COMPOSITIO MATHEMATICA

ANDRÁS NÉMETHI

Generalized local and global Sebastiani-Thom type theorems

Compositio Mathematica, tome 80, nº 1 (1991), p. 1-14

<http://www.numdam.org/item?id=CM_1991__80_1_1_0>

Compositio Mathematica **80**: 1–14, 1991.

Generalized local and global Sebastiani-Thom type theorems

ANDRÁS NÉMETHI

Institute of Mathematics, Bd. Păcii 220, R-79622 Bucharest, Romania

Received 3 June 1990; accepted in revised form 26 November 1990

1. Introduction

Let $g: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ and $h: (\mathbf{C}^m, 0) \to (\mathbf{C}, 0)$ be isolated singularities. We define the direct sum (resp. direct product) $g \oplus h$ (resp. $g \odot h$): $(\mathbf{C}^{n+m}, 0) \to (\mathbf{C}, 0)$ by $g \oplus h(x, y) = g(x) + h(y)$ (resp. $g \odot h(x, y) = g(x) \cdot h(y)$). Then the Milnor fiber of $g \oplus h$ is the join space of the Milnor fibers of g and h. In particular, $\mu(g \oplus h) = \mu(g) \cdot \mu(h)$. Moreover, M. Sebastiani and R. Thom [14] proved that the monodromy operator of the singularity $f \oplus g$ is equal to the tensor product of the monodromy operators of the singularities h and g. A. M. Gabrielov [3] obtained a description of the intersection matrix of the singularity $f \oplus g$ in terms of the intersection matrices of g and h (with respect to distinguished bases). P. Deligne proved (see [1]) the tensor product formula for the variation operator $\mathrm{Var}_{g \oplus h} = (-1)^{nm} \mathrm{Var}_g \otimes \mathrm{Var}_h$, which is equivalent to the tensor product formula of the Seifert forms proved by K. Sakamoto [13], who extended the Sebastiani-Thom result to nonisolated singularities and considered also the direct product case [12].

In this paper we consider the following general situation: let $g: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ and $h: (\mathbf{C}^m, 0) \to (\mathbf{C}, 0)$ be arbitrary germs of analytic functions and let $p: (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be an analytic germ in two variables. Then we determine the topological type of the Milnor fiber of $f = p(h, g): (\mathbf{C}^{n+m}, 0) \to (\mathbf{C}, 0)$ in terms of the Milnor fiber of g, h and p. (Theorem L1) and the zeta function of f in terms of the algebraic monodromies of g and h and the Alexander polynomial of the (algebraic) link determined by p (Theorem L2).

It is surprising that if we consider arbitrary global polynomials $g: \mathbf{C}^n \to \mathbf{C}$, $h: \mathbf{C}^m \to \mathbf{C}$ and $p: \mathbf{C}^2 \to \mathbf{C}$ (with some minor restrictions), we reobtain the same statements about the topological type of the generic fiber of $f = p(h, g)$ (in terms of the generic fiber of g, h resp. p) and the zeta function of the global monodromy operator (around all bifurcation points) of f (in terms of the global monodromies of g and h and the Alexander polynomial of the fiberable link at infinity of p).

In the global case M. Oka [11] studied the direct sum and direct product of weighted homogeneous polynomials. In his particular case the only bifurcation point of a polynomial map is $\{0\}$. In the general case it is very hard to determine the bifurcation set of a polynomial P (i.e., the minimal set Λ_p such that P is locally trivial over $\mathbf{C} - \Lambda_p$) and the generic fiber of P. This happens because Λ_p contains beside the critical values also some other "atypical values", and the behaviour of the fibers near the atypical fibers depends not only on the local data on these fibers but on the behaviour of P at infinity as well.

Therefore it is important to emphasize the fact that the generic fiber and the zeta function of $f = p(g, h)$ can be computed without studying the behaviour of g resp. h around each bifurcation point separately.

This paper can be considered as a continuation of [9] in which the author consider the global direct sum case proving that the generic fiber of $f = g \oplus h$ is the join of the generic fibers of g and h and the global algebraic monodromy (over $\mathbf{Z}$) is induced by the join of the global geometric monodromies of g and h.

The technique of the proofs is more or less similar to the proof of Theorems A, B and C in [8], where we considered the case of singularities of type $f = p(g, h)$ where $(g, h): (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^2, 0)$ is an isolated complete intersection singularity. Instead of repeating parts of these proofs, we shall refer the reader to that paper at some steps in our proofs.

2. The main results. Local case

2.1. Let $g: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ resp. $h: (\mathbf{C}^m, 0) \to (\mathbf{C}, 0)$ be analytic germs with Milnor fiber G resp. H. Let $p: (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be an analytic germ in two variables (denoted by c and d) with Milnor fiber P. In this note we suppose that P is connected, i.e., if $p = p_1^{m_1} \cdots p_r^{m_r}$ is the prime decomposition of p, then g.c.d. $(m_1, \ldots, m_r) = 1$. In this case P has the homotopy type of a bouquet of $\mu_P = 1 - \chi_P$ circles.

We define $n_c = 0$ if c is a factor of p and $n_c =$ the intersection multiplicity $m_0(p, c)$ otherwise. Symmetrically we define n_d. In fact, n_c is the number of points of the intersection $P \cap \{c = 0\}$.

Our first result is the following

THEOREM L1. *The Milnor fiber F of the analytic germ $f = p(h, g): (\mathbf{C}^n \times \mathbf{C}^m, 0) \to (\mathbf{C}, 0)$ defined by $f(x, y) = p(g(x), h(y))$ has the homotopy type of a space obtained from the total space of a fiber bundle with base space P and fiber $G \times H$ by gluing with the natural applications to a fiber $G \times H$ n_c copies of* $\mathrm{Con}\, G \times H$ *and n_d copies of $G \times$* $\mathrm{Con}\, H$ *(here* $\mathrm{Con}\, X$ *denotes the cone over X).*

The proof is given in §4.

2.2. REMARK. The assumption about the connectedness of P is not essential because if P has k connected components then $p = (p')^k$, hence we can use our theorem for $f' = p'(h, g)$, and the Milnor fiber of f is composed of k disjoint copies of the Milnor fiber of f'. (Moreover, for the zeta functions we have $\zeta_f(\lambda) = \zeta_{f'}(\lambda^k)$).

2.3. COROLLARY. *The Euler characteristic of the fiber F can be computed by*:

$$\chi(F) = \chi(P)\cdot\chi(G)\cdot\chi(H) + n_c(\chi(H) - \chi(G)\cdot\chi(H)) + n_d(\chi(G) - \chi(G)\cdot\chi(H))$$
$$= \chi(P - \{cd = 0\})\cdot\chi(G)\cdot\chi(H) + n_c\cdot\chi(H) + n_d\cdot\chi(G).$$

2.4. EXAMPLES. (a) If we take $p = c + d$, then we obtain the direct sum case

$$F \sim \operatorname{Con} G \times H \bigcup_{G\times H} G \times \operatorname{Con} H \sim G * H$$

(the join space of G and H).

(b) If $p = c\cdot d$ then $n_c = n_d = 0$, hence F is a fiber bundle over $P = S^1 \times \mathbf{R} \sim S^1$ with fiber $G \times H$. The characteristic map of the fiber bundle can be identified with $m_g \times m_h^{-1}$ (m_g and m_h are the geometric monodromies of g resp. h) (see [12], or our proof in §4).

2.5. REMARK. We denote by $G * H(n_c, n_d)$ the space obtained from $G \times H$ by gluing naturally n_c copies of $\operatorname{Con} G \times H$ and n_d copies of $G \times \operatorname{Con} H$ (e.g. $G * H(1,1) = G * H$). In our construction given in Theorem L1 we identify the fiber $G \times H$ with the fiber over the base point $*$ of P. Since $(P, *) \sim (\bigvee_{\mu_p} S^1, *)$, the total space of the fiber bundle over P can be identified with the total space of a fiber bundle (with projection map u) over $\bigvee_{\mu_p} S^1$.

Suppose that $n_c \geqslant 1$ and $n_d \geqslant 1$. Then $G * H(n_c, n_d)$ is connected; let $*'$ be a base point in it. Since the natural inclusion $G \times H \hookrightarrow G * H(n_c, n_d)$ is homotopical to the trivial map $G \times H \to *'$, the spaces $u^{-1}(S^1)$ over each circle can be identified with the nonreduced suspension of $G \times H$ with both vertices in $*'$. Therefore

$$F \sim \left[\bigvee_{\mu_p} S^1\right] \vee \left[\bigvee_{\mu_p} S(G \times H)\right] \vee [G * H(n_c, n_d)] \quad (S = \text{reduced suspension}).$$

In particular, the homotopy type of F *does not depend* on the characteristic maps of the fiber bundle u.

We note that in this case (i.e., if $n_c \cdot n_d \neq 0$) F is connected even if G or H is not. If G and H are connected spaces, then $\pi_1(F) \approx \pi_1(\bigvee_{\mu_p} S^1) \approx$ the free group with μ_p generators. (In fact we have an isomorphism at level π_1 induced by $u = (g, h)\colon F \to P$.)

2.6. In what follows we want to determine the zeta function of the germ f. For this, we introduce some notations.

Let $(M_g)_*\colon H^*(G, \mathbf{C})\circlearrowleft$ resp. $(M_h)_*\colon (H, \mathbf{C})\circlearrowleft$ be the algebraic monodromies (induced by the geometric monodromies m_g resp. m_h) of g resp. h, and ζ_g resp. ζ_h

the corresponding zeta functions defined by

$$\zeta_l(\lambda) = \prod_q \det(1 - \lambda M_{l,q})^{(-1)^{q+1}} \quad \text{where } l = g \text{ or } h.$$

Define $E_{q,i} \in \operatorname{Aut} H^q(G \times H, \mathbf{C})$ $(i = 1, 2)$ by

$$E_{q,1} = (m_g \times \mathrm{id}_h)_{*,q} = \bigoplus\nolimits_{i+j=q} (M_g)_i \otimes (I_h)_j$$

and

$$E_{q,2} = (\mathrm{id}_g \times m_h)_{*,q} = \bigoplus\nolimits_{i+j=q} (I_g)_i \otimes (M_h)_j \qquad (I = \text{identity}).$$

Consider the irreducible decomposition of p in the form $p = p_1^{m_1} p_2^{m_2} p_3^{m_3} \cdots p_r^{m_r}$ where $p_1 = c$, $p_2 = d$, $m_1 \geqslant 0$, $m_2 \geqslant 0$, $m_i \geqslant 1$ if $i \geqslant 3$ (i.e., $m_1 = 0$ iff c is not a factor of p). In a sufficiently small sphere we can consider the associated multilink $L(m) = (S^3, m_1 K_1 \cup m_2 K_2 \cup \cdots \cup m_r K_r)$ with natural orientation, as in [2]. Let $\Delta(\lambda_1, \ldots, \lambda_r)$ be the Alexander polynomial of the link $L = (S^3, K_1 \cup K_2 \cup \cdots \cup K_r)$ (i.e., $\cup_i K_i$ is composed of the link of p completed by the link components determined by $\{\mathrm{cd} = 0\}$ if those are not components of p).

THEOREM L2. *The zeta function of f is determined by*

$$\zeta_f(\lambda) = \zeta_g(\lambda^{n_d}) \cdot \zeta_h(\lambda^{n_c}) \cdot \prod_q \det \Delta(\lambda^{m_1} E_{q,1}, \lambda^{m_2} E_{q,2}, \lambda^{m_3} \cdot I, \ldots, \lambda^{m_r} \cdot I)^{(-1)^q}$$

(if n_d resp. $n_c = 0$ then $\zeta_g(\lambda^{n_d}) = 1$ resp. $\zeta_h(\lambda^{n_c}) = 1$).

We note that Δ is well defined only up to multiplication by monomials $\pm \lambda_1^{i_1} \cdots \lambda_r^{i_r}$, therefore the above equality is modulo $\pm \lambda^i$.

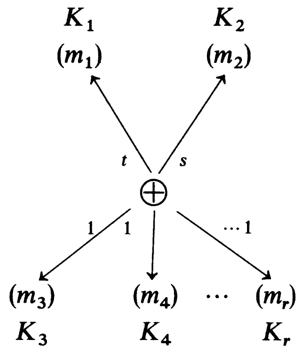
The proof will be given in §4.

2.7. EXAMPLES. If $p = c + d$, then $\Delta(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 \lambda_2 \lambda_3 - 1$, hence

$$\zeta_f(\lambda) = \zeta_g(\lambda) \cdot \zeta_h(\lambda) \cdot \prod_q \det(\lambda(m_g \times m_h)_{*,q} - 1)^{(-1)^q}$$
$$= \zeta_g(\lambda) \cdot \zeta_h(\lambda) \cdot \zeta_{g \times h}^{-1}(\lambda) = \zeta_{g*h}(\lambda).$$

(b) If $p = c \cdot d$, then $\Delta = 1$, $n_d = n_c = 0$. Therefore $\zeta_f = 1$.

(c) Let us take $p = c^{m_1} d^{m_2} \prod_{i=3}^r (d^s - a_i c^t)^{m_i} (a_i \neq a_j) (m_1 \geqslant 0, m_2 \geqslant 0, m_i \geqslant 1$ if $i \geqslant 3)$. Then the Eisenbud-Neumann diagram of the multilink $L(m)$ is of Seifert type:



The Alexander polynomial is given by $\Delta(\lambda_1, \ldots, \lambda_r) = (\lambda_1^{l_1}\lambda_2^{l_2} \cdots \lambda_r^{l_r} - 1)^{r-2}$ where $l_1 = s$, $l_2 = t$, $l_3 = \cdots = l_r = st$ [2, p. 96]. Therefore

$$\Pi = \prod_q \det \Delta(\lambda^{m_1}E_{q,1}, \lambda^{m_2}E_{q,2}, \ldots, \lambda^{m_r} \cdot I)^{(-1)^q}$$

$$= \prod_q \det(\lambda^{\Sigma m_k l_k}(m_g^s \times m_h^t)_{*,q} - 1)^{(-1)^q(r-2)}$$

$$= \prod_{i,j} \det(\lambda^{\Sigma m_k l_k}(M_g^s)_i \otimes (M_h^t)_j - 1)^{(-1)^{i+j}(r-2)}.$$

To each monic polynomial $P(\lambda) = (\lambda - \alpha_1) \cdots (\lambda - \alpha_r)$ with $\alpha_1, \ldots, \alpha_r \in \mathbf{C}^*$ we assign the divisor $D(P) = \langle\alpha_1\rangle + \cdots + \langle\alpha_r\rangle$ in the integral group ring $\mathbf{ZC}^*$. Define $D(P/Q) = D(P) - D(Q)$ and $D_l = D(\zeta_l)$ for $l = g, h$ and f. The application $\alpha \to \alpha^s (s \in \mathbf{N}^*)$ induces a $\mathbf{Z}$-linear map $(\cdot)^s: \mathbf{ZC}^* \to \mathbf{ZC}^*$ defined by $(\Sigma\, n_k\langle\alpha_k\rangle)^s = \Sigma\, n_k\langle\alpha_k^s\rangle$. We define also the $\mathbf{Z}$-linear map $(\cdot)^{1/s}: \mathbf{ZC}^* \to \mathbf{ZC}^*$ $(s \in \mathbf{N}^*)$ by $(\Sigma n_k\langle\alpha_k\rangle)^{1/s} = \Sigma_k \Sigma_{j=1}^s n_k\langle\alpha_{k_j}\rangle$, where $\{\alpha_{k_j}\}_j = \{\alpha : \alpha^s = \alpha_k\}$. With these notations we have $D(\Pi) = (r-2)\cdot(D_g^s \cdot D_h^t)^{1/\Sigma m_k l_k}$.

Since $(\Sigma\, m_k l_k)(r-2) = -\chi(P - \{cd - 0\}) = -\chi$, we obtain the formula

$$D_f = (r-2)(D_g^s \cdot D_h^t)^{(r-2)/-\chi} + (D_g)^{1/n_d} + (D_h)^{1/n_c}$$

If $m_1 = m_2 = 0$, then $n_d = t\,\Sigma\, m_k$ and $n_c = s\,\Sigma\, m_k$.

3. The main results. Global case

3.1. Let $g: \mathbf{C}^n \to \mathbf{C}$ and $h: \mathbf{C}^m \to \mathbf{C}$ be polynomial maps. Then there exists a finite set $\Lambda_g = \{c_1, \ldots, c_t\}$ (resp. $\Lambda_h = \{d_1, \ldots, d_s\}$) such that $g: \mathbf{C}^n - g^{-1}(\Lambda_g) \to \mathbf{C} - \Lambda_g$

(resp. $h: \mathbf{C}^m - h^{-1}(\Lambda_h) \to \mathbf{C} - \Lambda_h$) is a C^∞ locally trivial fibration. Let $p: \mathbf{C}^2 \to \mathbf{C}$ be a polynomial map in two variables (denoted by c and d) such that p depends effectively on both variables.

In this case the degree of the projective closure $\overline{p^{-1}(e)}$ of $p^{-1}(e)$, the multiplicity numbers $m_{[0:1:0]}\overline{p^{-1}(e)}$, $m_{[1:0:0]}\overline{p^{-1}(e)}$ and the tangent cones of $\overline{p^{-1}(e)}$ in [0:1:0] resp. [1:0:0] are independent of the choice of $e \in \mathbf{C}$. Denote:

$$T_c = \{c_0\text{: the line } \overline{\{c = c_0\}} \text{ is in the tangent cone of } \overline{p^{-1}(e)} \text{ in } [0:1:0]\},$$

$$T_d = \{d_0\text{: the line } \overline{\{d = d_0\}} \text{ is in the tangent cone of } \overline{p^{-1}(e)} \text{ in } [1:0:0]\}.$$

We work with the following

3.2. ASSUMPTIONS

$$A_c\colon T_c \cap \Lambda_g = \varnothing,$$

$$A_d\colon T_d \cap \Lambda_h = \varnothing.$$

Note that if $[0:1:0] \notin \overline{p^{-1}(e)}$ (resp. $[1:0:0] \notin \overline{p^{-1}(e)}$) then the tangent cone in this point is considered to be the void set. Hence in this case the assumption A_c (resp. A_d) is automatically fulfilled.

Define the following numbers:

$$n_c = \deg \overline{p^{-1}(e)} - m_{[0:1:0]}\overline{p^{-1}(e)},$$

$$n_d = \deg \overline{p^{-1}(e)} - m_{[1:0:0]}\overline{p^{-1}(e)}.$$

Then a generic fiber $p^{-1}(e)$ has exactly n_c intersection points with a line $\{c = c_i\}$.

3.3. Let G(resp. H) denote the generic fiber of g (resp. h), and P the generic fiber of p. Suppose that P is connected, i.e., p cannot be factored as $q \circ p'$ for some polynomials $p': \mathbf{C}^2 \to \mathbf{C}$ and $q: \mathbf{C} \to \mathbf{C}$, with q of degree $k \geqslant 2$.

With the above notations and assumptions we have the following

THEOREM G1. *The generic fiber F of the polynomial map $f = p(g, h)\colon \mathbf{C}^n \times \mathbf{C}^m \to \mathbf{C}$ defined by $f(x, y) = p(g(x), h(y))$ has the same construction as the local fiber described by Theorem L1, i.e., F has the homotopy type of a space obtained from the total space of a fiber bundle with base space P and fiber $G \times H$ by gluing naturally to a fiber $G \times H$ the space $G * H(n_c, n_d)$.*

The proof is given in §5.

3.4. EXAMPLES. (a) If $p = c + d$ then our assumptions are fulfilled, $n_c = n_d = 1$, hence $F \sim G * H$. This result was, in fact, our starting point in the study of global Sebastiani-Thom type theorems [9].

(b) Let $c_0 \notin \Lambda_g$ and $d_0 \notin \Lambda_h$. Then $p = (c - c_0)(d - d_0)$ verifies our assumptions. In this case also $n_c = n_d = 1$. Therefore, by Remark 2.5,

$$F \sim S^1 \vee S(G \times H) \vee G * H.$$

3.5. REMARK. A careful inspection of the proof of Theorem G1 shows that we have a nice description of the generic fiber F also in some cases when the Assumptions are not fulfilled.

Suppose that the assumption A_c is not fulfilled, but $\Lambda_g = \{c_1\}$ contains only one point. In this case Theorem G1 is also true if we replace n_c by $\deg \overline{p^{-1}(e)} - m_{[0:1:0]}(\overline{c - c_1}, \overline{p^{-1}(e)})$ (the second term denotes the intersection multiplicity of the projective curves at [0:1:0]) where e is a generic value in $\mathbf{C}$.

In particular, if $\Lambda_g = \{c_1\}$ and $\Lambda_h = \{d_1\}$ and we take $p = (c - c_1)(d - d_1)$, then $n_c = n_d = 0$ and we reobtain the global analogue of (2.3.b).

3.6. Consider a large circle $S_g = \{z: |z| = R_g\}$ such that $\Lambda_g \subset \{z: |z| < R_g\}$. Then g is a locally trivial fibration over S_g with characteristic map m_g. This global geometric monodromy induces the global algebraic monodromy operator $(M_g)_*: H^*(g^{-1}(R_g), \mathbf{C}) \circlearrowleft$ with zeta function $\zeta_g(\lambda)$. In the same way we define $(M_h)_*$ and ζ_h. With the same formulae as in the local case we define $E_{q,i} \in \operatorname{Aut} H^q(g^{-1}(R_g) \times h^{-1}(R_h), \mathbf{C})$, $(i = 1, 2)$.

3.7. The main obstructions in the computation of the zeta function of $f = p(h, g)$ in the global case (if we want to follow the local model) are:

(i) the Milnor fibration of p at infinity in general does not exist (e.g., for any $\delta \in \mathbf{C}$, the link at infinity determined by $p = c^2(c-1)^2 d - c = \delta$ is not fiberable),

(ii) even if the link determined by $p^{-1}(\delta)$ is fiberable for some δ, its fiber (minimal Seifert surface of the link) is topologically different from the generic fiber of p (e.g. if $p = c(cd - 1)$, then only the link of $p^{-1}(0)$ is fiberable, and the fiber of this bundle is the threefold punctured 2-sphere; but the generic fiber is the twice punctured 2-sphere.)

For this reason we consider only "good" polynomials p [6][7].

3.8. DEFINITION. [6][7]. The polynomial map $p: \mathbf{C}^2 \to \mathbf{C}$ is called *good* if, for any $\delta \in \mathbf{C}$, for some disk $D \ni \delta$ and some compact subset K of $\mathbf{C}^2$ $p \mid p^{-1}(D) - K$ is a trivial fibration.

If p is good, then all the fibers $p^{-1}(\delta)$ define the same link at infinity (up to isotopy), denoted by $\mathscr{L}(p, \infty)$. Moreover, there is a *Milnor fibration at infinity* (defined by $\phi_p = p/|p|: S_R^3 - p^{-1}(\delta_0) \to S^1$, $R \gg 0$, $\delta_0 \in \mathbf{C}$). The fiber of this fibration can be identified with the generic fiber of p (modulo a collar) [7].

If we fix a compact $C_c \subset \mathbf{C} - T_c$, then it is easy to verify that for R sufficiently large, the fibers $\phi_p^{-1}(e^{2\pi i \theta})$ meet the circles $\{c = c_0\} \cap S_R^3$ (for all $c_0 \in C_c$) transversely in S_R^3. In particular, we can consider the circles $K_c = \{c = c_0\} \cap S_R^3 (c_0 \notin T_c)$ and $K_d = \{d = d_0\} \cap S_R^3$ $(d_0 \notin T_d)$, $R \gg 0$, hence the isotopy type of the link $K_c \cup K_d \cup \mathscr{L}(p, \infty)$ is well defined and the fibers of ϕ_p meet transversally the (virtual) components K_c and K_d. Therefore, if we replace the link $(S^3_R, \mathscr{L}(p, \infty))$ by the multilink

$$L(\mathbf{m}) = (S^3_R, m_c K_c \cup m_d K_d \cup \mathscr{L}(p, \infty))$$

where we take $m_c = m_d = 0$ and $m = 1$ for the multilicities of the components of $\mathscr{L}(p, \infty)$ (we have no singular components!), then the fibers of the (fiberable) multilink $L(\mathbf{m})$ can be identified with $\{\phi_p^{-1}(e^{2\pi i\theta}) - K_c \cup K_d\}_\theta$.

Let $\Delta(\lambda_1, \lambda_2, \ldots, \lambda_r)$ be the Alexander polynomial of the link $(S^3_R, K_c \cup K_d \cup \mathscr{L}(p, \infty))$, $\mathscr{L}(p, \infty)$ has $r - 2$ components, λ_1 resp. λ_2 corresponding to K_c resp. K_d).

THEOREM G2. *Let g, h and p be as above such that A_c, A_d are fulfilled and p is good. Then the zeta function of the global monodromy operator of f is determined by the same formula as in the local case:*

$$\zeta_f(\lambda) = \zeta_g(\lambda^{n_d}) \cdot \zeta_h(\lambda^{n_c}) \cdot \prod_q \det \Delta(E_{q,1}, E_{q,2}, \lambda I, \ldots, \lambda I)^{(-1)^q}$$

3.9. REMARK. We note that $(S^3_R, \mathscr{L}(p, \infty))$ (and also $(L(\mathbf{m}))$ has an RPI-splice diagram [7], hence Δ can be computed by [2]. Moreover, n_c (resp. n_d) can also be determined by the splice diagram of $\mathscr{L}(p, \infty)$, where K_c(resp. K_d) appears as virtual link component, as $n_c = |m(K_c)|$(resp. $n_d = |m(K_d)|$).

3.10. EXAMPLE. If $p(0) = 0$, p is convenient and has a nondegenerate Newton principal part at infinity, then it is good, and the data n_c, n_d and Δ depend only on the Newton principal part at infinity [10].

For example, if $p = c^t + d^s + \sum_{si+tj<st} a_{ij} c^i d^j$, then ζ depends only on the principal part $c^t + d^s$. Therefore

$$D(\zeta_f) = -(D(\zeta_g)^s \cdot D(\zeta_h)^t)^{1/st} + D(\zeta_g)^{1/t} + D(\zeta_h)^{1/s}.$$

4. Proof of the results

Proof of Theorem L1. We choose $\varepsilon_1 > 0$ and $0 < \eta_1 \ll \varepsilon_1$ sufficiently small such that $B^n_{\varepsilon_1} = \{z \in \mathbf{C}^n : |z| \leqslant \varepsilon_1\}$ (resp. $B^m_{\varepsilon_1}$) is a Milnor-ball for g (resp. h) and

$$g: B^n_{\varepsilon_1} \cap g^{-1}(D_{\eta_1} - \{0\}) \to D_{\eta_1} - \{0\} \ (D_{\eta_1} = \{w \in \mathbf{C} : |w| \leqslant \eta_1\})$$

(resp. $h: B^m_{\varepsilon_1} \cap h^{-1}(D_{\eta_1} - \{0\}) \to D_{\eta_1} - \{0\}$) is a C^∞-fiber bundle.

Let $0 < \varepsilon < \varepsilon_1$ be so small that $g(B^n_\varepsilon) \subset D_{\eta_1}$, $h(B^m_\varepsilon) \subset D_{\eta_1}$ and finally we choose $0 < \eta < \eta_1$ small enough so that:

(i) $B_\varepsilon^n \cap g^{-1}(D_\eta - \{0\}) \to D_\eta - \{0\}$ resp. $B_\varepsilon^m \cap h^{-1}(D_\eta - \{0\}) \to D_\eta - \{0\}$ are C^∞-fiber bundles,

(ii) $D_\eta \times D_\eta$ is equivalent with a Milnor-ball for p, i.e., for $|\delta|$ small, $(D_\eta \times D_\eta, p^{-1}(\delta) \cap D_\eta \times D_\eta)$ is homeomorphical with the standard pair (Milnor-ball, $p^{-1}(\delta)$)

(iii) there exists a (deformation) retract

$$r_g: (B_{\varepsilon_1}^n - \operatorname{int} B_\varepsilon^n) \cap g^{-1}(D_\eta) \to \partial B_\varepsilon^n \cap g^{-1}(D_\eta)$$
$$(\text{resp. } r_h: (B_{\varepsilon_1}^m - B_\varepsilon^m) \cap h^{-1}(D_\eta) \to \partial B_\varepsilon^m \cap h^{-1}(D_\eta))$$

such that

$$r_g|_{\partial B_\varepsilon^n \cap g^{-1}(D_\eta)} = \mathrm{id},\ g \circ r_g = g$$
$$(\text{resp. } r_h|_{\partial B_\varepsilon^m \cap h^{-1}(D_\eta)} = \mathrm{id},\ h \circ r_h = h).$$

The existence of such a (deformation) retract can be proved in the usual way (see for example [8]) by integration of a vector field $v(z)$ with the following properties: $\mathrm{Re}\langle v(z), z\rangle \langle 0$ and $v(z)$ is tangent to $g^{-1}(g(z))$. The existence of such a vector field is locally ensured by the Curve Selection Lemma [5], then by a partition of unity we glue to a global field).

Denote $F_{\varepsilon,\delta} = f^{-1}(\delta) \cap B_\varepsilon^n \times B_\varepsilon^m$. Obviously $u(F_{\varepsilon,\delta}) \subset D_{\eta_1} \times D_{\eta_1}$ where $u = g \times h$. We take $\delta > 0$ sufficiently small so that:

(a) $p^{-1}(\delta)$ is a Milnor fiber in both squares $D_{\eta_1} \times D_{\eta_1}$ and $D_\eta \times D_\eta$,

(b) $p^{-1}(\delta) \cap \{cd = 0\} \subset D_\eta \times D_\eta$.

Firstly we prove that $u^{-1}(D_\eta \times D_\eta) \cap F_{\varepsilon,\delta}$ is a deformation retract in $F_{\varepsilon,\delta}$. Indeed, the restricted product map

$$u: B_{\varepsilon_1}^n \times B_{\varepsilon_1}^m \cap u^{-1}(p^{-1}(\delta)^0) \to p^{-1}(\delta)^0$$
$$(p^{-1}(\delta)^0 = p^{-1}(\delta) \cap D_{\eta_1} \times D_{\eta_1} - \operatorname{int} D_\eta \times D_\eta)$$

is a locally trivial fibration. Therefore the natural deformation retract $D_t: p^{-1}(\delta)^0 \to p^{-1}(\delta)^0$, $t \in [0, 1]$, with $D_0 = \mathrm{id}$, $\operatorname{im} D_1 \in p^{-1}(\delta) \cap \partial(D_\eta \times D_\eta)$ has a lifting

$$\tilde{D}_t: B_{\varepsilon_1}^n \times B_{\varepsilon_1}^m \cap u^{-1}(p^{-1}(\delta)^0) \to B_{\varepsilon_1}^n \times B_{\varepsilon_1}^m \cap u^{-1}(p^{-1}(\delta)^0).$$

Define $\tilde{r}_g: B_{\varepsilon_1}^n \cap g^{-1}(D_\eta) \to B_\varepsilon^n \cap g^{-1}(D_\eta)$ by

$$\tilde{r}_g = \begin{cases} r_g & \text{if } |x| \geqslant \varepsilon \\ \mathrm{id} & \text{if } |x| \leqslant \varepsilon. \end{cases}$$

Similarly we define $\tilde{r}_h$.

Then the composed map $(x, y) \mapsto (\tilde{r}_g \times \tilde{r}_h) \circ \tilde{D}_t(x, y)$ defines a deformation retract of $u^{-1}(D_\eta \times D_\eta) \cap F_{\varepsilon,\delta}$ in $F_{\varepsilon,\delta}$.

If we take $0 < \eta' < \eta$, $0 < \varepsilon' < \varepsilon$, (η', ε') with the same properties as (η, ε) then using the above argument again we obtain that, for δ sufficiently small, the inclusion $u^{-1}(D_{\eta'} \times D_{\eta'}) \cap F_{\varepsilon',\delta} \subset u^{-1}(D_\eta \times D_\eta) \cap F_{\varepsilon,\delta}$ admits a deformation retract, hence the inclusion $F_{\varepsilon',\delta} \hookrightarrow F_{\varepsilon,\delta}$ is a homotopy equivalence.

Therefore $\{B^n_\varepsilon \times B^m_\varepsilon\}_\varepsilon$ is a system of Milnor neighbourhoods for f [4]. In particular, $F_{\varepsilon,\delta}$ has the homotopy type of the "standard" Milnor fiber of f [op. cit.].

Consider the map $u: u^{-1}(D_\eta \times D_\eta) \cap F_{\varepsilon,\delta} \to D_\eta \times D_\eta \cap p^{-1}(\delta)$.

The restricted map is locally trivial over $p^{-1}(\delta) - \{cd = 0\}$ with fiber $G \times H$. Over the points $p^{-1}(\delta) \cap \{c = 0\}$ the special fiber of u is the product space (central fiber of g) $\times$ (Milnor fiber of h). Let $P' \subset p^{-1}(\delta) \cap D_\eta \times D_\eta$ be such that the inclusion is a homotopy equivalence (P' can be chosen to be a bouquet of circles with base point $*$) and $P' \cap \{cd = 0\} = \emptyset$. Then by a standard argument [8, 3.2.6] F has the homotopy type of a space obtained from $u^{-1}(P')$ by gluing to the fiber $u^{-1}(*)$ $\# p^{-1}(\delta) \cap \{c = 0\}$ copies $\operatorname{Con} G \times H$ and $\# p^{-1}(\delta) \cap \{d = 0\}$ copies of $G \times \operatorname{Con} H$.

Proof of Theorem L2. We denote $D = \{cd = 0\} \subset D_\eta \times D_\eta$. Then for $\eta > 0$ sufficiently small and $0 < \delta \ll \eta$,

$$p: (D_\eta \times D_\eta \cap p^{-1}(\partial B^1_\delta), p^{-1}(\partial B^1_\delta) \cap D) \to \partial B^1_\delta,$$

$$p: (B^2_\eta \cap p^{-1}(\partial B^1_\delta), p^{-1}(\partial B^1_\delta) \cap D) \to \partial B^1_\delta,$$

$$\phi_p = p/|p|: (\partial B^2_\eta - p^{-1}(0), \partial B^2_\eta \cap D - p^{-1}(0)) \to S^1$$

are (fiber-isomorphic) locally trivial fibrations of pairs of spaces. (The proof is similar to the one in [8, 2.2.1].)

Let $P_\phi = \phi_p^{-1}(1)$ and let $*$ be a point on $P_\phi - D$. Then we have the following exact sequence of groups:

$$1 \to \pi_1(P_\phi - D, *) \xrightarrow{i_*} \pi_1(\partial B^2_\eta - p^{-1}(0) \cup D,_*) \xrightarrow{\phi_*} \mathbf{Z} \to 0.$$

Since $u: B^n_\varepsilon \times B^m_\varepsilon \cap u^{-1}(B^2_\eta - D) \to B^2_\eta - D$ is a locally trivial fibration, we have a monodromy representation

$$\rho_q: \pi_1(B^2_\eta - D) = \pi_1(\partial B^2_\eta - D) = \mathbf{Z}^2 \to \mathbf{Aut}(H^q(G \times H), \mathbf{C}).$$

The generators of $\mathbf{Z}^2$ are chosen so that $(1, 0)$ (resp. $(0, 1)$) is a small meridian of the link component $\{c = 0\}$ (resp. $\{d = 0\}$).

By the inclusion $\partial B_\eta^2 - p^{-1}(0) \cup D \hookrightarrow B_\eta^2 - D$, $A^q = H^q(G \times H, \mathbf{C})$ becomes a $\pi = \pi_1(\partial B_\eta^2 - p^{-1}(0) \cup D, *)$ module, hence also a $\pi' = \pi_1(P_\phi - D, *)$ module.

Let $g \in \pi$ be such that $\phi_*(g) = 1$. Then the maps $\rho_q(g)\colon A^q \to A^q$ and $c_q(g)\colon \pi' \to \pi'$, $h \mapsto g^{-1}hg$ induce an automorphism of the exact sequence:

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^0(\pi', A^q) & \longrightarrow & A^q & \longrightarrow & \mathbf{Der}(\pi', A^q) & \longrightarrow & H^1(\pi', A^q) & \longrightarrow 0 \\
& \downarrow g_0^* & & \downarrow \rho(g) & & \downarrow g_{\mathrm{Der}} & & \downarrow g_1^* & \\
0 \longrightarrow & H^0(\pi', A^q) & \longrightarrow & A^q & \longrightarrow & \mathbf{Der}(\pi', A^q) & \longrightarrow & H^1(\pi', A^q) & \longrightarrow 0
\end{array}$$

Let us define

$$\begin{aligned}
[(\zeta_{p,D})_q(\lambda)]^{(-1)^{q+1}} &= \det(1 - \lambda g_0^*)/\det(1 - \lambda g_1^*) \\
&= \det(1 - \lambda \rho(g))/\det(1 - \lambda g_{\mathrm{Der}}).
\end{aligned}$$

Then repeating the arguments of the proof of Theorem B [8] we obtain that

$$\zeta_f(\lambda) = \zeta_g(\lambda^{n_d}) \cdot \zeta_h(\lambda^{n_c}) \cdot \prod_q (\zeta_{p,D})_q.$$

But similarly to the proof of Theorem C [op. cit.] we have

$$(\zeta_{p,D})_q^{(-1)^q} = \det \Delta(\lambda^{m_1} E_{q,1}, \lambda^{m_2} E_{q,2}, \lambda^{m_3} \cdot I, \ldots, \lambda^{m_r} \cdot I).$$

5. Proof of the global results

Proof of Theorem G1. We start with some notations:

Λ_p = the bifurcation set of p, i.e., Λ_p is the minimal set such that $p\colon \mathbf{C}^2 - p^{-1}(\Lambda_p) \to \mathbf{C} - \Lambda_p$ is a locally trivial fibration.

$\Lambda_{c_i} = \{p(c_i, d) | (\partial p/\partial d)(c_i, d) = 0\}, \quad i = 1, \ldots, t.$

$\Lambda_{d_j} = \{p(c, d_j) | (\partial p/\partial c)(c, d_j) = 0\}, \quad j = 1, \ldots, s.$

$\Lambda_f = \Lambda_p \cup \bigcup_i \Lambda_{c_i} \cup \bigcup_j \Lambda_{d_j} \cup \bigcup_{i,j} \{p(c_i, d_j)\},$

$\Lambda = \mathbf{C} \times \Lambda_h \cup \Lambda_g \times \mathbf{C}.$

Obviously, $u = g \times h\colon \mathbf{C}^n \times \mathbf{C}^m \to \mathbf{C} \times \mathbf{C}$ is a locally trivial fibration over $\mathbf{C}^2 - \Lambda$.

Let $e \notin \Lambda_f$ and consider the restricted projections

$$\pi_g: pr_1|p^{-1}(e): p^{-1}(e) \to \mathbf{C} \text{ with bufurcation set } S_g \subset \mathbf{C},$$

$$\pi_h: pr_2|p^{-1}(e): p^{-1}(e) \to \mathbf{C} \text{ with bufurcation set } S_h \subset \mathbf{C}.$$

From the definition of the set Λ_f and our assumptions A_c and A_d we get that $S_g \cap \Lambda_g = \varnothing$ and $S_h \cap \Lambda_h = \varnothing$. Therefore $\pi_g: p^{-1}(e) - \pi_g^{-1}(S_g) \to \mathbf{C} - S_g$ is an (unramified) covering space of degree n_c such that $\Lambda_g \subset \mathbf{C} - S_g$.

Let $D_1, \ldots, D_t$ be small closed C^∞-embedded disks in $\mathbf{C} - S_g$ with centers at the points $\{c_i\}_{i=1,\ldots,t}$ and with radius so small that they are mutually disjoint. For all $i = 1, \ldots, t$ let l_1 be a C^∞-embedded interval in $\mathscr{D} = \mathbf{C} - (S_g \cup \bigcup_i \operatorname{int} D_i)$ from a base point $c_0 \in \mathscr{D}$ (with property: $\mathrm{pr}_2 \pi_g^{-1}(c_0) \cap \Lambda_h = \varnothing$) to a point $\bar{c}_i$ on ∂D_i such that $l = \bigcup_i l_i$ can be contracted within itself to c_0, $B_g = l \cup \bigcup_i D_i$ is a deformation retract of $\mathbf{C}$. Then for each $P_i \in \pi_g^{-1}(c_0) = \{(c_0, d_i') \in p^{-1}(e), i = 1, \ldots, n_c\}$ we have a section $s_{g,i}: B_g \to p^{-1}(e)$, $s_{g,i}(c_0) = P_i$, $\pi_g \circ s_{g,i} = \mathrm{id}_{B_g}$. From the definition of the set Λ_f and from the choice of the value c_0 we can construct the set B_g such that $\mathrm{pr}_2 \circ s_{g,i}(B_g) \cap \Lambda_h = \varnothing$ for each $i = 1, \ldots, n_c$.

5.1. LEMMA.

$$(u^{-1}(s_{g,i}(B_g)), u^{-1}(P_i)) \sim (g^{-1}(B_g) \times H, G \times H) \sim (\mathbf{C}^n \times H, G \times H)$$
$$\sim (\operatorname{Con} G \times H, G \times H).$$

Proof of the Lemma. As a first step we prove that $(u^{-1}(s_{g,i}(D_j)), u^{-1}(s_{g,i}(\bar{c}_j))) \approx (g^{-1}(D_j) \times H, G \times H)(j = 1, \ldots, t; i = 1, \ldots, n_c)$.

Indeed, if D_j is sufficiently small then $\mathrm{pr}_2 \circ s_{g,i}(D_j)$ can be considered as a subset of a contractible set in $\mathbf{C} - \Lambda_h$. Therefore h is trivial over $\mathrm{pr}_2 \circ s_{g,i}(D_j)$, hence there exists a diffeomorphism $(h, \psi^h): h^{-1}(\mathrm{pr}_2 \circ s_{g,i}(D_j)) \to \mathrm{pr}_2 \circ s_{g,i}(D_j) \times H$ $(H = h^{-1}(\mathrm{pr}_2 \circ s_{g,i}(\bar{c}_j)))$ such that $\psi^h | H = \mathrm{id}_H$. Then

$$(p_1, \psi^h): u^{-1}(s_{g,i}(D_j)) \to g^{-1}(D_j) \times H \quad (p_1, \psi^h)(x, y) = (x, \psi^h(y))$$

is the wanted diffeomorphism. Its inverse is

$$(x', y') \mapsto (x', (h, \psi^h)^{-1}(\mathrm{pr}_2 \circ s_{g,i}(g(x')), y'))$$

Now the first equivalence follows from the fact that $s_{g,i}(l) \cap \Lambda = \varnothing$, $s_{g,i}(l)$ is contractible and u is locally trivial over $\mathbf{C}^2 - \Lambda$. The facts that $\Lambda_g \subset B_g$ and the inclusion $B_g \subset \mathbf{C}$ admits a deformation retract imply the second equivalence. The third one is trivial.

Similarly we can construct the set B_h with base point d_0 and sections $s_{h,j}: B_h \to p^{-1}(e)$ such that $s_{h,j}(d_0) = Q_j$, $\pi_h \circ s_{h,j} = \mathrm{id}_{B_h}$ $(j = 1, \ldots, n_d)$.

Let $\mathscr{B} = \bigcup_i s_{g,i}(B_g) \cup \bigcup_j s_{h,j}(B_h)$. Note that we can suppose (by the definition of Λ_f) that the intersections $s_{g,i}(B_g) \cap s_{h,j}(B_h)$ are void.

It is easy to see that there exists a subspace $\mathscr{C} = \{$bouquet of circles with base space $*\}$ in $p^{-1}(e) - \mathscr{B}$ such that $\mathscr{C}$ is a deformation retract in $p^{-1}(e)$. Let $\bar{l}_k (k = 1, \ldots, n_c + n_d)$ be C^∞-embedded intervals in $p^{-1}(e)$ from the base point $*$ to the points P_i $(i = 1, \ldots, n_c)$ and Q_j $(j = 1, \ldots, n_d)$ such that $\bar{l} = \bigcup_k \bar{l}_k$ can be contracted within itself to $*$, $\mathscr{C} \cup \bar{l}$ can be contracted within itself to $\mathscr{C}$ and the inclusion $\mathscr{C} \cup \bar{l} \cup \mathscr{B} \hookrightarrow p^{-1}(e)$ admits a (strong) deformation retract. Then using the above lemma and the fact that u is locally trivial over $\mathbf{C}^2 - \Lambda$ we get that $(u^{-1}(\bar{l}_i \cup s_{g,i}(B_g)), u^{-1}(*)) \sim (\operatorname{Con} G \times H, G \times H)$, $(i = 1, \ldots, n_c)$;

$$(u^{-1}(\bar{l}_{n_c+j} \cup s_{h,j}(B_h)), u^{-1}(*)) \sim (G \times \operatorname{Con} H,\ G \times H) \qquad (j = 1, \ldots, n_d),$$

where $G \times H$ is identified with $u^{-1}(*)$. Since $\mathscr{C}$ is a deformation retract in $p^{-1}(e)$ and u is locally trivial over $\mathscr{C}$, the result of Theorem G1 follows.

Proof of Theorem G2. Similarly to the local case [5] it can be proven that if p is good then the fibration determined by the restriction of p over a large circle (of radius R) is equivalent to the Milnor fibration at infinity in a sphere whose radius is sufficiently large in comparison with R. [6], [7], [10]. Hence we can identify the spaces $p^{-1}(e) - \mathscr{B}$, $p^{-1}(e) - \bigcup_i p_i \cup \bigcup_j Q_j$ and $\phi_p^{-1}(1) - \{c = c_0\} \cup \{d = d_0\}$. If we observe that the monodromy "around B_g" (i.e., the one induced by the path $l_1 \circ \partial D_1 \circ l_1^{-1} \circ l_2 \circ \partial D_2 \circ l_2^{-1} \circ \cdots$) is exactly the global monodromy of g, then the theorem follows from a combination of the proof in the local case (§4) with our global construction.

References

1. Demazure, M.: Classification des germs à points critiques isolés et à nombres de modules 0 ou 1 (d'après V. I. Arnold), Séminaire Bourbaki, 26-e année, 1973/74, 443, Février 1974.
2. Eisenbud, D. and Neumann, W.: *Three-dimensional link theory and invariants of plane curve singularities*, Annals of Math. Studies, Princeton Univ. Press., 110 (1985).
3. Gabrielov, A. M.: Intersection matrices for certain singularities, *Funktional'nyi Analiz i ego Prilozheniya*, 1973, 7:3, 18–32; FAA (Eng. Tranl. of FAP) 7, 182–193.
4. Massey, D. B.: The Lê Varieties, II: submitted.
5. Milnor, J.: *Singular points of complex hypersurfaces*, Annals of Math. Studies, Princeton Univ. Press, 61 (1968).
6. Neumann, W. D., Rudolf, L.: Unfoldings in knot theory. *Math. Ann.* 278, 409–439 (1987) and Corrigendum, ibid. 282, 349–351 (1988).
7. Neumann, W. D.: Complex algebraic plane curves via their links at infinity. *Invent. Math.* 98, 445–489 (1989).
8. Némethi, A.: The Milnor fiber and the zeta function of the singularities of type $f = P(h, g)$; to appear in *Compositio Math.*
9. Némethi, A.: Global Sebastiani-Thom theorem for polynomial maps; to appear in *J. Math. Soc. Japan.*

10. Némethi, A., Zaharia, A.: *The topology of some affine hypersurfaces* (preprint, INCREST, 65/1986).
11. Oka, M.: On the homotopy types of hypersurfaces defined by weighted homogeneous polynomials, *Topology*, 12, 19–32 (1973).
12. Sakamoto, K.: *Milnor fiberings and their characteristic maps*, Proc. Intern. Conf. on manifolds and Related Topics in Topology, Tokyo (1973).
13. Sakamoto, K.: The Seifert matrices of Milnor fiberings defined by holomorphic functions, *J. Math. Soc. Japan*, 26, 714–721 (1974).
14. Sebastiani, M., Thom, R.: Un résultat sur la monodromie, *Invent. Math.* 1971, 13: 1–2, 90–96.