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FIONA MURNAGHAN

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Asymptotic behaviour of supercuspidal characters of p-adic GSp(4)

FIONA MURNAGHAN*

Department of Mathematics, University of Toronto, Toronto, M5S 1A1 Canada

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1. Introduction

Let F be a p-adic field of characteristic zero. The purpose of this paper is to study the singular behaviour of the character Θ_π of an irreducible supercuspidal representation π of $G = \mathbf{GSp}_4(F)$ near the identity. The method used involves a comparison of Harish-Chandra's results expressing Θ_π as a linear combination of Fourier transforms of nilpotent measures on the Lie algebra with Arthur's germ expansion for Θ_π as a weighted orbital integral of a sum of matrix coefficients of π. Some relations between the constants in Harish-Chandra's theorem and unipotent weighted orbital integrals of matrix coefficients are obtained. In addition, as a consequence of explicit calculations carried out for certain representations, we find that Θ_π need not exhibit all possible types of singular behaviour.

Let G_{reg} be the open set consisting of $x \in G$ such that the coefficient of λ^3 in $\det(\lambda + 1 - \operatorname{Ad} x)$ is nonzero. Harish-Chandra showed in [HC] that Θ_π is a locally constant function on G_{reg} and

$$\Theta_\pi(\exp X) = \sum_{\mathcal{O} \in (\mathcal{N}_G)} c_{\mathcal{O}}(\pi)\hat{\mu}_{\mathcal{O}}(X),$$

for $X \in \mathfrak{g} = \mathrm{Lie}(G)$ close to zero and such that $\exp X \in G_{\mathrm{reg}}$. $(\mathcal{N}_G)$ is the set of nilpotent $\operatorname{Ad} G$-orbits in $\mathfrak{g}$, $c_{\mathcal{O}}(\pi)$ is a constant, and $\hat{\mu}_{\mathcal{O}}$ is the Fourier transform of the orbital integral over $\mathcal{O}$. The asymptotic behaviour of $\Theta_\pi(x)$ as $x \in G_{\mathrm{reg}}$ approaches 1 is determined by the homogeneity properties of those $\hat{\mu}_{\mathcal{O}}$'s for which $c_{\mathcal{O}}(\pi) \neq 0$. Section 2 includes the definition of the Fourier transform, the general statement of Harish-Chandra's result, and some remarks and notation concerning Levi subgroups, unipotent conjugacy classes, and induced representations.

Let f be a finite sum of matrix coefficients of π. Arthur [A3] showed that if $x \in G_{\mathrm{reg}}$ is elliptic in a Levi subgroup M, then $\Theta_\pi(x)$ is a constant multiple of the

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weighted orbital integral $J_M(x, f)$. In [A2], he derived a germ expansion for weighted orbital integrals on neighbourhoods of singular points, that is, points not in G_{reg}. The constants in the germ expansion around 1 are unipotent weighted orbital integrals and the germs have homogeneity properties. These results, which are summarized in section 3, combine to produce a second asymptotic expansion for Θ_π.

Section 4 consists of a lemma describing how to match up the terms having a fixed homogeneity in the two asymptotic expansions for Θ_π around the identity.

Normalizations of various measures are specified in section 5. Also contained in this section are the calculations of those volumes and weight factors which will be required in section 8 for evaluation of unipotent weighted orbital integrals.

Moy [Mo1-2] has classified the irreducible admissible representations of G in terms of nondegenerate representations of subgroups of parahoric subgroups of G. Let $\mathfrak{O}_F$ be the ring of integers of F. In section 6, we consider those irreducible supercuspidal representations π such that the parahoric subgroup is $K = \mathbf{GSp}_4(\mathfrak{O}_F)$. The inducing data for π, as determined by Jabon [J], is used to compute the values of the sums of matrix coefficients which will be needed for calculations in section 8.

In section 7 the values of Fourier transforms $\hat{\mu}_{\mathcal{O}}$ are computed at certain points in $\mathfrak{g}$.

The main results of the paper appear in section 8. In Theorem 8.1, relations between the constants $c_{\mathcal{O}}(\pi)$ and various weighted orbital integrals are derived using values of the functions $\hat{\mu}_{\mathcal{O}}$ from section 7 together with formulas from section 5. These relations hold for all irreducible supercuspidal representations π. Then, for π as in section 6, in Proposition 8.2, all of the coefficients in Arthur's germ expansion for Θ_π are calculated. Theorem 8.1 and Proposition 8.2 are combined to obtain Theorem 8.3 where explicit values for most of the coefficients $c_{\mathcal{O}}(\pi)$ are computed. For some representations, $c_{\mathcal{O}}(\pi) = 0$ for particular orbits $\mathcal{O}$. This contrasts with the results of [Mu], where every coefficient in the asymptotic expansions of characters of the analogous representations of $\mathbf{GL}_3(F)$ and $\mathbf{GL}_4(F)$ was found to be nonzero.

2. Characters as Fourier transforms of nilpotent measures

In this section, we state Harish-Chandra's result giving an asymptotic expansion for Θ_π near a singular point in terms of Fourier transforms on the Lie algebra. Following that are some remarks on unipotent classes in G. Lemma 2.6 relates some of the functions appearing in the expansion to the characters of induced representations.

We will assume that the residual characteristic of F is odd. G is realized as the set of $x \in \mathbf{GL}_4(F)$ satisfying $x^t H x = \lambda H$ for some $\lambda \in F^*$, where

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

and x^t is the transpose of x. Similarly, $\mathfrak{g} = \mathrm{Lie}(G)$ consists of those $X \in \mathbf{M}_4(F)$ such that $X^t H + HX$ is a multiple of H. If R is a ring, $\mathbf{M}_4(R)$ denotes the 4×4 matrices with entries in T.

If $\mathcal{O}$ is an $\mathrm{Ad}\, G$-orbit in $\mathfrak{g}$, then $\mu_{\mathcal{O}}$ denotes the distribution on $\mathfrak{g}$ given by integration over the orbit $\mathcal{O}$. The Fourier transform $\hat{\mu}_{\mathcal{O}}$ of $\mu_{\mathcal{O}}$ is defined by $\hat{\mu}_{\mathcal{O}}(f) = \mu_{\mathcal{O}}(\hat{f})$, $f \in C_c^\infty(\mathfrak{g})$. Here, $C_c^\infty(\mathfrak{g})$ is the space of locally constant, compactly supported, complex-valued functions on $\mathfrak{g}$. Recall that $\hat{f} \in C_c^\infty(\mathfrak{g})$ is given by:

$$\hat{f}(X) = \int_{\mathfrak{g}} \psi(\mathrm{tr}(XY)) f(Y)\, \mathrm{d}Y, \qquad X \in \mathfrak{g},$$

where ψ is a nontrivial character of F. Let $|\cdot|$ denote the norm on F which satisfies $|\varpi| = q^{-1}$ for any prime element ϖ, where q is the order of the residue class field $\mathbf{F}_q$ of F. The set $\mathfrak{g}_{\mathrm{reg}}$ of $X \in \mathfrak{g}$ having the property that λ^3 has nonzero coefficient in $\det(\lambda - \mathrm{ad}\, X)$ is an open dense subset of $\mathfrak{g}$.

LEMMA 2.1. [HC]. *With $\mathcal{O}$ as above,*

(1) *There exists a locally integrable function $\hat{\mu}_{\mathcal{O}}\colon \mathfrak{g} \to \mathbf{C}$ which is locally constant on $\mathfrak{g}_{\mathrm{reg}}$ such that $\hat{\mu}_{\mathcal{O}}(f) = \int_{\mathfrak{g}} \hat{\mu}_{\mathcal{O}}(X) f(X)\, \mathrm{d}X$, for $f \in C_c^\infty(\mathfrak{g})$.*
(2) *If $t \in F^*$, $\hat{\mu}_{\mathcal{O}}(t^2 X) = |t|^{-\dim \mathcal{O}} \hat{\mu}_{\mathcal{O}}(X)$.*

If $\gamma \in G$, let G_γ be the centralizer of γ in G, and let $\mathfrak{g}_\gamma$ be the Lie algebra of G_γ.

THEOREM 2.2. [HC, Theorem 5]. *For any irreducible admissible representation π of G, there exist unique complex numbers $c_{\mathcal{O}}(\pi)$, one for each nilpotent G_γ-orbit in $\mathfrak{g}_\gamma$, such that*

$$\Theta_\pi(\gamma \exp X) = \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi) \hat{\nu}_{\mathcal{O}}(X),$$

for $X \in \mathfrak{g}_\gamma \cap \mathfrak{g}_{\mathrm{reg}}$ sufficiently near 0. Here $\nu_{\mathcal{O}}$ is the G_γ-invariant measure on $\mathfrak{g}_\gamma$ corresponding to $\mathcal{O}$, and $\hat{\nu}_{\mathcal{O}}$ is the Fourier transform of $\nu_{\mathcal{O}}$ on $\mathfrak{g}_\gamma$.

The functions $\hat{\mu}_{\mathcal{O}}$, for nilpotent G-orbits $\mathcal{O}$ in $\mathfrak{g}$, are linearly independent on $V \cap \mathfrak{g}_{\mathrm{reg}}$, for any neighbourhood V of 0 in $\mathfrak{g}$ [HC, Theorem 4]. Therefore the functions $\{\hat{\mu}_{\mathcal{O}} \mid c_{\mathcal{O}}(\pi) \neq 0\}$ determine the singular behaviour of Θ_π near 1. Let $\mathcal{U}_G$ be the set of unipotent elements in G, and let $(\mathcal{U}_G)$ be the unipotent conjugacy classes in G. As a result of the correspondence between the set of nilpotent Ad G-orbits in $\mathfrak{g}$ and the set $(\mathcal{U}_G)$, the constants $c_{\mathcal{O}}(\pi)$ and the functions $\hat{\mu}_{\mathcal{O}}$ will be referred to as corresponding to some $\mathcal{O} \in (\mathcal{U}_G)$.

Fix a prime element $\varpi \in F$ and choose $\varepsilon \in F$ such that $|\varepsilon| = 1$ and $\varepsilon \notin (F^*)^2$. The following matrices are representatives for the nontrivial classes in $(\mathcal{U}_G)$.

$$u_\tau = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -\tau & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \tau = 0,\, 1,\, \varepsilon,\, \varpi,\, \varepsilon\varpi$$

$$u_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.3}$$

Let $\mathcal{O}_0$, $\mathcal{O}_1$, etc., be the corresponding unipotent classes. id will stand for the trivial unipotent class. The notation $\hat{\mu}_{\text{id}}$, $\hat{\mu}_0$, etc. ($c_{\{1\}}(\pi)$, $c_0(\pi)$, etc.) will be used for the functions $\hat{\mu}_{\mathcal{O}}$, (resp. the coefficients $c_{\mathcal{O}}(\pi)$), $\mathcal{O} \in (\mathcal{U}_G)$.

If M is a Levi subgroup of G, let $\mathscr{P}(M)$ be the set of parabolic subgroups having Levi component M. For $P \in \mathscr{P}(M)$, N_P denotes the unipotent radical of P. Given $\gamma \in M$, Arthur [A3, p. 255] defines the induced space of orbits $\gamma_M^G = \gamma^G$ in G as the finite union of all conjugacy classes in G which intersect γN_P in an open set, for any $P \in \mathscr{P}(M)$. This is a generalization of the definition of Lusztig and Spaltenstein [LS]. If $\mathcal{O}$ is such a class, we write $\mathcal{O} \in \gamma^G$.

Representatives for the conjugacy classes of proper Levi subgroups in G are:

$$M_0 = \left\{ \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{bmatrix} \middle|\, a_1 a_4 = a_2 a_3 \neq 0 \right\}$$

$$M_1 = \left\{ \begin{bmatrix} y_1 & & & \\ & x_1 & x_2 & \\ & x_3 & x_4 & \\ & & & y_2 \end{bmatrix} \middle|\, y_1 y_2 = x_1 x_4 - x_2 x_3 \neq 0 \right\} \tag{2.4}$$

$$M_2 = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \middle|\, A \in \mathrm{GL}_2(F),\ B = \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (A^{-1})^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mu \in F^* \right\}$$

LEMMA 2.5.

(1) $\mathcal{O}_1 = 1_{M_1}^G$

(2) $\mathcal{O}_1 \cup \mathcal{O}_\varepsilon \cup \mathcal{O}_\varpi \cup \mathcal{O}_{\varepsilon\varpi} = 1_{M_2}^G$

(3) $\mathcal{O}_R = 1_{M_0}^G = \tilde{u}_{M_1}^G = u'^G_{M_2}$, *where* $\tilde{u}$ *and* u' *are representatives for the regular unipotent classes in* M_1 *and* M_2, *respectively.*

Proof. (1) Let $P \in \mathscr{P}(M_1)$. It is easy to check that all unipotent elements of maximal dimension in N_P are conjugate to u_1.

(2) Let $P \in \mathscr{P}(M_2)$ such that N_P is upper triangular. Then

$$N_P = \left\{ u = \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & z & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \, x, y, z \in F \right\},$$

$u \in N_P$ is in $\mathcal{O}_\tau$ if and only if $x^2 - yz \in \tau(F^*)^2$, $\tau = 1, \varepsilon, \varpi, \varepsilon\varpi$, and the set of $u \in N_P$ such that $x^2 - yz \neq 0$ is an open dense subset of N_P.

(3) is immediate because there is only one regular unipotent class in G. □

If $P \in \mathscr{P}(M)$, for some Levi subgroup M, and $\mathfrak{n}$ is the Lie algebra of N_P, set $\delta_P(mn) = |\det(\mathrm{Ad}\, m)_\mathfrak{n}|$. Let Θ_P be the character of the representation π_P of G which is induced (unitarily) from the one-dimensional representation $\delta_P^{-1/2}$ of P. The next lemma gives the relation between the functions $\hat{\mu}_\mathcal{O}$, $\mathcal{O} \in 1_M^G$ and the character Θ_P.

LEMMA 2.6. *With appropriate normalizations of measures, for* $X \in \mathfrak{g}_{\mathrm{reg}}$ *sufficiently close to* 0,

(1) $\hat{\mu}_R(X) = \Theta_P(\exp X)$, $\quad P \in \mathscr{P}(M_0)$.

(2) $\hat{\mu}_1(X) = \Theta_P(\exp X)$, $\quad P \in \mathscr{P}(M_1)$.

(3) $(1/2)\hat{\mu}_1(X) + \hat{\mu}_\varepsilon(X) + \hat{\mu}_\varpi(X) + \hat{\mu}_{\varepsilon\varpi}(X) = \Theta_P(\exp X)$, $\quad P \in \mathscr{P}(M_2)$.

Proof. This follows from Lemma 2.5, and I.8 of [MW]. □

REMARKS. (1) The choice of normalizations of measures on nilpotent orbits is made in Lemma 5.8.

(2) Explicit formulas for the characters Θ_P are given in van Dijk [D].

(3) Moeglin and Waldspurger [MW] have generalized a result of Rodier [Ro] and shown that, if $\mathcal{O}$ has maximal dimension among classes such that $c_\mathcal{O}(\pi) \neq 0$, $c_\mathcal{O}(\pi)$ is equal to the dimension of a certain degenerate Whittaker model.

(4) The values of the functions $\hat{\mu}_\tau$, $\tau = 0, \varepsilon, \varpi, \varepsilon\varpi$, at certain points in $\mathfrak{g}$ will be computed in section 7.

3. Supercuspidal characters and weighted orbital integrals

Throughout the remainder of this paper, Θ_π will denote the character of an irreducible supercuspidal representation π of G. In this section, we summarize Arthur's results relating Θ_π to weighted orbital integrals of matrix coefficients of π and giving a germ expansion for weighted orbital integrals.

Let M be a Levi subgroup of G. For $\gamma \in M$ and $f \in C_c^\infty(G)$, let $J_M(\gamma, f)$ be the weighted orbital defined in [A2, §5]. Note that $J_M(\gamma, f)$ is also well-defined if f is compactly supported modulo the centre of G, for example, if f is a matrix coefficient of π. A_M denotes the split component of M and M_{ell} is the set of γ in M which lie in some elliptic Cartan subgroup of M. If $\mathfrak{m}$ is the Lie algebra of M, let $D_M(\gamma) = \det(1 - \mathrm{Ad}(\sigma))_{\mathfrak{m}/\mathfrak{m}_\sigma}$, where σ is the semisimple part of γ. $D(\gamma)$ will often be used in place of $D_G(\gamma)$.

THEOREM 3.1. [A3]. *Suppose f is a finite sum of matrix coefficients of π. For $\gamma \in M_{\mathrm{ell}} \cap G_{\mathrm{reg}}$,*

$$(-1)^{\dim(A_M/A_G)}\Theta_\pi(f)|D(\gamma)|^{1/2}\Theta_\pi(\gamma) = J_M(\gamma, f),$$

where $\Theta_\pi(f) = \int_{A_G\backslash G} \Theta_\pi(x) f(x)\,dx$.

In section 5, more will be said about the definitions of the weighted orbital integrals and the necessary normalizations of measures.

In order to describe the germ expansion for weighted orbital integrals, we recall some definitions from [A2]. Suppose ϕ_1 and ϕ_2 are functions defined on an open subset Σ of M which contains an M-invariant neighbourhood of a point $\sigma \in M$. ϕ_1 is *(M, σ)-equivalent* to ϕ_2, $\phi_1(\gamma) \overset{(M,\sigma)}{\sim} \phi_2(\gamma)$, if $\phi_1(\gamma) - \phi_2(\gamma) = J_M^M(\gamma, h)$ for $\gamma \in \Sigma \cap U$, where U is a neighbourhood of σ in M, and $h \in C_c^\infty(M)$. Let $(\sigma\mathscr{U}_{M_\sigma})$ be the set of orbits in $\sigma\mathscr{U}_{M_\sigma}$ under conjugation by M_σ. $\gamma \mapsto J_M(\gamma, f)$ is a class function on M, so $J_M(\mathcal{O}, f)$ is well-defined for $\mathcal{O} \in (\sigma\mathscr{U}_{M_\sigma})$. Let $\mathscr{L}(M)$ be the set of Levi subgroups in G which contain M.

THEOREM 3.2. [A2, Prop. 9.1, Prop. 10.2]. (1) *There are uniquely determined (M, σ)-equivalence classes of functions $\gamma \mapsto g_M^G(\gamma, \mathcal{O})$, $\gamma \in \sigma M_\sigma \cap G_{\mathrm{reg}}$, parametrized by the classes $\mathcal{O} \in (\sigma\mathscr{U}_{G_\sigma})$, such that, for any $f \in C_c^\infty(G)$,*

$$J_M(\gamma, f) \overset{(M,\sigma)}{\sim} \sum_{L \in \mathscr{L}(M)} \sum_{\mathcal{O} \in (\sigma\mathscr{U}_{L_\sigma})} g_M^L(\gamma, \mathcal{O}) J_L(\mathcal{O}, f),$$

(2) *Let $t \in F^*$ and $\mathcal{O} \in (\mathscr{U}_G)$. Set $d^G(\mathcal{O}) = 1/2(\dim G - \mathrm{rank}\, G - \dim \mathcal{O})$. For $x = \exp X$, $X \in \mathfrak{g}$, define $x^t = \exp(tX)$. Let $\mathcal{O}^t \in (\mathscr{U}_G)$ be the class of u^t, where $u \in \mathcal{O}$.*

$$g_M^G(\gamma^t, \mathcal{O}^t) \overset{(M,1)}{\sim} |t|^{d^G(\mathcal{O})} \sum_{L \in \mathscr{L}(M)} \sum_{\delta \in (\mathscr{U}_L)} g_M^L(\gamma, \delta) c_L(\delta, t)[\delta^G : \mathcal{O}],$$

where the $c_L(\delta, t)$ are certain real-valued functions and $[\delta^G : \mathcal{O}]$ is 1 if $\mathcal{O} \in \delta^G$, 0 otherwise.

We finish with a lemma listing some properties of germs and weighted orbital integrals which will be used in later sections.

LEMMA 3.3. (1) *Let $\gamma \in M_{\text{ell}} \cap G_{\text{reg}}$. If l is the F-rank of $L \in \mathscr{L}(M)$ and $d(\mathrm{St}_L)$ is the formal degree of the Steinberg representation of L, then*

$$g_M^L(\gamma, 1) \overset{(M,1)}{\sim} \frac{(-1)^{(l-\dim A_M)}}{d(\mathrm{St}_L)} |D_L(\gamma)|^{1/2}.$$

(2) *Let $L \in \mathscr{L}(M)$ and $\gamma \in M$. Then*

$$J_L(\gamma^L, f) = \lim_{a \to 1} \sum_{L_1 \in \mathscr{L}(L)} r_L^{L_1}(\gamma, a) J_{L_1}(a\gamma, f), \qquad a \in A_{M,\text{reg}},$$

where $J_L(\gamma^L, f) \overset{\text{def}}{=} \sum_i J_L(\mathcal{O}_i, f)$, if $\gamma^L = \bigcup_i \mathcal{O}_i$. The constants $r_L^{L_1}(\gamma, a)$ appear in the definition of weighted orbital integrals and $A_{M,\text{reg}}$ is the set of elements in A_M whose centralizer in G equals M.

(3) *If f is a cusp form on G such that* supp *f is compact modulo A_G then, if $\gamma \in M$ is semisimple and $\gamma \notin M_{\text{ell}}$, $J_M(\gamma, f) = 0$.*

Proof. (1) [Mu, Prop. 3.7]. (2) [A2, Cor. 6.3]. (3) is due to Arthur. See [Mu, Prop. 3.9]. □

4. Preliminary results

Relations between the terms occurring in the two expansions for Θ_π are stated in Lemma 4.1. They are obtained from the homogeneity properties of the functions $\hat{\mu}_{\mathcal{O}}$ and g_M^G, together with vanishing of certain weighted orbital integrals of cusp forms. If M is a Levi subgroup of G, $d(\mathrm{St}_M)$ denotes the formal degree of the Steinberg representation of M.

LEMMA 4.1. *Let f be a finite sum of matrix coefficients of π such that $f(1) \neq 0$. Let $d(\pi)$be the formal degree of π.*

(1) *$c_{\{\text{id}\}}(\pi) = d(\pi)/d(\mathrm{St}_G)$.*

(2) *If Γ_0 is the Shalika germ associated to the unipotent class $\mathcal{O}_0$, and $X \in \mathfrak{g}_{\text{reg}}$ lies in a sufficiently small neighbourhood of 0 and is such that $\gamma = \exp X \in G_{\text{ell}}$, then*

$$\frac{d(\pi) J_G(\mathcal{O}_0, f)}{f(1)} \Gamma_0(\gamma) = c_0(\pi) |D(\gamma)|^{1/2} \hat{\mu}_0(X).$$

In particular, $c_0(\pi) = c J_G(\mathcal{O}_0, f)$ for some nonzero constant c.

(3) *Let M be a Levi subgroup. Suppose $X \in \mathfrak{g}_{\mathrm{reg}}$ is close to* 0 *and* $\gamma = \exp X \in M_{\mathrm{ell}}$. *Then*

$$|D(\gamma)|^{1/2} \sum_{\tau = 1,\varepsilon,\varpi,\varepsilon\varpi} c_\tau(\pi)\hat{\mu}_\tau(X) =$$

$$= \sum_{j=1,2} \frac{d(\pi)}{2f(1)d(\mathrm{St}_{M_j})} J_{M_j}(1, f)|D(\gamma)|^{1/2}\Theta_{P_j}(\gamma)$$

$$+(-1)^{(\dim(A_M)-1)}\frac{d(\pi)}{f(1)} \sum_{\tau=\varepsilon,\varpi,\varepsilon\varpi} J_G(\mathcal{O}_\tau, f)g_M^G(\gamma, \mathcal{O}_\tau),$$

where $P_j \in \mathscr{P}(M_j)$, $j = 1, 2$, and Θ_{P_j} is as defined in section 2.

(4) $c_R(\pi) = d(\pi)J_{M_0}(1, f)/8f(1)$.

Proof. Assume that $g_M^M(\gamma, \mathcal{O})$, $\gamma \in G_{\mathrm{reg}} \cap M$, is equal to the Shalika germ corresponding to the orbit $\mathcal{O} \in (\sigma\mathscr{U}_{M_\sigma})$ for all Levi subgroups M. Then it follows immediately from Arthur's derivation of the germ expansion for weighted orbital integrals in the proof of Proposition 9.1 of [A2] that the germ expansion is actually an equality, rather than an (M, σ)-equivalence. Thus the results of the lemma are given as equalities.

Note that certain of the coefficients in the germ expansion for $J_M(\gamma, f)$ vanish because f is a cusp form. Indeed, if $M = M_1$ or M_2 and $\mathcal{O}$ is the nontrivial unipotent class in M, then $\mathcal{O} = 1_{M_0}^M$. By Lemma 3.3(2),

$$J_M(\mathcal{O}, f) = \lim_{a\to 1} (J_M(a, f) + r_M^G(1, a)J_G(a, f)), \qquad a \in A_{M_0,\mathrm{reg}}.$$

But $a \in A_{M_0,\mathrm{reg}}$ is not elliptic in M or G. Thus, by Lemma 3.3(3), $J_M(a, f) = J_G(a, f) = 0$, which implies that $J_M(\mathcal{O}, f) = 0$. Similarly, it follows from Lemma 2.5(1), (3) and Lemma 3.3(2), (3) that $J_G(\mathcal{O}_1, f) = J_G(\mathcal{O}_R, f) = 0$.

Let $\gamma = \exp X$ with X as in (2). Note that $\Theta_\pi(f) = f(1)/d(\pi)$ [Mu, Lemma 3.6]. Since $J_G(\gamma, f)$ is the ordinary orbital integral of f at γ, $g_G^G(\gamma, \mathcal{O}) = \Gamma_{\mathcal{O}}(\gamma)$, the Shalika germ corresponding to $\mathcal{O} \in (\mathscr{U}_G)$. Matching of terms having the same homogeneity in the two expansions for Θ_π, results in:

$$c_{\mathcal{O}}(\pi)|D(\gamma)|^{1/2}\hat{\mu}_{\mathcal{O}}(X) = \frac{J_G(\mathcal{O}, f)d(\pi)}{f(1)}\Gamma_{\mathcal{O}}(\gamma), \qquad \mathcal{O} = \mathrm{id}, \mathcal{O}_0.$$

Note that $\hat{\mu}_{\mathrm{id}} \equiv 1$, and recall Rogawski's formula in [Rog] for $\Gamma_{\mathrm{id}}(\gamma)$. This proves (1) and the first part of (2). Since $\dim \mathcal{O}_0 = 4 < \dim G - \dim M_i = 6$, $i = 1, 2$, it follows from Corollary 1, p. 311 of [HC] that Γ_0 is zero at any nonelliptic point in G_{reg}. The germs $\{\Gamma_{\mathcal{O}} \mid \mathcal{O} \in (\mathscr{U}_G)\}$ are linearly independent on any open neighbourhood of 1 intersected with G_{reg} [HC, Lemma 24], so there are points in $G_{\text{reg}} \cap G_{\text{ell}}$ where $\Gamma_0 \neq 0$. If $J_G(\mathcal{O}_0, f) \neq 0$ and $\gamma = \exp X$ is taken to be one of those points, then the left side, hence also the right side, of the equation in (2) is nonzero, so $c_{\mathcal{O}}(\pi) = d(\pi)J_G(\mathcal{O}_0, f)\Gamma_0(\gamma)/f(1)\hat{\mu}_0(X) \neq 0$. If $J_G(\mathcal{O}_0, f) = 0$, then, for any M, $J_M(\gamma, f)$, $\gamma \in M_{\text{ell}}$, has no term with the same homogeneity as $|D|^{1/2}\hat{\mu}_{\mathcal{O}}$ in its germ expansion. Since $\hat{\mu}_{\mathcal{O}} \not\equiv 0$, this implies $c_{\mathcal{O}}(\pi) = 0$.

To prove (3) we use Van Dijk's formula for $\Theta_{P_j}, j = 1, 2$ and the formula for $g_M^G(\gamma, 1)$ to write the right hand side of (3) as:

$$(-1)^{(\dim(A_M)-1)} \frac{d(\pi)}{f(1)} \times \left(\sum_{\{L \in \mathcal{L}(M), \dim A_L = 2\}} J_L(1, f) g_M^L(\gamma, 1) + \sum_{\tau = \varepsilon, \varpi, \varepsilon\varpi} J_G(\mathcal{O}_\tau, f) g_M^G(\gamma, \mathcal{O}_\tau) \right)$$

Taking into account the vanishing of the coefficients mentioned at the beginning of the proof, the above expression is the term in the germ expansion having the same homogeneity as $|D|^{1/2}\hat{\mu}_\tau$, $\tau = 1, \varepsilon, \varpi, \varepsilon\varpi$.

To obtain (4), note that homogeneity of the terms in $\Theta_\pi(a)$, $a = \exp X \in A_{M_0,\text{reg}}$, results in

$$\frac{d(\pi)}{f(1)} J_{M_0}(1, f) = c_R(\pi)|D(\gamma)|^{1/2}\hat{\mu}_R(X).$$

From Lemma 2.6(1) and [D], $\hat{\mu}_R(X) = 8|D(\gamma)|^{-1/2}$. □

In order to compute the $c_\tau(\pi)$'s, the values $J_{M_j}(1, f)$, $j = 0, 1, 2$, $J_G(\mathcal{O}_\tau, f)$, $\tau = 0, \varepsilon, \varpi, \varepsilon\varpi$ are required, along with enough knowledge of the values of the functions $\hat{\mu}_\tau$ to separate the terms in Lemma 4.1(3) and to determine the constant c in Lemma 4.1(2). The remainder of this paper is devoted to obtaining much of this information.

5. Weight factors and normalization of measures

The weight factors v_M, the integrals $J_M(\gamma, f)$, the formal degree $d(\pi)$, and the germs $g_M^G(\gamma, \mathcal{O})$ depend on various volumes and measures. We choose normalizations of measures so that Theorem 3.1 holds. In Lemma 5.4, we compute $v_M(x)$ for upper triangular unipotent elements x in G. This leads to integral formulas for certain weighted orbital integrals in Proposition 5.5. At the end of the section, the invariant measures on the unipotent classes $(\mathcal{U}_G)$ are specified. In addition, Lemma 5.8 gives the measures on the nilpotent orbits in $\mathfrak{g}$ required for Lemma 2.6.

The group $K = GSp_4(\mathfrak{O}_F)$ sia special maximal compact subgroup of G which is in good position relative to the Levi subgroups M_0, M_1 and M_2 (see (2.4)). Let dx be the Haar measure on G which assigns volume 1 to K. The Haar measure dk on K is taken to be the restriction of dx to K. If $P = MN$ is a parabolic subgroup with $G = KP$, the measures on M and N are normalized so that the measures of $M \cap K$ and $N \cap K$ equal one. With these choices,

$$\int_G f(x)\,dx = \int_K \int_M \int_N f(mnk)\,dk\,dm\,dn, \qquad f \in C_c^\infty(G).$$

Given a Levi subgroup M of G, let $\mathbf{M}$ be the Levi subgroup of GSp_4 such that $M = \mathbf{M}(F)$, and let $X(\mathbf{M})_F$ be the group of characters of $\mathbf{M}$ which are defined over F. The real vector space $\boldsymbol{a}_M = \mathrm{Hom}(X(\mathbf{M})_F, \mathbf{R})$ plays a role in the definition of the weight factor v_M in the weighted orbital integral J_M. In fact, $J_M(\gamma, f) = \int_{A_M\backslash G} f(x^{-1}\gamma x)v_M(x)\,dx$, $\gamma \in M_{\mathrm{ell}} \cap G_{\mathrm{reg}}$, depends on invariant measures on $A_M\backslash G$ and $\boldsymbol{a}_M/\boldsymbol{a}_G$. $\Theta_\pi(f)$ depends on an invariant measure on $A_G\backslash G$. In order for Theorem 3.1 to hold, certain compatibility requirements must be satisfied by these measures [A2, p. 5]. In the next paragraphs, we define a measure on $\boldsymbol{a}_M$ an 1 state the conditions relating the various measures.

For convenience, we assume that $M = M_l$, $l = 0, 1, 2$ or $M = G$. If $P \in \mathscr{P}(M)$ and $x = n_P(x)m_P(x)k(x)$, with $n_P(x) \in N_P$, $m_P(x) \in M$ and $k \in K$, define $H_P(x) = H_M(m_P(x))$. The function $H_M\colon M \to \boldsymbol{a}_M$ is given by:

$$e^{\langle H_M(x), \chi\rangle} = |\chi(m)|, \qquad m \in M, \quad \chi \in X(\mathbf{M})_F.$$

Let $\boldsymbol{a}_M^G$ be the kernel of the canonical map from $\boldsymbol{a}_M$ onto $\boldsymbol{a}_G$. There is a compatible embedding of $\boldsymbol{a}_G$ into $\boldsymbol{a}_M$ resulting from the embeddings of $X(\mathbf{M})_F$ and $X(\mathbf{G})_F$ into the character groups $X(A_\mathbf{M})$ and $X(A_\mathbf{G})$, respectively. Therefore, $\boldsymbol{a}_M = \boldsymbol{a}_M^G \oplus \boldsymbol{a}_G$. The restriction of a fixed Weyl-invariant norm $\|\cdot\|$ on $\boldsymbol{a}_{M_0}$ to $\boldsymbol{a}_M$ yields a measure on $\boldsymbol{a}_M$. Let $\kappa_M = K \cap A_M$. The function H_M maps A_M/κ_M bijectively onto a lattice in $\boldsymbol{a}_M$. The measure of κ_M in A_M must equal the volume of $\boldsymbol{a}_M/H_M(A_M)$. This fixes a Haar measure on A_M. The measures on $A_M\backslash G$, $A_G\backslash G$ and $\boldsymbol{a}_M^G \simeq \boldsymbol{a}_M/\boldsymbol{a}_G$ are the ones induced by those on G, A_M, A_G, $\boldsymbol{a}_M$, and $\boldsymbol{a}_G$.

Let $\boldsymbol{a}_{M_0}$ be realized in such a way that

$$H_{M_0}(\mathrm{diag}(a_1, a_2, a_3, a_1^{-1}a_2a_3)) = (\log|a_1|, \log|a_2|, \log|a_3|, \log|a_1^{-1}a_2a_3|).$$

The set $\{e_1 = (1, 0, 0, -1), e_2 = (0, 1, -1, 0), e_3 = (1, 1, 1, 1)\}$ is a basis of $\boldsymbol{a}_{M_0}$ and it is easy to check that an inner product $(\ ,\ )$ on $\boldsymbol{a}_{M_0}$ is Weyl-invariant if and only if $(e_i, e_j) = 0$, $i \neq j$ and $(e_1, e_1) = (e_2, e_2)$. Let $c_1 = (e_1, e_1)$ and $c_2 = (e_3, e_3)$. The measure on $M_0 = A_{M_0}$ has been normalized so that $\mathrm{vol}(\kappa_{M_0}) = 1$. Thus c_1 and c_2 must be chosen so that $\mathrm{vol}(\boldsymbol{a}_{M_0}/H_{M_0}(A_{M_0})) = 1$. We choose $c_1 = 2\log^{-2} q$

and $c_2 = \log^{-2} q$. The formal degree $d(\mathrm{St}_M)$ of the Steinberg representation of M appearing in $g_L^M(\gamma, 1)$ depends on $\mathrm{vol}(\kappa_M)$.

LEMMA 5.1.

(1) $\mathrm{vol}(\kappa_{M_0}) = 1$
(2) $\mathrm{vol}(\kappa_{M_1}) = \sqrt{2}$
(3) $\mathrm{vol}(\kappa_{M_2}) = 1$
(4) $\mathrm{vol}(\kappa_G) = 1$

Proof. (1) Let $M = M_0$. $\{u_1 = c_1^{-1/2} e_1,\ u_2 = c_1^{-1/2} e_2,\ u_3 = c_2^{-1/2}\}$ is an orthonormal basis of $\boldsymbol{a}_{M_0}$. $H_M(A_M)$ is generated over $\mathbf{Z}$ by $\log q(1,0,0,-1)$, $\log q(0,1,0,1)$ and $\log q(0,0,1,1)$. It follows that $\mathrm{vol}(\boldsymbol{a}_M/H_M(A_M)) = c_1 c_2^{1/2} \log^3 q/2 = 1$.

(2) Let $M = M_1$. The embedding of $\boldsymbol{a}_M$ into $\boldsymbol{a}_{M_0}$ is given by:

$$(x_1, x_2, x_2 - x_1) \mapsto (x_1, x_2/2, x_2/2, x_2 - x_1).$$

Furthermore, the image of $H_M(A_M)$ in $\boldsymbol{a}_{M_0}$ is generated by $\log q(1,0,0,1)$ and $\log q(0,1,1,2)$. Since $\{u_1, u_3\}$ is orthonormal in $\boldsymbol{a}_M$, $\mathrm{vol}(\boldsymbol{a}_M/H_M(A_M)) = (c_1 c_2)^{1/2} \log^2 q = \sqrt{2}$.

(3) and (4) are proved in a similar manner. □

The weight $v_M(x)$, $x \in G$, is the volume of the convex hull of the projection of the points $\{-H_P(x) \mid P \in \mathscr{P}(M)\}$ onto $\boldsymbol{a}_M^G$. Some definitions are required to give a formula for $v_M(x)$. Let $P \in \mathscr{P}(M)$. The roots of (P, A_M) are viewed as characters of A_M or as elements of the dual space $\boldsymbol{a}_M^*$ of $\boldsymbol{a}_M$. If Δ_P is the set of simple roots of (P, A_M), and $\alpha \in \Delta_P$, the co-root $\alpha^\vee \in \boldsymbol{a}_M^G$ is defined as follows. If $P_0 \in \mathscr{P}(M_0)$ and $P_0 \in P$, there is exactly one root $\beta \in \Delta_{P_0}$ such that $\beta|\boldsymbol{a}_{M_0} = \alpha$. $\alpha^\vee$ is the projection of $\beta^\vee \in \mathrm{Hom}(X(A_{M_0}), \mathbf{Z}) \subset \boldsymbol{a}_{M_0}$ onto $\boldsymbol{a}_M^G$. The lattice $\mathbf{Z}(\Delta_P)$ in $\boldsymbol{a}_M^G$ generated by $\Delta_P^\vee = \{a^\vee \mid \alpha \in \Delta_P\}$ is independent of the choice of $P \in \mathscr{P}(M)$ [A3, p. 12]. Define $\eta_M = \mathrm{vol}(\boldsymbol{a}_M^G/\mathbf{Z}(\Delta_P^\vee))$ and $\theta_P(\lambda) = \eta_M^{-1} \prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee)$, for $\lambda \in i\boldsymbol{a}_M^*$. Then, for $x \in G$, [A1, p. 36]:

$$v_M(x) = \lim_{\lambda \to 0} \sum_{P \in \mathscr{P}(M)} e^{-\lambda(H_P(x))} \Theta_P(\lambda)^{-1}, \qquad \lambda \in i\boldsymbol{a}_M^*.$$

For computations, the formula given in [A1, p. 46] is useful:

$$v_M(x) = \frac{1}{r!} \sum_{P \in \mathscr{P}(M)} (-\lambda(H_P(x))^r \Theta_P(\lambda)^{-1}, \qquad \text{where } r = \dim(A_M/A_G). \tag{5.2}$$

Fix $P_0 \in \mathscr{P}(M_0)$ having an upper triangular unipotent radical. The characters $\alpha = (1,-1,1,-1)$, $\beta = (0,2,-2,0)$, $\alpha + \beta$, $2\alpha + \beta \in \boldsymbol{a}_M^*$ are the roots of

(P_0, A_{M_0}). The corresponding co-roots are $\alpha^\vee = (1, -1, 1, -1)$, $\beta^\vee = (0, 1, -1, 0)$, $(\alpha + \beta)^\vee = (1, 1, -1, -1)$ and $(2\alpha + \beta)^\vee = (1, 0, 0, -1)$.

LEMMA 5.3. *With measures normalized as above,*

(1) $\eta_{M_0} = 2\log^{-2} q$
(2) $\eta_{M_1} = \sqrt{2}\log^{-1} q$
(3) $\eta_{M_2} = \log^{-1} q$

Proof. (1) follows from $\alpha^\vee = c_1^{1/2}(u_1 - u_2)$ and $\beta^\vee = c_1^{1/2}u_2$.

(2) Let $M = M_1$ and let $P \in \mathscr{P}(M)$ have upper triangular unipotent radical. $\alpha|A_M$ is the simple root of (P, A_M). The image in $\boldsymbol{a}_{M_0}$ of the projection of $\alpha^\vee$ onto $\boldsymbol{a}_M^G$ is $(1, 0, 0, -1) = c_1^{1/2}u_1$. Thus $\eta_M = c_1^{1/2}$.

(3) Let $M = M_2$. The image in $\boldsymbol{a}_{M_0}$ of the projection of $\beta^\vee$ onto $\boldsymbol{a}_M^G$ is $(1/2)(1, 1, -1, -1) = (c_1^{1/2}/2)(u_1 - u_2)$ and $(1/\sqrt{2})(u_1 - u_2)$ has norm one in $\boldsymbol{a}_M^G$. Therefore $\eta_M = (c_1/2)^{1/2}$. □

For any integer $d \geqslant 1$, define

$$A_{M,d} = \{a = \text{diag}(a_1, a_2, a_3, a_4) \in A_{M,\text{reg}} |\, |a_i - 1| = q^{-d} \forall i,$$
$$|a_i - a_j| = q^{-d} \text{ if } a_i \neq a_j\}.$$

If $x \in F^*$, $v(x)$ is defined by $|x| = q^{-v(x)}$. We now compute $v_M(n)$ for certain $n \in \mathscr{U}_G$.

LEMMA 5.4. *For $M = M_l$, $l = 0, 1, 2$, choose $P = MN \in \mathscr{P}(M)$ such that N is upper triangular. Let $a \in A_{M,d}$.*

(1) *If $M = M_2$ and $n \in N$ is defined by $u = a^{-1}n^{-1}an$ for*

$$u = \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & z & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in N \cap K,$$

then $v_M(n) = \log^{-1} q \log(\max\{1, q^d|x|, q^d|y|, q^d|z|, q^{2d}|x^2 - yz|\})$.

(2) *If $M = M_1$ and $n \in N$ is defined by $u = a^{-1}n^{-1}an$ for*

$$u = \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix} \in N \cap K,$$

then $v_M(n) = \sqrt{2}\log^{-1} q \log(\max\{1, q^d|x|, q^d|y|, q^d|z|\})$.

(3) *If $M = M_0$ and $n \in N$ is defined by $u = a^{-1}n^{-1}an$ for*

$$u = \begin{bmatrix} 1 & w & x & y \\ 0 & 1 & z & x - wz \\ 0 & 0 & 1 & -w \\ 0 & 0 & 0 & 1 \end{bmatrix} \in N \cap K,$$

such that $w, z \neq 0$, then, for large d,

$$v_M(n) = 2[7d^2 - 4d(v(w) + v(z)) + 2(v(w) + v(z))^2 - v(z)^2].$$

(4) *Let $M = M_0$ and define u as in* (3). *If $\sigma \in A_{M_1,t}$ for some positive integer t and $b = a\sigma$, let $n \in N$ be defined by $u = b^{-1}n^{-1}bn$. Define*

$$\begin{aligned}
\tilde{w} &= (1 - b_1^{-1}b_2)^{-1}w \\
\tilde{x} &= (1 - b_1^{-1}b_3)^{-1}(x + b_1^{-1}b_2(1 - b_1^{-1}b_2)^{-1}wz) \\
\tilde{y} &= (1 - b_1^{-1}b_4)^{-1}(y + (1 - b_1^{-1}b_2)^{-1}(1 - b_1^{-1}b_3)^{-1}b_1^{-1}b_2 w \\
&\qquad \times ((1 - b_2^{-1}b_3)x - wz)) \\
\mathscr{A} &= \max\{0, \ -v(\tilde{w}), \ -v(\tilde{x}), \ -v(\tilde{y})\} \\
\mathscr{B} &= \max\{0, \ -2v(\tilde{w}), \ -v(\tilde{x}\tilde{w} + \tilde{y})\} \\
\mathscr{C} &= \max\{0, \ -v(\tilde{w})\}
\end{aligned}$$

If d is sufficiently large,

$$v_M(n) = 2d\mathscr{A} - 2(\mathscr{A} - \mathscr{B})^2 + 4\mathscr{C}(\mathscr{B} - \mathscr{C}) - 2\mathscr{A}v(z).$$

Proof. The following is useful for computing $H_P(x)$, $x \in G$. Let $P_0 = M_0 N \in \mathscr{P}(M_0)$ with N upper triangular. Suppose $x = nak$, $n \in N$, $a = \operatorname{diag}(a_1, a_2, a_3, a_4) \in A_{M_0}$, $k \in K$. Then, for $1 \leqslant j \leqslant 4$, $|a_j \cdots a_4|$ is equal to the maximum of the norms of the determinants of all $5 - j \times 5 - j$ matrices which can be formed from the last $5 - j$ rows of x. For $P \supset P_0$, $H_P(x)$ is the projection of $H_{P_0}(x)$ onto $\mathfrak{a}_M$.

We begin by computing $H_Q(n)$, $Q \in \mathscr{P}(M_0)$, for

$$n = \begin{bmatrix} 1 & \tilde{w} & \tilde{x} & \tilde{y} \\ 0 & 1 & \tilde{z} & \tilde{x} - \tilde{w}\tilde{z} \\ 0 & 0 & 1 & -\tilde{w} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in N.$$

Modulo A_{M_0}, the Weyl group W of G is equal to

$$\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, (s_1s_2)^2\},$$

where

$$s_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For each $Q \in \mathscr{P}(M_0)$, there exists $s \in W$ such that $N = N_Q^s = sN_Qs^{-1}$. If $n = n_Qa_Qk_Q$, with $n_Q \in N_Q$, $a_Q \in A_{M_0}$ and $k_Q \in K$, then $n^s = n_Q^s a_Q^s k_Q^s$ satisfies $n_Q^s \in N$, $a_Q^s \in A_{M_0}$, and $k_Q^s \in K$ (since s_1, $s_2 \in K$). Thus $H_Q(n) = H_{M_0}(a_Q) = H_{P_0}(n^s)^{s^{-1}}$, where $H_{M_0}(a)^s = H_{M_0}(a^s)$, $a \in A_{M_0}$, $s \in W$.

Each group $Q \in \mathscr{P}(M_0)$ is identified by the set Δ_Q of simple roots of (Q, A_{M_0}). If $\Delta_Q = \{-\alpha, -\beta\}$, then $Q = \bar{P}_0$ is the parabolic opposite to P_0, $N = N_Q^s$, $s = (s_1s_2)^2$, and

$$n^s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\tilde{w} & 1 & 0 & 0 \\ -\tilde{x} + \tilde{w}\tilde{z} & -\tilde{z} & 1 & 0 \\ -\tilde{y} & -\tilde{x} & \tilde{w} & 1 \end{bmatrix}.$$

Suppose $a_Q^s = \operatorname{diag}(a_1, a_2, a_3, a_4)$. Then $H_Q(n) = H_{M_0}(a_Q) = (\log|a_4|, \log|a_3|, \log|a_2|, \log|a_1|)$.

Applying the comment at the beginning of the proof, we obtain

$$|a_4| = \max\{1, |\tilde{w}|, |\tilde{x}|, |\tilde{y}|\}$$
$$|a_3a_4| = \max\{1, |\tilde{z}|, |\tilde{x} - \tilde{w}\tilde{z}|, |\tilde{w}(\tilde{x} - \tilde{w}\tilde{z}) - \tilde{y}|, |\tilde{x}(\tilde{x} - \tilde{w}\tilde{z}) - \tilde{y}\tilde{z}|\}.$$

Note that $|\det a_Q| = |\det n| = 1$. This implies $|a_1| = |a_4|^{-1}$ and $|a_2| = |a_3|^{-1}$. For convenience, let $X = X(\tilde{w}, \tilde{x}, \tilde{y}, \tilde{z})$ be the quantity appearing as $|a_3a_4|$. Then

$$H_Q(n) = \log(\max\{1, |\tilde{w}|, |\tilde{x}|, |\tilde{y}|\})\alpha^\vee + \log(X)\beta^\vee.$$

In the case $\Delta_Q = \{\alpha, -2\alpha - \beta\}$, $n = n_1 n_2$, where

$$n_1 = \begin{bmatrix} 1 & \tilde{w} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\tilde{w} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in N_Q$$

and

$$n_2 = \begin{bmatrix} 1 & 0 & \tilde{x} - \tilde{w}\tilde{z} & \tilde{y} - \tilde{w}(\tilde{x} - \tilde{w}\tilde{z}) \\ 0 & 1 & \tilde{z} & \tilde{x} - \tilde{w}\tilde{z} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In addition, $N = N_Q^s$, $s = s_1 s_2 s_1$ and

$$n_2^s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\tilde{x} + \tilde{w}\tilde{z} & -\tilde{y} + \tilde{w}(\tilde{x} - \tilde{w}\tilde{z}) & 1 & 0 \\ -\tilde{z} & -\tilde{x} + \tilde{w}\tilde{z} & 0 & 1 \end{bmatrix}.$$

It follows that, if $a_Q^s = \operatorname{diag}(a_1, a_2, a_3, a_4)$,

$$|a_4| = \max\{1, |\tilde{z}|, |\tilde{x} - \tilde{w}\tilde{z}|\} \qquad \text{and} \qquad |a_3 a_4| = X,$$

and

$$\begin{aligned} H_Q(n) = H_Q(n_2) &= \log|a|^4(0, 1, -1, 0) + \log|a_3|(1, 0, 0, -1) \\ &= \log(X)(2\alpha + \beta)^\vee - \log(\max\{1, |\tilde{z}|, |\tilde{x} - \tilde{w}\tilde{z}|\})\alpha^\vee. \end{aligned}$$

For other Δ_Q's, $H_Q(n)$ is obtained similarly. The values of $H_Q(n)$, $Q \in \mathscr{P}(M_0)$ are:

Δ_Q	$H_Q(n)$
$\{\alpha, \beta\}$	0
$\{-\alpha, -\beta\}$	$\log(\max\{1, \lvert\tilde{w}\rvert, \lvert\tilde{x}\rvert, \lvert\tilde{y}\rvert\})\alpha^\vee + \log(X)\beta^\vee$
$\{\alpha, -2\alpha-\beta\}$	$-\log(\max\{1, \lvert\tilde{z}\rvert, \lvert\tilde{x} - \tilde{w}\tilde{z}\rvert\})\alpha^\vee + \log(X)(2\alpha + \beta)^\vee$
$\{-\alpha, 2\alpha + \beta\}$	$\log(\max\{1, \lvert\tilde{w}\rvert\})\alpha^\vee$
$\{\alpha + \beta, -\beta\}$	$\log(\max\{1, \lvert\tilde{z}\rvert\})\beta^\vee$
$\{-\alpha - \beta, \beta\}$	$\log(\max\{1, \lvert\tilde{w}\rvert, \lvert\tilde{x}\rvert, \lvert\tilde{y}\rvert\})(\alpha + \beta)^\vee - \log(\max\{1, \lvert\tilde{w}\rvert^2, \lvert\tilde{x}\tilde{w} + \tilde{y}\rvert\})\beta^\vee$
$\{-\alpha - \beta, 2\alpha + \beta\}$	$\log(\max\{1, \lvert\tilde{z}\rvert, \lvert\tilde{x} - \tilde{w}\tilde{z}\rvert\})(\alpha + \beta)^\vee - \log(\max\{1, \lvert\tilde{z}\rvert\})(2\alpha + \beta)^\vee$
$\{\alpha + \beta, -2\alpha - \beta\}$	$-\log(\max\{1, \lvert\tilde{z}\rvert\})(\alpha + \beta)^\vee + \log(\max\{1, \lvert\tilde{w}\rvert^2, \lvert\tilde{x}\tilde{w} + \tilde{y}\rvert\})(2\alpha + \beta)^\vee$

Here $X = \max\{1, |\tilde{z}|, |\tilde{x} - \tilde{w}\tilde{z}|, |\tilde{w}(\tilde{x} - \tilde{w}\tilde{z}) - \tilde{y}|, |\tilde{x}(\tilde{x} - \tilde{w}\tilde{z}) - \tilde{y}\tilde{z}|\}$.

Now let $M = M_2$. $\mathscr{P}(M) = \{P, \bar{P}\}$, where N_P is taken to be upper triangular. If $n \in N_P$, then n can be written as above with $\tilde{w} = 0$. If $Q \in \mathscr{P}(M_0)$ with $\Delta_Q = \{-\alpha, -\beta\}$, then $\bar{P} \subset Q$ and $H_{\bar{P}}(n)$ is equal to the projection of $H_Q(n)$ onto $\mathfrak{a}_M$, that is, $H_{\bar{P}}(n) = \log(X)(1, -1)$. $X(0, \tilde{x}, \tilde{y}, \tilde{z}) = \max\{1, |\tilde{x}|, |\tilde{y}|, |\tilde{z}|, |\tilde{x}^2 - \tilde{y}\tilde{z}|\}$ and $v_M(n)$ is the volume in $\mathfrak{a}_M^G$ of the convex hull of $H_P(n) = 0$ and $H_{\bar{P}}(n)$. It now follows, using Lemma 5.3(3), that

$$v_M(n) = \log^{-1} q \log(\max\{1, |\tilde{x}|, |\tilde{y}|, |\tilde{z}|, |\tilde{x}^2 - \tilde{y}\tilde{z}|\}).$$

To finish the proof of (1), $v_M(n)$ must be expressed in terms of the entries of the matrix $u = a^{-1}n^{-1}an$, $a = \mathrm{diag}(a_1, a_1, a_2, a_2) \in A_{M,d}$. It is easily verified that $\tilde{r} = (1 - a_1^{-1}a_2)^{-1} r$ for $r = x, y, z$.

The proof of (2) involves the same type of argument as that of (1) and is omitted.

Returning to the proofs of (3) and (4), let $M = M_0$, $b = a\sigma = \mathrm{diag}(b_1, b_2, b_3, b_4) \in A_{M,\mathrm{reg}}$, and $u = b^{-1}n^{-1}bn$. Then

$$\begin{aligned}
\tilde{w} &= (1 - b_1^{-1}b_2)^{-1} w,\\
\tilde{x} &= (1 - b_1^{-1}b_3)^{-1}(x + b_1^{-1}b_2(1 - b_1^{-1}b_2)^{-1} wz),\\
\tilde{z} &= (1 - b_2^{-1}b_3)^{-1} z,\\
\tilde{x} - \tilde{w}\tilde{z} &= (1 - b_1^{-1}b_3)^{-1}(x - (1 - b_2^{-1}b_3)^{-1} wz),\\
\tilde{y} &= (1 - b_1^{-1}b_4)^{-1}(y + (1 - b_1^{-1}b_2)^{-1}(1 - b_1^{-1}b_3)^{-1} b_1^{-1}b_2 w\\
&\quad ((1 - b_2^{-1}b_3)x - wz)).
\end{aligned}$$

If d is large enough,

$$\begin{aligned}
\max\{1, |\tilde{z}|\} &= q^d |z|\\
\max\{1, |\tilde{z}|, |\tilde{x} - \tilde{w}\tilde{z}|\} &= q^d |z| \max\{1, |\tilde{w}|\}\\
X &= q^d |z| \max\{1, |\tilde{w}|^2, |\tilde{x}\tilde{w} + \tilde{y}|\}
\end{aligned}$$

It is now possible to express each $H_Q(n)$, $Q \in \mathscr{P}(M)$, in terms of $\mathscr{A}, \mathscr{B}, \mathscr{C}, d$ and $|z|$. For example,

$$H_Q(n) = \log q \mathscr{A} \alpha^\vee + \log q (d - v(z) + \mathscr{B}) \beta^\vee,$$

for $\Delta_Q = \{-\alpha, -\beta\}$. Let $\lambda = (i\lambda_1, i\lambda_2, i\lambda_3, i(-\lambda_1 + \lambda_2 + \lambda_3)) \in i\mathfrak{a}_M^*$, where λ_1, λ_2

and λ_3 are distinct real numbers. After substituting η_M and $H_Q(n)$, $Q \in \mathscr{P}(M)$ into (5.2) and simplifying, we obtain $v_M(n)$.

To prove (3), observe that if $u = a^{-1}n^{-1}an$, $a \in A_{M,d}$, then $\tilde{w}$, $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ can be expressed in terms of w, x, y and z as above, with b_j replaced by a_j, $1 \leqslant j \leqslant 4$. Then, for d sufficiently large,

$$|\tilde{w}| = q^d|w|, \quad |\tilde{x}| = |\tilde{x} - \tilde{w}\tilde{z}| = q^{2d}|wz|, \quad |\tilde{z}| = q^d|z|$$

$$|\tilde{y}| = |\tilde{x}\tilde{w} + \tilde{y}| = |\tilde{w}(\tilde{x} - \tilde{w}\tilde{z}) - \tilde{y}| = q^{3d}|w^2z|$$

$$|\tilde{x}(\tilde{x} - \tilde{w}\tilde{z}) - \tilde{y}\tilde{z}| = q^{4d}|wz|^2$$

This allows us to write $H_Q(n)$, $Q \in \mathscr{P}(M)$, in terms of $v(w)$, $v(z)$, and d. For example, $X = q^{4d}|wz|^2$, and $\max\{1, |\tilde{w}|, |\tilde{x}|, |\tilde{y}|\} = q^{3d}|w^2z|$, which implies that

$$H_Q(n) = \log q(3d - 2v(w) - v(z))\alpha^\vee + \log q(4d - 2v(w) - 2v(z))\beta^\vee,$$

for $\Delta_Q = \{-\alpha, \ -\beta\}$. Proceeding as for (4), let $\lambda = (i\lambda_1, \ i\lambda_2, \ i\lambda_3, \ i(-\lambda_1 + \lambda_2 + \lambda_3)) \in i\mathfrak{a}_M^*$ and substitute η_M and $H_Q(n)$, $Q \in \mathscr{P}(M)$, into (5.2). □

If f is a locally constant function on G, define $f_K(x) = \int_K f(k^{-1}xk)\,dk$, $x \in G$.

PROPOSITION 5.5. *Let f be a cusp form on G such that* $\operatorname{supp} f \subset KA_G$. *For $M = M_l$, $l = 0, 1, 2$, choose $P \in \mathscr{P}(M)$ such that N is upper triangular. Let $u \in N$ be as in Lemma* 5.4(3), (2) *and* (1), *respectively.*

(1) *If $a \in A_{M_2,d}$, and d is sufficiently large, then*

$$J_{M_2}(a, f) = J_{M_2}(1, f) + \frac{q^{-3d}}{q^3 - 1} f(1) + \frac{q^{-d}}{q - 1} J_G(\mathcal{O}_0, f),$$

where $J_G(\mathcal{O}, f)$ is given by Lemma 5.7.

(2) $J_{M_2}(1, f) = -\int_N f_K(u)v(x^2 - yz)\,du.$

(3) *If $a \in A_{M_1,d}$, and d is sufficiently large, then*

$$J_{M_1}(a, f) = J_{M_1}(1, f) + \frac{\sqrt{2}q^{-3d}}{q^3 - 1} f(1).$$

(4) $J_{M_1}(1, f) = -\sqrt{2}\int_N f_K(u)\min\{v(x), v(y), v(z)\}\,du.$

(5) $J_{M_0}(1, f) = 2\int_N f_K(u)(2(v(w) + v(z))^2 - v(z)^2)\,du.$

(6) *Let $\sigma \in A_{M_1,t}$, $b \in A_{M_0,\mathrm{reg}}$, and $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be as in Lemma* 5.4(4). *Then*

$$J_{M_0}(\sigma, f) = \int_N f_K(\sigma u)(-2(\mathscr{A} - \mathscr{B})^2 + 4\mathscr{C}(\mathscr{B} - \mathscr{C}) - 2\mathscr{A}v(z)\,du.$$

Proof. Recall from p. 254 of [A2] that

$$J_M(1, f) = \lim_{a\to 1} \textstyle\sum_{L\in\mathscr{L}(M)} r_M^L(1, a) J_L(a, f),\ a \in A_{M,\mathrm{reg}},$$

for certain functions r_M^L. Because f is a cusp form, and $a \notin L_{\mathrm{ell}}$ for $L \in \mathscr{L}(M)$, $L \neq M$, it follows from Lemma 3.3(3) that $J_L(a, f) = 0$. Also, $r_M^M \equiv 1$. Thus $J_M(a, f) = \lim_{a\to 1} J_M(a, f)$. Using $G = MNK$ and the change of variables $n \in N \mapsto u = a^{-1}n^{-1}an \in N$, $J_M(a, f) = \int_{M\backslash G} f(x^{-1}ax)\,dx$ can be rewritten as $\int_N f_K(au) v_M(n)\,du$ [Mu, Lemma 6.1]. To prove (2), (4), (5) and (6), use Lemma 5.4 for the values of the weight and argue in the same way as in the proof of Proposition 6.5 of [Mu]. (2) and (4) also follow immediately from the proofs of (1) and (3), respectively.

From Lemma 5.4(1) and the above remarks, noting that if d is large, $f_K(au) = f_K(u)$, for all $u \in N$,

$$\begin{aligned} J_{M_2}(a, f) &= \int_N f_K(u) v_M(n)\,du \\ &= \sum_{j=0}^{d-1} \int_{N(j)} f_K(u) \max(2d - v(x^2 - yz), d - j)\,du \end{aligned}$$

where $N(j) = \{u \in N \mid \min(v(x), v(y), v(z)) = j\}$. For $j \geqslant 0$, define

$$\begin{aligned} A_d(j) &= \{u \in N(j) \mid v(x) = v(y) = j,\ v(x^2 - yz) \geqslant d + j + 1\} \\ B_d(j) &= \{u \in N(j) \mid v(z) = j,\ v(x) \geqslant j + 1,\ v(x^2 - yz) \geqslant d + j + 1\} \\ C_d(j) &= \{u \in N(j) \mid v(y) = j,\ v(x) \geqslant j + 1,\ v(x^2 - yz) \geqslant d + j + 1\} \end{aligned}$$

Let $\mathscr{A}_d(j) = A_d(j) \cup B_d(j) \cup C_d(j)$. Note that $2d - v(x^2 - yz) < d - j$ if and only if $u \in \mathscr{A}_d(j)$. Assume that d is large enough that $f_K(u) = f_K(1) = f(1)$ for $u \in N(j), j \geqslant d$. The above expression for $J_{M_2}(a, f)$ can be rewritten as

$$\begin{aligned} J_{M_2}(a, f) = &-\int_N f_K(u) v(x^2 - yz)\,du - f(1) \sum_{j=d}^{\infty} \int_{N(j)} (2d - v(x^2 - yz))\,du \\ &+ \sum_{j=0}^{d-1} \int_{\mathscr{A}_d(j)} f_K(u)(-d + v(x^2 - yz) - j))\,du. \end{aligned}$$

Here we have used the fact that $\int_N f_K(u)\,du = 0$ because f is a cusp form. Let $u_0^j = 1 + \varpi^j(u_0 - 1), j \geqslant 0$, where u_0 is given by (2.3). If $u \in \mathscr{A}_d(j)$, it is easy to see that there exists $k \in K$ such that $k^{-1}uk \in u_0^j \bigcup_{l \geqslant d} N(l)$. This implies $f_K(u) = f_K(u_0^j)$. Evaluation of all but the first integral in the above formula for $J_{M_2}(a, f)$ results in

$$\int_{A_d(j)} (-d + v(x^2 - yz) - j)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = q^{-2j-d-2}(q-1)$$

$$\int_{B_d(j)\cup C_d(j)} (-d + v(x^2 - yz) - j)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 2q^{-2j-d-2}$$

$$\int_{N(j)} (2d - v(x^2 - yz))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 2(d-j)q^{-3j-3}(q^3-1) - q^{-3j-2}(q+1)$$

After substitution, summing over j, and some rearrangement, we obtain

$$J_{M_2}(a, f) = -\int_N f_K(u)v(x^2 - yz)\,\mathrm{d}u + \frac{q^{-3d}}{q^3-1} f(1) + \left[\frac{q^{-3d}}{q-1} f(1) + q^{-d-2}(q+1)\sum_{j=0}^{d-1} q^{-2j} f_K(u_0^j)\right].$$

Using Lemma 5.7, the term in square brackets is easily recognized as $q^{-d}(q-1)^{-1}J_G(\mathcal{O}_0, f)$. Let $d \to \infty$ to get (2).

The proof of (3) is along the same lines as that of (1), but the algebra is much simpler. □

Next we discuss normalizations of the invariant measures on unipotent classes. Let $P_l = M_l N_l \in \mathscr{P}(M_l)$, $l = 0, 1, 2$, with $\mathrm{d}n_l$ the Haar measure on N_l. For $l = 0$ (resp. 1) $f \mapsto \int_{N_l} f_K(n_l)\,\mathrm{d}n_l$, $f \in C_c^\infty(G)$ defines a G-invariant measure on $\mathcal{O}_R$ (resp. $\mathcal{O}_1$) (see Lemma 2.5).

From Lemma 2.5(2), it follows that the restriction of $\mathrm{d}n_2$ to the open subset $\mathcal{O} \cap N_2$ of N_2 defines an invariant measure on $\mathcal{O} = \mathcal{O}_\tau$, $\tau = 1$, ε, $\varepsilon\varpi$, or ϖ, via $f \mapsto \int_{N_2 \cap \mathcal{O}} f_K(n_2)\,\mathrm{d}n_2$. We use this as a choice of measure on $\mathcal{O}_\tau$, $\tau = \varepsilon$, $\varepsilon\varpi$ and ϖ. However, $\int_{N_1} f_K(n_1)\,\mathrm{d}n_1 = 2\int_{\mathcal{O}_1 \cap N_2} f_K(n_2)\,\mathrm{d}n_1$. This can be verified by evaluating both integrals for f equal to the characteristic function of K.

Summarizing, for $f \in C_c^\infty(G)$,

$$\begin{aligned} J_G(\mathcal{O}_R, f) &= \int_{N_0} f_K(n_0)\,\mathrm{d}n_0 \\ J_G(\mathcal{O}_1, f) &= \int_{N_1} f_K(n_1)\,\mathrm{d}n_1 = 2\int_{N_2 \cap \mathcal{O}_1} f_K(n_2)\,\mathrm{d}n_2 \\ J_G(\mathcal{O}_\tau, f) &= \int_{N_2 \cap \mathcal{O}_\tau} f_K(n_2)\,\mathrm{d}n_2, \qquad \tau = \varepsilon, \varepsilon\varpi, \varpi \end{aligned} \tag{5.6}$$

$$\int_{N_2} f_K(n_2)\,\mathrm{d}n_2 = (1/2)J_G(\mathcal{O}_1, f) + \sum_{\tau = \varepsilon, \varepsilon\varpi, \varpi} J_G(\mathcal{O}, f).$$

Now we choose the measure on $\mathcal{O}_0$.

LEMMA 5.7. *Let* d*x be the standard Haar measure on F which assigns measure one to $\mathfrak{O}_F$. Then*

$$f \mapsto \frac{q+1}{q} \int_{F^*} |x| f_K(1 + x(u_0 - 1))\,\mathrm{d}x, \qquad f \in C_c^\infty(G)$$

defines a G-invariant measure on $\mathcal{O}_0$.

Proof. Let $X_0 = u_0 - 1$, where u_0 is defined in (2.3). Then $e^{X_0} = 1 + X_0 = u_0$ and $G_{X_0} = G_{u_0}$. Thus Ranga Rao's formula in Theorem 1 of [R] gives an invariant measure on $\mathcal{O}_0 \simeq G_{u_0}\backslash G$. The above expression is that formula for this particular case. The constant $(q+1)/q$ is chosen so that $J_G(\mathcal{O}_0, f) = 1$ for f equal to the characteristic function of K. □

Finally, measures must be specified on the nilpotent orbits in $\mathfrak{g}$ so that Lemma 2.6 holds. Assume that the character ψ appearing in the definition of the Fourier transform on $\mathfrak{g}$ has conductor equal to $\mathfrak{O}_F$. Fix Haar measures on $\mathfrak{g}$, Levi subalgebras $\mathfrak{m}$, and nilradicals $\mathfrak{n}$ so that $\mathfrak{g}(\mathfrak{O}_F)$, $\mathfrak{m}(\mathfrak{O}_F)$ and $\mathfrak{n}(\mathfrak{O}_F)$ have volume one.

LEMMA 5.8. *Let $f \in C_c^\infty(\mathfrak{g})$. Choose $\mathfrak{n}_l$, such that $M_l \exp(\mathfrak{n}_l) \in \mathscr{P}(M_l)$, $l = 0, 1, 2$.*

$$\mu_R(f) = q^{-4}(q+1)^2(q^2+1) \int_{\mathfrak{n}_0} f_K(X)\,\mathrm{d}X$$

$$\mu_1(f) = q^{-3}(q+1)(q^2+1) \int_{\mathfrak{n}_1} f_K(X)\,\mathrm{d}X$$

$$\mu_\tau(f) = q^{-3}(q+1)(q^2+1) \int_{\mathfrak{n}_2 \cap \mathcal{O}_G(u_\tau - 1)} f_K(X)\,\mathrm{d}X, \quad \tau = \varepsilon, \varpi, \varepsilon\varpi$$

$$\mu_0(f) = \tfrac{1}{2} q^{-1}(q+1)(q^2+1) \int_{F^*} |x| f_K(x(u_0 - 1))\,\mathrm{d}x$$

define G-invariant measures on the non-trivial nilpotent orbits in $\mathfrak{g}$, and the functions $\hat{\mu}_\mathcal{O}$ satisfy Lemma 2.6.

Proof. That the measures are G-invariant is clear. Thus it is sufficient to check that, if $P \in \mathscr{P}(M)$, $\sum_{\mathcal{O} \in 1_M^G} \hat{\mu}_\mathcal{O}(X) = \Theta_P(\exp X)$ for X near 0. Let $\mathfrak{n}$ be the nilradical of the Lie algebra of $P = MN$. Let μ_P be the distribution on $\mathfrak{g}$ given by integration over $\mathfrak{n}$. From I.8 of [MW], there exists a constant $c > 0$ such that $\Theta_P(\exp X) = c\hat{\mu}_P(X)$ for X in some small neighbourhood of 0 in $\mathfrak{g}$. For $j \geqslant 1$, let K_j be the set of $k \in K$ such that the entries of the matrix $k - 1$ lie in $\mathfrak{p}^j$, where $\mathfrak{p} = \varpi\mathfrak{O}_F$. Log will indicate the inverse of exp, defined on a neighbourhood of 0.

If H is a finite group, $|H|$ denotes the order of H. If f_j is the characteristic function of K_j, $j \geqslant 1$, then for large enough j,

$$\begin{aligned} c^{-1}\Theta_P(f_j) &= \int_{K_j} \hat{\mu}_P(\log x)\,\mathrm{d}x = q^{\dim G}|\mathbf{G}(\mathbf{F}_q)|^{-1} \int_{\mathfrak{g}(\mathfrak{p}^j)} \hat{\mu}_P(X)\,\mathrm{d}X \\ &= q^{\dim G}|\mathbf{G}(\mathbf{F}_q)|^{-1}\hat{\mu}_P(h_j) = q^{(1-j)\dim G}|\mathbf{G}(\mathbf{F}_q)|^{-1}\mu_P(h_{-j}). \\ &= q^{(1-j)\dim G + j\dim N}|\mathbf{G}(\mathbf{F}_q)|^{-1} \end{aligned}$$

Here, h_j denotes the characteristic function of $\mathfrak{g}(\mathfrak{p}^j)$, j any integer. It is easy to verify that $\hat{h}_j = q^{-j\dim G}h_{-j}$. By Theorem 2 of [D],

$$\Theta_P(f_j) = \int_M \int_N (f_j)_K(mn)\,\mathrm{d}m\,\mathrm{d}n = q^{(1-j)\dim M - j\dim N}|\mathbf{M}(\mathbf{F}_q)|^{-1}.$$

Therefore, $c = q^{-2\dim N}|\mathbf{M}(\mathbf{F}_q)|^{-1}|\mathbf{G}(\mathbf{F}_q)|$. Evaluation of c for $M = M_0$, M_1, and M_2 yields the constants given in the statement of the lemma. □

6. Matrix coefficients of supercuspidal representations

Let f be a finite sum of matrix coefficients of π such that $f(1) \neq 0$. The coefficients occurring in the weighted orbital integral germ expansion of the character Θ_π are weighted orbital integrals of f over unipotent classes in Levi subgroups of G. As can be seen from the formulas appearing in Proposition 5.5 and in (5.6), the values

$$f_K(u) = \int_K f(k^{-1}uk)\,\mathrm{d}k, \qquad u \in \mathcal{U}_G$$

are required in order to compute these coefficients. In the main results of this section, Proposition 6.3 and Lemma 6.6, we evaluate $f_K(u)$, $u \in \mathcal{U}_G$ for certain types of supercuspidal representations (described below). Lemma 6.7 gives the formal degrees of these representations.

In [Mo1] and [Mo2], Moy defined nondegenerate representations, a set of irreducible representations of open compact mod centre subgroups of G. Up to twisting by a one-dimensional character of G, each irreducible admissible representation of G contains a nondegenerate representation. Using Hecke algebra isomorphisms, Moy classified the irreducible admissible representations containing a given nondegenerate representation. He identified the supercuspidal representations and proved that they are all induced from representations of open compact mod centre subgroups. Jabon [J] then used Moy's results to explicitly determine the inducing data for each supercuspidal representation. This inducing data will be used to determine f_K on $\mathcal{U}_G$. Morris [M] has also found inducing data for some of the supercuspidal representations of G.

Each nondegenerate representation is a representation of a filtration subgroup of some parahoric subgroup of G. In this section, we deal with those supercuspidal representations of G which contain a nondegenerate representation of a filtration subgroup of the parahoric subgroup K. As shown in Proposition 6.3, for such π the function f can be chosen so that the computation of $f_K(u)$, $u \in \mathcal{U}_G$ reduces to a sum over a unipotent conjugacy class in $\mathbf{GSp_4(F}_q)$, and there are only five possible distinct nonzero values for $f_K(u)$. We remark that in cases where a parahoric subgroup other than K is involved, the inducing subgroup may not behave well under conjugation by K, and the inducing representation may be a tensor product of several different representations, making f_K more difficult to compute.

Let $\mathfrak{p} = \varpi\mathfrak{O}_F$. The filtration subgroups of K are $K_0 = K$ and $K_j = \{x \in G \mid x - 1 \in \mathbf{M}_4(\mathfrak{p}^j)\}, j \geqslant 1$. In Lemma 6.1, we outline those properties of the inducing data which will be used to compute f_K on $\mathcal{U}_G$ for supercuspidal representations containing a nondegenerate representation of $K_j, j \geqslant 0$. Let ψ be a character of the additive group F having conductor $\mathfrak{O}_F$, that is, ψ is non-trivial on $\varpi^{-1}\mathfrak{O}_F$ and trivial on $\mathfrak{O}_F$. Suppose $\alpha \in \mathfrak{g}$ and $\varpi^{j+1}\alpha \in \mathfrak{g}(\mathfrak{O}_F)$ for some $j \geqslant 1$. Then

$$\Omega_\alpha(c(X)) = \psi(-\mathrm{tr}(X\alpha)/2)$$

defines a character of $K_j = c(\mathfrak{g} \cap \mathbf{M}_4(\mathfrak{p}^j))$, where $c(X) = (1 - X)(1 + X)^{-1}$ is the Cayley transform of X.

LEMMA 6.1. *Suppose the supercuspidal representation π contains a nondegenerate representation of K_j.*

(1) *If $j = 0$, then $\pi = \mathrm{Ind}_{KZ}^G(\rho \otimes \chi)$ for some ρ arising from a cuspidal representation of* $\mathrm{GSp}_4(\mathrm{F}_q)$ *and some character χ of the centre Z of G which is trivial on $Z \cap K_1$. Furthermore, for every such ρ and χ the corresponding induced representation is irreducible and supercuspidal.*
(2) *If j is odd, then $\pi = \mathrm{Ind}_{TK_{(j+1)/2}}^G \rho$ where $\rho | K_j = \Omega_\alpha$, for some $\alpha \in \mathfrak{g}$ having the property that the image of $\varpi^{j+1}\alpha$ in $\mathfrak{g}(\mathrm{F}_q)$ is regular and elliptic. T is the unique Cartan subgroup of G such that $\alpha \in \mathrm{Lie}(T)$.*
(3) *If $j > 0$ is even, then $\pi = \mathrm{Ind}_{TK_{j/2}}^G \tilde{\rho}$ where $\tilde{\rho}$ is the unique extension of a certain representation ρ of $TK_{j/2+1}$ to $TK_{j/2}$. ρ has the property that $\rho | K_j = \Omega_\alpha$, and α and T are as in (2).*

Proof. [J] Prop. 4.3, Prop. 4.6, and Prop. 4.7. □

There are two conjugacy classes of Cartan subalgebras of $\mathfrak{g}$ whose intersections with $\mathfrak{g}(\mathfrak{O}_F)$ project to elliptic Cartan subalgebras of $\mathfrak{g}(\mathbf{F}_q)$ [Mo2, (4.1)]. Representatives $\mathfrak{h}^\varepsilon$ and $\mathfrak{h}^{\mathfrak{Q}}$ are given below. Let T^ε and $T^{\mathfrak{Q}}$ be the corresponding Cartan subgroups of G.

$$\mathfrak{h}^\varepsilon = \left\{ \begin{bmatrix} a & 0 & 0 & b \\ 0 & a & c & 0 \\ 0 & \varepsilon c & a & 0 \\ \varepsilon b & 0 & 0 & a \end{bmatrix} \middle| \, a, b, c \in F \right\} \tag{6.2a}$$

Fix $\mathscr{A}, \mathscr{B} \in \mathfrak{O}_F^*$ such that $\mathscr{A} + \mathscr{B}\sqrt{\varepsilon}$ is a non-square in $F(\sqrt{\varepsilon})$.

$$\mathfrak{h}^{\mathfrak{Q}} = \left\{ \begin{bmatrix} c & 0 & a & b \\ 0 & c & \varepsilon b & a \\ a' & b' & c & 0 \\ \varepsilon b' & a' & 0 & c \end{bmatrix} \middle| \, a, b, c \in F,\ a' + b'\sqrt{\varepsilon} = (\mathscr{A} + \mathscr{B}\sqrt{\varepsilon})(a + b\sqrt{\varepsilon}) \right\} \tag{6.2b}$$

Suppose $\pi = \mathrm{Ind}_H^G \sigma$ for some representation σ of an open compact mod centre subgroup H. Let χ_σ be the character of σ. Define

$$f(x) = \begin{cases} \chi_\sigma(x), & \text{if } x \in H, \\ 0, & \text{otherwise.} \end{cases}$$

Because χ_σ is a finite sum of matrix coefficients of σ, f is a finite sum of matrix coefficients of π. Note that $f(1) = \dim \sigma \neq 0$. By definition, the support of f is compact modulo Z and f is locally constant. This particular f is chosen because it is invariant under conjugation by H.

PROPOSITION 6.3. *Let π be as in Lemma* 6.1 *with $j \geqslant 1$ and f as above. Set l equal to the greatest integer in $(j + 1)/2$. Let $u \in \mathscr{U}_G$.*

(1) *If $u \notin K_j$, $f_K(u) = 0$.*
(2) *If $u \in K_j$, $f_K(u)$ is given in the following table. The left column gives a representative for the conjugacy class of the image of $u - 1$ in $K_{j+1} \backslash K_j \simeq \mathfrak{g}(\mathbf{F}_q)$.*

	$f_K(u)$, $T = T^\varepsilon$	$f_K(u)$, $T = T^{\Omega}$
0	$q^{2(j+1-2l)}$	$q^{2(j+1-2l)}$
$u_0 - 1$	$\dfrac{1}{q^2+1} q^{2(j+1-2l)}$	$\dfrac{-1}{q^2-1} q^{2(j+1-2l)}$
$u_1 - 1$	$\dfrac{-1}{(q-1)(q^2+1)} q^{2(j+1-2l)}$	$\dfrac{-1}{(q+1)(q^2-1)} q^{2(j+1-2l)}$
$u_\varepsilon - 1$	$\dfrac{-3q+1}{(q-1)^2(q^2+1)} q^{2(j+1-2l)}$	$\dfrac{1}{(q-1)(q^2-1)} q^{2(j+1-2l)}$
$u_R - 1$	$\dfrac{1}{(q-1)^2(q^2+1)} q^{2(j+1-2l)}$	$\dfrac{1}{(q^2-1)^2} q^{2(j+1-2l)}$

Proof. (1) If $u \notin K_l$, then $\{k^{-1}uk \mid k \in K\} \cap TK_l = \varnothing$. Thus there is no loss of generality in assuming $u \in K_l$. Suppose j is even. The extension $\tilde{\rho}$ of ρ to TK_l is obtained from a Heisenberg group and, as can be deduced from details of the proof of Lemma 6.7, $f(u) = \chi_{\tilde{\rho}}(u) = q^4\rho(u)$ for $u \in TK_{l+1}$. Also, $\chi_{\tilde{\rho}}(u) = 0$ for $u \in K_l - K_{l+1}$ because $\{k^{-1}uk \mid k \in K\} \cap TK_{l+1} = \varnothing$. If j is odd, by definition, $f(u) = \rho(u)$, $u \in K_l$. Therefore,

$$f(u) = q^{2(j+1-2l)}\rho(u), \qquad u \in K_h \cap \mathcal{U}_G,$$

where h is the greatest integer in $(j+2)/2$, and $f(u) = 0$ for all other $u \in \mathcal{U}_G$. Let $u \in K_i - K_{i+1}$ where $j > i \geqslant h$. $f_K(u) = \int_K \rho(k^{-1}uk)\,dk$ is a nonzero multiple of

$$\int_K \int_{K_{j-1}} \rho(\kappa^{-1}k^{-1}uk\kappa^{-1})\,d\kappa\,dk.$$

For each $k \in K$, $k^{-1}uk \in (K_i - K_{i+1}) \cap \mathcal{U}_G$. To prove (1), it suffices to show that

$$\int_{K_{j-i}} f(\kappa^{-1}u\kappa)\,d\kappa = 0, \qquad u \in (K_i - K_{i+1}) \cap \mathcal{U}_G.$$

Let $u = c(Y)$, $\kappa = c(X)$, $Y \in \mathfrak{g}(\mathfrak{p}^i)$, $X \in \mathfrak{g}(\mathfrak{p}^{j-i})$. Then

$$u^{-1}\kappa^{-1}u\kappa = 1 + 4(YX - XY) + Z$$

for some $Z \in \mathbf{M}_4(\mathfrak{p}^{j+1})$. This implies that

$$u^{-1}\kappa^{-1}u\kappa - c(2(XY - YX)) \in \mathbf{M}_4(\mathfrak{p}^{j+1}).$$

Thus

$$\int_{K_{j-i}} f(\kappa^{-1}u\kappa)\,d\kappa = \rho(u)\int_{K_{j-i}} \Omega_\alpha(u^{-1}\kappa^{-1}u\kappa)\,d\kappa$$
$$= \rho(u)\int_{\mathfrak{g}(\mathfrak{p}^{j-i})} \psi(\mathrm{tr}(\alpha(YX - XY)))\,dX$$
$$= \rho(u)\int_{\mathfrak{g}(\mathfrak{p}^{j-i})} \psi(\mathrm{tr}((\alpha Y - Y\alpha)X))\,dX.$$

Here dX is the measure on $\mathfrak{g}(\mathfrak{p}^{j-i})$ which transfers to $d\kappa$ under the Cayley transform c. If $\alpha Y - Y\alpha \in \mathfrak{g}(\mathfrak{p}^{j-i})$, then the images of $\varpi^{j+1}\alpha$ and $\varpi^{-i}Y$ in $\mathfrak{g}(\mathbf{F}_q)$ must commute. This is impossible because these images are, respectively, regular semisimple and nontrivial nilpotent. Therefore $X \mapsto \psi(\mathrm{tr}((\alpha Y - Y\alpha)X))$ is a nontrivial character of $\mathfrak{g}(\mathfrak{p}^{j-i})$. This implies that the above integral is zero.

Proof of (2). Let $u \in K_j$. We wish to compute

$$f_K(u) = q^{2(j+1-2l)}[K:K_1]^{-1} \sum_{k\in K_1\backslash K} \Omega_\alpha(k^{-1}uk). \tag{6.4}$$

Let

$$u_R^j = \begin{bmatrix} 1 & \varpi^j & 0 & 0 \\ 0 & 1 & \varpi^j & -\varpi^{2j} \\ 0 & 0 & 1 & -\varpi^j \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad u_\tau^j = 1 + \varpi^j(u_\tau - 1), \quad \tau = 0, 1, \varepsilon.$$

Given $u \in K_j - K_{j+1}$, there exists a unique τ and some $k \in K$ such that $k^{-1}uk \in K_j u_\tau^j$. Thus $f_K(u) = f_K(u_\tau^j)$. The next step is to show that it suffices to calculate (6.4) for u_0^j. Define

$$t_\tau = q^{-2(j+1-2l)} f_K(u_\tau^j), \qquad \tau = 0, 1, \varepsilon, R.$$

Let $P = MN \in \mathscr{P}(M)$, $M = M_i$, $i = 0, 1, 2$. In terms of the constants t_τ, $\int_N f_K(u)\,du$ is equal to:

$$q^{2(j+1-2l)}q^{-3(j+1)}(t_1 q(q^2-1)/2 + t_0(q^2-1) + t_\varepsilon q(q-1)^2/2 + 1), \quad i = 2,$$
$$q^{2(j+1-2l)}q^{-3(j+1)}(t_0(q-1) + t_1 q(q^2-1) + 1), \quad i = 1,$$
$$q^{2(j+1-2l)}q^{-4(j+1)}(t_R q^2(q-1)^2 - t_1 q(q-1) - t_0(q-1) - 1), \quad i = 0.$$

Because f is a cusp form, $\int_N f_K(u)\,du = 0$. It follows that

$$t_1 = \frac{-1 - t_0(q-1)}{q(q^2-1)} \tag{6.5}$$

$$t_\varepsilon = \frac{-1 - t_0(q-1)(2q+1)}{q(q-1)^2}$$

$$t_R = \frac{1 + t_0(q-1)}{q(q-1)(q^2-1)}$$

Thus to determine the values of f_K on $\mathscr{U}_G \cap K_j$, it is sufficient to compute

$$t_0 = [K:K_1]^{-1} \sum_{k \in K_1 \backslash K} \Omega_\alpha(k^{-1}u_0^j k)$$

$$= [K:K_1]^{-1} \sum_{k \in K_1 \backslash K} \psi(\varpi^j \operatorname{tr}(\alpha k^{-1}(u_0 - 1)k)).$$

In Lemma 6.9, t_0 is found to equal $(q^2+1)^{-1}$ in the case $T = T^\varepsilon$ and $-(q^2-1)^{-1}$ in the case $T = T^\Omega$. (6.5) can be used to produce the other values of f_K given in the statement of the proposition. □

LEMMA 6.6. *Let π be as in Lemma 6.1 with $j = 0$ and f as above. There are three families $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$ of such representations π. The values of $f_K(u)$, $u \in K$, are listed in the table below. The left column gives a representative for the conjugacy class of the image of u in* $\mathbf{GSp}_4(\mathbf{F}_q)$.

	$f_K(u)$, $\mathscr{F} = \mathscr{F}_1$	$f_K(u)$, $\mathscr{F} = \mathscr{F}_2$	$f_K(u)$, $\mathscr{F} = \mathscr{F}_3$
1	$(q-1)^2(q^2+1)$	$(q^2-1)^2$	$q(q-1)^2/2$
u_0	$(q-1)^2$	$-(q^2-1)$	$-q(q-1)/2$
u_1	$-(q-1)$	$-(q-1)$	0
u_ε	$-3q+1$	$q+1$	q
u_R	1	1	0

Proof. Reid [Re] has computed the characters of $\mathbf{GSp}_4(\mathbf{F}_q)$. The three families $\mathscr{F}_i$, $i = 1, 2, 3$ of cuspidal representations contain $(q-1)^3/8$, $(q-1)(q^2-1)/4$ and 1 representations, respectively, and are called $\varphi(\hat{t}_3)$, $\varphi(\hat{t}_5)$ and $\eta_4(\hat{\lambda})$ in [Re]. From Lemma 6.1 (1), if ρ is one of these representations $\pi = \operatorname{Ind}_{KZ}^G(\rho \otimes \chi)$, where ρ is inflated to a representation of K. Note that $\chi|\mathscr{U}_G \equiv 1$. □

LEMMA 6.7. *Let π be as in Lemma 6.1. Let $d(\pi)$ be the formal degree of π.*

(1) *If $j = 0$,*

$$d(\pi) = \begin{cases} [\mathrm{vol}(Z\backslash KZ)]^{-1}(q-1)^2(q^2+1), & \pi \in \mathscr{F}_1 \\ [\mathrm{vol}(Z\backslash KZ)]^{-1}(q^2-1)^2, & \pi \in \mathscr{F}_2 \\ [\mathrm{vol}(Z\backslash KZ)]^{-1}q(q-1)^2/2, & \pi \in \mathscr{F}_3. \end{cases}$$

(2) If $j \geqslant 1$,

$$\mathrm{d}(\pi) = \begin{cases} [\mathrm{vol}(Z\backslash KZ)]^{-1}q^{4j}(q-1)^2(q^2+1), & \text{if } T = T^{\varepsilon}, \\ [\mathrm{vol}(Z\backslash KZ)]^{-1}q^{4j}(q^2-1)^2, & \text{if } T = T^{\mathfrak{Q}}. \end{cases}$$

Proof. In each case, $\pi = \mathrm{Ind}_L^G \sigma$ for some open compact mod centre subgroup L. Thus, $d(\pi) = [\mathrm{vol}(Z\backslash L)]^{-1} \dim \sigma$. (1) now follows from Lemma 6.6. We copy the procedure used by Jabon for computing formal degrees. (See, for example, [J, p. 23].) Let $T_i = T \cap K_i$, $i \geqslant 0$. We begin by calculating $\mathrm{vol}(Z\backslash TK_i) = \mathrm{vol}(Z\backslash KZ)[KZ : TK_i]^{-1}$, $i \geqslant 1$. From the exact sequence

$$1 \to T_i\backslash T_0 \to K_i\backslash K \to TK_i\backslash KZ \to 1$$

it follows that $[KZ : TK_i] = [K : K_i][T_0 : T_i]^{-1}$. It is easy to see that $T_{l+1}\backslash T_l \simeq \mathfrak{h}(\mathbf{F}_q)$, $l \geqslant 1$, so $[T_l : T_{l+1}] = q^3$. From $K_{l+1}\backslash K_l \simeq \mathfrak{g}(\mathbf{F}_q)$, we have $[K_l : K_{l+1}] = q^{11}$. Thus

$$\mathrm{vol}(Z\backslash TK_i) = [\mathrm{vol}(Z\backslash KZ)]q^{-4(2i-1)}(q-1)^{-1}(q^2+1)^{-1}(q^2-1)^{-2}[T_0 : T_1].$$

If j is odd, then $\dim \rho = 1$. So $d(\pi) = [\mathrm{vol}(Z\backslash TK_{(j+1)/2})]^{-1}$. Suppose j is even. Let $N_\rho = \ker \rho$. ρ (resp. $\tilde{\rho}$) can be viewed as a representation of the group $A = T_1 K_{(j/2)+1}/N_\rho$ (resp. $H_1 = TK_{j/2}/N_\rho$). Let $\tilde{\rho}_0 = \tilde{\rho}|H$, where $H = T_1 K_{j/2}/N_\rho$. The induced representation $\mathrm{Ind}_A^H \rho$ is the direct sum of $[H : A]^{1/2}$ copies of $\tilde{\rho}_0$. Therefore $\dim \tilde{\rho} = \dim \tilde{\rho}_0 = [H : A]^{1/2} = [T_1 K_{j/2} : T_1 K_{(j/2)+1}]^{1/2}$. From the exact sequence $1 \to T_{(j/2)+1}\backslash T_{j/2} \to K_{(j/2)+1}\backslash K_{j/2} \to T_1 K_{(j/2)+1}\backslash T_1 K_{j/2} \to 1$,

$$[H : A] = [K_{j/2} : K_{(j/2)+1}][T_{j/2} : T_{(j/2)+1}]^{-1} = q^8.$$

Therefore,

$$d(\pi) = q^4[\mathrm{vol}(Z\backslash TK_{j/2})]^{-1}.$$

To complete the proof, it remains to find $[T_0 : T_1]$. From [J, *p.* 74] it follows that

$$T_1^{\varepsilon}\backslash T_0^{\varepsilon} \simeq \{(x, y) \in \mathbf{F}_{q^2}^* \times \mathbf{F}_{q^2}^* \mid N_{\mathbf{F}_{q^2}/\mathbf{F}_q}(x) = N_{\mathbf{F}_{q^2}/\mathbf{F}_q}(y)\}.$$

Given an arbitrary $x \in \mathbf{F}_{q^2}^*$, there are $q+1$ elements $y \in \mathbf{F}_{q^2}^*$ having the same norm as x. Thus $[T_0 : T_1] = (q^2-1)(q+1)$. Similarly,

$$T_1^{\mathfrak{Q}} \backslash T_0^{\mathfrak{Q}} \simeq \{x \in \mathbf{F}_{q^4}^* \mid N_{\mathbf{F}_{q^4}/\mathbf{F}_{q^2}}(x) \in \mathbf{F}_q\}.$$

For each $a \in \mathbf{F}_q^* \subset \mathbf{F}_{q^2}^*$, there are q^2+1 elements in $\mathbf{F}_{q^4}^*$ having norm equal to a. This implies that $[T_0^{\mathfrak{Q}} : T_1^{\mathfrak{Q}}] = (q-1)(q^2+1)$. □

REMARKS. (1) Note that $\chi_{\tilde{\rho}_0} | A = q^4 \chi_\rho$ and $\chi_{\tilde{\rho}_0} | H - A \equiv 0$. This fact was used in the proof of Proposition 6.3.

(2) Let $u \in K_j, j \geqslant 0$. Comparing the first columns of the tables in Proposition 6.3 and Lemma 6.6, the values $f_K(1 + \varpi^j(u_\tau - 1))/f(1)$, $\tau = 0, 1, \varepsilon, R$ agree when $T = T^\varepsilon$ and $\mathscr{F} = \mathscr{F}_1$. Up to multiplication by q^{4j}, the formal degrees agree. This means that the calculations of the coefficients for all of these representations can be done simultaneously. From now on, this will be referred to as case 1. Similarly, case 2 will refer to $T = T^{\mathfrak{Q}}$ or, if $j = 0$, $\mathscr{F} = \mathscr{F}_2$. Finally, case 3 is the third possibility when $j = 0$, that is, $\mathscr{F} = \mathscr{F}_3$.

The remainder of the section is devoted to computing the constant t_0 which appears in the proof of Proposition 6.3. Let $x \in \mathfrak{O}_F^*$. $y \mapsto \psi(\varpi^{-1}xy)$ is a function on $\mathfrak{O}_F$ which can be viewed as a function on $\mathbf{F}_q \simeq \mathfrak{O}_F/\mathfrak{p}$. Define

$$\varphi(x) = \sum_{y \in \mathbf{F}_q^*} \psi(\varpi^{-1}xy^2).$$

For $z \in \mathfrak{O}_F^*$, let

$$\kappa_0(z) = \begin{cases} 1 & z \in \mathfrak{O}_F^{*2}, \\ -1 & \text{otherwise.} \end{cases}$$

LEMMA 6.8. *There exists $c_0 \in \mathbf{C}$ such that $c_0^2 = \kappa_0(-1)$ and*

$$\varphi(x) = -1 + \kappa_0(x) c_0 q^{1/2}, \qquad x \in \mathfrak{O}_F^*.$$

Proof. $\kappa = \kappa_0 |\cdot|$ is a character of F^*. As in [ST], define $\Gamma(\kappa) = \int_F \psi(z)\kappa(z)|z|^{-1}\,dz$. By Theorem 1 of [ST], there exist constants c_0 and c_0' depending on κ_0 and κ_0^{-1} respectively, such that $\Gamma(\kappa) = c_0 q^{1/2}$ and $\Gamma(\kappa^{-1}) = c_0' q^{1/2}$ and $c_0 c_0' = \kappa_0(-1)$. But $\kappa_0^{-1} = \kappa_0$, so $c_0' = c_0$.

$$c_0 q^{1/2} = \sum_{j=0}^{\infty} \int_{|z| = q^{-j}} \kappa_0(\varpi^{-j}z)\,dz + \sum_{j=1}^{\infty} \int_{|z| = q^j} \psi(z)\kappa_0(\varpi^j z)\,dz.$$

If $j \geqslant 0$, then $\int_{|z| = q^{-j}} \kappa_0(\varpi^{-j}z)\,dz = 0$ because κ_0 is a nontrivial character of $\mathfrak{O}_F^*$. For $j \geqslant 1$, $\int_{|z| = q^j}$, $\psi(z)\kappa_0(\varpi^j z)\,dz = q^j \int_{\mathfrak{O}_F^*} \psi(\varpi^{-j}z)\kappa_0(z)\,dz$, which, by Lemma

1 of [ST], is equal to zero unless $j = 1$. Thus

$$\begin{aligned} c_0 q^{1/2} &= \int_{|z|=q} \psi(z)\kappa_0(\varpi z)\, dz \\ &= \int_{\varpi z \in \mathfrak{O}_F^*} \psi(z)\, dz - \int_{\varpi z \in \varepsilon \mathfrak{O}_F^{*2}} \psi(z)\, dz \\ &= (\varphi(\varpi^{-1}) - \varphi(\varpi^{-1}\varepsilon))/2. \end{aligned}$$

From the definition of φ,

$$\varphi(\varpi^{-1}) + \varphi(\varpi^{-1}\varepsilon) = 2 \sum_{y \in F_q^*} \psi(\varpi^{-1} y) = -2.$$

Solving,

$$\varphi(\varpi^{-1}) = -1 + c_0 q^{1/2} \quad \text{and} \quad \varphi(\varpi^{-1}\varepsilon) = -1 - c_0 q^{1/2}. \qquad \square$$

LEMMA 6.9. *Let $j \geq 1$. If $\alpha \in \mathfrak{h}^{\Omega}$ or $\mathfrak{h}^{\varepsilon}$ is such that the image of $\varpi^{j+1}\alpha$ in $\mathfrak{g}(\mathbf{F}_q)$ is regular, then*

$$[K:K_1]^{-1} \sum_{k \in K_1 \backslash K} \psi(\varpi^j \operatorname{tr}(\alpha k^{-1}(u_0 - 1)k)) = \begin{cases} (q^2+1)^{-1} & \text{if } \alpha \in \mathfrak{h}^{\varepsilon}, \\ -(q^2-1)^{-1} & \text{if } \alpha \in \mathfrak{h}^{\Omega}. \end{cases}$$

Proof. We will use the Bruhat decomposition of $K_1\backslash K \simeq \bar{G} = \mathbf{GSp}_4(\mathbf{F}_q)$ to evaluate the sum. Let B be the standard Borel subgroup of $\bar{G}$. We identify the Weyl group W of G with that of $\bar{G}$. $\bar{G} = \coprod_{s \in W} BsB$ and $BsB = Bs \cdot (B \cap s^{-1}Bs)\backslash B$. An arbitary element of B will be written as:

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} 1 & w & x & y \\ 0 & 1 & z & x - wz \\ 0 & 0 & 1 & -w \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $w, x, y, z \in \mathbf{F}_q$, $\lambda_i \in \mathbf{F}_q^*$, $i = 1, 2, 3, 4$ and $\lambda_1\lambda_4 = \lambda_2\lambda_3$. Let s_1 and s_2 be the generators of W given in the proof of Lemma 5.4. The representatives for $(B \cap s^{-1}Bs)\backslash B$, $s \in W$ can be realized as subgroups of the unipotent radical of B, each of which is given by certain restrictions on the coordinates w, x, y, and z:

1	$w=x=y=z=0$	$s_2 s_1$	$w=y=0$	(6.10)
s_1	$w=x=y=0$	$s_2 s_1 s_2$	$z=0$	
s_2	$x=y=z=0$	$s_1 s_2 s_1$	$w=0$	
$s_1 s_2$	$x=z=0$	$(s_1 s_2)^2$	no restrictions	

The unipotent radical of B commutes with u_0, and

$$\mathrm{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}, \lambda_4^{-1})(u_0 - 1)\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1^{-1}\lambda_4(u_0 - 1).$$

The sum becomes

$$\eta \sum_{s \in W} \sum_{v \in (B \cap s^{-1}Bs)\backslash B} \sum_{\lambda_1, \lambda_4 \in F_q^*} \psi(\varpi^j \lambda_1^{-1}\lambda_4 \operatorname{tr}(\alpha v^{-1}(u_0 - 1)v)), \tag{6.11}$$

where $\eta = q^4(q-1)[K:K_1]^{-1} = (q^2-1)^{-2}(q^2+1)^{-1}$. After conjugating $\varpi^j(u_0 - 1)$ by Bs and $(B \cap s^{-1}Bs)\backslash B$, the result is:

$$\varpi^j\lambda_1^{-1}\lambda_4(u_0-1)_s \quad 1,\ s_2,$$

$$\varpi^j\lambda_1^{-1}\lambda_4 \begin{bmatrix} 0 & 0 & -w & w^2 \\ 0 & 0 & 1 & -w \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad s_1,\ s_1s_2,$$

$$\varpi^j\lambda_1^{-1}\lambda_4 \begin{bmatrix} 0 & x & xz & x^2 \\ 0 & z & z^2 & xz \\ 0 & -1 & -z & -x \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad s_2s_1,\ s_1s_2s_1,$$

$$\varpi^j\lambda_1^{-1}\lambda_4 \begin{bmatrix} y & wy & xy & y^2 \\ x & wx & x^2 & xy \\ -w & -w^2 & -wx & -wy \\ -1 & -w & -x & -y \end{bmatrix}, \quad s_2s_1s_2,\ (s_2s_1)^2 \tag{6.12}$$

Let $\alpha \in \mathfrak{h}^\varepsilon$ be as in (4.2a) with $|b| = |c| = q^{j+1}$ and $|a| \leqslant q^{j+1}$. Using (6.10) and (6.12), (6.11) can be rewritten as:

$$\eta \sum_{\lambda_1,\lambda_4 \in \mathbf{F}_q^*} \Bigg[(1+q)\psi(\varpi^j\lambda_1^{-1}\lambda_4 b\varepsilon) + (1+q) \sum_{w \in \mathbf{F}_q} \psi(\varpi^j\lambda_1^{-1}\lambda_4\varepsilon(bw^2 + c))$$
$$+ (1+q) \sum_{x,z \in \mathbf{F}_q} \psi(\varpi^j\lambda_1^{-1}\lambda_4(b\varepsilon x^2 + c(-1 + \varepsilon z^2)))$$
$$+ (1+q) \sum_{w,x,y \in \mathbf{F}_q} \psi(\varpi^j\lambda_1^{-1}\lambda_4(b(\varepsilon y^2 - 1) + c(-w^2 + \varepsilon x^2))) \Bigg]$$

After repeated applications of Lemma 6.8, this reduces to:

$$\eta(1+q)\big(-(q-1) + q(q-1)\kappa_0(-\varpi^{2j+2}bc) - q(q-1)\kappa_0(-\varpi^{2j+2}bc) + q^2(q-1)\big)$$
$$= (q^2+1)^{-1}.$$

Let $\alpha \in \mathfrak{h}^{\Omega}$ be as in (6.2b) with $|a| = |b| = q^{j+1}$. In this case, (6.11) becomes:

$$\eta \sum_{\lambda_1,\lambda_4 \in \mathbf{F}_q^*} \Big[(1+q)\psi(\varpi^j \lambda_1^{-1}\lambda_4 b'\varepsilon) + \sum_{w \in \mathbf{F}_q} (1+q)\psi(\varpi^j \lambda_1^{-1}\lambda_4(-2a' + b'(1+\varepsilon w^2)))$$

$$+ \sum_{x,z \in \mathbf{F}_q} (1+q)\psi(\varpi^j \lambda_1^{-1}\lambda_4(-b\varepsilon + 2a'xz + b'(z^2+\varepsilon x^2)))$$

$$+ \sum_{w,x,y \in \mathbf{F}_q} (1+q)\psi(\varpi^j \lambda_1^{-1}\lambda_4(-2aw - b(1+\varepsilon w^2)$$

$$+ 2a'xy + b'(x^2+\varepsilon y^2))) \Big].$$

Use of Lemma 6.8 produces

$$\eta(1+q)(-(q-1) + (q-1)\kappa_0(\varpi^{2j+2}(a'^2 - b'^2\varepsilon))$$

$$-(q-1)\kappa_0(\varpi^{2j+2}(a'^2 - b'^2\varepsilon)) + q^2(q-1)\kappa_0(\varpi^{4j+4}(a'^2 - b'^2\varepsilon)(a^2 - b^2\varepsilon)))$$

$$= \eta(q^2-1)(-1 + q^2\kappa_0(\mathscr{A}^2 - \beta^2\varepsilon)) = -(q^2-1)^{-1}.$$

7. Some values of $\hat{\mu}_{\mathcal{O}}$

Recall that if $\mathcal{O} \in (\mathscr{U}_G)$, $\mu_{\mathcal{O}}$ denotes the distribution given by integration over the corresponding nilpotent orbit in $\mathfrak{g}$.

Sally [S] has derived a formula for the Fourier transform $\hat{\mu}_{\mathcal{O}}$ as follows. For $X \in \mathfrak{g}_{\text{reg}}$, let

$$\eta_X(Y) = \int_K \psi(\operatorname{tr}(Xk^{-1}Yk))\,\mathrm{d}k, \qquad Y \in \mathfrak{g}.$$

Let $\mathcal{O} \in (\mathscr{U}_G)$. After showing that $\mu_{\mathcal{O}}(\eta_X)$ is well-defined, a change in the order of integration and the definition of $\hat{h}$ can be used to prove that

$$\int_{\mathfrak{g}} h(X)\mu_{\mathcal{O}}(\eta_X)\,\mathrm{d}X = \hat{\mu}_{\mathcal{O}}(h), \qquad h \in C_c^\infty(\mathfrak{g}).$$

This implies

$$\hat{\mu}_{\mathcal{O}}(X) = \mu_{\mathcal{O}}(\eta_X), \qquad X \in \mathfrak{g}_{\text{reg}}.$$

Evaluating Sally's formula for $\hat{\mu}_{\mathcal{O}}$ yields explicit values of $\hat{\mu}_\tau$, $\tau = 0, \varepsilon, \varpi, \varepsilon\varpi$ at certain points. Let r and t be positive integers. For a, $b \in F^*$ such that

$|a| = q^{-t} > |b| = q^{-r}$, define

$$X_0 = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & -a \end{bmatrix}, \quad X_1 = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & \varepsilon b & 0 & 0 \\ 0 & 0 & 0 & -a \end{bmatrix},$$

$$X_2 = \begin{bmatrix} a & b & 0 & 0 \\ \varepsilon b & a & 0 & 0 \\ 0 & 0 & -a & -b \\ 0 & 0 & -\varepsilon b & -a \end{bmatrix}.$$

Let $X_\varepsilon \in \mathfrak{h}^\varepsilon$ be given by (6.2a) with $|a| = |b| = q^{-t}$, and let $X_\mathfrak{Q} \in \mathfrak{h}^\mathfrak{Q}$ be given by (6.2b) with $|b| = |c| = q^{-t}$.

LEMMA 7.1.

X	$\hat\mu_0(X)$	$\hat\mu_1(X)$	$\hat\mu_\varepsilon(X)$	$\hat\mu_\varpi(X) = \hat\mu_{\varepsilon\varpi}(X)$
X_0	q^{t+r}	$2q^{3t}+2q^{2t+r}$	$-\frac{q(q-1)}{q^3-1}q^{3t}+q^{2t+r}$	$-\frac{(q-1)(q^2+1)}{2(q^3-1)}q^{3t}+q^{2t+r}$
X_1	0	$2q^{3t}$	$-\frac{q(q-1)}{q^3-1}q^{3t}$	$-\frac{(q-1)(q^2+1)}{2(q^3-1)}q^{3t}$
X_2	q^{2t}	0	$\frac{(q-1)(q^2+1)}{q^3-1}q^{3t}$	$\frac{(q-1)(q+1)^2}{2(q^3-1)}q^{3t}$
X_ε	q^{2t}	0	$-\frac{(q-1)(q+1)^2}{q^3-1}q^{3t}$	$\frac{(q-1)(q+1)^2}{2(q^3-1)}q^{3t}$
$X_\mathfrak{Q}$	0	0	$\frac{(q-1)(q^2+1)}{q^3-1}q^{3t}$	$-\frac{(q-1)(q^2+1)}{2(q^3-1)}q^{3t}$

Proof. By Lemma 2.6(2), $\hat\mu_1(X) = \Theta_P(\exp X)$, $P \in \mathscr{P}(M_1)$, so $\hat\mu_1(X)$ is obtained by evaluating van Dijk's character formula [D].

Choose $P_2 = M_2 N_2 \in \mathscr{P}(M_1)$ so that N_2 is upper triangular. Let $\mathfrak{n}_2 = \mathrm{Lie}(N_2)$. By Lemma 5.8 and the comments preceding the lemma, for $X \in \mathfrak{g}_{\mathrm{reg}}$,

$$\hat\mu_\tau(X) = q^{-3}(q+1)(q^2+1)\int_{\mathfrak{n}_2 \cap \mathcal{O}_\tau}\int_K \psi(\mathrm{tr}(Xk^{-1}Yk))\,dk\,dY, \qquad \tau = \varepsilon,\ \varpi,\ \varepsilon\varpi \tag{7.2}$$

$$\hat\mu_0(X) = \tfrac{1}{2}q^{-1}(q+1)(q^2+1)\int_{F^*}\int_K |x|\psi(\mathrm{tr}\,Xk^{-1}(x(u_0-1))k))\,dk\,dx \tag{7.3}$$

Let $X = X_\varepsilon$ or $X = X_{\mathfrak{Q}}$ and $Y \in \mathfrak{n}_2$. By Proposition 6.3(1), $\eta_X(Y) = 0$ if $Y \notin \mathfrak{g}(\mathfrak{p}^{-t-1})$. If $Y = (Y_{ij})$, define $\delta = Y_{13}Y_{24} - Y_{14}Y_{23}$. From (6.5) and Lemma 6.9,

$$\eta_X(Y) = \begin{cases} \dfrac{-1-t_0(q-1)(2q+1)}{q(q-1)^2} & \text{if } \delta \in \varepsilon(F^*)^2 \text{ and } |\delta| = q^{2t+2}, \\ t_0, & \text{if } |\delta| \leqslant q^{2t+1} \text{ and } Y \notin \mathfrak{g}(\mathfrak{p}^{-t}), \\ 1, & \text{if } Y \in \mathfrak{g}(\mathfrak{p}^{-t}), \end{cases}$$

where $t_0 = (q^2+1)^{-1}$ if $X = X_\varepsilon$, and $t_0 = -(q^2-1)^{-1}$ if $X = X_{\mathfrak{Q}}$. (7.2) and (7.3) can now be evaluated to produce $\hat{\mu}_\tau$, $\tau = 0, \varepsilon, \varpi$ and $\varepsilon\varpi$. In fact it is clear from the values of η_X that $\hat{\mu}_\varpi(X) = \hat{\mu}_{\varepsilon\varpi}(X)$. Also, by Lemma 2.6(3),

$$\hat{\mu}_\varepsilon(X) + \hat{\mu}_\varpi(X) + \hat{\mu}_{\varepsilon\varpi}(X) = \Theta_{P_2}(\exp X) - \hat{\mu}_1(X)/2.$$

Thus we need only evaluate (7.2) for one of $\tau = \varepsilon$ and ϖ and then use this relation to solve for the other $\hat{\mu}_\tau(X)$. This will also be the case for $X = X_l$, $l = 0, 1, 2$.

Let $X = X_2$. Suppose $Y \notin \mathfrak{g}(\mathfrak{p}^{-t-1})$. If $\delta \notin (F^*)^2$, that is, $\exp Y \notin \mathcal{O}_1$, then $\eta_X(Y) = 0$. This is proved the same way as Proposition 6.3(1), using the fact that any nontrivial nilpotent element $Y' \in \mathfrak{g}$ which commutes with $\operatorname{diag}(a, a, -a, -a)$ satisfies $\exp Y' \in \mathcal{O}_1$. Note that if $Y \in \mathfrak{g}(\mathfrak{p}^{-t-1})$, $k \mapsto \psi(\operatorname{tr}(Xk^{-1}Yk))$ can be viewed as a function on $\mathbf{G}(\mathbf{F}_q) \simeq K_1 \backslash K$. Thus the Bruhat decomposition of $\mathbf{G}(\mathbf{F}_q)$ can be used in computing $\eta_X(Y)$, $Y \in \mathfrak{g}(\mathfrak{p}^{-t-1})$, as in Lemma 6.9 for the case X_ε or $X_{\mathfrak{Q}}$. Given $Y \in \mathfrak{g}(\mathfrak{p}^{-t-1}) - \mathfrak{g}(\mathfrak{p}^{-t})$, there exists $k \in K$ such that $k^{-1}Yk - \varpi^{-t-1}(u_\tau - 1) \in \mathfrak{g}(\mathfrak{p}^{-t})$ for some τ, so there is no loss of generality in taking $Y = \varpi^{-t-1}(u_\tau - 1)$.

$$\eta_X(\varpi^{-t-1}(u_\tau - 1)) = \begin{cases} \dfrac{q-1}{q^4-1} & \text{if } \tau = \varepsilon, \\ -\dfrac{q+1}{q^4-1} & \text{if } \tau = 1, \\ \dfrac{q^2-1}{q^4-1} & \text{if } \tau = 0, \varpi, \varepsilon\varpi. \end{cases}$$

$\hat{\mu}_\tau(X)$ can now be obtained by a simple calculation.

Let $X = X_0$ or X_1. For $Y \in \mathfrak{g}(\mathfrak{p}^{-t-1})$, $\eta_{X_0}(Y) = \eta_{X_1}(Y)$ can be computed using the methods described above for X_2:

$$\eta_X(\varpi^{-t-1}(u_\tau - 1)) = \begin{cases} \dfrac{q^2-1}{q^4-1} & \text{if } \tau = 1, \varepsilon, \\ \dfrac{q^3-1}{q^4-1} & \text{if } \tau = 0, \varpi, \varepsilon\varpi. \end{cases}$$

Let $Y \in \mathfrak{g}(\mathfrak{p}^{-l}) - \mathfrak{g}(\mathfrak{p}^{-l+1})$, $t + 2 \leqslant l \leqslant r$. Using an argument similar to that in the proof of Proposition 6.3(1), it can be shown that $\eta_X(Y) = 0$ if $|\delta| > q^{t+l+1}$. An explicit calculation involving the Bruhat decomposition of K yields:

$$\eta_X(Y) = \begin{cases} q^{-l+t+2}\dfrac{q-1}{q^4-1} & \text{if } |\delta| = q^{t+l+1}, \\ q^{-l+t+2}\dfrac{q^2-1}{q^4-1} & \text{if } |\delta| \leqslant q^{t+l}. \end{cases}$$

Finally, let $Y \in \mathfrak{g}(\mathfrak{p}^{-r-1}) - \mathfrak{g}(\mathfrak{p}^{-r})$. In this case, X_0 and X_1 must be considered separately. Another calculation results in:

$$\eta_X(Y) = \begin{cases} -q^{t-r+1}\dfrac{q+1}{q^4-1} & \text{if } X = X_1, \\ q^{t-r+1}\dfrac{q-1}{q^4-1} & \text{if } X = X_0, \end{cases}$$

if $|\delta| \leqslant q^{t+r+1}$, and zero otherwise. For $Y \notin \mathfrak{g}(\mathfrak{p}^{-r-1})$ it is easy to show that $\eta_X(Y) = 0$. We now have all of the values of η_X necessary for evaluating (7.2) and (7.3) to obtain $\hat{\mu}_\tau(X)$. □

8. Evaluation of coefficients

We conclude the paper with results concerning the coefficients in the two asymptotic expansions for Θ_π, both when π is an arbitrary supercuspidal representation (Theorem 8.1) and when π is one of the representations discussed in section 6 (Proposition 8.2, Theorem 8.3). Throughout this section, we assume that all measures are normalized as in section 5.

THEOREM 8.1. *Let π be a supercuspidal representation of G. If f is a finite sum of matrix coefficients of π such that $f(1) \neq 0$, then*

(1) $c_{id}(\pi) = \dfrac{d(\pi)}{d(\mathrm{St}_G)}$.

(2) $c_0(\pi) = \dfrac{d(\pi) J_G(\mathcal{O}_0, f)}{(q-1) d(\mathrm{St}_{M_2}) f(1)}$.

(3) $\dfrac{(q-1)}{(q^3-1)}\left[(q^2+1)c_\varepsilon(\pi) + \tfrac{1}{2}(q+1)^2(c_\varpi(\pi) + c_{\varepsilon\varpi}(\pi))\right] = \dfrac{d(\pi) J_{M_2}(1, f)}{d(\mathrm{St}_{M_2}) f(1)}$.

(4) $-2c_1(\pi) + c_\varepsilon(\pi) + c_\varpi(\pi) + c_{\varepsilon\varpi}(\pi) = \dfrac{d(\pi)}{f(1)}\left[\dfrac{J_{M_2}(1, f)}{d(\mathrm{St}_{M_2})} - \dfrac{J_{M_1}(1, f)}{d(\mathrm{St}_{M_1})}\right]$.

(5) $c_R(\pi) = \dfrac{d(\pi)}{8f(1)} J_{M_0}(1, f)$.

Proof. (1) and (5) are proved in Proposition 4.1.

Let X_1 and X_2 be as in section 7. Define $\sigma = \exp(\mathrm{diag}(a, a, -a, -a))$, where $a \in F$ and $|a| = q^{-t}$. Then $\sigma \in A_{M_2,t}$. Let $\gamma = \exp X_2$. If $|b|$ is sufficiently small, then, by Theorem 3.2(1) and the remark at the beginning of the proof of Lemma 4.1,

$$\Theta_\pi(\gamma) = \frac{d(\pi)|D_{M_2}(\gamma)|^{1/2}}{d(\mathrm{St}_{M_2})f(1)|D(\gamma)|^{1/2}} J_{M_2}(\sigma, f).$$

To see that the terms corresponding to $M = G$ are zero, apply Lemma 3.3(2) and (3). Using Proposition 5.5(2) for $J_{M_2}(\sigma, f)$ and noting that $|D_{M_2}(\gamma)|^{1/2}|D(\gamma)|^{-1/2} = q^{3t}$, $\Theta_\pi(\gamma)$ becomes

$$\Theta_\pi(\gamma) = \frac{d(\pi)}{d(\mathrm{St}_{M_2})}\left[(q^3 - 1)^{-1} + \frac{J_G(\mathcal{O}_0, f)q^{2t}}{(q-1)f(1)} + \frac{J_{M_2}(1, f)q^{3t}}{f(1)}\right].$$

If t is large, then Harish-Chandra's expansion for Θ_π around 1 is valid at γ. By Lemma 7.1, it is equal to

$$\Theta_\pi(\gamma) = c_{id}(\pi) + 2c_0(\pi)\frac{q^{2t+2}}{(q^2+1)}$$
$$+\left[(q^2+1)c_\varepsilon(\pi) + \frac{1}{2}(q+1)^2(c_\varpi(\pi) + c_{\varepsilon\varpi}(\pi))\right]\frac{(q-1)q^{3t}}{q^3-1}.$$

Equating terms in the two expressions for $\Theta_\pi(\gamma)$ yields (2) and (3) of the statement of the theorem.

To obtain (4), define $\sigma = \exp(\mathrm{diag}(a, 0, 0, -a))$, $|a| = q^{-t}$, and $\gamma = \exp X_1$. Replace M_2 by M_1 and repeat the argument used for (3), except that Proposition 5.5(4) should be used for $J_{M_1}(\sigma, f)$. This results in

$$\frac{d(\pi)J_{M_1}(1, f)}{d(\mathrm{St}_{M_1})f(1)} = 2c_1(\pi) - \frac{(q-1)}{q^3-1}\left[qc_\varepsilon(\pi) + \frac{q^2+1}{2}(c_\varpi(\pi) + c_{\varepsilon\varpi}(\pi))\right].$$

Subtracting this expression from (3) yields (4). □

PROPOSITION 8.2. *Let π be an irreducible supercuspidal representation containing a non-degenerate representation of K_j, $j \geqslant 0$. Choose f as described* *preceding Proposition* 6.3 *and let cases* 1, 2 *and* 3 *be as defined in remark* (2) *following Lemma* 6.7.

	Case 1	Case 2	Case 3
$\frac{d(\pi)}{d(\mathrm{St}_G)}$	$\frac{2q^{4j}(q-1)(q^2+1)}{q^3-1}$	$\frac{2q^{4j}(q+1)(q^2-1)}{q^3-1}$	$\frac{q(q-1)}{q^3-1}$
$\frac{d(\pi)}{f(1)} J_G(\mathcal{O}_0, f)$	$2q^{2j}(q-1)^2$	0	$\frac{-(q-1)^2}{2}$
$\frac{d(\pi)}{f(1)} J_G(\mathcal{O}_\varepsilon, f)$	$\frac{-q^j(q-1)^2(q^2-1)}{q^3-1}$	$\frac{q^j(q-1)^2(q^2-1)}{q^3-1}$	$\frac{(q-1)^2(q^2-1)}{2(q^3-1)}$
$\frac{d(\pi)}{f(1)d(\mathrm{St}_{M_2})} J_{M_2}(1, f)$	$\frac{-2q^j(q-1)(3q^2+2q+3)}{q^3-1}$	$\frac{-2q^j(q+1)(q^2-1)}{q^3-1}$	$\frac{(q-1)(q^2+1)}{q^3-1}$
$\frac{d(\pi)}{f(1)d(\mathrm{St}_{M_1})} J_{M_1}(1, f)$	$\frac{-2q^j(q-1)(q^2+1)}{q^3-1}$	$\frac{-2q^j(q+1)(q^2-1)}{q^3-1}$	$\frac{-q(q-1)}{q^3-1}$
$\frac{d(\pi)}{8f(1)} J_{M_0}(1, f)$	1	1	0

In addition, $J_G(\mathcal{O}_\varpi, f) = J_G(\mathcal{O}_{\varepsilon\varpi}, f) = -\frac{1}{2} J_G(\mathcal{O}_\varepsilon, f)$.

Proof. For $j \geqslant 1$ let u_τ^j, $\tau = 0, 1, \varepsilon, R$ be defined as in the proof of Proposition 6.3. Let $u_\tau^0 = u_\tau$. Recall $t_\tau = f_K(u_\tau^j)/f(1)$, $\tau = 0, 1, \varepsilon, R$. Note that $f_K(1 + \varpi^j(u_\tau - 1)) = f_K(u_0^j)$, $\tau = \varpi, \varepsilon\varpi$, $j \geqslant 0$. The above integrals can be evaluated and expressed in terms of the constants t_τ, using the formulas for the unipotent orbital integrals given at the end of section 5 and using Proposition 5.5 for $J_M(1, f)$. The details are omitted. The procedure is straightforward. In the case of $J_{M_0}(1, f)$, the calculation is simplified by the observation that $\int_N f_K(u)v(w)^2\,du$ and $\int_N f_K(u)v(z)^2\,du$ are zero.

The second step is to use (6.5) to express each integral in terms of t_0. This results in:

$$\frac{J_G(\mathcal{O}_0, f)}{f(1)} = q^{-2j-2}(q^2-1)\left[t_0 + \frac{1}{q^2-1}\right]$$

$$\frac{J_G(\mathcal{O}_\varepsilon, f)}{f(1)} = -q^{-3j-2}(q-1)\left[t_0 + \frac{1}{q^3-1}\right]$$

$$\frac{J_G(\mathcal{O}_\tau, f)}{f(1)} = \frac{1}{2}q^{-3j-2}(q-1)\left[t_0 + \frac{1}{q^3-1}\right], \qquad \tau = \varpi, \varepsilon\varpi$$

$$\frac{J_{M_2}(1, f)}{f(1)} = -q^{-3j-2}\left[(q+1)t_0 + \frac{2q^2+q+1}{q^3-1}\right]$$

$$\frac{J_{M_1}(1,f)}{f(1)} = -\frac{\sqrt{2}q^{-3j}}{q^3-1}$$

$$\frac{J_{M_0}(1,f)}{f(1)} = 8q^{-4j-1}\left[\frac{1+(q-1)t_0}{(q-1)(q^2-1)}\right]$$

$d(\mathrm{St}_M)$ depends on $\mathrm{vol}(A_M\backslash(K\cap M)A_M) = [\mathrm{vol}\,\kappa_M]^{-1}$. Recall that $\mathrm{vol}\,\kappa_M$ is given in Lemma 5.1. Thus

$$d(\mathrm{St}_G) = \tfrac{1}{2}(q^3-1)(q-1)[\mathrm{vol}\,\kappa_G] = \tfrac{1}{2}(q^3-1)(q-1)$$

$$d(\mathrm{St}_{M_1}) = \tfrac{1}{2}(q-1)[\mathrm{vol}\,\kappa_{M_1}] = \frac{1}{\sqrt{2}}(q-1)$$

$$d(\mathrm{St}_{M_2}) = \tfrac{1}{2}(q-1)[\mathrm{vol}\,\kappa_{M_2}] = \tfrac{1}{2}(q-1)$$

Using Lemma 6.7 for the value of $d(\pi)$ and observing from Proposition 6.3 and Lemma 6.6 that $t_0 = (q^2+1)^{-1}$, $-(q^2-1)^{-1}$ and $-(q-1)^{-1}$ in cases 1, 2 and 3, respectively, completes the proof. □

THEOREM 8.3. *Let* π *and* f *be as in Proposition* 8.2.

	Case 1	Case 2	Case 3
$c_{id}(\pi)$	$\frac{2q^{4j}(q-1)(q^2+1)}{q^3-1}$	$\frac{2q^{4j}(q+1)(q^2-1)}{q^3-1}$	$\frac{q(q-1)}{q^3-1}$
$c_0(\pi)$	$4q^{2j}$	0	-1
$c_1(\pi)$	$-2q^j$	$-2q^j$	0
$c_\varepsilon(\pi)$	$-4q^j$	0	1
$c_\varpi(\pi)+c_{\varepsilon\varpi}(\pi)$	$-4q^j$	$-4q^j$	0
$c_R(\pi)$	1	1	0

Proof. $c_{id}(\pi)$ and $c_R(\pi)$ are given by Theorem 8.1(1), (5) and Proposition 8.2. Similarly, $c_0(\pi)$ is given by Theorem 8.1(2) and Proposition 8.2.

Let X_0, X_ε and $X_\mathfrak{Q}$ be as in section 7.

Let $X = X_\varepsilon$ or $X_\mathfrak{Q}$. Note (see Lemma 7.1) that $\hat{\mu}_\varpi(X) = \hat{\mu}_{\varepsilon\varpi}(X) = -\hat{\mu}_\varepsilon(X)/2 \neq 0$ and $J_G(\mathcal{O}_\varpi, f) = J_G(\mathcal{O}_{\varepsilon\varpi}, f) = -J_G(\mathcal{O}_\varepsilon, f)/2 \neq 0$. Let Γ_τ denote the Shalika germ corresponding to $\mathcal{O}_\tau \in (\mathcal{U}_G)$. Lemma 4.1(3) implies

$$c_\varepsilon(\pi) - c_\varpi(\pi)/2 - c_{\varepsilon\varpi}(\pi)/2$$

$$= \frac{d(\pi)}{f(1)} J_G(\mathcal{O}_\varepsilon, f)\hat{\mu}_\varepsilon(X)^{-1}[\Gamma_\varepsilon(\exp X) - \tfrac{1}{2}\Gamma_\varpi(\exp X) - \tfrac{1}{2}\Gamma_{\varepsilon\varpi}(\exp X)]$$

Define

$$\lambda = \hat{\mu}_\varepsilon(X)^{-1}[\Gamma_\varepsilon(\exp X) - \tfrac{1}{2}\Gamma_\varpi(\exp X) - \tfrac{1}{2}\Gamma_{\varepsilon\varpi}(\exp X)].$$

Then $c_\varepsilon(\pi) + c_\varpi(\pi)/2 + c_{\varepsilon\varpi}(\pi)/2 = \lambda d(\pi)J_G(\mathcal{O}_\varepsilon, f)/f(1)$ can be combined with Theorem 8.1(3) and (4) to produce the following:

$$\begin{aligned}
c_1(\pi) &= \frac{d(\pi)}{f(1)}\left[\frac{J_{M_1}(1,f)}{d(\mathrm{St}_{M_1})} + \left(\frac{-\lambda(q-1)^3}{4(q^3-1)} + \frac{q+1}{2(q-1)^2}\right) J_G(\mathcal{O}_\varepsilon, f)\right] \\
c_\varepsilon(\pi) &= \frac{d(\pi)}{f(1)}\left[\frac{J_{M_1}(1,f)}{d(\mathrm{St}_{M_1})} + \left(\frac{\lambda(q-1)(q+1)^2}{2(q^3-1)} + \frac{q+1}{(q-1)^2}\right) J_G(\mathcal{O}_\varepsilon, f)\right] \qquad (8.4) \\
&c_\varpi(\pi) + c_{\varepsilon\varpi}(\pi) \\
&= \frac{2d(\pi)}{f(1)}\left[\frac{J_{M_1}(1,f)}{d(\mathrm{St}_{M_1})} + \left(\frac{-\lambda(q-1)(q^2+1)}{2(q^3-1)} + \frac{q+1}{(q-1)^2}\right) J_G(\mathcal{O}_\varepsilon, f)\right]
\end{aligned}$$

Here we have used the fact that

$$\frac{J_{M_2}(1,f)}{d(\mathrm{St}_{M_2})} = \frac{2J_{M_1}(1,f)}{d(\mathrm{St}_{M_1})} + \frac{2(q+1)J_G(\mathcal{O}_\varepsilon, f)}{(q-1)^2},$$

which can be seen from Proposition 8.2.

The next step is to find λ. Let $\sigma \in A_{M_1,t} \cap K_1$. Using Lemma 6.6 and Proposition 5.5(6) we compute $J_{M_0}(\sigma, f)$ in case 3. The calculation is lengthy, but many parts of the integral cancel and the answer is simple:

$$J_{M_0}(\sigma, f) = q^{-t} - q^{-2t} \quad \text{(case 3)}$$

If r and t are sufficiently large and $\gamma = \exp(X_0)$, then

$$\begin{aligned}
\Theta_\pi(\gamma) &= \frac{d(\pi)}{f(1)}|D(\gamma)|^{-1/2}\left[J_{M_0}(\sigma, f) + \frac{J_{M_1}(\sigma, f)}{d(\mathrm{St}_{M_1})}|D_{M_1}(\gamma)|^{1/2}\right] \\
&= q^{2t+r} - q^{t+r} + \frac{q(q-1)}{q^3-1}(-q^{3t}+1)
\end{aligned}$$

Here we have used Proposition 5.5(3) and Proposition 8.2 for the value of $J_{M_1}(\sigma, f)/d(\mathrm{St}_{M_1})$. If X_0 is close to 0, then Harish-Chandra's expansion around 1 holds. Using Lemma 7.1 for the values of $\hat{\mu}_\tau$ at X_0 and equating the coefficients of q^{2t+r} in the two expansions, we conclude that $2c_1(\pi) + \sum_{\tau=\varepsilon,\varpi,\varepsilon\varpi} c_\tau(\pi) = 1$ in case 3. Subtracting Theorem 8.1(4) (after substituting for $d(\pi)J_{M_1}(1,f)/$

$d(\mathrm{St}_{M_l})f(1)$, $l = 1, 2$), we obtain $c_1(\pi) = 0$ in case 3. It now follows from (8.4) and Proposition 8.2 that

$$\lambda = \frac{2(q^3 - 1)}{(q - 1)^2(q^2 - 1)}.$$

The proof is completed upon substitution of λ and the constants computed in Proposition 8.2 into (8.4). □

Given $\mathcal{O} \in (\mathcal{U}_G)$, Rodier [Ro] and Moeglin and Waldspurger [MW] have defined a quotient W of the representation space of π in terms of a pair (X, φ), $\exp X \in \mathcal{O}$ and φ a one-parameter subgroup which satisfies certain conditions. In [Ro], $\mathcal{O}$ is regular, and W is called a Whittaker model, and in [MW], $\mathcal{O}$ may not be regular, in which case W is called a degenerate Whittaker model. We say that π admits a (degenerate) Whittaker model relative to $\mathcal{O}$ if the corresponding W is non-zero. In some cases, $c_{\mathcal{O}}(\pi)$ if the dimension of a Whittaker model.

COROLLARY 8.5. (1) *In cases* 1 *and* 2, π *admits a Whittaker model of dimension* 1 *relative to* $\mathcal{O}_R$.

(2) In case 3, π *does not admit a Whittaker model relative to* $\mathcal{O}_R$.

(3) *In case* 3, π *admits a degenerate Whittaker model of dimension* 1 *relative to* $\mathcal{O}_\varepsilon$ *and does not admit a degenerate Whittaker model relative to* $\mathcal{O}_\tau$, $\tau = 1, \varpi, \varepsilon\varpi$.

Proof. (1) and (2) are immediate from Theorem 8.3 and [Ro]. To prove (3), note that in case 3, because $c_R(\pi) = 0$, results of [MW] imply that, up to a nonzero constant depending on normalizations of measures, $c_\tau(\pi)$ is the dimension of a degenerate Whittaker model relative to $\mathcal{O}_\tau$, $\tau = 1, \varepsilon, \varpi, \varepsilon\varpi$. Thus $c_\varpi(\pi) + c_{\varepsilon\varpi}(\pi) = 0$ implies $c_\varpi(\pi) = c_{\varepsilon\varpi}(\pi) = 0$. Upon comparison of the normalization of μ_ε in section 5 with that in [MW], (3) follows.

References

[A1] J. Arthur, The trace formula in invariant form, *Ann. of Math.* 114 (1981), 1–74.

[A2] J. Arthur, The local behaviour of weighted orbital integrals, *Duke Math. J.* 56 (1988), 223–293.

[A3] J. Arthur, The characters of supercuspidal representations as weighted orbital integrals, *Proc. Indian Acad. Sci.* 97 (1988), 3–19.

[D] G. van Dijk, Computation of certain induced characters of p-adic groups, *Math. Ann.* 199 (1972), 229–240.

[HC] Harish-Chandra, "Admissible distributions on reductive p-adic groups," in *Lie Theories and Their Applications*, Queen's Papers in Pure and Applied Mathematics, Queen's University, Kingston, Ontario (1978), 281–347.

[J] D. Jabon, *The supercuspidal representations of* U(2, 1) *and* $\mathbf{GSp}_4$ *over a local field via Hecke algebra isomorphisms*, thesis, University of Chicago, 1989.

[LS] G. Lusztig and N. Spaltenstein, Induced unipotent classes, *J. London Math. Soc.* 19 (1979), 41–52.

[MW] C. Moeglin and J. L. Waldspurger, Modèles de Whittaker dégénérés pour des groupes p-adiques, *Math. Z.* 196 (1987), 427–452.

[M] L. Morris, *Tamely ramified supercuspidal representations of some classical groups*, preprint, 1988.

[Mo1] A. Moy, Representations of **GSp**(4) over a p-adic field: part 1, *Comp. Math.* 66 (1988), 237–284.

[Mo2] A. Moy, Representations of **GSp**(4) over a p-adic field: part 2, *Comp. Math.* 66 (1988), 285–328.

[Mu] F. Murnaghan, Asymptotic behaviour of supercuspidal characters of p-adic $\mathbf{GL}_3$ and $\mathbf{GL}_4$: the generic unramified case, *Pacific J. Math*, to appear.

[R] R. Ranga Rao, Orbital integrals in reductive groups, *Ann. of Math.* 96 (1972), 505–510.

[Re] S. Reid, thesis, University of Warwick, 1974.

[Ro] F. Rodier, "Modèle de Whittaker et caractères de reprèsentations," in *Non Commutative Harmonic Analysis*, Lecture Notes in Math. 466, Springer-Verlag, Berlin, Heidelberg, New York (1974), 151–171.

[Rog] J. Rogawski, An application of the building to orbital integrals, *Comp. Math.* 42 (1981), 417–423.

[S] P. J. Sally, Jr., in preparation.

[ST] P. J. Sally, Jr. and M. H. Taibleson, *Special functions on locally compact fields*, *Acta Math.* 116 (1966), 279–309.