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Actions of group schemes (I)*

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0. Introduction

Let G be a separated group scheme over a base scheme S. Let $\omega_{G/S}$ be the sheaf of left invariant differentials of G. Let $\tau: X \to S$ be a separated morphism. Our main result of section 1 is the following

THEOREM. *Let $\rho: G \times_S X \to X$ be an action of G on X. Then*

(i) *There is a canonical complex induced by ρ:*

$$\mathrm{DR}_\rho: O_X \to \tau^* \omega_{G/S} \to \tau^* \wedge^2 \omega_{G/S} \to \tau^* \wedge^3 \omega_{G/S} \cdots$$

(ii) *The identity map of O_X induces a (unique) canonical O_X-linear map $\Omega^{\cdot}_{X/S} \to \mathrm{DR}_\rho$, which is surjective when ρ is free.*
(iii) *Let $\mathscr{D}_\rho$ be the sheaf of ρ-invariant derivations. Then ρ induces a canonical map $\mathrm{Lie}(G/S) \to \mathscr{D}_\rho$ of sheaves of Lie algebras over O_S.*

Let k be an algebraically closed field of characteristic $p > 0$. A finite group scheme G over k is called *infinitesimal* (or "local", in the terminology of [9]) if it has only one point. The main result of section 2 is

THEOREM. *Let G be an infinitesimal group scheme over k. Let X be a smooth projective variety over k with a free action of G such that* $\dim(X) \leqslant \mathrm{rank}_k \omega_G$. *Suppose that either*

(i) *X is ordinary and $p > 2$; or*
(ii) $\mathrm{Pic}(X)$ *is reduced, G is commutative and the Cartier dual of G is also infinitesimal.*

Then X is an abelian variety.

There is an example which shows that the condition $p > 2$ is necessary in case (i).

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1. Calculus of group scheme actions

Let $\pi: G \to S$ be a group scheme together with multiplication $m: G \times_S G$, unit section $o: S \to G$ and inverse $\iota: G \to G$ (over S). We will always assume that π is separated, or equivalently, that o is a closed immersion. Let $\mathscr{M}$ be the ideal sheaf of o. Denote by $\omega_{G/S} \simeq o^*\mathscr{M}$ the sheaf of (left) invariant differentials.

Let $\tau: X \to S$ be a separated morphism. By an *action* of G on X we mean a morphism $\rho: G \times_S X \to X$ such that

(i) $\rho \circ (\mathrm{id}_G \times_S \rho) = \rho \circ (m \times_S \mathrm{id}_X): G \times_S X \times_S X \to X$;
(ii) $\rho \circ (o \times_S \mathrm{id}_X) = \mathrm{id}_X: X \simeq S \times_S X \to X$.

We will always denote $\alpha = (\rho, \mathrm{pr}_2): G \times_S X \to X \times_S X$. We say ρ is *free* if α is a closed immersion, and ρ is *transitive* if α is smooth and onto.

LEMMA 1.1. *Let $\tau: X \to S$ be a separated morphism and $\pi: G \to S$ be a group scheme. Then an action $\rho: G \times_S X \to X$ induces a canonical map of O_X-modules*

$$\Omega^1_{X/S} \to \tau^*\omega_{G/S} \tag{1}$$

which is surjective when ρ is free.

Proof. Look at the following commutative diagram

$$\begin{array}{ccccc} S & \xleftarrow{\ \tau\ } & X & & \\ \downarrow{\scriptstyle o} & & \downarrow{\scriptstyle i} & \searrow{\scriptstyle \Delta} & \\ G & \xleftarrow{\ \mathrm{pr}_1\ } & G \times_S X & \xrightarrow{\ \alpha\ } & X \times_S X \end{array} \tag{2}$$

where $i = o \times_S \mathrm{id}_X$. Let $\mathscr{I}$ (resp., $\mathscr{I}'$, $\mathscr{M}$) be the ideal sheaf of Δ (resp., i, o). Then we have an exact sequence

$$0 \longrightarrow \mathscr{M} \longrightarrow O_G \longrightarrow o_*O_S \longrightarrow 0 \tag{3}$$

Apply pr_1^*. Since the left square of (2) is cartesian and o has a section, (3) splits locally over $\pi^{-1}O_S$. Hence the first row of the following diagram is again exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{pr}_1^*\mathscr{M} & \longrightarrow & \mathrm{pr}_1^*O_G & \longrightarrow & \mathrm{pr}_1^*o_*O_S & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \\ 0 & \longrightarrow & \mathscr{I}' & \longrightarrow & O_{G\times_S X} & \longrightarrow & i_*O_X & \longrightarrow & 0 \end{array} \tag{4}$$

Therefore $\mathrm{pr}_1^*\mathscr{M} \simeq \mathscr{I}'$. Next we apply α^* to $0 \to \mathscr{I} \to O_{X \times_S X} \to \Delta_* O_X \to 0$. Since $\Delta_* O_X \simeq \alpha_*(i_* O_X)$ we have

$$\begin{array}{ccccccc}
0 & & & & & & \\
 & \searrow & & & & & \\
 & & \mathscr{I}' & & & & \\
 & & & \searrow & & & \\
 & \alpha^*\mathscr{I} & \longrightarrow & O_{G \times_S X} & \longrightarrow & \alpha^*\Delta_* O_X & \longrightarrow 0 \\
 & & & & \searrow & \downarrow f & \\
 & & & & & i_* O_X & \\
 & & & & & & \searrow \\
 & & & & & & \quad 0
\end{array} \tag{5}$$

This induces $\alpha^*\mathscr{I} \to \mathscr{I}'$. Now applying i^* we get

$$\begin{array}{ccc}
i^*(\alpha^*\mathscr{I}) & \longrightarrow & i^*\mathscr{I}' \\
\downarrow \simeq & & \downarrow \simeq \\
\Delta^*\mathscr{I} & & i^*(\mathrm{pr}_1^*\mathscr{M}) \\
\downarrow \simeq & & \downarrow \simeq \\
\Omega^1_{X/S} & \longrightarrow & \tau^*(o^*\mathscr{M}) \simeq \tau^*\omega_{G/S}
\end{array} \tag{6}$$

Finally, when α is a closed immersion, f is an isomorphism, hence $\alpha^*\mathscr{I} \twoheadrightarrow \mathscr{I}'$ which implies that $\Omega^1_{X/S} \twoheadrightarrow \tau^*\omega_{G/S}$. □

Suppose that an action $\rho: G \times_S X \to X$ is given. For an open affine subset $V \subset S$, let $U = \tau^{-1}(V)$. We say that a derivation $D \in \mathscr{D}er_S(O_X)(U)$ is *ρ-invariant* if the following diagram is commutative:

$$\begin{array}{ccc}
O_U & \xrightarrow{D} & O_U \\
\downarrow \rho^* & & \downarrow \rho^* \\
\rho_* O_{G \times_S U} & \xrightarrow{\rho_* \mathrm{pr}_2^*(D)} & \rho_* O_{G \times_S U}
\end{array} \tag{7}$$

The sheaf of ρ-invariant derivations (as a sheaf of O_S-modules) will be denoted by $\mathscr{D}_\rho$. Clearly $\mathscr{D}_\rho$ is quasi-coherent.

LEMMA 1.2. *An action ρ induces a canonical map of O_S-Lie algebras:*

$$\mathrm{Lie}(G/S) \to \mathscr{D}_\rho. \tag{8}$$

Proof. Let $d: O_X \to \Omega^1_{X/S}$ be the universal derivation. Let $\beta = (\rho \circ \mathrm{pr}_{12}, \rho \circ \mathrm{pr}_{13}): G \times_S X \times_S X \to X \times_S X$. Then the following diagram is cartesian:

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\ \rho\ } & X \\ \Big\downarrow{\scriptstyle \mathrm{id}_G \times \Delta} & & \Big\downarrow{\scriptstyle \Delta} \\ G \times_S X \times_S X & \xrightarrow{\ \beta\ } & X \times_S X \end{array} \tag{9}$$

Imitating the proof of Lemma 1.1, we see that β induces a map

$$\Omega^1_{X/S} \to \rho_* \mathrm{pr}_2^* \Omega^1_{X/S}, \tag{10}$$

denoted by β^*, by abuse of notation. It is easy to check that the following diagram is commutative:

$$\begin{array}{ccc} O_X & \xrightarrow{\ d\ } & \Omega^1_{X/S} \\ \Big\downarrow{\scriptstyle \rho^*} & & \Big\downarrow{\scriptstyle \beta^*} \\ \rho_* O_{G \times_S X} & \xrightarrow{\ \rho_* \mathrm{pr}_2^*(d)\ } & \rho_* \mathrm{pr}_2^* \Omega^1_{X/S} \end{array} \tag{11}$$

Indeed, since d is induced by $\mathrm{pr}_1^* - \mathrm{pr}_2^*: O_X \to \Delta^{-1} O_{X \times_S X}$, we need only check that $\rho \circ \mathrm{pr}_{12} = \mathrm{pr}_1 \circ \beta$ and $\rho \circ \mathrm{pr}_{13} = \mathrm{pr}_2 \circ \beta$.

Now consider an open subset $U \subset X$ as in the above definition of ρ-invariant derivations. Without losing generality we may assume that $U = X$. By (11), a section $D \in \mathscr{D}_\rho(X)$ corresponds to an O_X-linear map $\tilde{D}: \Omega^1_{X/S} \to O_X$ such that the following diagram is commutative:

$$\begin{array}{ccc} \Omega^1_{X/S} & \xrightarrow{\ \tilde{D}\ } & O_X \\ \Big\downarrow{\scriptstyle \beta^*} & & \Big\downarrow{\scriptstyle \rho^*} \\ \rho_* \mathrm{pr}_2^* \Omega^1_{X/S} & \xrightarrow{\ \rho_* \mathrm{pr}_2^*(\tilde{D})\ } & \rho_* O_{G \times_S X} \end{array} \tag{12}$$

Let $\bar{D}$ be a global section of $\mathscr{H}om_{O_S}(\omega_{G/S}, O_S) \simeq \mathrm{Lie}(G/S)$. Define $\tilde{D} =$

$\tau^*(\bar{D})\circ\alpha^*\colon \Omega^1_{X/S}\to\tau^*\omega_{G/S}\to O_X$. Then $\tilde{D}$ is ρ-invariant by the following commutative diagram:

$$\begin{array}{ccccc}
\Omega^1_{X/S} & & \xrightarrow{\quad\tilde{D}\quad} & & O_X \\
 & \searrow^{\alpha^*} & & \nearrow_{\tau^*(\bar{D})} & \\
\downarrow{\scriptstyle\beta^*} & & \tau^*\omega_{G/S} & & \downarrow{\scriptstyle\rho^*} \\
\rho_*\mathrm{pr}_2^*\Omega^1_{X/S} & & \xrightarrow{\quad\rho_*\mathrm{pr}_2^*(\tilde{D})\quad} & & \rho_* O_{G\times_S X} \\
 & \searrow^{\nu^*} & \downarrow{\scriptstyle\lambda^*} & \nearrow_{\rho_*(\pi\times\tau)^*(\bar{D})} & \\
 & & \rho_*(\pi\times\tau)^*\omega_{G/S} & &
\end{array}\tag{13}$$

where λ^* is induced by $\lambda = (\mathrm{pr}_2, \rho\circ\mathrm{pr}_{13})\colon G\times_S G\times_S X\to G\times_S X$ together with the following commutative diagram:

$$\begin{array}{ccc}
G\times_S X & \xrightarrow{\quad\rho\quad} & X \\
{\scriptstyle \mathrm{id}_G\times o_G\times\mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle o_G\times\mathrm{id}_X} \\
G\times_S G\times_S X & \xrightarrow{\quad\lambda\quad} & G\times_S X
\end{array}\tag{14}$$

and ν^* is induced by $\nu = (\mathrm{pr}_1, \rho\circ(m\circ(\iota\circ\mathrm{pr}_1, \mathrm{pr}_2), \rho\circ\mathrm{pr}_{13}), \mathrm{pr}_3)$:

$$G\times_S G\times_S X\to G\times_S X\times_S X$$

together with the following cartesian diagram:

$$\begin{array}{ccc}
G\times_S X & \xrightarrow{\quad\mathrm{id}\quad} & G\times_S X \\
{\scriptstyle \mathrm{id}_G\times o_G\times\mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_G\times\Delta_X} \\
G\times_S G\times_S X & \xrightarrow{\quad\nu\quad} & G\times_S X\times_S X
\end{array}\tag{15}$$

The upper triangle of (13) commutes by the definition of $\tilde{D}$. The three maps of the lower triangle are all O_G-linear, so we need only check the commutativity of the lower triangle on $1\otimes\Omega^1_{X/S}$, which also comes from the definition of $\tilde{D}$. The left parallelogram commutes since $\alpha\circ\lambda = \beta\circ\nu$. The commutativity of the right parallelogram is obvious (a change of coefficients). Thus the commutativity of (13) is checked.

Now we get a map $\mathrm{Lie}(G/S)\to\mathscr{D}_\rho$ sending $\bar{D}$ to $\tilde{D}\circ d$. Clearly this map is O_S-linear. To check that this is a map of Lie algebras, we first note that for any global section $\tilde{D}$ of $\mathscr{H}om_{O_S}(\omega_{G/S}, O_S)$ corresponding to D of $\mathscr{D}_\rho$ and D' of Lie(G/S), the following diagram is commutative:

$$\begin{array}{ccc} O_X & \xrightarrow{-D} & O_X \\ \rho^* \downarrow & & \downarrow \rho^* \\ \rho_* O_{G\times_S X} & \xrightarrow{\rho_* \mathrm{pr}_1^*(D')} & \rho_* O_{G\times_S X} \end{array} \tag{16}$$

Since ρ^* is injective, given D', D is uniquely determined by (16). By (13), to check the commutativity of (16), we need only check the commutativity of the following diagram:

$$\begin{array}{ccc} O_X & \xrightarrow{d_X} & \Omega^1_{X/S} \\ \rho^* \downarrow & & \\ \rho_* O_{G\times_S X} & & \Big\downarrow \lambda^*\circ\alpha^* \\ \rho_*\mathrm{pr}_1^*(d_G) \downarrow & & \\ \rho_*\mathrm{pr}_1^*\Omega^1_{G/S} & \xrightarrow{-\sigma^*} & \rho_*(\pi\times\tau)^*\omega_{G/S} \end{array} \tag{17}$$

where σ^* is induced by $\sigma = (\mathrm{pr}_1, m\circ(\mathrm{pr}_2, \mathrm{pr}_1), \mathrm{pr}_3)$:

$$G \times_S G \times_S X \to G \times_S G \times_S X.$$

It reduces to checking that $\rho\circ\mathrm{pr}_{13}\circ\sigma = \mathrm{pr}_2\circ\alpha\circ\lambda$ and that $\rho\circ\mathrm{pr}_{23}\circ\sigma = \mathrm{pr}_1\circ\alpha\circ\lambda$.

Suppose $\tilde{D}_1, \tilde{D}_2 \in \mathrm{Hom}_{O_S}(\omega_{G/S}, O_S)$ correspond to $D_1, D_2 \in \Gamma(\mathscr{D}_\rho)$ and $D'_1, D'_2 \in \Gamma(\mathrm{Lie}(G/S))$ respectively. Then (16) shows that $D_1\circ D_2 = (-D_1)\circ(-D_2)$ is uniquely determined by $\rho_*\mathrm{pr}_2^*(D'_1)\circ\rho_*\mathrm{pr}_2^*(D'_2) = \rho_*\mathrm{pr}_2^*(D'_1\circ D'_2)$. Also $D_2\circ D_1$ is determined by $\rho_*\mathrm{pr}_2^*(D'_2\circ D'_1)$. Hence $[D_1, D_2]$ is determined by $\rho_*\mathrm{pr}_2^*([D'_1, D'_2])$, or in other words, $[D'_1, D'_2]$ maps to $[D_1, \mathrm{D}_2]$ under $\mathrm{Lie}(G/S)\to\mathscr{D}_\rho$. □

REMARK 1.3. In the case when $X = G$ and $\rho = m$, Lemma 1.1 recovers the well-known isomorphism $\Omega^1_{G/S} \simeq \pi^*\omega_{G/S}$. Also Lemma 1.2 recovers the well-known isomorphism $\mathscr{H}om_{O_S}(\omega_{G/S}, O_S) \xrightarrow{\sim} \mathrm{Lie}(G/S)$, whose inverse is defined by $D \mapsto o^*(\tilde{D})(D = \tilde{D}\circ d)$. We leave to the reader to check the details of this.

To set up the complex DR_ρ, we need to use the following definition of exterior differentials.

Let $P'_X = O_X \otimes_{O_S} O_X$, $P_X = \Delta^{-1} O_{X\times_S X}$. Then there is an obvious homomorphism $t: P'_X \to P_X$ sending $a\otimes b$ to $\Delta^{-1}(\mathrm{pr}_1^*(a)\cdot\mathrm{pr}_2^*(b))$. Thus the following diagram commutes:

$$\begin{array}{ccc} P'_X & \xrightarrow{\ t\ } & P_X \\ & {}_{\mu}\searrow \quad \nearrow_{\Delta^*} & \\ & O_X & \end{array} \tag{18}$$

where μ is the multiplication map: $\mu(a \otimes b) = ab$. Let $\mathscr{I} = \ker(\Delta^*)$, $\mathscr{I}' = \ker(\mu)$. Then for any positive integer n, the map $P'_X/\mathscr{I}'^n \to P_X/\mathscr{I}^n$ induced by t is an isomorphism. Furthermore, $P_X/\mathscr{I}^n$ is quasi-coherent (and is coherent if X is noetherian). Denote $P^n_X = P_X/\mathscr{I}^n$. In the following, P_X, P'_X and P^n_X will be viewed as left O_X-modules.

Let $\tilde{d}_n$: $\bigotimes^{n+1}_{O_S} O_X \to \bigotimes^{n+2}_{O_S} O_X$ be defined by

$$\tilde{d}_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n+1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n. \tag{19}$$

Since $\bigotimes^{n+1}_{O_S} O_X \simeq \bigotimes^{n}_{O_X} P'_X$, $\tilde{d}_n$ can be viewed as a map $\bigotimes^n_{O_X} P'_X \to \bigotimes^{n+1}_{O_X} P'_X$. It is easy to check that $\tilde{d}_n$ induces a map $\bar{d}_n$: $\bigotimes^n_{O_X} P^1_X \to \bigotimes^{n+1}_{O_X} P^1_X$. Also $\bar{d}_n$ induces a map $\hat{d}_n$: $\bigwedge^n_{O_X} P^1_X \to \bigwedge^{n+1}_{O_X} P^1_X$. Clearly we have a canonical exact sequence:

$$0 \to \Omega^1_{X/S} \to P^1_X \xrightarrow{\ \Delta^*\ } O_X \to 0 \tag{20}$$

which splits over O_X. Hence we have an exact sequence

$$0 \to \Omega^n_{X/S} \to \bigwedge^n_{O_X} P^1_X \to \Omega^{n-1}_{X/S} \to 0. \tag{21}$$

We now show that $\hat{d}_n(\Omega^n_{X/S}) \subset \Omega^{n+1}_{X/S}$. We need only check on a set of generators of $\Omega^n_{X/S}$ over O_S. Let ω be a section of $\bigotimes^{n-1}_{O_X} \mathscr{I}$, and a be a section of O_X. Then $\omega' = 1 \otimes a - a \otimes 1$ is a section of $\mathscr{I}$. We have

$$\begin{aligned} \tilde{d}_n(\omega \otimes \omega') &= \tilde{d}_n(\omega \otimes a - a\omega \otimes 1) \\ &= \tilde{d}_{n-1}(\omega) \otimes a + (-1)^{n+1}\omega \otimes a \otimes 1 - a\tilde{d}_{n-1}(\omega) \otimes 1 \\ &\quad -\tilde{d}_0(a) \otimes \omega \otimes 1 - (-1)^{n+1} a\omega \otimes 1 \otimes 1 \\ &= \tilde{d}_{n-1}(\omega) \otimes \omega' + (-1)^{n+1}\omega \otimes \omega' \otimes 1 - \omega' \otimes \omega \otimes 1. \end{aligned} \tag{22}$$

Passing to $\hat{d}_n$, we get $\hat{d}_n(\hat{\omega} \wedge da) = \hat{d}_{n-1}(\hat{\omega}) \wedge da$ for any section $\hat{\omega}$ of $\Omega^{n-1}_{X/S}$. This shows that $\hat{d}_n(\Omega^n_{X/S}) \subset \Omega^{n+1}_{X/S}$ by induction. Thus d_n induces d_n: $\Omega^n_{X/S} \to \Omega^{n+1}_{X/S}$ which obviously coincides with the classical definition of the exterior differential map. It is also easy to check that the map $\Omega^{n-1}_{X/S} \to \Omega^n_{X/S}$ induced by $\hat{d}_n$ on (21) is equal to d_{n-1}, but we will not use this fact.

REMARK 1.4. Let $P_{n,X} = \Delta_n^{-1} O_{X_n}$, where X_n is the fiber product of $(n+1)$ copies of X (indexed from 0 to n) over S, and $\Delta_n: X \to X_n$ is the diagonal map. Let $\tau_{n,i}: X_{n+1} \to X_n$ $(0 \leqslant i \leqslant n+1)$ be the projection to all except the ith factors. Then $\hat{d}_n$ is also induced by $\Sigma_{i=0}^{n+1}(-1)^i \tau^*_{n,i}: P_{n,X} \to P_{n+1,X}$.

Now we go back to the action ρ. We define the following morphisms from $G \times_S \overset{n+1}{\cdots} \times_S G \times_S X$ (the copies of G's being indexed from 0 to n) to $G \times_S \overset{n}{\cdots} \times_S G \times_S X$. Denote by $p_{n,i}$ $(0 \leqslant i \leqslant n)$ the projection to all except the ith factors. Let $v_n = (m \circ (\mathrm{pr}_0, \iota \circ \mathrm{pr}_n), \ldots, m \circ (\mathrm{pr}_{n-1}, \iota \circ \mathrm{pr}_n), \rho \circ \mathrm{pr}_{n,n+1})$.

Again denote $\mathscr{M} = \ker(o^*)$. Let $O^1_G = o^{-1}(O_G/\mathscr{M}^2)$. Then we have an exact sequence

$$0 \to \omega_{G/S} \to O^1_G \to O_S \to 0 \tag{23}$$

which gives an exact sequence

$$0 \to \bigwedge_{O_S}^{n} \omega_{G/S} \to \bigwedge_{O_S}^{n} O^1_G \to \bigwedge_{O_S}^{n-1} \omega_{G/S} \to 0. \tag{24}$$

Let $\bar{\delta}_n: \bigotimes^n_{O_S} O^1_G \otimes_{O_S} O_X \to \bigotimes^{n+1}_{O_S} O^1_G \otimes_{O_S} O_X$ be the map induced by $\Sigma_{i=0}^n (-1)^i p^*_{n,i} + (-1)^{n+1} v^*_n$ (using the trick of Remark 1.4). We leave to the reader to check that this definition makes sense.

It is easy to see that $\bar{\delta}_n$ induces a map

$$\hat{\delta}_n: \bigwedge_{O_S}^{n} O^1_G \otimes_{O_S} O_X \to \bigwedge_{O_S}^{n+1} O^1_G \otimes_{O_S} O_X$$

Let us show that

$$\hat{\delta}_n\left(\bigwedge_{O_S}^{n} \omega_{G/S} \otimes_{O_S} O_X\right) \subset \bigwedge_{O_S}^{n+1} \omega_{G/S} \otimes_{O_S} O_X$$

LEMMA 1.5. *Let a be a section of $\mathscr{M}$. Then*

(i) *$m^*(a) - a \otimes 1 - 1 \otimes a$ is a section of* $\mathrm{pr}_1^*(\mathscr{M}) \cdot \mathrm{pr}_2^*(\mathscr{M})$;
(ii) *$\iota^*(a) + a$ is a section of $\mathscr{M}^2$.*

Proof. (i) Since (3) splits locally over $\pi^{-1} O_S$, it is easy to see that the following sequence is exact:

$$\begin{aligned} 0 &\to \mathrm{pr}_1^*(\mathscr{M}) \cdot \mathrm{pr}_2^*(\mathscr{M}) \to \ker(o^*_{G \times_S G}) \\ &\to (\mathrm{id}_G \times_S o)_* \mathscr{M} \oplus (o \times_S \mathrm{id}_G)_* \mathscr{M} \to 0 \end{aligned} \tag{25}$$

Since a is a section of $\mathscr{M}$, $m^*(a)$ is a section of $\ker(o^*_{G\times_S G})$, so are $a\otimes 1$ and $1\otimes a$. But

$$(\mathrm{id}_G \times_S o)^*(m^*(a) - a\otimes 1 - 1\otimes a) = (o\times_S \mathrm{id}_G)^*(m^*(a) - a\otimes 1 - 1\otimes a) = 0$$

Hence $m^*(a) - a\otimes 1 - 1\otimes a$ is a section of $\mathrm{pr}_1^*(\mathscr{M})\cdot\mathrm{pr}_2^*(\mathscr{M})$.

(ii) Let $\mu = m\circ(\mathrm{id}_G, \iota)$. Then $\mu\circ\Delta_G = o\circ\pi$. By (i), $\mu^*(a) = a\otimes 1 + 1\otimes \iota^*(a) + b$, where b is a section of $\mathrm{pr}_1^*(\mathscr{M})\cdot\mathrm{pr}_2^*(\mathscr{M})$. Therefore

$$0 = \Delta_G^*\circ\mu^*(a) = \Delta_G^*(a\otimes 1 + 1\otimes\iota^*(a) + b) \equiv a + \iota^*(a) \pmod{\mathscr{M}^2}. \tag{26}$$

□

Let $\omega_0,\ldots,\omega_{n-1}$ be sections of $\omega_{G/S}$, and b be a section of O_X. By direct calculation using Lemma 1.5 we get

$$\begin{aligned}
&\bar{\delta}_n(\omega_0\otimes\cdots\otimes\omega_{n-1}\otimes b)\\
&\quad\equiv \sum_{i=0}^{n}(-1)^i\omega_0\otimes\cdots\otimes\omega_{i-1}\otimes 1\otimes\omega_i\otimes\cdots\otimes\omega_{n-1}\otimes b\\
&\qquad +(-1)^{n+1}(\omega_0\otimes\cdots\otimes\omega_{n-1}\otimes 1\otimes 1\\
&\qquad -\sum_{i=0}^{n-1}\omega_0\otimes\cdots\otimes\omega_{i-1}\otimes 1\otimes\omega_{i+1}\otimes\cdots\otimes\omega_{n-1}\otimes\omega_i\otimes 1)\\
&\qquad \times(1\otimes \overset{n+1}{\cdots}\otimes 1\otimes(1\otimes b - \bar{\delta}_0(b)))\left(\mathrm{mod}\ \bigotimes_{O_S}^{n}\omega_{G/S}\otimes_{O_S}O_X\right)\\
&\quad\equiv \sum_{i=0}^{n-1}((-1)^i\omega_0\otimes\cdots\otimes\omega_{i-1}\otimes 1\otimes\omega_i\otimes\cdots\otimes\omega_{n-1}\\
&\qquad +(-1)^n\omega_0\otimes\cdots\otimes\omega_{i-1}\otimes 1\otimes\omega_{i+1}\otimes\cdots\otimes\omega_{n-1}\otimes\omega_i)\otimes b\\
&\qquad \left(\mathrm{mod}\ \bigotimes_{O_S}^{n}\omega_{G/S}\otimes_{O_S}O_X\right).
\end{aligned}\tag{27}$$

The last row of (27) maps to 0 in $\bigwedge_{O_S}^{n+1}O_G^1\otimes_{O_S}O_X$. This shows that

$$\hat{\delta}_n\left(\bigwedge_{O_S}^{n}\omega_{G/S}\otimes_{O_S}O_X\right)\subset\bigwedge_{O_S}^{n+1}\omega_{G/S}\otimes_{O_S}O_X$$

Hence $\hat{\delta}_n$ induces a map

$$\delta_n\colon \bigwedge_{O_S}^{n}\omega_{G/S}\otimes_{O_S}O_X\to\bigwedge_{O_S}^{n+1}\omega_{G/S}\otimes_{O_S}O_X$$

Next we check that $\delta_n \circ \delta_{n-1} = 0$. It is enough to check that $\bar{\delta}_n \circ \bar{\delta}_{n-1} = 0$. By the definition of $\bar{\delta}_n$, $\bar{\delta}_n \circ \bar{\delta}_{n-1}$ is equal to the sum of the following terms

$$\sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n-1} (-1)^j p_{n,i}^* \circ p_{n-1,j}^*; \tag{28}$$

$$\sum_{i=0}^{n-1} (-1)^{n+1+i} v_n^* \circ p_{n-1,i}^*; \tag{29}$$

$$\sum_{i=0}^{n-1} (-1)^{n+i} p_{n,i}^* \circ v_{n-1}^*; \tag{30}$$

$$p_{n,n}^* \circ v_{n-1}^*; \tag{31}$$

$$-v_n^* \circ v_{n-1}^*. \tag{32}$$

(28) is obviously equal to 0. (29) cancels (30) since $v_{n-1} \circ p_{n,i} = p_{n-1,i} \circ v_n$ $(0 \leqslant i \leqslant n-1)$. Finally, one checks that $v_{n-1} \circ p_{n,n} = v_{n-1} \circ v_n$, which shows that (31) cancels (32).

Now we can define a map $\Omega^{\cdot}_{X/S} \to \mathrm{DR}_\rho$ by letting the map of degree n be

$$\mu_n^*: \Omega^n_{X/S} \to \bigwedge_{O_S}^{n} \omega_{G/S} \bigotimes_{O_S} O_X, \tag{33}$$

where μ_n^* is induced by

$$\mu_n = (\rho \circ \mathrm{pr}_{0n}, \ldots, \rho \circ \mathrm{pr}_{n-1,n}, \mathrm{pr}_n): G \times_S \overset{n}{\cdots} \times_S G \times_S X \to X \times_S \overset{n+1}{\cdots} \times_S X. \tag{34}$$

To check the commutativity $\mu_{n+1}^* \circ d_n = \delta_n \circ \mu_n^*$, one need only check that $\tau_{n,i} \circ \mu_{n+1} = \mu_n \circ p_{n,i}$ $(0 \leqslant i \leqslant n)$ and $\tau_{n,n+1} \circ \mu_{n+1} = \mu_n \circ v_n$, which are clear. Since $\Omega^{n+1}_{X/S}$ is generated by $d_n(\Omega^n_{X/S})$ over O_X, we see that there is only one O_X-linear map $\Omega^{\cdot}_{X/S} \to \mathrm{DR}_\rho$ whose degree zero map is the identity.

Summarizing, we get

THEOREM 1.6. *Let $\rho: G \times_S X \to X$ be an action of G on X. Then*

(i) *There is a canonical complex induced by ρ:*

$$\mathrm{DR}_\rho: O_X \to \tau^*\omega_{G/S} \to \tau^* \bigwedge_{O_S}^{2} \omega_{G/S} \to \tau^* \bigwedge_{O_S}^{3} \omega_{G/S} \cdots$$

(ii) *The identity map of O_X induces a (unique) canonical O_X-linear map $\Omega^{\cdot}_{X/S} \to \mathrm{DR}_\rho$, which is surjective when ρ is free.*

(iii) *Let $\mathscr{D}_\rho$ be the sheaf of ρ-invariant derivations. Then ρ induces a canonical map* $\mathrm{Lie}(G/S) \to \mathscr{D}_\rho$ *of sheaves of Lie algebras over O_S.*

COROLLARY 1.7. *If $\omega_{G/S}$ is flat, then ρ induces a canonical spectral sequence*

$$E_1^{i,j} = \bigwedge_{O_S}^{i} \omega_{G/S} \otimes_{O_S} R^j\tau_* O_X. \tag{35}$$

REMARK 1.8. We conjecture that

$$\omega_{G/S} \otimes_{O_S} 1 \subset \ker(\delta_1) \tag{36}$$

if G is commutative. When $X = G$ and $\rho = m$, (36) is well-known in some special cases. In fact, (36) is true at least in most of the cases. Indeed, if ω is a section of $\omega_{G/S}$, then $\bar{\delta}_1(\omega \otimes 1) = (m^*(\omega) - \omega \otimes 1 - 1 \otimes \omega) \otimes 1$ which is symmetric when G is commutative. Hence $\delta_1(\omega \otimes 1) = 0$ when either S has characteristic away from 2 (i.e., 2 has an inverse in $\Gamma(O_S)$) or $\omega_{G/S}$ is locally free.

2. Free actions

Let S be a scheme over a field k of characteristic $p > 0$. Then a noetherian group scheme G over S is called *infinitesimal* (or "local", in the terminology of [9, p. 136]) if $\mathcal{M} = \ker(o^*)$ is nilpotent, or equivalently, $F^n_{G/S} = 0$ for some n, where $F^n_{G/S}: G \to G^{(p^n)}$ is the (relative) nth power of the Frobenius morphism.

EXAMPLE 2.1. Suppose G is an infinitesimal group scheme over k such that $\operatorname{rank}_k(\omega_G) = 1$, and k is algebraically closed. Then G must be isomorphic to one of the following $G^r_{1,n}$'s (here we follow the notation of [5]):

$$G^r_{1,n} \simeq \operatorname{Spec} k[x]/(x^{p^r}),\ o^*(x) = 0,$$

$$m^*(x) = x \otimes 1 + 1 \otimes x + \sum_{\substack{i+j=p \\ i,j>0}} \frac{1}{i!\,j!} x^{ip^n} \otimes x^{jp^n} + \text{terms of higher degree}, \tag{1}$$

and $\iota^*(x) = -x$ when $p > 2$ (see [12] for another description). In particular, G is commutative. We have some special cases: $\mu_p \simeq G^1_{1,0}$, $\alpha_p \simeq G^1_{1,1}$, and $\ker(p_E) \simeq G^2_{1,1}$ for a supersingular elliptic curve E.

PROPOSITION 2.2. *Let X be a smooth complete curve over k. Let G be a nontrivial connected group scheme over k. If there exists a free action of G on X, then $g(X) = 1$. In particular, when k is algebraically closed, if X has a free action of μ_p* (resp., *α_p*)*, then X is an ordinary* (resp., *supersingular*) *elliptic curve.*

Proof. Since G is non-trivial, $\omega_G \neq 0$. But Ω^1_X is locally free of rank 1. Hence by Lemma 1.1 we must have $\operatorname{rank}_k(\omega_G) = 1$ and (1) of section 1 is an isomorphism, i.e., $\Omega^1_X \simeq O_X$. For the last statement we need the following

LEMMA 2.3. *Let $X \to S$ be an abelian scheme and $G \to S$ be a noetherian group scheme with connected fibers. Let ρ be an action of G on X. Then $h = \rho \circ (\mathrm{id}_G \times_S o_X): G \to X$ is a homomorphism and $\rho = m_X \circ (h \times_S \mathrm{id}_X)$. In particular, if there is a section $s: S \to X$ of $X \to S$ such that $\rho \circ (\mathrm{id}_G \times_S s)$ is a closed immersion, then h embeds G into X as a closed subgroup scheme.*

Proof. Let $g = \rho - m_X \circ (h \times_S \mathrm{id}_X): G \times_S X \to X$. Then we have $g \circ (o_G \times_S \mathrm{id}_X) = 0: X \to X$. Since $G \to S$ has connected fibers, by rigidity, we see that g factors through pr_1. However, for the zero section o_X, we have $g \circ (\mathrm{id}_G \times_S o_X) = h - m_X \circ (h \times_S o_X) = 0: G \to X$. Hence $g = 0$, or $\rho = m_X \circ (h \times_S \mathrm{id}_X)$.

Now we check that h is a homomorphism. We have

$$\begin{aligned} h \circ m_G &= \rho \circ (m_G \times_S o_X) = \rho \circ (m_G \times_S \mathrm{id}_X) \circ (\mathrm{id}_{G \times_S G} \times_S o_X) \\ &= \rho \circ (\mathrm{id}_G \times_S \rho) \circ (\mathrm{id}_{G \times_S G} \times_S o_X) = \rho \circ (\mathrm{id}_G \times_S h) \\ &= m_X \circ (h \times_S \mathrm{id}_X) \circ (\mathrm{id}_G \times_S h) = m_X \circ (h \times_S h). \end{aligned} \tag{2}$$

Finally, if we have a closed immersion $\rho \circ (\mathrm{id}_G \times_S s)$, then applying the translation by s we see that h is also a closed immersion since $\rho \circ (\mathrm{id}_G \times_S s) = m_X \circ (h \times_S s)$. □

We now try to generalize Proposition 2.2.

LEMMA 2.4. *Let $e: \tilde{X} \to X$ be an étale covering over S. Let G be an infinitesimal group scheme over S. Suppose that ρ is an action of G on X. Then ρ can be (uniquely) lifted to an action of G on $\tilde{X}$.*

Proof. Let Y be the pull-back of $e \times_S e: \tilde{X} \times_S \tilde{X} \to X \times_S X$ and

$$\alpha = (\rho, \mathrm{pr}_2): G \times_S X \to X \times_S X.$$

Then we have a cartesian diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\beta} & \tilde{X} \times_S \tilde{X} \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle e \times_S e} \\ G \times_S X & \xrightarrow{\alpha} & X \times_S X \end{array} \tag{3}$$

Hence $\beta(Y) = (e \times_S e)^{-1}(\Delta_X) \supset \Delta_{\tilde{X}}$. Let Y_0 be the component of Y such that $\beta(Y_0) = \Delta_{\tilde{X}}$. Then $\gamma|_{Y_0}$ is étale of degree n, where $n = \deg(e)$. Let

$$\mu = (\mathrm{pr}_1 \circ \gamma|_{Y_0}, \mathrm{pr}_2 \circ \beta|_{Y_0}): Y_0 \to G \times_S \tilde{X}. \tag{4}$$

We claim that μ is a closed immersion. To show this we need only check on closed fibers over X, which is clear.

Since Y_0 and $G \times_S \tilde{X}$ are both flat over $G \times_S X$ of the same degree, we see that μ is an isomorphism. Let

$$\tilde{\rho} = \mathrm{pr}_1 \circ \beta|_{Y_0} \circ \mu^{-1}: G \times_S \tilde{X} \to \tilde{X}. \tag{5}$$

Then it is easy to see that the following diagram is commutative:

$$\begin{array}{ccc} G \times_S \tilde{X} & \xrightarrow{(\tilde{\rho}, pr_2)} & \tilde{X} \times_S \tilde{X} \\ {\scriptstyle \mathrm{id}_G \times_S e} \downarrow & & \downarrow {\scriptstyle e \times_S e} \\ G \times_S X & \xrightarrow{\alpha} & X \times_S X \end{array} \tag{6}$$

Let us check that $\tilde{\rho}$ is an action. We need to show that

$$(\tilde{\rho} \circ (m \times_S \mathrm{id}_{\tilde{X}}))^* = (\tilde{\rho} \circ (\mathrm{id}_G \times_S \tilde{\rho}))^*$$

as maps of O_X-modules. By the standard argument of formal completion, this is reduced to

$$(\rho \circ (m \times_S \mathrm{id}_X))^* = (\rho \circ (\mathrm{id}_G \times_S \rho))^*. \qquad \square$$

LEMMA 2.5. *Let $X \to S$ be an abelian scheme of dimension g. Let f be an endomorphism of X such that* $(\mathrm{id}_X + f)^n = \mathrm{id}_X$ *for some* $n > 0$. *Suppose that f factors through $F^r_{X/S}$ for some $r > g + 1$. Then* $f = 0$.

Proof. We may assume that n is prime, by induction.

Suppose $f \neq 0$. Then there exists $m \geqslant r$ such that f factors through $F^m_{X/S}$ but not $F^{m+1}_{X/S}$. Let $f = h \circ F^m_{X/S}$. Expand $(\mathrm{id}_X + f)^n$:

$$\mathrm{id}_X = \mathrm{id}_X + nh \circ F^m_{X/S} + \frac{n(n-1)}{2}(h \circ F^m_{X/S})^2 + \cdots + (h \circ F^m_{X/S})^n \tag{7}$$

Cancel id_X. Then we can cancel $F^m_{X/S}$ since it is an isogeny. We get

$$-nh = \frac{n(n-1)}{2} h_1 \circ F^m + \cdots + h_{n-1} \circ F^{m(n-1)} \tag{8}$$

for some $h_1 \cdots h_{n-1}$. Hence $\ker(F) \subset \ker(nh)$. There are two possible cases:

(i) $n \neq p$. Take $s, t \in \mathbb{Z}$ such that $tn + sp = 1$. Then

$$\ker(F) \subset \ker(tnh) \cap \ker(sph) \subset \ker(tnh + sph) = \ker(h), \tag{9}$$

contrary to our choice of h.

(ii) $n = p$. Denote by V the Verschiebung morphism of X. Since $\ker(V)$ is contained in the kernel of every but possibly the last term in (8), it must be contained in $\ker(h_{n-1} \circ F^{m(n-1)})$ also. Clearly $\ker(V) \subset \ker(V)_{\mathrm{red}} \times \ker(F^g)$. Hence the kernel of the right hand side of (8) is contained in $\ker(F^2 \circ V) = \ker(pF)$. Canceling p in (8) we see that h factors through F, again contrary to our choice of h. □

REMARK 2.6. In any case, it is never necessary to assume that $r > g + 1$. For example, if X is ordinary, then we can take $r = 1$ when $p > 2$, and $r = 2$ when $p = 2$.

In the following, we assume that k is algebraically closed. We will use the definition of an *ordinary* variety over k introduced by Illusie and Raynaud ([4]). If X is smooth projective of dimension g over k such that $\Omega_X^1 \simeq O_X^{\oplus g}$, then X is ordinary if and only if its Frobenius induces a non-degenerate map on $H^g(X, O_X)$ ([7, p. 193]).

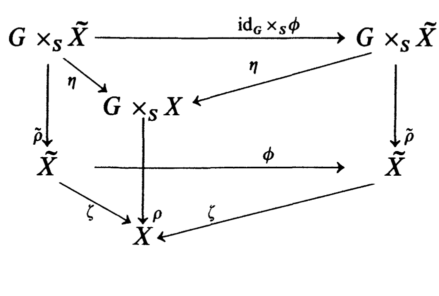
THEOREM 2.7. *Let X be an ordinary smooth projective variety of dimension g over a field k of characteristic $p > 2$. Let G be a connected group scheme over k such that* $\mathrm{rank}_k \omega_G \geqslant g$. *Suppose there is a free action ρ of G on X. Then*

(i) *X is an ordinary abelian variety;*
(ii) *G can be viewed as a closed subgroup scheme of X acting on X by translation. In particular, G is projective and commutative;*
(iii) *If G is infinitesimal, then $G \simeq \mu_{p^{l_1}} \times \cdots \times \mu_{p^{l_g}}$* for some positive integers $l_1, \ldots, l_g$.

If $\mathrm{ch}(k) = 2$, *the statement is also true with an additional assumption that the structure ring of* $\ker(F^2_{G/k})$ *has rank at least p^{2g} over k.*

Proof. By taking $\ker(F^2_{G/k})$ instead of G, we may assume that G is infinitesimal. (By Lemma 2.3, ρ induces a homomorphism $G \to X$, which is a closed immersion $\Leftrightarrow \omega_X \twoheadrightarrow \omega_G \simeq \omega_{\ker(F^2_{G/k})} \Leftrightarrow \ker(F^2_{G/k}) \to X$ is a closed immersion.) By Lemma 1.1, we have $\Omega_X^1 \simeq O_X^{\oplus g}$. Then by [7, Theorem 1], there is an ordinary abelian variety $\tilde{X}$ together with a free action of a finite étale group scheme G' such that $X \simeq \tilde{X}/G'$. Let $\zeta: \tilde{X} \to X$ be the projection. By Lemma 2.4, ρ can be lifted to an action $\tilde{\rho}$ on $\tilde{X}$. Then by Lemma 2.4, G can be viewed as a closed subgroup scheme of $\tilde{X}$ such that $\tilde{\rho} = m|_{G \times_S \tilde{X}}$, and $\ker(F_{\tilde{X}/k}) \subset G$. (In the case when $p = 2$, the additional assumption guarantees that $\ker(F^2_{\tilde{X}/k}) \subset G$.)

Let $\phi \in G'$. Then the following diagram is commutative:



(10)

where $\eta = \mathrm{id}_G \times_S \zeta$. Indeed, to check that $\phi \circ \tilde{\rho} = \tilde{\rho} \circ (\mathrm{id}_G \times_S \phi)$, we need only check that $\zeta \circ \phi \circ \tilde{\rho} = \zeta \circ \tilde{\rho} \circ (\mathrm{id}_G \times_S \phi)$ by the standard argument of formal completion.

Therefore $\phi|_G = \phi(0) + \mathrm{id}_G$. Let $\psi = \phi - \phi(0)$. Suppose $\phi^n = \mathrm{id}_{\tilde{X}}$. Then since $\psi(0) = 0$ and $\psi^n = \phi^n +$ constant, we must have $\psi^n = \mathrm{id}_{\tilde{X}}$. Let $f = \psi - \mathrm{id}_{\tilde{X}}$. Then $f|_G = 0$, so f factors through $F_{\tilde{X}/k}$ (resp., $F^2_{\tilde{X}/k}$ when $p = 2$). Now by Lemma 2.5 and Remark 2.6, we have $f = 0$. Hence ϕ is a translation. Therefore G' can be viewed as a subgroup scheme of $\tilde{X}$ acting via $m_{\tilde{X}}$. Hence $\tilde{X}/G'$ is also an abelian variety. The remaining statements come from Lemma 2.3. □

EXAMPLE 2.8 (cf. [3]). Let E be an ordinary elliptic curve over a field k of characteristic $p = 2$. Let $a \in E$ be a closed point of order 2. Let $X = E \times E$. Then X has a closed subgroup scheme $G \simeq \mu_p \times \mu_p$. Let $G' = \mathbb{Z}/2\mathbb{Z} = (\bar{0}, \bar{1})$. Let $\bar{0}$ correspond to id_X and $\bar{1}$ correspond to the isomorphism $(x, y) \mapsto (-x, y + a)$ of X. This defines a free action of G' on X. The action of G (by translation) commutes with the action of G' since $\mathrm{id}_{\mu_p} = -\mathrm{id}_{\mu_p}$. Let $Y = X/G'$. Then the action of G on X induces a free action of G on Y. But clearly Y is not an abelian variety.

Therefore we really need the additional condition in Theorem 2.7 in the case when $p = 2$.

For a smooth projective variety X over k, we denote by $\mathrm{Pic}^\tau(X)$ (following [10, p. 85]) the subscheme of $\mathrm{Pic}(X)$ representing the following functor

$$((k\text{-schemes})) \to ((\text{abelian groups}))$$
$$T \mapsto \{\text{invertible sheaves } \mathscr{F} \text{ on } X \times T \text{ with numerical class } 0 \text{ such that } \mathscr{F}|_{\{x\} \times T} \simeq O_T\}$$

where x is a fixed closed point of X. Denote by $\hat{X}$ the component of $\mathrm{Pic}(X)$ containing 0 with the reduced induced scheme structure. Then $\mathrm{Pic}^\tau(X)$ is a projective group scheme and $\hat{X}$ is an abelian variety. There is an invertible sheaf $\mathscr{F}_X$ on $X \times \hat{X}$ representing the following functor:

$$((k\text{-varieties})) \to ((\text{abelian groups}))$$
$$T \mapsto \{\text{invertible sheaves } \mathscr{F} \text{ on } X \times T \text{ with Néron-Severi class } 0 \text{ such that } \mathscr{F}|_{\{x\} \times T} \simeq O_T\}.$$

Denote by $\tilde{X} = \mathrm{Pic}^\tau(\hat{X})$ and $\mathscr{P}_X$ the Poincaré sheaf on $\tilde{X} \times \hat{X}$. Then $\mathscr{F}_X$ induces a canonical morphism $\mu_X : X \to \tilde{X}$ such that $(\mu_X \times \mathrm{id}_{\hat{X}})^* \mathscr{P}_X \simeq \mathscr{F}_X$.

Let $f: X \to Y$ be a morphism of smooth projective varieties over k. Then f induces $\hat{f}: \hat{Y} \to \hat{X}$ such that $(\mathrm{id}_X \times \hat{f})^* \mathscr{F}_X \simeq (f \times \mathrm{id}_{\hat{Y}})^* \mathscr{F}_Y$, and $\hat{f}$ induces $\tilde{f}: \tilde{X} \to \tilde{Y}$ such that $(\mathrm{id}_{\tilde{X}} \times \hat{f})^* \mathscr{P}_X \simeq (\tilde{f} \times \mathrm{id}_{\hat{Y}})^* \mathscr{P}_Y$. The following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_Y} \\ \tilde{X} & \xrightarrow{\ \tilde{f}\ } & \tilde{Y} \end{array} \tag{11}$$

Indeed, we have

$$\begin{aligned} (\mu_Y \circ f \times \mathrm{id}_{\hat{Y}})^* \mathscr{P}_Y &\simeq (f \times \mathrm{id}_{\hat{Y}})^* \circ (\mu_Y \times \mathrm{id}_{\hat{Y}})^* \mathscr{P}_Y \simeq (f \times \mathrm{id}_{\hat{Y}})^* \mathscr{F}_Y \\ &\simeq (\mathrm{id}_X \times \hat{f})^* \mathscr{F}_X \simeq (\mathrm{id}_X \times \hat{f})^* \circ (\mu_X \times \mathrm{id}_{\hat{X}})^* \mathscr{P}_X \\ &\simeq (\mu_X \times \mathrm{id}_{\hat{Y}})^* \circ (\mathrm{id}_{\tilde{X}} \times \hat{f})^* \mathscr{P}_X \\ &\simeq (\mu_X \times \mathrm{id}_{\hat{Y}})^* \circ (\tilde{f} \times \mathrm{id}_{\hat{Y}})^* \mathscr{P}_Y \\ &\simeq (\tilde{f} \circ \mu_X \times \mathrm{id}_{\hat{Y}})^* \mathscr{P}_Y. \end{aligned} \tag{12}$$

Therefore $\mu_Y \circ f = \tilde{f} \circ \mu_X$ by the universality of $\mathscr{P}_Y$.

In particular, if f is the relative Frobenius $F_{X/k}: X \to X^{(p)}$, then $\hat{Y} \simeq \hat{X}^{(p)}$, $\tilde{Y} \simeq \tilde{X}^{(p)}$ and $\tilde{f} = F_{\tilde{X}/k}$. Indeed, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ F_{X/k}\ } & X^{(p)} \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_{X^{(p)}}} \\ \tilde{X} & \xrightarrow{\ F_{X/k}\ } & \tilde{X}^{(p)} \end{array} \tag{13}$$

Hence

$$\begin{aligned} (F_{X/k} \times \mathrm{id}_{\hat{X}^{(p)}})^* \mathscr{F}_{X^{(p)}} &\simeq (\mu_X \times \mathrm{id}_{\hat{X}^{(p)}})^* \circ (F_{\tilde{X}/k} \times \mathrm{id}_{\hat{X}^{(p)}})^* \mathscr{P}_{X^{(p)}} \\ &\simeq (\mu_X \times \mathrm{id}_{\hat{X}^{(p)}})^* \circ (\mathrm{id}_{\tilde{X}} \times V_{\hat{X}/k})^* \mathscr{P} \\ &\simeq (\mathrm{id}_X \times V_{\hat{X}/k})^* \circ (\mu_X \times \mathrm{id}_{\hat{X}})^* \mathscr{P}_X \\ &\simeq (\mathrm{id}_X \times V_{\hat{X}/k})^* \mathscr{F}_X \end{aligned} \tag{14}$$

where $V_{\hat{X}/k}$ is the relative Verschiebung morphism. Therefore $\widehat{F_{X/k}} = V_{\hat{X}/k}$ by the universality of $\mathscr{F}_X$.

Now suppose that X has a free action ρ of a finite commutative group scheme G and f is the quotient of ρ. In this case we have

LEMMA 2.9. Let $K = \ker(\tilde{f})$. Then there is an epimorphism $h: G \to K$ such that the following diagram is commutative:

$$\begin{array}{ccc} G \times X & \xrightarrow{\rho} & X \\ {\scriptstyle h \times \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ K \times \tilde{X} & \xrightarrow{m_{\tilde{X}}} & \tilde{X} \end{array} \tag{15}$$

where $m_{\tilde{X}}$ denotes the multiplication of $\tilde{X}$ restricted to $K \times \tilde{X}$, by abuse of notation. Furthermore, $\hat{G}$ (the Cartier dual of G) can be viewed as a closed subgroup scheme of $\mathrm{Pic}^{\tau}(Y)$, and h is an isomorphism if $\hat{G}$ is contained in $\hat{Y}$.

Proof. We know that there exists $\lambda: G \times X \xrightarrow{\sim} X \times_Y X$ such that $\mathrm{pr}_1 \circ \lambda = \rho$, $\mathrm{pr}_2 \circ \lambda = \mathrm{pr}_2$. Also there exists $\tilde{\lambda}: K \times \tilde{X} \xrightarrow{\sim} \tilde{X} \times_{\tilde{Y}} \tilde{X}$ such that $\mathrm{pr}_1 \circ \tilde{\lambda} = m_{\tilde{X}}$, $\mathrm{pr}_2 \circ \tilde{\lambda} = \mathrm{pr}_2$ ([9, p. 112]). Therefore we get $\xi: G \times X \to K \times \tilde{X}$ such that the following diagram is commutative:

$$\begin{array}{ccc} G \times X & \xrightarrow{\lambda} & X \times_Y X \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \mu_X \times \mu_X} \\ K \times \tilde{X} & \xrightarrow{\tilde{\lambda}} & \tilde{X} \times_{\tilde{Y}} \tilde{X} \end{array} \tag{16}$$

The above says that $\mathrm{pr}_2 \circ \xi = \mu_X \circ \mathrm{pr}_2$, $m_{\tilde{X}} \circ \xi = \mu_X \circ \rho$. The induced morphism $G \times X \to \mathrm{Spec}(\Gamma(O_{G \times X})) \simeq G$ is simply the first projection. Similarly, $K \times \tilde{X} \to \mathrm{Spec}(\Gamma(O_{K \times \tilde{X}})) \simeq K$ is the first projection. Therefore we get a morphism $h: G \to K$ induced by $\Gamma(O_{K \times \tilde{X}}) \to \Gamma(O_{G \times X})$ such that $\mathrm{pr}_1 \circ \xi = h \circ \mathrm{pr}_1$. This means that $\xi = h \times \mu_X$ and the following diagram is commutative:

$$\begin{array}{ccc} G \times X & \xrightarrow{\rho} & X \\ {\scriptstyle h \times \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ K \times \tilde{X} & \xrightarrow{m_X} & \tilde{X} \end{array} \tag{17}$$

We claim that h is a homomorphism. We have

$$\begin{aligned} & m_{\tilde{X}} \circ (m_K \times \mathrm{id}_{\tilde{X}}) \circ (h \times h \times \mu_X) \\ & \quad = m_{\tilde{X}} \circ (\mathrm{id}_K \times m_{\tilde{X}}) \circ (h \times h \times \mu_X) \\ & \quad = m_{\tilde{X}} \circ (h \times m_{\tilde{X}} \circ (h \times \mu_X)) = m_{\tilde{X}} \circ (h \times \mu_X \circ \rho) \\ & \quad = m_{\tilde{X}} \circ (h \times \mu_X) \circ (\mathrm{id}_G \times \rho) = \mu_X \circ \rho \circ (\mathrm{id}_G \times \rho) \\ & \quad = \mu_X \circ \rho \circ (m_G \times \mathrm{id}_X) = m_{\tilde{X}} \circ (h \times \mu_X) \circ (m_G \times \mathrm{id}_X). \end{aligned} \tag{18}$$

Hence $m_{\tilde{X}} \circ (m_K \circ (h \times h) \times \mu_X) = m_{\tilde{X}} \circ ((h \circ m_G) \times \mu_X)$. Canceling μ_X on both sides we get $m_K \circ (h \times h) = h \circ m_G$, which shows that h is a homomorphism.

Now we show that h is an epimorphism. Using the argument of [9, p. 144], one sees that there exists a canonical isomorphism $\eta\colon \hat{G} \xrightarrow{\sim} \ker(\mathrm{Pic}(Y) \to \mathrm{Pic}(X))$. Since $\hat{G}$ is torsion, $\eta(\hat{G}) \subset \mathrm{Pic}^\tau(Y)$. Hence $\hat{G} \simeq \ker(\mathrm{Pic}^\tau(Y) \to \mathrm{Pic}^\tau(X))$. Therefore $\ker(\hat{f}) = \eta(\hat{G}) \cap \hat{Y} \simeq \hat{K}$. Hence $\widehat{\ker(\hat{f})} \simeq K$ is a quotient of G. Let $H = \ker(G \to \widehat{\ker(\hat{f})})$. Let $X' = X/H$. Then by functoriality we have $\ker(\hat{X}' \to \hat{X}) = 0$, hence $\hat{X}' \simeq \hat{X}$, $\tilde{X}' \simeq \tilde{X}$. Therefore μ_X factors through X', hence $H \subset \ker(h)$. It is enough to show that $\ker(h) = H$. Let $Y' = X/\ker(h)$. Let $g\colon X \to Y'$, $f'\colon Y' \to Y$ be the projections. We have a commutative diagram:

$$\begin{array}{ccc} \ker(h) \times X & \xrightarrow{\rho} & X \\ \Big\downarrow{\scriptstyle \mathrm{pr}_2} & & \Big\downarrow{\scriptstyle \mu_X} \\ X & \xrightarrow{\mu_X} & \tilde{X} \end{array} \tag{19}$$

Hence μ_X factors through Y'. Therefore $\hat{\mu}_X$ factors through $\hat{g}$. However, clearly we have $\hat{\mu}_X = \mathrm{id}_{\hat{X}}$, so $\hat{g}$ is an isomorphism. Hence $\ker(\hat{f}) = \ker(\hat{f}')$. Since $\deg(f') \geqslant \deg(\hat{f}')$, we have $\deg(g) \leqslant \deg(X \to X')$. Hence $\ker(h) = H$.

Finally, if η factors through $\hat{Y}$, then $H = 0$ and h is an isomorphism. □

REMARK 2.10. When h is an isomorphism, diagram (11) is cartesian. Indeed, in this case we have

$$\begin{aligned} (\tilde{X} \times_{\tilde{Y}} Y) \times_Y X &\simeq \tilde{X} \times_{\tilde{Y}} X \simeq (\tilde{X} \times_{\tilde{Y}} \tilde{X}) \times_{\tilde{X}} X \\ &\simeq (K \times \tilde{X}) \times_{\tilde{X}} X \simeq G \times X \simeq X \times_Y X. \end{aligned} \tag{20}$$

Since X is faithfully flat over Y, this shows that $X \to \tilde{X} \times_{\tilde{Y}} Y$ is an isomorphism.

COROLLARY 2.11 (see [8, p. 47]). *Let G be a finite commutative group scheme over k. Let X be a smooth projective variety with a free action of G. If $Y = X/G$ is an abelian variety, so is X.*

Proof. In this case $\mathrm{Pic}^\tau(Y)$ is an abelian variety. Hence the homomorphism h in Lemma 2.9 is an isomorphism. Now (11) is cartesian by Remark 2.10, and μ_Y is an isomorphism. Hence μ_X is also an isomorphism. □

THEOREM 2.12. *Let X be a smooth projective variety and G be a commutative infinitesimal group scheme over k such that* $\dim(X) \leqslant \mathrm{rank}_k(\omega_G)$. *Suppose that X has a free action of G. Then*

(i) If $\mathrm{Pic}^\tau(X)$ *is reduced and connected, then X is an abelian variety;*
(ii) *If $\hat{G}$ is also infinitesimal and* $\mathrm{Pic}(X)$ is *reduced, then X is a very special* (*i.e., having no closed point of order p*) *abelian variety.*

Proof. We may assume that $F_{G/k} = 0$, otherwise we can take $\ker(F_{G/k})$ instead of G. Again use the above notation. By Lemma 1.1, we have $\Omega^1_{X/k} \simeq \omega_G \otimes O_X$ and hence $\dim(X) = \mathrm{rank}_k(\omega_G)$. By the functoriality of Frobenius, we have the following commutative diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\ \rho\ } & X \\ {\scriptstyle F_{X/k}\circ \mathrm{pr}_2}\Big\downarrow & & \Big\downarrow{\scriptstyle F_{X/k}} \\ X^{(p)} & \xrightarrow{\ \mathrm{id}\ } & X^{(p)} \end{array} \tag{21}$$

since $F_{G\times X/k}$ factors through X. Hence $F_{X/k}$ factors through $Y = X/G$. However, X is flat over both $X^{(p)}$ and Y of the same degree, so $X^{(p)} \simeq Y$.

Under condition (i) or (ii), G can be identified with a subgroup scheme of $\tilde{X}$ by Lemma 2.9. Let $i: G \to \tilde{X}$ be the inclusion morphism. Then i induces an isomorphism $\omega_{\tilde{X}} \simeq \omega_G$, and hence an isomorphism $\mu_X^* \Omega^1_{\tilde{X}} \simeq \Omega^1_X$ by Lemma 1.1 again. Therefore μ_X is finite. By [6, Theorem 51], μ_X is flat, hence étale. Therefore X is an abelian variety by Serre-Lang's Theorem ([9, p. 167]). Hence μ_X is an isomorphism.

Finally, since $\hat{G} \simeq \ker(V_{\hat{X}/k})$, X is very special under condition (ii). □

REMARK 2.13. In Example 2.8, we can take $X = E' \times E$ instead of $E \times E$ and let $G = \ker(F^2_{X/k})$, where E' is a supersingular elliptic curve. Then G satisfies the additional assumption of Theorem 2.7, but $Y = X/G'$ is still not an abelian variety. Furthermore, in this case $\mathrm{Pic}(Y)$ must be non-reduced by Theorem 2.12, since α_p is a subgroup scheme of $\mathrm{Pic}(Y)$ and $\hat{Y}$ is an ordinary elliptic curve.

EXAMPLE 2.14. Let G be a direct product of g copies of α_p. Let X be a smooth projective variety of dimension g over k such that $\mathrm{Pic}(X)$ is reduced. If X has a free action of G, then $X \simeq E^g$, where E is a supersingular elliptic curve (see [11, Theorem 2]).

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