. This completes the proof.

REMARK. Equalities (3) and (4), on the tangent space, may have independent interest; the latter improves Proposition 4.2 of (I).

3. The *e*-function and invariant differential operators

3.1. The algebra $\mathbf{D}(S_0)$ of H-invariant constant coefficients differential operators on S_0 is canonically isomorphic to the algebra $I(\mathfrak{s})$ of H-invariant elements in the (complexified) symmetric algebra of $\mathfrak{s}$, that is H-invariant polynomial functions on the dual space $\mathfrak{s}^*$. We shall write $p(\partial_X)$, or p, or $p(\xi)$, an element of this algebra, with $X \in \mathfrak{s}$, $\xi \in \mathfrak{s}^*$; thus

$$p(\xi) = e^{-\langle \xi, X \rangle} p(\partial_X)(e^{\langle \xi, X \rangle}).$$

Let δ, resp. δ_0, be the Dirac measures of the symmetric space S, resp. its tangent space S_0, at the origin. The map $\tilde{}$ of Section 2 gives a linear isomorphism $p \to \tilde{p}$ between $\mathbf{D}(S_0)$ and the algebra $\mathbf{D}(S)$ of G-invariant differential operators on S, according to:

$$({}^t p \delta_0)^{\sim} = {}^t \tilde{p} \delta$$

or, more generally ([9] §1):

$$\alpha * ({}^t p \delta_0)^{\sim} = {}^t \tilde{p} \alpha$$

for any distribution α on S. Here ^{t} means transpose with respect to the duality between distributions and functions, on S and on S_0. Explicitly

$$\tilde{p}\phi(\bar{x}) = p(\partial_Y)(J(Y)^{1/2} \phi(x \,.\, \mathrm{Exp}\ Y))_{Y=0}$$

for $\phi \in C_c^\infty(S)$, $x \in G$, $\bar{x} = xH \in S$; inversely, with $\mathrm{Log} = \mathrm{Exp}^{-1}$ near the origin,

$$pf(X) = \tilde{p}(J(\mathrm{Log}\ \bar{x})^{-1/2} f(X + \mathrm{Log}\ \bar{x}))_{\bar{x}=o}$$

for $f \in C^\infty(\mathfrak{s})$, $X \in \mathfrak{s}$. Up to the factor $J^{1/2}$, the map $p \to \tilde{p}$ is therefore the correspondence considered by Helgason [7] p. 269, or [8] p. 287.

Note that $\tilde{p}\tilde{f}(o) = pf(0)$, but this equality does not hold, in general, at other points; the next theorem provides the appropriate generalization.

3.2. Let V be a finite-dimensional vector space over $\mathbf{R}$, V^* its dual space. As above, we denote by p, or $p(\partial_X)$, or $p(\xi)$ an element of the (complexified) symmetric algebra $S(V)$, with $X \in V$, $\xi \in V^*$. The nondegenerate bilinear form

$$\langle p \,|\, q \rangle = \langle p(\xi) \,|\, q(X) \rangle = (p(\partial_X) q)(0),$$

with $p \in S(V)$, $q \in S(V^*)$ defines a duality between these spaces. This bilinear form is symmetric:

$$(p(\partial_X) q)(0) = (q(\partial_\xi) p)(0),$$

which easily implies that, for $p \in S(V)$, q, $r \in S(V^*)$,

$$\langle p(\xi) \,|\, q(X) r(X) \rangle = \langle q(\partial_\xi) p(\xi) \,|\, r(X) \rangle \tag{3}$$

(cf. e.g. [8] p. 347). This equality still holds when q is a formal series, and r is a smooth function on V, near 0. These facts will be needed in the proof of the following theorems.

THEOREM 3.1. *Let S be a symmetric space, and $p \in \mathbf{D}(S_0) = I(\mathfrak{s})$, and let U be an H-invariant open subset of $\mathfrak{s}''$.*

(i) *For any H-invariant distribution u on U, one has*

$$ {}^t\tilde{p}\tilde{u} = ({}^t(p(X,\partial_X))u)^{\sim} \quad \textit{on } \operatorname{Exp} U, $$

where $p(X, \partial_X)$ *is the differential operator with analytic coefficients on* $\mathfrak{s}''$ *corresponding to the symbol*

$$ p(X,\xi) = e(X,\partial_\xi)p(\xi) = \sum \frac{1}{\alpha!}\partial_Y^\alpha e(X,0)\,.\,\partial_\xi^\alpha p(\xi); $$

here $X \in \mathfrak{s}''$, $\xi \in \mathfrak{s}^*$, *and* Σ *is a (finite) summation over* $\alpha \in \mathbf{N}^n$.

(ii) *If H is compact, and f is any H-invariant* C^∞ *function on U, one has*

$$ \tilde{p}\tilde{f} = (p(X,\partial_X)f)^{\sim} \quad \textit{on } \operatorname{Exp} U. $$

The set $\mathfrak{s}''$ has been introduced at the beginning of Section 2. If $Y_1, \ldots, Y_n$ are coordinates with respect to some basis of $\mathfrak{s}$, and $\xi_1, \ldots, \xi_n$ the dual coordinates on $\mathfrak{s}^*$, then ∂_Y^α and ∂_ξ^α mean the corresponding partial derivatives, in multi-index notation.

Proof. (i) Applying Theorem 2.1 with $v = {}^t p\delta_0$, $f \in \mathscr{D}(U)$, we obtain

$$ \begin{aligned} \langle {}^t\tilde{p}\tilde{u}, \tilde{f}\rangle &= \langle u(X) \otimes {}^t p\delta_0(Y), e(X,Y)f(X+Y)\rangle \\ &= \langle u(X), p(\partial_Y)(e(X,Y)f(X+Y))_{Y=0}\rangle. \end{aligned} $$

By (3), with $V = \mathfrak{s}$, fixed X, and Y as the variable, we have

$$ \begin{aligned} p(\partial_Y)(e(X,Y)f(X+Y))_{Y=0} &= \langle p \mid e(X,.)f(X+.)\rangle \\ &= \langle e(X,\partial)p \mid f(X+.)\rangle = p(X,\partial_X)f(X), \end{aligned} $$

where, by Taylor expansion of e, the differential operator $p(X, \partial_X)$ is obtained by putting $\xi = \partial_X$ in

$$ \begin{aligned} p(X,\xi) = e(X,\partial_\xi)p(\xi) &= \sum_\beta \frac{1}{\beta!}\partial_Y^\beta e(X,0)\,.\,\partial_\xi^\beta p(\xi) \\ &= \sum_{\alpha,\beta} \frac{1}{\alpha!\beta!}\partial_X^\alpha \partial_Y^\beta e(0,0)\,.\,X^\alpha\,.\,\partial_\xi^\beta p(\xi) \end{aligned} $$

(finite summation over β); note that $p(\xi) = p(0, \xi)$. Therefore

$$ \langle {}^t\tilde{p}\tilde{u}, \tilde{f}\rangle = \langle u(X), p(X,\partial_X)f(X)\rangle = \langle {}^t p(X,\partial)u, f\rangle = \langle ({}^t p(X,\partial)u)^{\sim}, \tilde{f}\rangle. $$

This equality, valid for all $f \in \mathscr{D}(U)$, implies (i).

(ii) The latter equations can be written as

$$ \langle \tilde{u}, \tilde{p}\tilde{f}\rangle = \langle \tilde{u}, (p(X,\partial)f)^{\sim}\rangle, $$

and they also hold, by Theorem 2.1, if $\operatorname{supp} u$ is a compact subset of U and $\operatorname{supp} f$ is arbitrary. When f is H-invariant, then

$$p(X, \partial_X)f(X) = p(\partial_Y)(e(X, Y)f(X + Y))_{Y=0}$$

is H-invariant too, by invariance of p and e. The invariant function $\tilde{p}\tilde{f} - (p(X, \partial)f)^{\sim}$ is then annihilated by any invariant compactly supported distribution on $\operatorname{Exp} U$. If H is compact, this function must vanish identically on $\operatorname{Exp} U$, as follows from the existence of H-invariant mean, whence (ii).

The next theorem relates the structure of the algebra $\mathbf{D}(S)$ to the e-function.

THEOREM 3.2. *Let S be a symmetric space. The map $p \to \tilde{p}$ is an isomorphism of algebras of $\mathbf{D}(S_0) = I(\mathfrak{s})$ onto $\mathbf{D}(S)$, if $I(\mathfrak{s})$ is equipped with the product $\times$ defined by*

$$(p \times q)(\xi) = e(\partial_\xi, \partial_\eta)(p(\xi)q(\eta))_{\eta=\xi} = (q(\partial_\xi, \eta)p(\xi))_{\eta=\xi}$$

$$= \sum \frac{1}{\alpha!\beta!} \partial_X^\alpha \partial_Y^\beta e(0,0) \,.\, \partial_\xi^\alpha p(\xi) \,.\, \partial_\xi^\beta q(\xi),$$

where p, $q \in I(\mathfrak{s})$, ξ, $\eta \in \mathfrak{s}^*$, and Σ runs over all α, $\beta \in \mathbf{N}^n$ (finite sum).

The differential operator $q(\partial_\xi, \eta)$ in this theorem is obtained from $q(X, \eta)$ in Theorem 3.1 by substituting ∂_ξ for X, so that

$$q(X, \partial_X)e^{\langle \xi, X\rangle} = q(X, \xi)e^{\langle \xi, X\rangle} = (q(\partial_\xi, \eta)e^{\langle \xi, X\rangle})_{\eta=\xi}.$$

Proof. We already know that $\tilde{}$ is an isomorphism of vector spaces, whence the existence of a unique $r \in I(\mathfrak{s})$ such that $\tilde{p} \circ \tilde{q} = \tilde{r}$. To make r explicit, we may apply Theorem 3.1(i) with u supported at the origin, or compute directly as follows. Let f be any smooth function on $\mathfrak{s}$, supported near the origin. Then

$$\begin{aligned} r(\partial)f(0) &= \langle {}^t r\delta_0, f\rangle = \langle ({}^t r\delta_0)^{\sim}, \tilde{f}\rangle = \langle {}^t\tilde{r}\delta, \tilde{f}\rangle \\ &= \langle {}^t\tilde{q}\,{}^t\tilde{p}\delta, \tilde{f}\rangle = \langle {}^t\tilde{p}\delta * {}^t\tilde{q}\delta, \tilde{f}\rangle \\ &= \langle {}^t p\delta_0(X) \otimes {}^t q\delta_0(Y), e(X, Y)f(X + Y)\rangle \quad \text{by Theorem 2.1} \\ &= \langle p(\xi)q(\eta) \,|\, e(X, Y)f(X + Y)\rangle \quad \text{on the space } V = \mathfrak{s} \times \mathfrak{s} \\ &= \langle e(\partial_\xi, \partial_\eta)p(\xi)q(\eta) \,|\, f(X + Y)\rangle \quad \text{by (3).} \end{aligned}$$

Since the derivatives ∂_X and ∂_Y have the same effect on $f(X+Y)$, this equality is

$$\langle r(\xi) \,|\, f(X)\rangle = \langle e(\partial_\xi, \partial_\eta)(p(\xi)q(\eta))_{\eta=\xi} \,|\, f(X)\rangle,$$

on the space $V = \mathfrak{s}$. This implies the first result in the theorem. The other two are obvious by Taylor expansion of e, and observing that

$$q(\partial_\xi, \eta) = e(\partial_\xi, \partial_\eta)q(\eta).$$

The proof is complete.

REMARKS. (1) Let $r = p \times q$. Combining Theorems 3.1 and 3.2, it follows that ${}^t(r(X, \partial_X))$ and ${}^t(p(X, \partial_X) \circ q(X, \partial_X))$ define the same operators on the space of H-invariant distributions on $\mathfrak{s}$. The same is true, for compact H, for $r(X, \partial_X)$ and $p(X, \partial_X) \circ q(X, \partial_X)$ on H-invariant functions.

(2) Theorem 3.2 implies the associativity of the law $\times$, which was not clear from its definition.

3.3. By Theorem 3.2, the vector space $I(\mathfrak{s})$ is now endowed with two different structures of associative graded algebras: the usual structure, with product

$$(p \cdot q)(\xi) = p(\xi)q(\xi),$$

and the new structure

$$(p \times q)(\xi) = p(\xi)q(\xi) + \text{lower order terms};$$

both structures are the same for special spaces ($e = 1$). Replacing e by $e_t(X, Y) = e(tX, tY)$ (cf. (I) Proposition 3.17), it is clear that the product

$$(p \times_t q)(\xi) = e(t\partial_\xi, t\partial_\eta)(p(\xi)q(\eta))_{\eta = \xi}$$

gives a contraction of $(\times)$ (for $t = 1$) onto $(.)$ (for $t = 0$). Obviously $p \times_t q = (p_t \times q_t)_{1/t}$, setting $p_t(\xi) = p(t\xi)$, with $t \neq 0$; thus the laws $\times_t$ are isomorphic for $t \neq 0$.

A bijection of $I(\mathfrak{s})$ onto itself is an algebra isomorphism $\mathscr{A}: (I(\mathfrak{s}), \times) \to (I(\mathfrak{s}), .)$ if and only if the composed map $\gamma: \mathbf{D}(S_0) \to \mathbf{D}(S)$ defined by $\gamma p = (\mathscr{A}^{-1}p)^\sim$ is an algebra isomorphism. We shall call such a γ a *generalized Duflo's isomorphism* for S. In fact, the classical Duflo's isomorphism corresponds to the case $S = G \times G/\text{diagonal}$, a Lie group G considered as a symmetric space. Then $\mathbf{D}(S)$ is the algebra of bi-invariant differential operators on G, and $\mathbf{D}(S_0)$ consists of all Ad G-invariant constant coefficients differential operators on $\mathfrak{g}$. Duflo has constructed an isomorphism γ between these algebras, by means of the theory of primitive ideals in enveloping algebras (cf. [2] §10.4); later he proved that $\gamma p = \tilde{p}$ (cf. [3], also [8] p. 339).

Here we know that this choice of γ is suited to all special symmetric spaces, such as $S = G/H$ with solvable G; a new proof of Duflo's results would follow from the conjectures stated in (I), implying that all $G \times G/\text{diagonal}$ are special too. Besides those cases, the following easy corollary applies to all semi-simple symmetric spaces.

COROLLARY 3.3. *Assume $\mathbf{D}(S)$ is commutative, and $I(\mathfrak{s})$ is a polynomial algebra in the generators $p_1, \ldots, p_m$. Then there exists a unique morphism $\mathscr{A}: (I(\mathfrak{s}), \times) \to (I(\mathfrak{s}), .)$ such that $\mathscr{A}p_j = p_j$ for $j = 1, \ldots, m$. This $\mathscr{A}$ is a degree preserving isomorphism onto, and $\gamma p = (\mathscr{A}^{-1}p)^\sim$ is a generalized Duflo's isomorphism for S, which is given by its e-function.*

Note that $\mathscr{A}$ may depend on the choice of generators p_j! See Proposition 3.5 for an example, and Section 5.1 for a different approach to $\mathscr{A}$.

Proof. The polynomials $p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, for $\alpha_1, \ldots, \alpha_m \in \mathbf{N}$, form a basis of $I(\mathfrak{s})$, and the same holds for $p_1^{\times\alpha_1} \times \cdots \times p_m^{\times\alpha_m}$, by the properties of $\times$. Clearly $\mathscr{A}$ must map the latter basis onto the former; thus $\mathscr{A}p = p +$lower order, and $\mathscr{A}^{-1}$ can be expressed in terms of e. The rest follows from Theorem 3.2.

3.4. If we could prove the conjectured symmetry $e(X, Y) = e(Y, X)$ under assumption (A) in Section 1.1, then Theorem 3.2 would also give a new, 'constructive' proof of Duflo's theorem for commutativity of $\mathbf{D}(S)$ (cf. [4] p. 136). For lack of a proof, I can only make a few observations:

- The symmetry holds up to sixth order terms in the Taylor expansion of e (cf. (1′) §1.1).
- The symmetry holds in the only case where e is explicitly known, i.e. $S = SO_0(n, 1)/SO(n)$ (Flensted–Jensen, unpublished; cf. (I) p. 258).
- The function $\mathrm{tr}_{\mathfrak{h}} E(X, Y)$ used in the construction of e is symmetric, assuming (A): this easily follows from the definition of $E(X, Y)$ in (I) p. 254 and the fact that $Z(X, Y)$ and $Z(Y, X)$ belong to the same H-orbit in $\mathfrak{s}$.
- The latter remark shows that both sides of (3) Section 2, written with $g = 1$,

$$\left\langle u(X)v(Y), \left(\frac{J(X)J(Y)}{J(Z(X, Y))}\right)^{1/2}\right\rangle = \langle u(X)v(Y), e(X, Y)\rangle$$

are symmetric with respect to permutation of u, v. In particular, all derivatives $p(\partial_X)q(\partial_Y)e(0, 0)$ are symmetric with respect to p, $q \in I(\mathfrak{s})$; in other words:

$$(p \times q)(0) = (q \times p)(0),$$

without assuming (A).

Theorem 3.2 implies the following partial converse of Duflo's theorem.

PROPOSITION 3.4. *Assume* $\mathbf{D}(S)$ *is a commutative algebra. For* $\xi \in \mathfrak{s}^*$, *let* $\mathfrak{s}_\xi$ *denote the linear subspace of* $\mathfrak{s}$ *spanned by all* $Dp(\xi)$, *for* $p \in I(\mathfrak{s})$. *Then*

$$\mathrm{tr}_{\mathfrak{h}}\, \mathrm{ad}[X, Y] = 0$$

for any $\xi \in \mathfrak{s}^*$ *and any* X, $Y \in \mathfrak{s}_\xi$.

Proof. By (1) Section 1.1 we have

$$e(X, Y) = 1 + \tfrac{1}{4}\mathrm{tr}_{\mathfrak{h}}\, \mathrm{ad}[X, Y] + \cdots = 1 + \sum_{i,j} \partial_{X_i}\partial_{Y_j} e(0, 0) X_i Y_j + \cdots.$$

By Theorem 3.2,

$$\begin{aligned}(p \times q)(\xi) &= p(\xi)q(\xi) + \sum_{i,j} \partial_{X_i}\partial_{Y_j} e(0, 0)\partial_{\xi_i}p(0)\partial_{\xi_j}q(0) + \cdots \\ &= p(\xi)q(\xi) + \tfrac{1}{4}\mathrm{tr}_{\mathfrak{h}}\, \mathrm{ad}[Dp(\xi), Dq(\xi)] + \cdots,\end{aligned}$$

where $Dp(\xi)$, $Dq(\xi) \in \mathfrak{s}^{**}$ are identified to elements of $\mathfrak{s}$. If p, q are homogeneous elements of $I(\mathfrak{s})$, with degrees a, b, the second term in this expansion of $p \times q$ is its homogeneous part with degree $a+b-2$. But commutativity of $\mathbf{D}(S)$ implies symmetry of this term with respect to p, q, hence

$$\mathrm{tr}_{\mathfrak{h}}\mathrm{ad}[Dp(\xi), Dq(\xi)] = 0.$$

The result extends immediately to arbitrary elements of $I(\mathfrak{s})$, whence the proposition.

3.5. This last section deals with second order operators, e.g. Laplacians. Theorems 3.1, 3.2 and Corollary 3.3 may then be given a simpler form.

PROPOSITION 3.5. *Let S be a symmetric space satisfying assumption (A) of Section* 1.1. *Let $q \in I(\mathfrak{s})$ be a homogeneous invariant of degree two, and let $e_q(X) = (q(\partial_Y)e)(X,0)$, which is an H-invariant, even, analytic function on $\mathfrak{s}''$, with $e_q(0) = 0$. Then*

(i) $q(X, \partial_X) = q(\partial_X) + e_q(X) = {}^t(q(X, \partial_X)$.
(ii) $(p \times q)(\xi) = (q(\xi) + e_q(\partial_\xi))p(\xi)$ *for* $p \in I(\mathfrak{s})$, $\xi \in \mathfrak{s}^*$.
(iii) *Assume that q generates $I(\mathfrak{s})$. Then there exists a unique linear map $\mathscr{A}: I(\mathfrak{s}) \to I(\mathfrak{s})$ such that $\mathscr{A}1 = 1$ and $[q(\xi), \mathscr{A}] = \mathscr{A} \circ e_q(\partial_\xi)$. This map is bijective, satisfies the equality*

$$\mathscr{A}^{-1}(P(q)) = P(q(\xi) + e_q(\partial_\xi))1,$$

for any polynomial P with complex coefficients, and the relation $\gamma p = (\mathscr{A}^{-1}p)^{\sim}$ defines a generalized Duflo's isomorphism $\gamma: \mathbf{D}(S_0) \to \mathbf{D}(S)$.

Here $q(\xi)$ is to be considered as a multiplication operator on $I(\mathfrak{s})$, and the bracket denotes the commutator of endomorphisms of $I(\mathfrak{s})$.

Proof. (i) From the proof of Theorem 3.1, we have

$$q(X, \partial_X)f(X) = q(\partial_Y)(e(X, Y)f(X + Y))_{Y=0}$$

for any smooth function f. Leibnitz' formula implies the first part of (i), since $e(X,0) = 1$ (cf. §1.1(ii)) and $D_Y e(X,0) = 0$ (cf. Proposition 1.1). The second equality follows, since ${}^t q(\partial_X) = q(-\partial_X) = q(\partial_X)$ and ${}^t e_q(X) = e_q(X)$.

(ii) Follows from (i) and Theorem 3.2.

(iii) The conditions imposed on $\mathscr{A}$ are, in view of (ii),

$$\mathscr{A}1 = 1 \quad \text{and} \quad \mathscr{A}(p \times q) = (\mathscr{A}p).q$$

for any $p \in I(\mathfrak{s})$. Therefore $\mathscr{A}q = q$, and $\mathscr{A}(p_1 \times p_2) = (\mathscr{A}p_1).(\mathscr{A}p_2)$ for any p_1, $p_2 \in I(\mathfrak{s})$. The rest follows from Corollary 3.3, whence the proposition.

Let S be a (*pseudo-*) *Riemannian symmetric space*; then its tangent space $\mathfrak{s}$ at the origin has an H-invariant (pseudo-) Euclidean structure $q(X)$ whence, by duality, a similar structure on $\mathfrak{s}^*$, still denoted by $q(\xi)$. Let $\Delta \in \mathbf{D}(S_0)$ be the corresponding (pseudo-) Laplacian; here $e_q(X) = \Delta_Y e(X,0)$. The Laplace–Beltrami operator $L \in \mathbf{D}(S)$ is symmetric with respect to the Riemannian measure of S (cf. [8] p. 245), which allows to write ${}^t L = L$.

COROLLARY 3.6. *For (pseudo-) Riemannian S, we have*

(i) $\tilde{\Delta} = L + \Delta J^{1/2}(0)$.
(ii) $L\tilde{u} = (\Delta u - J^{-1/2}.\Delta J^{1/2}.u)^{\sim}$ *on* $\operatorname{Exp} U$,

where U is an H-invariant open subset of $\mathfrak{s}''$, and u is any H-invariant function, or distribution, on U.

(iii) $\Delta J^{1/2}(X) = (\Delta J^{1/2}(0) - \Delta_Y e(X,0)).J^{1/2}(X)$ *on* $\mathfrak{s}''$.
(iv) *If S is semi-simple and $q(X) = B(X,X)$ (the Killing form of $\mathfrak{g}$), then*

$$\Delta J^{1/2}(0) = n/12 \quad \textit{with } n = \dim S.$$

Formula (ii) generalizes a result proved by Helgason for semi-simple Riemannian symmetric spaces of the noncompact type (cf. [8] p. 273). Formula (iii) relates the functions J and e; differential equations of this type will be studied in Section 4.

Proof. (i) Both sides are G-invariant operators on S, and it suffices to compute at the origin o of S. For $\phi \in C^\infty(S)$ we have, by Section 3.1,

$$\tilde{\Delta}\phi(o) = \Delta_{Y=0}(J^{1/2}(Y)\phi(\operatorname{Exp} Y)) = \Delta_{Y=0}\phi(\operatorname{Exp} Y) + \Delta J^{1/2}(0).\phi(o),$$

since $J(0) = 1$ and J is even. Now it is classical (and easily proved in 'normal coordinates' Exp) that the first term on the right is $L\phi(o)$, whence (i).

(ii) and (iii): By (i), Theorem 3.1, and Proposition 3.5(i) we have

$${}^tL\tilde{u} = ({}^t\Delta u + (\Delta_Y e(X,0) - \Delta J^{1/2}(0))u)^{\sim};$$

the transpose signs may be omitted, and the equality holds for distributions as well as functions. Taking $u = J^{1/2}$, we obtain $\tilde{u} = 1$, whence (iii) since $L1 = 0$; then (ii) follows at once.

(iv) For any S one has, for X near 0 in $\mathfrak{s}$,

$$J^{1/2}(X) = (\det_{\mathfrak{s}} \operatorname{sh}(\operatorname{ad} X)/\operatorname{ad} X)^{1/2} = \det_{\mathfrak{s}}\left(I + \frac{(\operatorname{ad} X)^2}{6} + O(X^4)\right)^{1/2}$$

$$= 1 + \frac{1}{12}\operatorname{tr}_{\mathfrak{s}}(\operatorname{ad} X)^2 + O(X^4);$$

besides $\operatorname{tr}_{\mathfrak{s}}(\operatorname{ad} X)^2 = \operatorname{tr}_{\mathfrak{h}}(\operatorname{ad} X)^2 = B(X,X)/2 = q(X)/2$ (cf. (I) Lemma 3.2). But $\Delta q = 2n$, whence the result.

Combining Proposition 3.5(iii) and Corollary 3.6 gives an explicit Duflo's isomorphism for all isotropic (pseudo-) Riemannian symmetric spaces.

4. The *e*-function and spherical functions

Throughout this section, H is a *compact* group, so that $S = G/H$ is a Riemannian symmetric space. Having chosen an H-invariant Euclidean norm

$X \to |X|$ on its tangent space S_0, let Δ denote the corresponding Euclidean Laplacian.

A spherical function is an H-invariant smooth function ϕ on S, which is a joint eigenfunction of all differential operators in $\mathbf{D}(S)$, and such that $\phi(o) = 1$. Let $\psi(X) = J(X)^{1/2}\phi(\operatorname{Exp} X)$, so that $\phi = \tilde{\psi}$ on the open set S'. We wish to relate ψ and e.

4.1. Let us consider first the simpler case of an *isotropic* space, that is when H acts transitively on the unit sphere of S_0. Then $\mathbf{D}(S_0)$ and $\mathbf{D}(S)$ are known to be polynomial algebras, respectively generated by Δ and the Laplace–Beltrami operator L, or $\tilde{\Delta}$ by Corollary 3.6; thus ϕ is spherical if and only if $\tilde{\Delta}\phi = \lambda\phi$ for some $\lambda \in \mathbf{C}$, and $\phi(o) = 1$. In terms of ψ this implies, by Theorem 3.1(ii) and Proposition 3.5(i),

$$\Delta\psi(X) = (\lambda - \Delta_Y e(X,0)).\psi(X), \qquad \psi(0) = 1 \tag{1}$$

on the neighborhood $\mathfrak{s}''$ of the origin. This Schrödinger equation may be considered as a perturbation of $\Delta\psi = \lambda\psi$, and solutions of (1) as perturbed Bessel functions.

Both functions $\psi(X)$ and $\Delta_Y e(X,0)$ are analytic and H-invariant, and their Taylor expansions at the origin may be written as

$$\psi(X) = \sum_{k \geqslant 0} a_k \frac{|X|^{2k}}{c_k}, \qquad \Delta_Y e(X,0) = \sum_{k \geqslant 1} b_k \frac{|X|^{2k}}{c_k},$$

where a_k, b_k are coefficients, and c_k is the constant $c_k = \Delta^k|X|^{2k}$, i.e.

$$c_0 = 1, \qquad c_k = 2^k k!\, n(n+2)\cdots(n+2k-2), \quad \text{with } k \geqslant 1, \quad n = \dim S.$$

Then $a_k = \Delta^k\psi(0)$, $b_k = \Delta_X^k \Delta_Y e(0,0)$, for $k \geqslant 0$, and (1) is equivalent to

$$a_{k+1} = \lambda a_k - \sum_{1 \leqslant j \leqslant k} \frac{c_k}{c_j c_{k-j}} b_j a_{k-j}, \qquad k \geqslant 1$$

$$a_0 = 1, \qquad a_1 = \lambda = \Delta\psi(0).$$

Solving this inductively, we obtain

$$\psi(X) = 1 + \lambda\frac{|X|^2}{c_1} + (\lambda^2 - b_1).\frac{|X|^4}{c_2} + \left(\lambda^3 - \left(3 + \frac{4}{n}\right)b_1.\lambda - b_2\right).\frac{|X|^6}{c_3} + \cdots. \tag{2}$$

In the case of an isotropic special space ($e(X, Y) = 1$), (2) reduces to the classical expansion

$$\psi(X) = \sum_{k \geqslant 0} \lambda^k \frac{|X|^{2k}}{c_k} = \Gamma\left(\frac{n}{2}\right)(2/\sqrt{-\lambda}\,|X|)^{n/2-1}.J_{n/2-1}(\sqrt{-\lambda}\,|X|)$$

of Bessel functions.

Convergence of the power series $\Sigma\, b_k|X|^{2k}/c_k$ implies the estimates

$|b_k| \leqslant Mc_kR^k$, with M, R positive constants. It is easily shown by induction that similar estimates must hold for the a_k: $|a_k| \leqslant M'c_kR^k$, with $M' > 0$ depending on λ.

4.2. Section 4.1 motivates the following extension to arbitrary Riemannian symmetric spaces.

Compactness of H implies that the algebra $I(\mathfrak{s}) = \mathbf{D}(S_0)$ is finitely generated ([8] p. 352). Let $p_1, \dots, p_m$ be a system of generators, homogeneous with degrees $d_1, \dots, d_m$. They may be algebraically dependent – I am grateful to A. Cerezo for showing me an example of that – but from the set of all

$$p_\alpha = p_1^{\alpha_1} \cdots p_m^{\alpha_m}, \qquad \alpha \in \mathbf{N}^m,$$

one can extract a basis of $I(\mathfrak{s})$ as a vector space. Let $B \subset \mathbf{N}^m$ be the corresponding subset of indices.

Let $(p_\alpha^*)_{\alpha\in B}$ be the dual basis of the algebra $I(\mathfrak{s}^*)$ of H-invariant polynomial functions on $\mathfrak{s}$, with respect to the duality $\langle | \rangle$ of Section 3.2:

$$\langle p_\alpha | p_\beta^* \rangle = \delta_{\alpha\beta}, \quad \alpha, \beta \in B \text{ (Kronecker symbol)}.$$

The existence of such a basis can be explained as follows. The polynomials $\xi^\sigma \in S(\mathfrak{s})$ and $X^\sigma/\sigma! \in S(\mathfrak{s}^*)$, with $\sigma \in \mathbf{N}^n$ (defined by means of coordinates with respect to some dual bases of $\mathfrak{s}$ and $\mathfrak{s}^*$), are clearly dual bases of the symmetric algebras $S(\mathfrak{s})$ and $S(\mathfrak{s}^*)$ for $\langle | \rangle$. It follows that $S(\mathfrak{s}^*)$ can be identified with the space of linear forms on $S(\mathfrak{s})$ which have finite degree, i.e. vanish on all homogeneous polynomials of degree greater than some value. The projection $S(\mathfrak{s}) \to I(\mathfrak{s})$ given by H-invariant mean preserves homogeneity and degrees. Therefore any linear form of finite degree on $I(\mathfrak{s})$ can be represented as $\langle | p^* \rangle$, with a unique $p^* \in S(\mathfrak{s}^*)$; furthermore $p^* \in I(\mathfrak{s}^*)$, by H-invariance of $\langle | \rangle$. Thus $I(\mathfrak{s}^*)$ identifies to the space of linear forms of finite degree on $I(\mathfrak{s})$, whence the dual basis $(p_\alpha^*)_{\alpha\in B}$.

Each p_α is homogeneous of degree $\alpha . d = \alpha_1 d_1 + \cdots + \alpha_m d_m$, and it follows that p_α^* is homogeneous with the same degree; indeed, for $\alpha, \beta \in B$, $t \in \mathbf{C}$ we have

$$\begin{aligned}\langle p_\beta(\partial_X) | p_\alpha^*(tX) \rangle &= t^{\beta . d}(p_\beta(\partial) p_\alpha^*)(0) = t^{\beta . d}\delta_{\alpha\beta} \\ &= t^{\alpha . d}\delta_{\alpha\beta} = t^{\alpha . d}\langle p_\beta(\partial_X) | p_\alpha^*(X) \rangle.\end{aligned}$$

LEMMA 4.1. *Any H-invariant function, analytic in a neighborhood of 0 in $\mathfrak{s}$, admits a unique H-invariant Taylor expansion*

$$\phi(X) = \sum_{\alpha\in B} a_\alpha p_\alpha^*(X),$$

with $a_\alpha = p_\alpha(\partial_X)\phi(0) = \langle p_\alpha | \phi \rangle$. The series converges in the following sense: there exists $r > 0$ such that

$$\sum_{q \geqslant 0} \left| \sum_{\alpha . d = q} a_\alpha p_\alpha^*(X) \right| < \infty \quad \textit{for } |X| < r.$$

One can take term by term derivatives of this expansion of ϕ, for $|X| < r$.

Proof. The classical Taylor expansion $\phi(X) = \Sigma c_\sigma X^\sigma$, $\sigma \in \mathbf{N}^n$, converges absolutely, and can be differentiated term by term, for $|X| < r$. Gathering terms of equal degrees, the same properties a fortiori hold for $\Sigma_{q \geqslant 0} (\Sigma_{|\sigma| = q} c_\sigma X^\sigma)$. Each term of this series, as an invariant q-homogeneous polynomial in X, is a linear combination of p_α^* 's, with $\alpha . d = q$, whence the expansion of ϕ. The formula for a_α comes out when applying $p_\alpha(\partial_X)$ term by term.

EXAMPLES. (a) Trivial case: $H = \{e\}$, $p_\alpha = \partial_X^\alpha$, $p_\alpha^* = X^\alpha/\alpha!$

(b) Isotropic case: $p_\alpha = \Delta^k$, $p_\alpha^* = |X|^{2k}/c_k$, with $\alpha = k \in \mathbf{N}$ (cf. §4.1).

(c) *Generalized Bessel functions*: for any Riemannian S, let $\xi \in \mathfrak{s}^*$, $X \in \mathfrak{s}$ and

$$\psi_{\xi,0}(X) = \int_H e^{\langle \xi, h.X \rangle} \, dh,$$

where dh is the normalized Haar measure on H. Then

$$p(\partial_X)\psi_{\xi,0}(X) = p(\xi)\psi_{\xi,0}(X)$$

for any $p \in \mathbf{D}(S_0)$, and $\psi_{\xi,0}$ is a spherical function for the flat space S_0. Lemma 4.1 gives

$$\psi_{\xi,0}(X) = \int_H e^{\langle \xi, h.X \rangle} \, dh = \sum_{\alpha \in B} p_\alpha(\xi) p_\alpha^*(X), \tag{3}$$

a convergent series for all X and ξ, in the sense of the lemma.

THEOREM 4.2. *Let ϕ be a spherical function on a Riemannian symmetric space S, and $\psi(X) = J(X)^{1/2}\phi(\operatorname{Exp} X)$ (so that $\phi = \tilde{\psi}$). Let $p_1, \ldots, p_m$ be a fixed system of homogeneous generators of $I(\mathfrak{s})$, and $\lambda_1, \ldots, \lambda_m$ be the corresponding eigenvalues: $\tilde{p}_j \phi = \lambda_j \phi$. Then ψ has the following H-invariant Taylor expansion at the origin*

$$\psi(X) = \sum_{\alpha \in B} a_\alpha(\lambda_1, \ldots, \lambda_m; e) p_\alpha^*(X).$$

Each coefficient a_α is a homogeneous polynomial in the λ_j's and the derivatives $e_{\sigma\tau} = \partial_X^\sigma \partial_Y^\tau e(0,0)/\sigma! \, \tau!$ of the e-function of S, homogeneous of degree $\alpha . d$ if one assigns $\deg(\lambda_j) = \deg(p_j) = d_j$, and $\deg(e_{\sigma\tau}) = |\sigma| + |\tau|$ for σ, $\tau \in \mathbf{N}^n$. The coefficients of a_α only depend on the structure of the algebra $I(\mathfrak{s})$ (and on the choice of the p_j's and a basis of $\mathfrak{s}$).

REMARKS. (a) The series in the theorem converges in the sense of Lemma 4.1, on a neighborhood of 0 in $\mathfrak{s}$.

(b) The proof will show that $a_\alpha = \lambda^\alpha +$lower order terms (with respect to the above graduation); the latter terms involve derivatives of e at $(0,0)$. In particular

$$\psi(X) = \sum_{\alpha \in B} \lambda^\alpha p_\alpha^*(X) \tag{4}$$

when S is a special space.

(c) Theorem 4.2 provides a more precise form to an expansion of spherical functions already obtained by Helgason [7] p. 277, as a consequence of his expansion of mean value operators.

Proof. The Laplace–Beltrami operator of S is elliptic, therefore ϕ is analytic on S, and ψ can be expanded as

$$\psi(X) = \sum_{B} a_\alpha p_\alpha^*(X) \tag{5}$$

on a neighborhood of 0 in $\mathfrak{s}$, by Lemma 4.1. We wish to show that the $a_\alpha = p_\alpha(\partial)\phi(0)$, for $\alpha \in B$, are determined inductively from the λ_j's.

From the definition of spherical functions, we have $\tilde{p}\tilde{\psi} = \lambda_p \tilde{\psi}$ near 0, for any $p \in I(\mathfrak{s})$, and the map $\tilde{p} \to \lambda_p = \tilde{p}\tilde{\psi}(o) = p(\partial)\psi(0) = \langle p \mid \psi \rangle$ (cf. §3.1) is a character of the algebra $\mathbf{D}(S)$. Taking $\tilde{p} = \tilde{p}_\beta \circ \tilde{p}_j$, it follows, in view of Theorem 3.2, that

$$\langle p_\beta \times p_j \mid \psi \rangle = \lambda_j \langle p_\beta \mid \psi \rangle, \tag{6}$$

for all $\beta \in \mathbf{N}^m$ and $j = 1, \dots, m$. Here

$$(p_\beta \times p_j)(\xi) = \sum e_{\sigma\tau} \partial_\xi^\sigma p_\beta(\xi) . \partial_\xi^\tau p_j(\xi),$$

where Σ runs over all σ, $\tau \in \mathbf{N}^n$; by (1′) Section 1.1, this expansion begins with $p_\beta(\xi)p_j(\xi) = p_{\beta+\varepsilon_j}(\xi)$ (setting $\varepsilon_j = (0, \dots, 0, 1, 0, \dots, 0)$), and all remaining terms correspond to $|\sigma| + |\tau| \geqslant 4$.

Let $\alpha \in B$, with $\alpha_j \geqslant 1$; then we may write $\alpha = \beta + \varepsilon_j$ and apply (6). If β happens to belong to B, the right-hand side of (6) is simply $\lambda_j a_\beta$. In general, replacing ψ by (5), we obtain the following explicit form of (6):

$$a_\alpha + \sum \langle \partial_\xi^\sigma p_\beta . \partial_\xi^\tau p_j \mid p_\gamma^* \rangle e_{\sigma\tau} a_\gamma = \lambda_j \sum \langle p_\beta \mid p_\delta^* \rangle a_\delta. \tag{7}$$

The left-hand summation runs over all σ, $\tau \in \mathbf{N}^n$ and all $\gamma \in B$ such that $|\sigma| + |\tau| \geqslant 4$ and $\gamma . d = \alpha . d - |\sigma| - |\tau|$; the right-hand summation runs over all $\delta \in B$ such that $\delta . d = \beta . d$; in fact the $\langle \mid \rangle$ product of homogeneous polynomials vanishes unless their degrees are equal. Both sums are finite. The scalar products in (7) are certain structure constants of the algebra $I(\mathfrak{s})$.

All a_α's for $\alpha \in B$ can now be obtained from (7) by induction on $\alpha . d$, starting from $\alpha . d = 0$, i.e. $\alpha = 0$, with $a_0 = \psi(0) = 1$. All claimed properties of $a_\alpha = a_\alpha(\lambda_1, \dots, \lambda_m; e)$ are easily checked inductively, whence the theorem.

5. Spherical functions and Bessel functions

Here again H is a *compact* group, and we keep to the notation of Section 4.2. The aim of this section is to investigate formal relations between spherical functions on S and on S_0, already glimpsed in Section 4.1 for isotropic spaces; here we meet again with the operator $\mathscr{A}$ introduced in Section 3.3.

5.1. Having chosen homogeneous generators $p_1, \ldots, p_m$ of $I(\mathfrak{s})$, *we assume* that, for any ξ in the complexified dual $\mathfrak{s}_c^*$, there exists a (necessarily unique) spherical function ϕ_ξ of S corresponding to the eigenvalues $\lambda_j = p_j(\xi)$, $1 \leqslant j \leqslant m$. As above, we write $\phi_\xi = \tilde{\psi}_\xi$. We also assume convergence of the Taylor expansion of $\psi_\xi(X)$ for $|X| < r$, with $r > 0$ independent of ξ.

These assumptions are clearly satisfied by Euclidean symmetric spaces: take $\phi_\xi = \psi_{\xi,0}$, as in Section 4.2 Example (c). They also hold for semi-simple spaces of the noncompact type. Indeed $\mathbf{D}(S)$ is then a polynomial algebra in the generators $\tilde{p}_1, \ldots, \tilde{p}_m$, and the map $\tilde{p}_j \to \lambda_j = p_j(\xi)$ extends to a character χ_ξ of this algebra. The existence of ϕ_ξ such that $\tilde{p}\phi_\xi = \chi_\xi(\tilde{p})\phi_\xi$ for all $p \in I(\mathfrak{s})$ is classical (cf. [8] p. 418), and the convergence assumption is clear from Harish–Chandra's integral formula for spherical functions. (One should keep aware that the present notation ϕ_ξ, which depends on the choice of generators, differs from Harish–Chandra's notation ϕ_λ). Finally the above assumptions are not satisfied by compact symmetric spaces, where integrality conditions are required on the parameter ξ.

Let $\mathscr{A}p(\xi)$ denote the eigenvalue of $\tilde{p}$ corresponding to the eigenfunction ϕ_ξ:

$$\tilde{p}\phi_\xi = \mathscr{A}p(\xi) . \phi_\xi, \quad \text{with } \xi \in \mathfrak{s}_c^*,\ p \in I(\mathfrak{s}). \tag{1}$$

PROPOSITION 5.1. *Under the above assumptions on the Riemannian symmetric space S, the operator $\mathscr{A}$ defined by* (1) *is an isomorphism of algebras of* $I(\mathfrak{s})$, $\times$ *onto* $I(\mathfrak{s}), (.)$. *Besides*

(i) $\mathscr{A}p(\xi) = p(\partial)\psi_\xi(0)$ *for* $\xi \in \mathfrak{s}_c^*$, $p \in I(\mathfrak{s})$,

(ii) $\mathscr{A}1 = 1$, $\mathscr{A}p_j = p_j$ *for* $j = 1, \ldots, m$ (*thus* $\mathscr{A}$ *agrees with the operator of Corollary* 3.3),

(iii) $\mathscr{A}p_\alpha(\xi) = a_\alpha(p_1(\xi), \ldots, p_m(\xi); e)$ *for* $\alpha \in B$ (*in the notation of Theorem* 4.2),

(iv) $\mathscr{A}p = p +$ *lower order terms for any* $p \in I(\mathfrak{s})$, *and* $\mathscr{A}$ *is the identity for special* S.

Proof. The proposition is an easy consequence of previous results. By (1), $\mathscr{A}p(\xi) = \tilde{p}\tilde{\psi}_\xi(o) = p(\partial)\psi_\xi(0)$, whence (i). (ii) Is obvious. (iii) Follows from Theorem 4.2, and shows that $\mathscr{A}p_\alpha \in I(\mathfrak{s})$ and $\mathscr{A}p_\alpha = p_\alpha +$ lower order (cf. Remark (b) following Theorem 4.2). Then (iv) follows, and implies that $\mathscr{A}$ is invertible, as the sum of the identity and a locally nilpotent operator. Finally, for $p, q \in I(\mathfrak{s})$, we have $\tilde{p}\tilde{q}\phi_\xi = \mathscr{A}p(\xi)\mathscr{A}q(\xi)\phi_\xi$, whence $\mathscr{A}(p \times q) = \mathscr{A}p . \mathscr{A}q$ by Theorem 3.2.

5.2. The map $\mathscr{A}$ extends in an obvious way to all series $\Sigma_{\alpha \in B} c_\alpha p_\alpha(\xi)$ with complex coefficients c_α satisfying the following condition:

(C) the series

$$\sum_{q \geqslant 0} \left(\sum_{\alpha.d = q} c_\alpha p_\alpha(\xi) \right) \quad \text{and} \quad \sum_{q \geqslant 0} \left(\sum_{\alpha.d = q} c_\alpha \mathscr{A}p_\alpha(\xi) \right)$$

converge for all $\xi \in \mathfrak{s}_c^*$.

We shall then set

$$\mathscr{A}\left(\sum_B c_\alpha p_\alpha(\xi)\right) = \sum_B c_\alpha \mathscr{A} p_\alpha(\xi).$$

Thus, for $|X| < r$,

$$\mathscr{A}\left(\sum p_\alpha^*(X) p_\alpha(\xi)\right) = \Sigma\, p_\alpha^*(X) \mathscr{A} p_\alpha(\xi),$$

which means, by Section 4.2 Example (c), that

$$\psi_\xi(X) = \mathscr{A}(\psi_{\xi,0}(X)). \tag{2}$$

The operator $\mathscr{A}$, acting on ξ, transforms spherical functions of S_0 (i.e. generalized Bessel functions) into spherical functions of S, up to the map $\tilde{}$.

The *spherical transform* of an H-invariant smooth function $\tilde{f}$ on S (with supp f a compact subset of $|X| < r$) may be defined by

$$(Ff)(\xi) = \int_S \tilde{f}(s)\tilde{\phi}_{-i\xi}(s)\,\mathrm{d}s = \int_{\mathfrak{s}} f(X)\phi_{-i\xi}(X)\,\mathrm{d}X,$$

where $\xi \in \mathfrak{s}^*$, $\mathrm{d}X$ is a Lebesgue measure on $\mathfrak{s}$, and $\mathrm{d}s = (J(X)^{1/2}\,\mathrm{d}X)^{\sim}$ is the corresponding G-invariant measure on S'. Let $(Mu)(\xi) = u(i\xi)$, where u is any function on $\mathfrak{s}_c^*$, and $\mathscr{A}' = M^{-1}\mathscr{A}M$. According to our assumptions at the beginning of this section, we may write

$$Ff = \mathscr{A}'F_0 f, \quad \text{where } (F_0 f)(\xi) = \int_{\mathfrak{s}} f(X)e^{-i\langle\xi,X\rangle}\,\mathrm{d}X$$

is the usual Fourier transform; indeed the coefficients

$$c_\alpha = \int_{\mathfrak{s}} f(X)p_\alpha^*(X)\,\mathrm{d}X$$

satisfy condition (C) above. Thus the operator $\mathscr{A}'$ relates the spherical transforms on the spaces S and S_0; a similar role is played by the Abel transform of a semi-simple space of the noncompact type. The classical inversion

$$f(0) = \int_{\mathfrak{s}^*} (F_0 f)(\xi)\,\mathrm{d}\xi$$

gives

$$f(0) = \int_{\mathfrak{s}^*} (\mathscr{A}'^{-1}Ff)(\xi)\,\mathrm{d}\xi,$$

and the spherical Plancherel measure of S is (formally)

$$\mathrm{d}\mu(\xi) = {}^t\mathscr{A}'^{-1}(\mathrm{d}\xi). \tag{3}$$

5.3. As in (I) Section 1 let S_t, $0 \leqslant t \leqslant 1$, be the *contracted symmetric space* of S corresponding to the Lie triple product $t^2[X, Y, Z]$ on $\mathfrak{s}$; in particular $S_1 = S$

and $\dots S_0 = S_0$. By (I) Proposition 3.17 and (1′) Section 1.1, its e-function is

$$e_t(X, Y) = e(tX, tY) = 1 + t^4 e_4(X, Y) + t^6 e_6(X, Y) + \cdots$$

where $e_4, e_6, \dots$ are homogeneous polynomials of degrees 4, 6, ... On S we had $\tilde{p}_j \tilde{\psi}_\xi = p_j(\xi)\tilde{\psi}_\xi$, which is equivalent, by Theorem 3.1(ii), to $p_j(X, \partial_X)\psi_\xi(X) = p_j(\xi)\psi_\xi(X)$. Replacing e by e_t it is easily seen that

$$\psi_{\xi,t}(X) = \psi_{\xi/t}(tX)$$

is the analogue of ψ_ξ for the space S_t, with the same eigenvalues $p_j(\xi)$, and this is coherent with our notation $\psi_{\xi,0}$.

Let $\mathscr{A}_t$ be the operator $\mathscr{A}$ corresponding to S_t (cf. Proposition 5.1):

$$\mathscr{A}_t p_\alpha(\xi) = a_\alpha(p_1(\xi), \dots, p_m(\xi); e_t) = t^{\alpha.d} a_\alpha(p_1(\xi/t), \dots, p_m(\xi/t); e).$$

Expanding into powers of t, we obtain the series

$$\mathscr{A}_t = I + t^4 \mathscr{A}_4 + t^6 \mathscr{A}_6 + \cdots,$$

a locally finite sum on $I(\mathfrak{s})$. Indeed, from the homogeneity properties in Theorem 4.2, it follows that $\mathscr{A}_k p_\alpha(\xi)$ is homogeneous with degree $\alpha.d - k$; therefore $\deg(\mathscr{A}_k p) \leqslant \deg(p) - k$ for all $p \in I(\mathfrak{s})$. Besides

$$\psi_{\xi,t}(X) = \psi_{\xi/t}(tX) = (I + t^4 \mathscr{A}_4 + t^6 \mathscr{A}_6 + \cdots)\psi_{\xi,0}(X). \tag{4}$$

Such an expansion has been proved (with precise estimates) by Stanton and Tomas [10] for rank one spaces of the noncompact type; its first term

$$\phi_{\xi,0}(X) = \lim_{t \to 0} \phi_{\xi/t}(tX)$$

is the generalized Mehler–Heine formula (Clerc [1], Stanton).

To give $\mathscr{A}_t$ a more specific form, let us consider the case of an *isotropic Riemannian space*, with $q(\xi) = |\xi|^2$. It is now convenient to work with the variable $\lambda = |\xi|^2$. The Laplacian with respect to ξ is

$$\Delta_\xi = 4\lambda \partial_\lambda^2 + 2n\partial_\lambda$$

(when acting on H-invariant functions, which is our only concern here), and

$$e_q(\partial_\xi) = \sum_{k \geqslant 0} \frac{b_k}{c_k} \Delta_\xi^k,$$

with $b_0 = 0$, $b_k = \Delta_X^k \Delta_Y e(0, 0)$, as in Section 4.1. The commutator equation of Proposition 3.5(iii), written for the space S_t, is

$$[\lambda, \mathscr{A}_t] = \mathscr{A}_t \circ \sum_{k \geqslant 0} t^{2(k+1)} \frac{b_k}{c_k} \Delta_\xi^k.$$

It can be solved by

$$\mathscr{A}_t = I + \sum t^{2(j+k)} a_{jk} \partial_\lambda^j \Delta_\xi^k, \tag{5}$$

where Σ runs over all integers $j \geqslant 1$, $k \geqslant 0$ such that $j + k \geqslant 2$, and the coefficients a_{jk} are obtained from the b_k by induction on j:

$$a_{00} = 1, \qquad a_{0k} = 0 \quad \text{for } k \geqslant 1,$$

$$(j+2k+1)a_{j+1,k} = (k+1)(2n-8-4k)a_{j,k+1} - \sum_{p+q=k} a_{jp}\frac{b_q}{c_q} \quad \text{for } j,\, k \geqslant 0.$$

This follows from straightforward calculations relying on the identity

$$[\lambda, \partial_\lambda^j \Delta_\xi^k] = -(j + 2k)\partial_\lambda^{j-1}\Delta_\xi^k + k(2n - 4 - 4k)\partial_\lambda^j \Delta_\xi^{k-1},$$

for $j \geqslant 1$, $k \geqslant 0$. The first terms are

$$\mathscr{A}_t = I - t^4 \frac{b_1}{6n}(\partial_\lambda \Delta_\xi + (n-4)\partial_\lambda^2) -$$

$$- t^6 \frac{b_2}{40n(n+2)}(\partial_\lambda \Delta_\xi^2 + (n-6)\partial_\lambda^2 \Delta_\xi + \tfrac{2}{3}(n-4)(n-6)\partial_\lambda^3) + o(t^8).$$

Of course, application of $\mathscr{A}_t$ (on the variable $\lambda = |\xi|^2$) to the Bessel function $\psi_{\xi,0}(X) = \Sigma\, \lambda^k |X|^{2k}/c_k$ gives back the expansion of $\psi_{\xi,t}(X)$ already obtained for $t = 1$ in Section 4.1; the Stanton–Tomas expansion is thus obtained inductively.

Remembering that $\psi_{\xi,0}(X) = \int_H e^{\langle \xi, h.X \rangle}\, \mathrm{d}h$, the latter result may also be written as an integral. After a few transformations, it can be shown that

$$\psi_{\xi,t}(X) = \mathscr{A}_t \psi_{\xi,0}(X) = \int_H e^{\langle \xi, h.X \rangle} a\left(t^2|X|^2,\, t^2\left\langle \frac{\xi}{|\xi|},\, h.X \right\rangle^2\right) \mathrm{d}h,$$

where $a(.,.)$ is some (formal) power series in two variables, related to the a_{jk}'s. This is the remarkable form Duistermaat [6] has given to the spherical functions for semi-simple spaces of the noncompact type. We shall not detail this here, hoping to study the links between a and e in a forthcoming paper.

Added on proof

The conjectured symmetry $e(X, Y) = e(Y, X)$ discussed in Section 3.4 can be proved from assumption (A) in Section 1.1 (to appear).

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